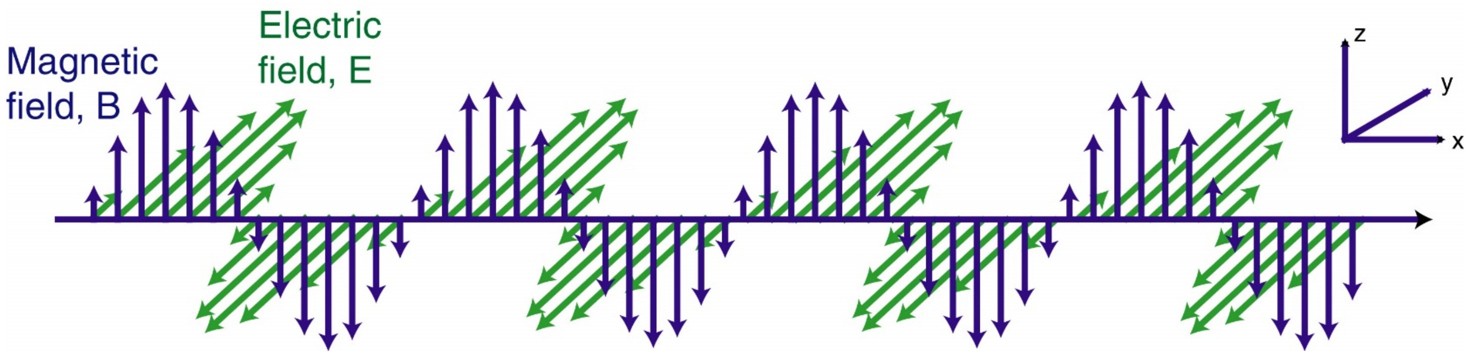
3. Maxwell's Equations and Light Waves



Vector fields, vector derivatives and the 3D Wave equation

Derivation of the wave equation from Maxwell's Equations

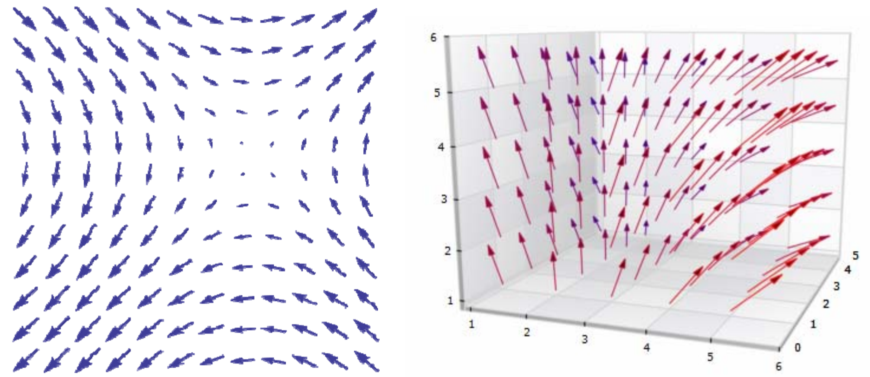
Why light waves are transverse waves

Why is the B-field so much ‘smaller’ than the E-field (and what that really means)

Vector fields

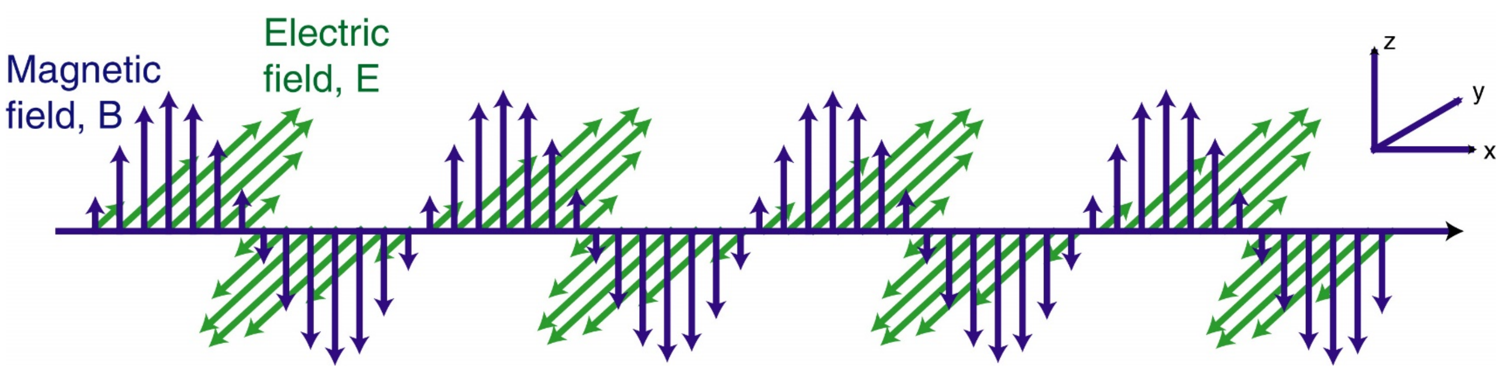
Light is a pair of 3D vector fields.

A 3D vector field $\vec{f}\,(\vec{r}\,)$ assigns a vector (i.e., an arrow having both direction and length) to each point $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ in 3D space.



Attempts to graphically represent vector fields in 2D and in 3D

A light wave has both electric and magnetic 3D vector fields.



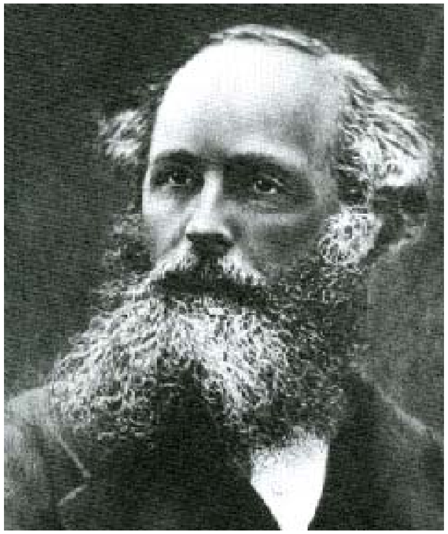
In this illustration, the vectors of the two fields are only shown at a few selected locations, equally spaced along a line. But the fields are defined at every point (x,y,z).

The equations of optics are Maxwell's equations.

(first written down in 1864)

$$\vec{\nabla} \cdot \vec{E} = \rho / \varepsilon \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu \vec{J} + \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$$


James Clerk Maxwell (1831-1879)

where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, ρ is the charge density, $\vec{J}$ is the current density, ε is the permittivity, and μ is the permeability of the medium.

Often, in optics, there are no free charges or currents, so mostly we can assume that $\rho = 0$ and $J = 0$.

Historical digression

$$\left.\begin{aligned} p' &= p + \frac{df}{dt} \\ q' &= q + \frac{dg}{dt} \\ r' &= r + \frac{dh}{dt} \end{aligned}\right\} \rightarrow \left.\begin{aligned} J_1 &= j_1 + \frac{\partial D_1}{\partial t} \\ J_2 &= j_2 + \frac{\partial D_2}{\partial t} \\ J_3 &= j_3 + \frac{\partial D_3}{\partial t} \end{aligned}\right\} \Rightarrow \mathbf{J} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \qquad (1.1)$$

$$\left.\begin{aligned} \mu\alpha &= \frac{dH}{dy} - \frac{dG}{dz} \\ \mu\beta &= \frac{dF}{dz} - \frac{dH}{dx} \\ \mu\gamma &= \frac{dG}{dx} - \frac{dF}{dy} \end{aligned}\right\} \rightarrow \left.\begin{aligned} \mu H_1 &= \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \\ \mu H_2 &= \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \\ \mu H_3 &= \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{aligned}\right\} \Rightarrow \mu\mathbf{H} = \nabla \times \mathbf{A} \qquad (1.2)$$

$$\left.\begin{aligned} \frac{d\gamma}{dy} - \frac{d\beta}{dz} &= 4\pi p' \\ \frac{d\alpha}{dz} - \frac{d\gamma}{dx} &= 4\pi q' \\ \frac{d\beta}{dx} - \frac{d\alpha}{dy} &= 4\pi r' \end{aligned}\right\} \rightarrow \left.\begin{aligned} \frac{\partial H_3}{\partial y} - \frac{\partial H_2}{\partial z} &= 4\pi J_1 \\ \frac{\partial H_1}{\partial z} - \frac{\partial H_3}{\partial x} &= 4\pi J_2 \\ \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} &= 4\pi J_3 \end{aligned}\right\} \Rightarrow \nabla \times \mathbf{H} = \mathbf{J} \qquad (1.3)$$

$$\left.\begin{aligned} P &= \mu\left(\gamma\frac{dy}{dt} - \beta\frac{dz}{dt}\right) - \frac{dF}{dt} - \frac{d\Psi}{dx} \\ Q &= \mu\left(\alpha\frac{dz}{dt} - \gamma\frac{dx}{dt}\right) - \frac{dG}{dt} - \frac{d\Psi}{dy} \\ R &= \mu\left(\beta\frac{dx}{dt} - \alpha\frac{dy}{dt}\right) - \frac{dH}{dt} - \frac{d\Psi}{dz} \end{aligned}\right\} \rightarrow \left.\begin{aligned} E_1 &= \mu(H_3 v_2 - H_2 v_3) - \frac{dA_1}{dt} - \frac{d\varphi}{dx} \\ E_2 &= \mu(H_1 v_3 - H_3 v_1) - \frac{dA_2}{dt} - \frac{d\varphi}{dy} \\ E_3 &= \mu(H_2 v_1 - H_1 v_2) - \frac{dA_3}{dt} - \frac{d\varphi}{dz} \end{aligned}\right\} \qquad (1.4)$$

$$\Rightarrow \mathbf{E} = \mu(\mathbf{v} \times \mathbf{H}) - \frac{\partial \mathbf{A}}{\partial t} - \nabla\varphi$$

$$\left.\begin{aligned} P &= kf \\ Q &= kg \\ R &= kh \end{aligned}\right\} \rightarrow \left.\begin{aligned} \varepsilon E_1 &= D_1 \\ \varepsilon E_2 &= D_2 \\ \varepsilon E_3 &= D_3 \end{aligned}\right\} \Rightarrow \varepsilon\mathbf{E} = \mathbf{D} \qquad (1.5)$$

$$\left.\begin{aligned} P &= -\zeta p \\ Q &= -\zeta q \\ R &= -\zeta r \end{aligned}\right\} \rightarrow \left.\begin{aligned} \sigma E_1 &= j_1 \\ \sigma E_2 &= j_2 \\ \sigma E_3 &= j_3 \end{aligned}\right\} \Rightarrow \sigma\mathbf{E} = \mathbf{j} \qquad (1.6)$$

$$e + \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0 \rightarrow \rho + \frac{\partial D_1}{\partial x} + \frac{\partial D_2}{\partial y} + \frac{\partial D_3}{\partial z} = 0 \Rightarrow -\rho = \nabla \cdot \mathbf{D} \qquad (1.7)$$

$$\frac{de}{dt} + \frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz} = 0 \rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial j_1}{\partial x} + \frac{\partial j_2}{\partial y} + \frac{\partial j_3}{\partial z} = 0 \Rightarrow -\frac{\partial \rho}{\partial t} = \nabla \cdot \mathbf{j} \qquad (1.8)$$

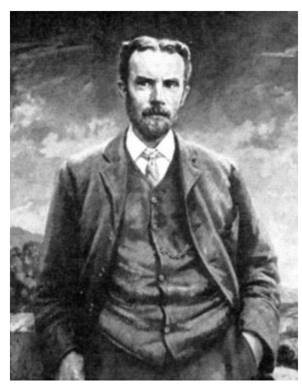
original form of
Maxwell's equations

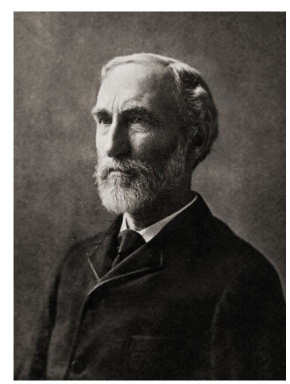
In Maxwell's original notation, the equations were not nearly so compact and easy to understand.

But, he was able to derive a value for the speed of light in empty space, which was within 5% of the correct answer.

The modern vector notation was introduced by Oliver Heaviside and Willard Gibbs in 1884. Heaviside is responsible for the currently accepted form of Maxwell's equations.

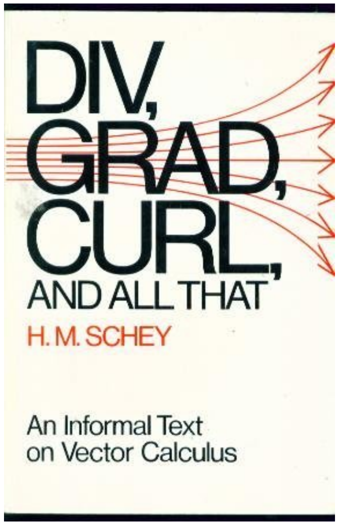
Oliver Heaviside
(1850 - 1925)





J. Willard Gibbs
(1839 - 1903)

Div, Grad, Curl, and all that



Types of 3D derivatives: "vector derivatives" that appear in Maxwell's equations

The "Del" operator: $\vec{\nabla} \equiv \left(\frac{\partial}{\partial x} , \frac{\partial}{\partial y} , \frac{\partial}{\partial z} \right)$

The "Gradient" of a scalar function f:

$$\vec{\nabla} f \equiv \left(\frac{\partial f}{\partial x} , \frac{\partial f}{\partial y} , \frac{\partial f}{\partial z} \right)$$

The gradient points in the direction of steepest ascent.

The "Divergence" of a vector function $\vec{G}$:

$$\vec{\nabla} \cdot \vec{G} \equiv \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}$$

Div, Grad, Curl, and more all that

The "Laplacian" of a scalar function :

$$\nabla^2 f \quad \equiv \quad \vec{\nabla} \cdot \vec{\nabla} f \quad = \quad \vec{\nabla} \cdot \left(\frac{\partial f}{\partial x} \ , \ \frac{\partial f}{\partial y} \ , \ \frac{\partial f}{\partial z} \right)$$

$$= \quad \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

The Laplacian of a vector function is the same,
but for each component :

$$\nabla^2 \vec{G} = \left(\frac{\partial^2 G_x}{\partial x^2} + \frac{\partial^2 G_x}{\partial y^2} + \frac{\partial^2 G_x}{\partial z^2} \ , \ \frac{\partial^2 G_y}{\partial x^2} + \frac{\partial^2 G_y}{\partial y^2} + \frac{\partial^2 G_y}{\partial z^2} , \frac{\partial^2 G_z}{\partial x^2} + \frac{\partial^2 G_z}{\partial y^2} + \frac{\partial^2 G_z}{\partial z^2} \right)$$

The Laplacian is related to the curvature of a function.

Div, Grad, Curl, and still more all that

The "Curl" of a vector function $\vec{G}$:

$$\vec{\nabla} \times \vec{G} \quad \equiv \quad \left(\frac{\partial G_z}{\partial y} - \frac{\partial G_y}{dz},\ \frac{\partial G_x}{\partial z} - \frac{\partial G_z}{dx},\ \frac{\partial G_y}{\partial x} - \frac{\partial G_x}{dy} \right)$$

The curl can computed from a matrix determinant :

$$\vec{\nabla} \times \vec{G} = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_x & G_y & G_z \end{bmatrix}$$

The curl measures the microscopic circulation of the vector field

(which is not the same as the macroscopic circulation).

See http://mathinsight.org/curl_idea for more information.

Generalizing from 1D to 3D: A vector wave equation for the electric field

in one dimension:

$$\frac{\partial^2 E}{\partial x^2} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0$$

in three dimensions:

$$\vec{\nabla}^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

Note the vector symbol over the E.

Expanding the Laplacian, we find:

$$\frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

This is really <u>three</u> independent wave equations, one each for the x-, y-, and z-components of E.

Derivation of the 3D Wave Equation from Maxwell's Equations

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$$

First: take $\vec{\nabla} \times$ of this one: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = \vec{\nabla} \times [-\frac{\partial \vec{B}}{\partial t}]$$

Next: change the order of differentiation on the right-hand side:

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\frac{\partial}{\partial t}[\vec{\nabla} \times \vec{B}]$$

Derivation of the Wave Equation from Maxwell's Equations (cont'd)

But: $$\vec{\nabla} \times \vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$$

Next: substituting for $\vec{\nabla} \times \vec{B}$, we have:

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\frac{\partial}{\partial t}[\vec{\nabla} \times \vec{B}] \quad \Rightarrow \quad \vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\frac{\partial}{\partial t}[\mu\varepsilon \frac{\partial \vec{E}}{\partial t}]$$

Or: $$\vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

assuming that μ and ε are both independent of time.

Derivation of the Wave Equation from Maxwell's Equations (cont'd)

We are up to here: $\vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\mu\varepsilon \dfrac{\partial^2 \vec{E}}{\partial t^2}$

Now, it can be shown that this: $\vec{\nabla} \times [\vec{\nabla} \times \vec{E}]$

is the same as this: $\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E}$

For any function at all,

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - \nabla^2 \vec{F} = \vec{\nabla} \times \left[\vec{\nabla} \times \vec{F} \right]$$

See: https://www.youtube.com/watch?v=P4edqL1r4DQ

If we now assume zero charge density: $\rho = 0$, then

$$\vec{\nabla} \cdot \vec{E} \quad = \quad 0$$

and we're left with the Wave Equation! $\nabla^2 \vec{E} = \mu\varepsilon \dfrac{\partial^2 \vec{E}}{\partial t^2}$

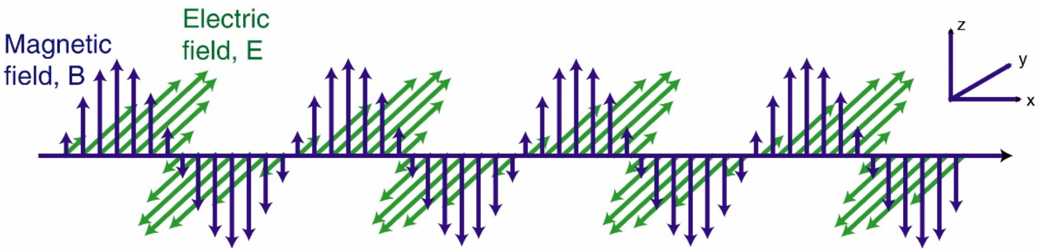
There are really two Wave Equations, one for *E* and one for *B*

We could also derive a wave equation for the magnetic field, using a very similar approach. The result is the same:

$$\vec{\nabla}^2 \vec{B} - \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

$\vec{E}$ and $\vec{B}$ satisfy the same equation. But that doesn't mean they're equal.

In fact, we will show that $\vec{E}$ and $\vec{B}$ are always perpendicular to each other.



An interesting aside

We only used 3 out of the 4 Maxwell equations to derive the wave equation for E.

$$\vec{\nabla} \cdot \vec{E} = \rho / \varepsilon \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

we assumed $\rho = 0$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu \vec{J} + \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$$

We never needed this one!

we assumed $J = 0$

Some physicists believe that this 4th equation may need to be modified one day, if anybody ever finds conclusive experimental evidence for the existence of magnetic 'charges'.

Doing so would be a very big deal, although it would not change the wave equation at all.

Waves in 3 dimensions

We must now allow the complex field $\underset{\sim}{E}$ and its amplitude $\underset{\sim}{E}_0$ to be vectors.

So our expression from Lecture 2: $\underset{\sim}{E}(x,t) = \underset{\sim}{E}_0 \exp\left[j\left(kx - \omega t\right)\right]$

becomes: $$\underset{\sim}{\vec{E}}(\vec{r},t) = \underset{\sim}{\vec{E}}_0 \exp\left[j\left(\vec{k}\cdot\vec{r} - \omega t\right)\right]$$

Note the arrows over the E's!

Notation: $\vec{r} \equiv x\hat{x} + y\hat{y} + z\hat{z}$

A complex vector amplitude has six numbers that must be specified to completely determine it!

$$\underset{\sim}{\vec{E}}_0 = (\mathrm{Re}\{E_x\} + j\,\mathrm{Im}\{E_x\}, \mathrm{Re}\{E_y\} + j\,\mathrm{Im}\{E_y\}, \mathrm{Re}\{E_z\} + j\,\mathrm{Im}\{E_z\})$$

Note that the quantity $\underset{\sim}{\vec{E}}(\vec{r},t)$ is now a function of FOUR variable: $\vec{r}$ and t.

For a 3D wave, which way is it traveling?

Well, for a wave in 1 dimension, it was easy to tell.

$$E(z,t) = E_0 \exp\left[j(kz - \omega t) \right]$$

This wave is traveling in the positive z direction.

For a wave in 3 dimensions, we have a vector which specifies the variable:

not just an x, y, or z, but an $\vec{r}$!

So it isn't surprising that k becomes a vector too. And its direction is the direction along which the wave is traveling.

dot product!

$$\vec{E}(\vec{r},t) = \vec{E}_0 \exp\left[j\left(\vec{k}\cdot\vec{r} - \omega t\right) \right] = \vec{E}_0 \exp\left[j\left(k_x x + k_y y + k_z z - \omega t\right) \right]$$

Note: the length of $\vec{k}$ still has the same meaning as it did in Lecture 2:

$$\left|\vec{k}\right| = \frac{2\pi}{\lambda} = \frac{\omega}{c}$$

A common mistake

Question: write down an expression for the electric field component of an electromagnetic wave.

Incorrect answer: $\vec{E}\left(\vec{r},t\right)=\vec{E}_0 \exp\left[\,j\left(k\vec{r}-\omega t\right)\right]$

Another incorrect answer: $\vec{E}\left(\vec{r},t\right)=\vec{E}_0 \exp\left[\,j\left(\vec{k}r-\omega t\right)\right]$

And another: $\vec{E}\left(\vec{r},t\right)=\vec{E}_0 \exp\left[\,j\left(kr-\omega t\right)\right]$

Remember: you cannot have a vector in the exponent!

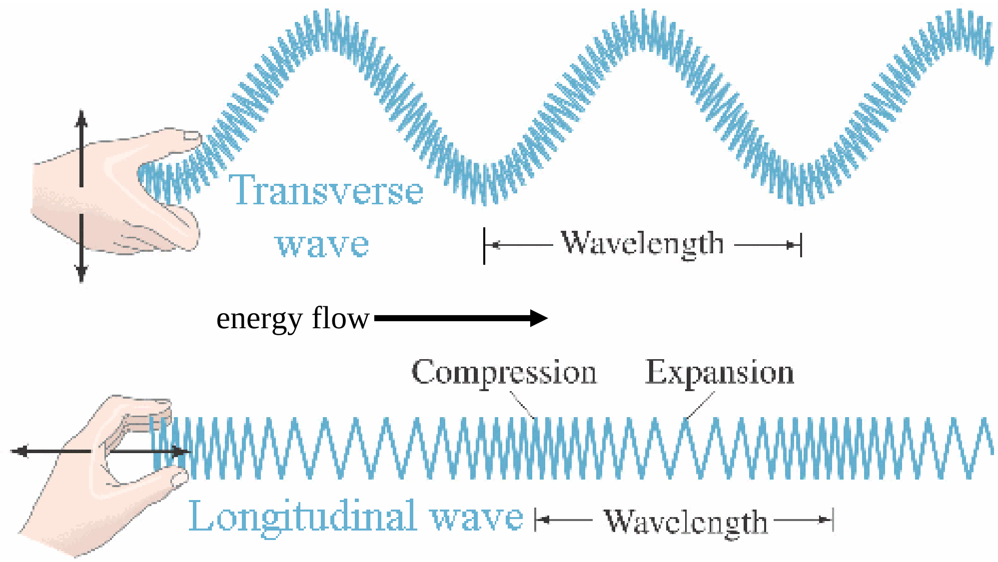
Correct answer: $\vec{E}\left(\vec{r},t\right)=\vec{E}_0 \exp\left[\,j\left(\boxed{\vec{k}\cdot\vec{r}}-\omega t\right)\right]$

this is not a vector

Longitudinal vs. Transverse waves

Transverse:

Longitudinal:

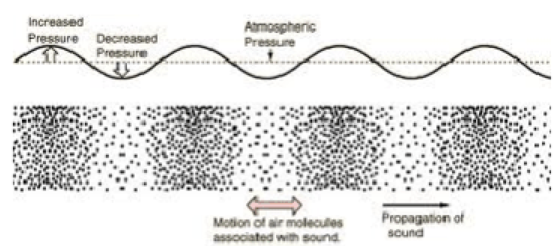


Motion is perpendicular to the direction of propagation

Motion is along the direction of propagation

Space has 3 dimensions, of which 2 directions are transverse to the propagation direction, so there are 2 different kinds of transverse waves in addition to the longitudinal one.

e.g., sound waves are longitudinal



Why light waves are transverse

Suppose a wave propagates in the x-direction. Then it's a function of x and t (and not y or z), so all y- and z-derivatives are zero:

$$\frac{\partial E_y}{\partial y} = \frac{\partial E_z}{\partial z} = \frac{\partial B_y}{\partial y} = \frac{\partial B_z}{\partial z} = 0$$

Recall, in a charge-free medium, $\vec{\nabla} \cdot \vec{E} = 0$ $\quad \vec{\nabla} \cdot \vec{B} = 0$

that is,

$$\frac{\partial E_x}{\partial x} + \cancel{\frac{\partial E_y}{\partial y}} + \cancel{\frac{\partial E_z}{\partial z}} = 0 \qquad \frac{\partial B_x}{\partial x} + \cancel{\frac{\partial B_y}{\partial y}} + \cancel{\frac{\partial B_z}{\partial z}} = 0$$

We find: $\frac{\partial E_x}{\partial x} = 0$ and $\frac{\partial B_x}{\partial x} = 0$

The component of the wave pointing parallel to the propagation direction does not vary along that direction. Only E_y and E_z can vary with x.

The magnetic-field direction in a light wave

Suppose a wave propagates in the x-direction and has its electric field along the y-direction [so, by assumption, $E_x = E_z = 0$].

What is the direction of the magnetic field?

Use:

$$-\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)$$

Why is this term zero? Because we've assumed propagation in the x direction. If the field was varying along the z direction, then it wouldn't be propagating solely along x.

So:

$$-\frac{\partial \vec{B}}{\partial t} = \left(0, 0, \frac{\partial E_y}{\partial x} \right)$$

In other words:

$$-\frac{\partial B_z}{\partial t} = \frac{\partial E_y}{\partial x}$$

The only non-zero component of B is the z component.

The magnetic-field strength in a light wave

Suppose a wave propagates in the x-direction and has its electric field in the y-direction. What is the *strength* of the magnetic field?

Start with: $$-\frac{\partial B_z}{\partial t} = \frac{\partial E_y}{\partial x}$$ and $$E_y(\vec{r},t) = \underset{\sim}{E}_0 \exp\left[j\left(kx - \omega t\right)\right]$$

We can integrate: $$B_z(x,t) = B_z(x,0) - \int_0^t \frac{\partial E_y}{\partial x}\, dt$$

Take $B_z(x,0) = 0$

So: $$B_z(x,t) = -\frac{jk}{-j\omega} \underset{\sim}{E}_0 \exp\left[j(kx - \omega t)\right]$$

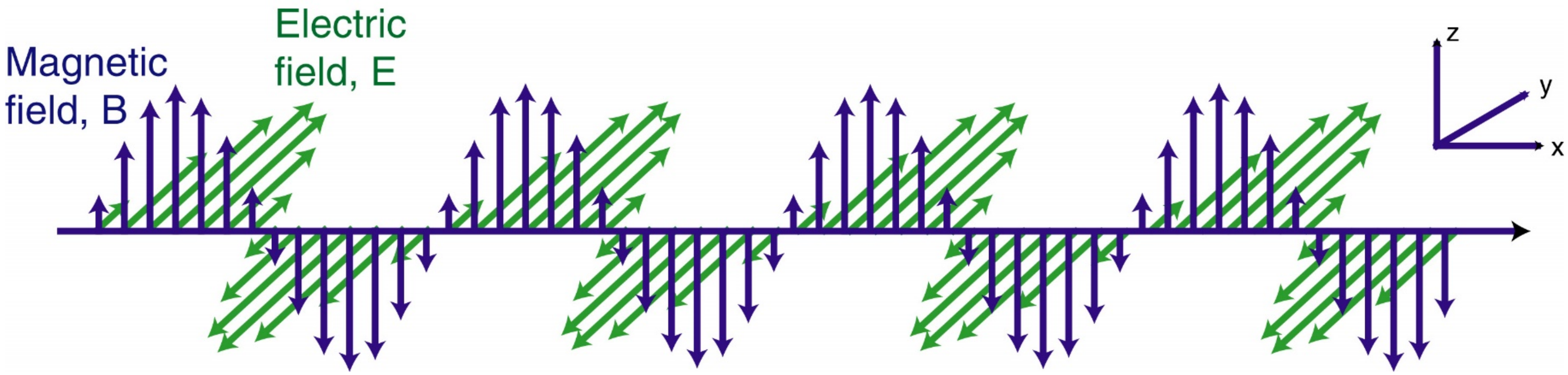
Differentiating E_y with respect to x yields a jk, and integrating with respect to t yields a $1/(-j\omega)$.

But $\omega / k = c$:

$$B_z(x,t) = \frac{1}{c} E_y(x,t)$$

An Electromagnetic Wave

To summarize: the electric and magnetic fields are **in phase**.



The electric field, the magnetic field, and the k-vector are all **perpendicular**:

$$\vec{E} \times \vec{B} \propto \vec{k}$$

And the magnitude of B_0 is **smaller** than the magnitude of E_0 by a factor of the wave velocity:

$$\left|\vec{B}_0\right| = \frac{\left|\vec{E}_0\right|}{c}$$

A note on units

The units of electric field are volts per meter:

$$\left[\vec{E}\right] = V/m$$

The strength of the magnetic field of an EM wave is related to the strength of the electric field by:

$$\left|\vec{B}_0\right| = \frac{\left|\vec{E}_0\right|}{c}$$

Therefore the units of B field must be:

$$\left[\vec{B}\right] = V \cdot s/m^2$$

This is called a 'Tesla'.

(not one of these)

It is not really accurate to say "the B field is much smaller than the E field", since they have different units. But people often say it anyway.

Free-space impedance

In optics, magnetic materials are not encountered too often.
So people mostly use E and B, (rather than E and H) and

$$\left|\vec{B}_0\right| = \left|\vec{E}_0\right| / c$$

At lower frequencies (i.e., rf and microwaves), electromagnetics engineers often use magnetic materials. It is often more convenient to use E and H, where H is defined by

$$\vec{H} = \vec{B} / \mu$$ which implies $$\left|\vec{E}_0\right| / \left|\vec{H}_0\right| = \mu \cdot c$$

But, since $c = 1 / \sqrt{\mu\varepsilon}$, this means that $\left|\vec{E}_0\right| / \left|\vec{H}_0\right| = \sqrt{\mu / \varepsilon}$

In empty space: $$\frac{\left|\vec{E}_0\right|}{\left|\vec{H}_0\right|} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377\Omega$$

This quantity is known as the "impedance of empty space."