

The Rendering Equation

Direct (*local*) illumination

- Light directly from light sources
- No shadows

Indirect (*global*) illumination

- Hard and soft shadows
- Diffuse interreflections (radiosity)
- Glossy interreflections (caustics)

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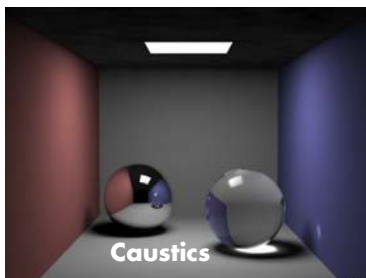
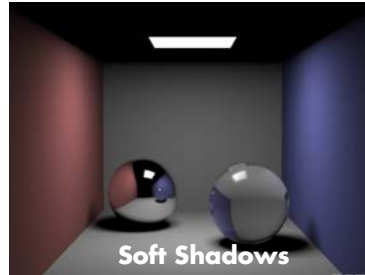
Radiosity



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Lighting Effects



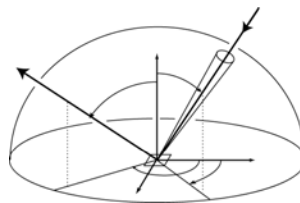
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Challenge

To evaluate the reflection equation
the incoming radiance must be known

$$L_r(x, \omega_r) = \int_{H^2} f_r(x, \omega_i \rightarrow \omega_r) L_i(x, \omega_i) \cos \theta_i d\omega_i$$



To evaluate the incoming radiance
the reflected radiance must be known

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To The Rendering Equation

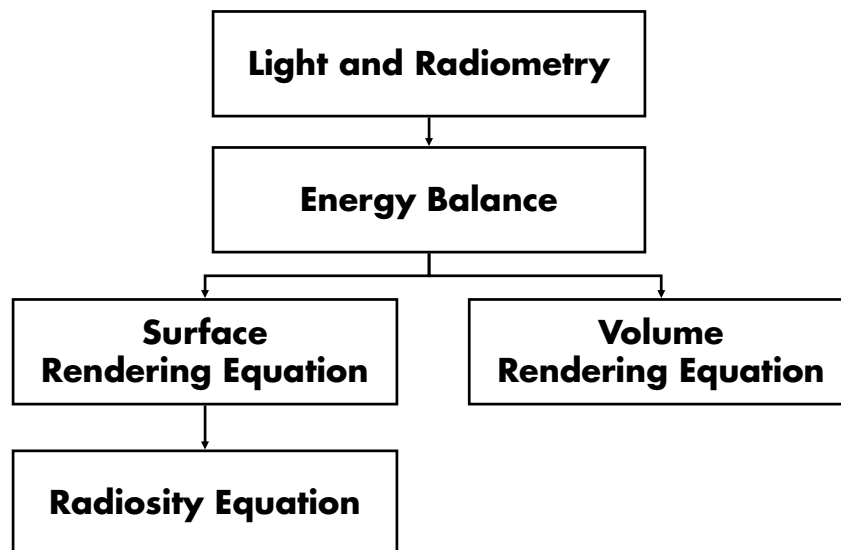
Questions

1. How is light measured?
2. How is the spatial distribution of light energy described?
3. How is reflection from a surface characterized?
4. What are the conditions for equilibrium flow of light in an environment?

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The Grand Scheme



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Balance Equation

Accountability

$$[\text{outgoing}] - [\text{incoming}] = [\text{emitted}] - [\text{absorbed}]$$

■ Macro level

The total light energy put into the system must equal the energy leaving the system (usually, via heat).

$$\Phi_o - \Phi_i = \Phi_e - \Phi_a$$

■ Micro level

The energy flowing into a small region of phase space must equal the energy flowing out.

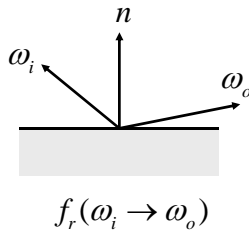
$$B(x) - E(x) = B_e(x) - E_a(x)$$

Surface Balance Equation

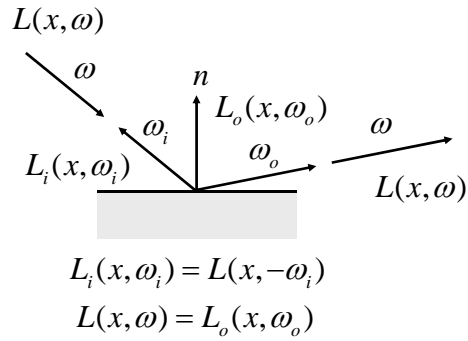
$$[\text{outgoing}] = [\text{emitted}] + [\text{reflected}]$$

$$\begin{aligned} L_o(x, \omega_o) &= L_e(x, \omega_o) + L_r(x, \omega_o) \\ &= L_e(x, \omega_o) + \int_{H^2} f_r(x, \omega_i \rightarrow \omega_o) L_i(x, \omega_i) \cos \theta_i d\omega_i \end{aligned}$$

Direction Conventions



BRDF



Surface vs. Field Radiance

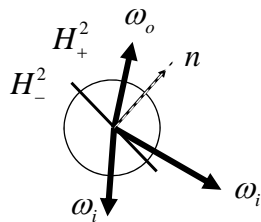
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Surface Balance Equation

[outgoing] = [emitted] + [reflected] + [transmitted]

$$L_o(x, \omega_o) = L_e(x, \omega_o) + L_r(x, \omega_o) + L_t(x, \omega_o)$$



$$L_r(x, \omega_o) = \int_{H_+^2} f_r(x, \omega_i \rightarrow \omega_o) L_i(x, \omega_i) \cos \theta_i d\omega_i$$

$$L_t(x, \omega_o) = \int_{H_-^2} f_t(x, \omega_i \rightarrow \omega_o) L_i(x, \omega_i) \cos \theta_i d\omega_i$$



BTDF

$$H_+^2(n) \quad \omega_o \cdot n(x) > 0$$

$$H_-^2(n) \quad \omega_o \cdot n(x) < 0$$

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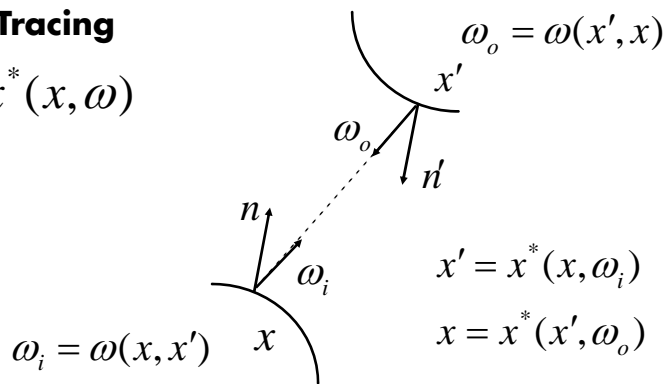
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Two-Point Geometry

$$\omega(x, x') = \omega(x \rightarrow x') = \frac{x' - x}{|x' - x|}$$

Ray Tracing

$$x^*(x, \omega)$$



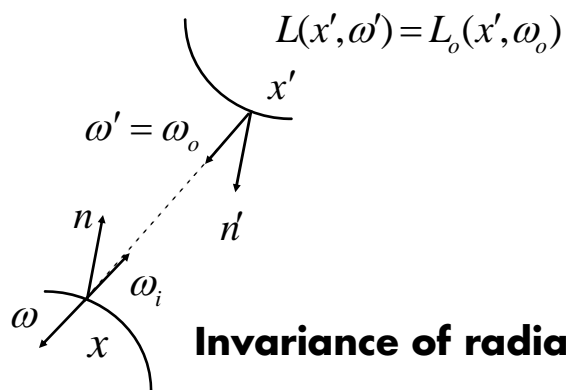
$$x' = x^*(x, \omega_i)$$

$$x = x^*(x', \omega_o)$$

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Coupling Equations



Invariance of radiance

$$L_i(x, \omega_i) = L(x, -\omega_i)$$

$$L(x, \omega) = L(x', \omega')$$

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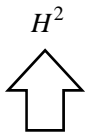
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The Rendering Equation

Directional form

$$L(x, \omega) = L_e(x, \omega) +$$

$$\int_{H^2} f_r(x, \omega' \rightarrow \omega) L(x, \omega'), -\omega' \cos \theta' d\omega'$$



Integrate over
hemisphere of
directions



Transport operator
i.e. ray tracing

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The Rendering Equation

Surface form

$$L(x', x) = L_e(x', x) +$$

$$\int_{M^2} f_r(x'', x', x) L(x'', x') G(x'', x') dA''(x'')$$

M^2

Geometry term



Integrate over
all surfaces

$$G(x'', x') = \frac{\cos \theta'' \cos \theta'}{\|x'' - x'\|^2} V(x'', x')$$

Visibility term



$$V(x'', x') = \begin{cases} 1 & \text{visible} \\ 0 & \text{not visible} \end{cases}$$

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The Radiosity Equation

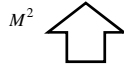
Assume diffuse reflection

1. $f_r(x, \omega_i \rightarrow \omega_o) = f_r(x) \Rightarrow \rho(x) = \pi f_r(x)$

2. $L(x, \omega) = B(x) / \pi$

$$B(x) = B_e(x) + \rho(x)E(x)$$

$$B(x) = B_e(x) + \rho(x) \int F(x, x') B(x') dA'(x')$$

$$F(x, x') = \frac{G(x, x')}{\pi}$$


Integral Equations

Integral equations of the 1st kind

$$f(x) = \int k(x, x') g(x') dx'$$

Integral equations of the 2nd kind

$$f(x) = g(x) + \int k(x, x') f(x') dx'$$

Linear Operators

Linear operators act on functions like matrices act on vectors

$$h(x) = (L \circ f)(x)$$

They are linear in that

$$L \circ (af + bg) = a(L \circ f) + b(L \circ g)$$

Types of linear operators

$$(K \circ f)(x) \equiv \int k(x, x') f(x') dx'$$

$$(D \circ f)(x) \equiv \frac{\partial f}{\partial x}(x)$$

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Solving the Rendering Equation

Rendering Equation

$$L = L_e + K \circ L$$

$$(I - K) \circ L = L_e$$

Solution

$$L = (I - K)^{-1} \circ L_e$$

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Formal Solution

Neumann series

$$(I - K)^{-1} = \frac{1}{I - K} = I + K + K^2 + \dots$$

Verify

$$\begin{aligned}(I - K) \circ (I - K)^{-1} &= (I - K) \circ (I + K + K^2 + \dots) \\ &= (I + K + \dots) - (K + K^2 + \dots) \\ &= I\end{aligned}$$

Successive Approximations

Successive approximations

$$L^1 = L_e$$

$$L^2 = L_e + K \circ L^1$$

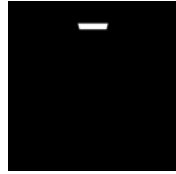
...

$$L^n = L_e + K \circ L^{n-1}$$

Converged

$$L^n = L^{n-1} \quad \therefore \quad L^n = L_e + K \circ L^n$$

Successive Approximation



L_e



$K \circ L_e$



$K \circ K \circ L_e$



$K \circ K \circ K \circ L_e$



L_e



$L_e + K \circ L_e$



$L_e + \dots K^2 \circ L_e$



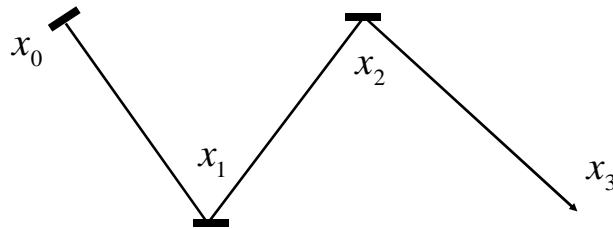
$L_e + \dots K^3 \circ L_e$

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Light Path

$$S(x_0, x_1) = L_e(x_0, x_1)$$

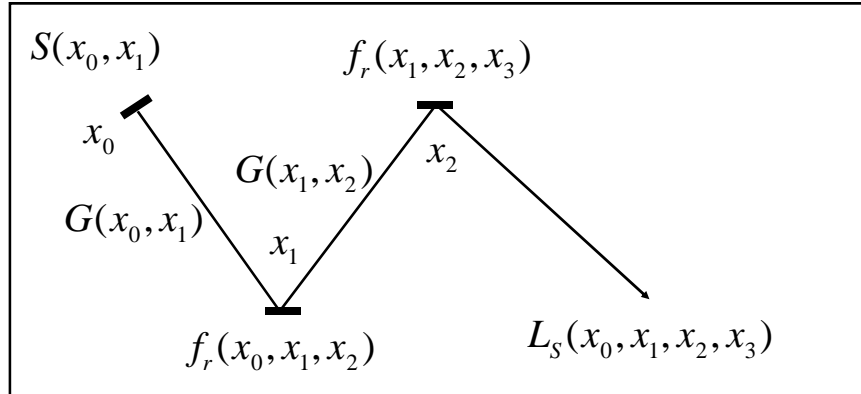


$$L_S(x_0, x_1, x_2, x_3)$$

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Light Path

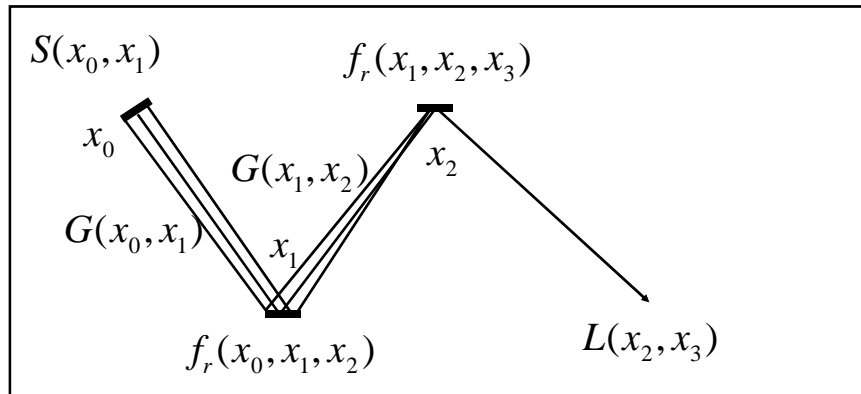


$$L_S(x_0, x_1, x_2, x_3) = S(x_0, x_1)G(x_0, x_1)f_r(x_0, x_1, x_2)G(x_1, x_2)f_r(x_1, x_2, x_3)$$

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Light Paths



$$L(x_2, x_3) = \int_{A_0} \int_{A_1} L_S(x_0, x_1, x_2, x_3) dA(x_0) dA(x_1)$$

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Light Transport

Integrate over all paths of all lengths

$$L(x_{k-1}, x_k) = \sum_{k=1}^{\infty} \int_{M^2} \cdots \int_{M^2} L_S(x_0, \dots, x_{k-2}, x_{k-1}, x_k) dA(x_0) \cdots dA(x_{k-2})$$

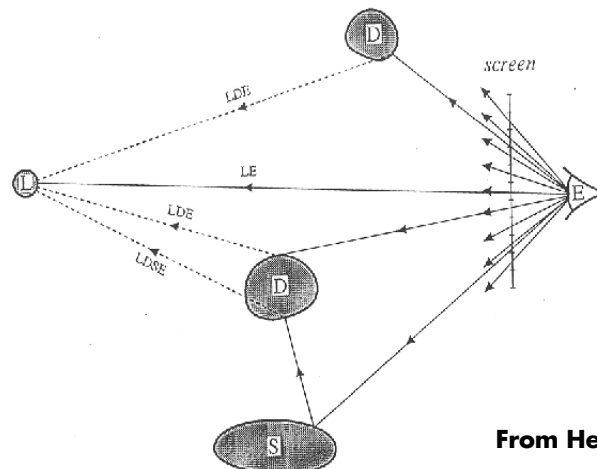
Question:

- How to sample space of paths?

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Classic Ray Tracing



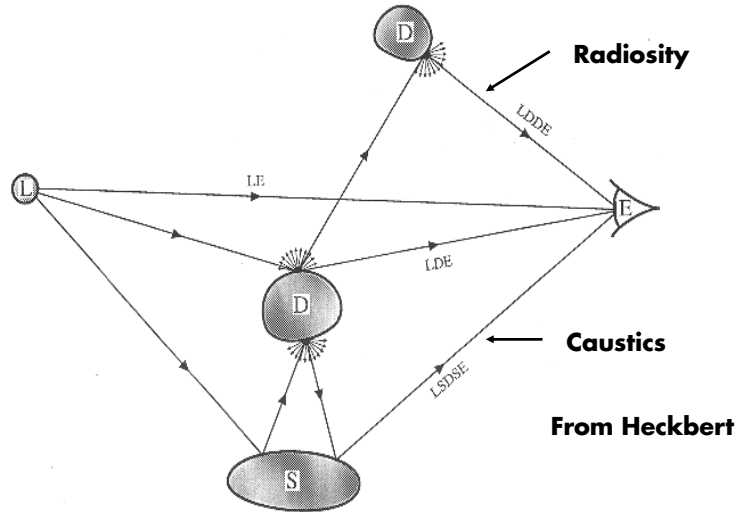
From Heckbert

Forward (from eye): E S* (D|G) L

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Photon Paths



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How to Solve It?

Finite element methods

- **Classic radiosity**
 - Mesh surfaces
 - Piecewise constant basis functions
 - Solve matrix equation
- **Not practical for rendering equation**

Monte Carlo methods

- **Path tracing (distributed ray tracing)**
- **Bidirectional ray tracing**
- **Photon mapping**

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