# Abstract Algebra

## A Comprehensive Introduction

## Volume 1: Linear Algebra

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by
Professor Steven Roman
**www.sroman.com**

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To Donna

# Preface To The Series

This is a multi-volume series on abstract algebra, designed for the serious undergraduate or beginning graduate student. Dependencies among the first three volumes of this series, with hopefully more volumes to come, are as follows.

1) Volume 1: *Linear Algebra* can be read independently of the other two volumes.
2) Volume 2: *Group Theory* can be read independently of the other two volumes.
3) Volume 3: *Ring and Field Theory* does reference material on both linear algebra and group theory, but any reader with a modest acquaintance with these subjects should be able to read this volume without difficulty.
4) Volume 4: *Order and Lattices* can be read indepent=dently of theother volumes.

Thus, one has the option of starting this series with either the volume on group theory or the volume on linear algebra.

As the title suggests, this series is a *comprehensive introduction* to abstract algebra, where by *introduction*, I mean that the book starts more or less at the beginning, assuming no prior knowledge of abstract algebra. The only prerequisites for this series are an understanding of basic mathematical tools as found in a typical "transition course" and a solid understanding of elementary linear algebra as taught in a "relatively serious" lower division course. Of course, some experience beyond these prerequisites will no doubt be of significant benefit in absorbing the material presented in this series.

Realizing that this may be a first exposure to rigorous abstract mathematics for some readers, I have tried to write the discussions and the proofs with an eye towards developing sound mathematical thinking patterns on the part of the reader. Proofs that appear in textbooks often do not show the *thought pattern* behind the discovery of the proof, often preferring elegance or conciseness over insight or motivation. However, showing a bit of the motivation behind a proof is in my opinion extremely valuable for students. Of course, one can easily get carried away with this idea, so I have tried to temper it with an eye towards not bloating these volumes or slowing down the flow of the books.

By *comprehensive*, I mean that the books in this series include a somewhat wider array of topics (some marked as optional or placed in topics chapters) than is often seen in elementary treatments, so that hopefully *all* readers will find something new and of interest in these books. My goal is for all readers to leave the series with a thorough grounding in the fundamentals of abstract algebra, well prepared to attack more advanced treatments if desired.

It seems that the current trend in mathematical education is to motivate abstract concepts by introducing *applications* as quickly as possible, in an effort to satisfy those students whose overriding question is: Of what use is this material?

While I certainly respect the views of those whose main concern is whether or not the subject matter at hand has applications to the "real" world, I have chosen to take a more abstract approach to the subject of abstract algebra in this series. I am a pure mathematician and appreciate mathematics as an *art form*, as well as the cornerstone of all science and technology. Merriam-Webster defines art as follows:

> something that is created with imagination and skill and that is beautiful or that expresses important ideas or feelings

What could possibly fit this description more accurately than mathematics and in particular, abstract algebra?

# Preface to this Volume

This book is a comprehensive introduction to linear algebra, designed for the serious upper division undergraduate or beginning graduate student. The prerequisites for this book consist of a firm grounding in topics generally covered in a typical "transition" course as well as a moderately serious lower division linear algebra course. This should include familiarity with *at least* the following topics:

1) Proof techniques, such as proof by contradiction, proof by contraposition and proof by induction.
2) The arithmetic of complex numbers.
3) The algebra of polynomials.
4) Equivalence relations and partitions (reviewed quickly in Chapter 0).
5) The basic properties of matrices and determinants, including the row and column spaces of a matrix, the rank of a matrix, elementary matrices and Gaussian and Gauss–Jordan elimination.
6) Finite-dimensional Euclidean space $\mathbb{R}^n$.

## Organization of the Book

The first chapter of this book (Chapter 0) is entitled *Preliminaries* and is divided into four parts: Background, Matrices, Equivalence Relations and Partitions and Canonical Forms and Invariants. While some of this material may be familiar to the reader, there may very well be some material that is not familiar and so I *strongly encourage* everyone to read or at least skim through this chapter before proceeding with the text proper.

The rest of the book is divided into four parts (not including the appendices):

1) Basic Theory
2) Inner Product Spaces
3) Spectral Theory and Canonical Forms
4) Topics

### *Part I: Basic Theory*

Part I cover the basic theory of vector spaces in what I hope is a comprehensive manner appropriate for this level of textbook.

Chapter 1: vector spaces
Chapter 2: linear transformations
Chapter 3: linear functionals, dual spaces, quotient spaces, the correspondence and isomorphism theorems
Chapter 4: eigenvalues and eigenvectors

Chapter 3 has essentially two parts. The first part covers linear functionals and dual spaces and the second part covers quotient spaces and the correspondence and isomorphism theorems. These two parts are independent of one another and so either one can be covered by itself if there is a need to reach Chapter 4 more quickly, as may be the case in a 10-week course. Also, Chapter 5 can be postponed until after the discussion of inner product spaces in

Chapters 6 and 7. On the one hand, the material in Chapter 5 is more related to that of Chapters 8 and 9 but on the other hand, it does provide a nice conclusion to a one-semester course in linear algebra.

Thus, a 10-week course could consist of Chapters 1, 2 and 4, with perhaps a sprinkling of Chapter 3 or Chapter 5 if time allows. A 15-week course should provide ample time for all of Chapters 1, 2 and 4 as well as much of Chapters 3 and 5.

I should also mention that the first part of the book concerns vector spaces over *any* field $F$, although the reader can easily assume that $F$ represents either the real or complex field if desired. The field $\mathbb{Z}_2$ is used in a few places to provide counterexamples. In much of the remainder of the book, we do restrict attention only to real and complex vector spaces.

## *Part II: Inner Product Spaces*

Part II covers the basic theory of inner product spaces. The topic choices here are, I believe, fairly standard. There are two optional sections in this part of the book: one on least squares solutions and the other on the QR factorization of a matrix.

## *Part III: Spectral Theory and Canonical Forms*

Part III is devoted to spectral theory and the theory of canonical forms. Chapter 7 contains a discussion of minimal polynomials, the primary decomposition and the Cayley–Hamilton theorem. In this chapter, I introduce the notion of the *operator module* of a linear operator. However, I do not pursue the study of modules *per se* in the book, except very briefly in an optional topics chapter, which I call the "vector space appreciation chapter." Frankly, all that is needed in this book is little more than the *definition* of an operator module and a submodule, since this provided a very convenient language in which to express ideas about linear operators concisely and crisply.

Chapter 8 contains a discussion of Schur's theorem and the spectral theorem. In an effort to increase flexibility, I first discuss the simpler case where the minimal polynomial of an operator splits over the base field, which of course always happens when the base field is the complex field. The slightly more difficult case where the minimal polynomial does not split (and the base field is the real field) is covered separately and can be omitted if either time or motivation is lacking.

In Chapter 8, I also discuss normal, self-adjoint and unitary operators as well as the optional topics of functional calculus, positive operators and their square roots. I include here a result that is not often found in textbooks on linear algebra: Two real matrices are *unitarily* similar in $\mathbb{C}$ if and only if they are *orthogonally* similar in $\mathbb{R}$.

Finally, Chapter 9 is devoted to the study of canonical forms, including both rational canonical form and Jordan canonical form, as well as a brief discussion of Smith normal form.

## *Part IV: Topics*

Part IV contains a set of more-or-less independent topics that can be inserted at various times in the course of study, or assigned as independent reading. The topics are as follows.

### Chapter 10—Uniqueness of Reduced Row Echelon Form

Here I give a proof of the uniqueness of the reduce row echelon form. This chapter can be read at any time.

### Chapter 11—Determinants

This chapter describes the three definitions of the determinant and their equivalence. The only prerequisite for this chapter is the *definition* of a vector space in the discussion of the determinant as a multilinear form. However, if the reader has not yet read the definition of a vector space, he or she can easily substitute $\mathbb{R}^n$ or $\mathbb{C}^n$ for any mention of an arbitrary vector space $V$.

### Chapter 12—LDU Factorization

Much of this chapter can be read at any time, but the later material on Sylverter's law of inertia uses the spectral theorem in its proof.

**Chapter 13—The Polar and Singular Value Decompositions**

As a continuation of Chapter 7, this topics chapter describes the polar decomposition, the singular value decomposition and the Moore–Penrose generalized inverse.

**Chapter 14—Tensor Products**

This is a *brief* introduction to the tensor product, describing its purpose, namely, bilinear-for-linear exchange and its construction, both basis free and otherwise. I also discuss a few basic properties of the tensor product, in particular, how to tell when a tensor is zero. The chapter can be read any time after the material on quotient spaces.

**Chapter 15—Modules**

Here I give a very brief introduction to abstract modules and their relationship to vector spaces. This is the *vector space appreciation* chapter. After seeing an example of a module that has a basis of every size $n \geq 1$, perhaps the reader will appreciate how nicely behaved vector spaces really are! This chapter can be read any time after Chapter 1, but might be better appreciated after the discussion of operator modules in Chapter 4.

## Appendices

The book contains an appendix covering the basic notions related to infinite cardinal numbers and an appendix on partially ordered sets.

### *Cardinality*

Throughout the book, I will refer to the size of a set as its *cardinality*. Also, I will state and prove a few results in the language of *arbitrary* cardinal numbers, both finite and infinite. This includes the following:

1) Any two bases for a vector space $V$ have the same cardinality.
2) If $S$ is a set of cardinality $\kappa$, then the vector space $F_0^S$ of all functions from $S$ to $F$ that have finite support has dimension $\kappa$. However, the vector space $F^S$ of all functions from $S$ to $F$ has dimension strictly greater than $\kappa$.
3) $\dim(V) \leq \dim(V^*)$, with equality if and only if $V$ is finite dimensional.

For those not familiar with cardinal numbers, I have provided a very short appendix that contains sufficient background on this fascinating subject which, contrary to the opinion of the mathematician Leopold Kronecker is *not* a corrupter of youth!

### *Partially-Ordered Sets and Lattices*

It has always been my opinion that the subject of partial order and in particular the concepts of extreme elements (maximum, maximal and their duals) as well as the lattice operations of meet and join are of great importance and deserve direct (albeit brief) attention in any algebra course. This is particularly true because the family of all substructures of an algebraic structure—in the present context, the family $\text{sub}(V)$ of all subspaces of a vector space $V$ is a complete lattice. Moreover, since in this case meet is just intersection and join is defined in terms of intersection, we can describe the lattice $\text{sub}(V)$ as an *intersection structure* and thereby avoid the need for a direct study of abstract lattices.

*I strongly suggest that all readers examine Appendix B before reading Chapter 1, for we will rely on its contents in the text proper.*

### *Zorn's Lemma*

The appendix on order is also the natural place to discuss Zorn's lemma, which is used at least thrice in the text. The first use of Zorn's lemma is to prove that every subspace of a vector space has a complement. I use Zorn's lemma in this proof for two reasons. First, I have no choice because the theorem comes before a discussion of bases, although I also give a later proof of the result using bases. Second, it provides a reasonably gentle place to introduce the reader to a typical use of Zorn's lemma (the proof is only two paragraphs long).

There are also two critical occasions where Zorn's lemma is employed: one to prove that *all* nontrivial vector spaces have a basis and one to prove that all nontrivial inner product spaces have a maximal orthonormal subset.

## To The Student: Are There Any Questions?

To study abstract mathematics, one needs to begin to think like a mathematician. There are a few aspects to thinking like a mathematician that can be articulated and with which I suspect few mathematicians would disagree.

1) As you read and think about mathematics, *always and repeatedly* ask the following question of yourself: How do I know that this is true? As you read this book, you should be asking this question many, many times, perhaps as often as once or twice in every paragraph when the going gets rough! In fact, you may encounter *several consecutive sentences* where you need to pause and ask this question about each sentence. I have been a mathematician for almost 50 years and I still do this. In short: *Be skeptical about everything you read until you thoroughly understand it*!
2) To help answer these questions, ask yourself another question: How else can I say this? To what is this equivalent? How can I rephrase this in different language? Hopefully, by rephrasing in enough different ways, one of those ways will turn a little light on in your head, guiding you to that understanding. Such is the way of a mathematician.
3) In order to be successful as a student of mathematics, you *must* commit to memory all *definitions* and *statements of theorems*, for these are the *tools* by which you can construct valid mathematical arguments. Of course, memorization is not always fun, but sometimes it is just simply required.

## Index of Symbols

There is an index of symbols at the back of the book, in case you encounter a symbol that you do not recognize.

It seems that mathematicians never have enough available symbols. In particular, the usual Roman alphabet does not supply enough symbols to denote variables of different types. Accordingly, mathematicians find it necessary to reach out to other alphabet systems. It is fair to say that all mathematicians (and many mathematics books) make considerable use of the Greek alphabet, shown in the table below. If you intend to study mathematics seriously, some familiarity with this alphabet is essential (with the possible exception of omicron and upsilon!).

| | | | |
|---|---|---|---|
| A $\alpha$ alpha | H $\eta$ eta | N $\nu$ nu | T $\tau$ tau |
| B $\beta$ beta | $\Theta$ $\theta$ theta | $\Xi$ $\xi$ xi | $\Upsilon$ $\upsilon$ upsilon |
| $\Gamma$ $\gamma$ gamma | I $\iota$ iota | O o omicron | $\Phi$ $\phi$ phi |
| $\Delta$ $\delta$ delta | K $\kappa$ kappa | $\Pi$ $\pi$ pi | X $\chi$ chi |
| E $\epsilon$ epsilon | $\Lambda$ $\lambda$ lambda | P $\rho$ rho | $\Psi$ $\psi$ psi |
| Z $\zeta$ zeta | M $\mu$ mu | $\Sigma$ $\sigma$ sigma | $\Omega$ $\omega$ omega |

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*Sections marked with an asterisk may be omitted on first reading.*