[6] **1.** Consider the eleven letters *ABRACADABRA*. A WORD is a sequence of these symbols, in any order, that uses all eleven of them.

a) How many words can be made with these letters? You may leave your final answer in terms of factorials.

b) How many of these words do not start with A?

c) How many of these words have five consecutive A's?

[4] **2.** Determine the value of $\sum_{k=0}^{2m+1} \binom{2m+1}{k}(-2)^k$. Your final answer should be a single number. (hint: binomial theorem)

[6] **3.** In each case, determine which is larger. Justify your answer!

a) The number of unit paths from the origin to the point $(a_1, a_2, \cdots, a_m)$ in $\mathbb{R}^m$, or the number of arrangements of the multiset $\{\!\{x_1^{(a_1)}, x_2^{(a_2)}, \cdots, x_m^{(a_m)}\}\!\}$?

b) The number of points in $\mathbb{R}^m$ that a unit path of length n that starts at the origin could end at, or the number of different multisets of size n that are composed entirely of 1's, 2's, 3's, ... and m's?

c) The sum of the number of unit paths to $(a_1, a_2, \cdots, a_m)$ in $\mathbb{R}^m$ over all points with $a_1 + a_2 + \cdots + a_m = n$, or m^n?

[4] **4.** Recall that a UNIT PATH OF LENGTH n is a path that consists of n steps, each of which is of length one in the direction of a positive axis.

Find an expression for the number of unit paths in $\mathbb{R}^4$ starting at the origin where for each point (a_i, b_i, c_i, d_i) we have $a_i \leq b_i$ and $c_i \leq d_i$ for $0 \leq i \leq n$.

You may use results we saw in class, but you should explain your reasoning. Note that we did *not* see this one in class, so if you just write down the correct answer with no explanation you will get zero marks.

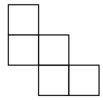
[6] **5.** Consider the set $D = \{6, 10, 14\}$.

a) Using inclusion-exclusion, give an expression for the number of integers in $\{1, 2, \cdots, N\}$ that are divisible by none of the numbers in D.

b) Using inclusion-exclusion, give an expression for the number of integers in $\{1, 2, \cdots, N\}$ that are divisible by exactly one of the numbers in D.

[4] **6.** Prove, using whichever technique you prefer, that $\binom{n+1}{t+1} = \sum_{j=t}^{n} \binom{j}{t}$.

Note that we saw three different proofs in class (there are others). It is possible to earn bonus marks by giving multiple *different* proofs (but note that giving one correct proof will earn you more marks than several partially correct ones).

[6] **7.** Consider the following board: 

a) Choose one square e, and express the rook polynomial $R_Q(X)$ in terms of $R_{Q\setminus e}(X)$ and $R_{Q\,/\,e}(X)$.
(hint: I would choose the square closest to the center)

b) Determine $R_{Q\setminus e}(X)$ and $R_{Q\,/\,e}(X)$ in any way you like, and then using your answer to the previous part, find R_Q.

[4] **8.** Consider that x_1, x_2, x_3, x_4 are constrained as follows.

$$\begin{cases} x_1 \geq 5, \quad x_2 \text{ odd}, \quad x_3 \text{ multiple of 5}, \quad x_4 \leq 20 \\ x_1, x_2, x_3, x_4 \in \mathbb{N} \end{cases}$$

a) Let a_n be the number of solutions to $x_1 + x_2 + x_3 + x_4 = n$ with the above constraints. Determine the generating function $A(X) = \sum_{n\geq 0} a_n X^n$ in closed form.

b) Let b_n be the number of solutions to $x_1 + x_2 + x_3 + x_4 \leq n$ with the above constraints. Determine the generating function $B(X) = \sum_{n\geq 0} b_n X^n$ in closed form.

[2] **9.** Consider the generating function $\displaystyle\sum_{n\geq 0} b_n X^n = \frac{X^5}{(1-X)^3(1+X)(1-X^7)}$. Give a value k and some restrictions such that b_n is the number of solutions to $x_1 + x_2 + \cdots + x_k = n$ with your restrictions.

[4] **10.** For each of the following generating functions, find a formula for a_n.

a) $\displaystyle A(X) = \sum_{n\geq 0} a_n X^n = \frac{X^2 + 3X}{(1-X)(1-X^2)}$

b) $\displaystyle A(X) = \sum_{n\geq 0} a_n X^n = \frac{X}{(1-X)^3} + \frac{X^2}{(1-X^3)} + \frac{X^3 + X^4}{1-(2X)}$

[5] **11.** Consider the recurrence $a_{n+3} - a_{n+2} - a_{n+1} + a_n = 0$ with $a_0 = 1$, $a_1 = 1$, $a_2 = 2$. Also let b_n be the number of solutions to $x_1 + x_2 = n$ with $x_1, x_2 \in \mathbb{N}$ and x_2 even.

a) Show that the two generating functions are equal. That is find a closed form for $A(X) = \sum_{n\geq 0} a_n X^n$ and $B(X) = \sum_{n\geq 0} b_n X^n$, and show that they are equal.

b) What can you conclude about a_n and b_n from the above?

[4] **12.** Consider the equality $(1+X)^{m+n} = (1+X)^m(1+X)^n$. By finding the the coefficient of X^k on both sides (without simplifying either side in any way), derive an identity on binomial coefficients.
(hint: follow the model we saw in class)

[**5**] **13.** Consider the recurrence $a_{n+2} - 2a_{n+1} + a_n = 3^n$, and define the generating function for this recurrence as $A(X) = \sum_{n \geq 0} a_n X^n$.

a) Find a closed form expression for $A(X)$. Also, give a closed form for $A^{(h)}(X)$ and $A^{(p)}(X)$, the homogeneous and particular generating functions.

b) Using your generating functions, or otherwise, compute a_n.

[**2**] **14.** For each of the following sequences of integers, determine if there is a graph with this exact degree sequence: either draw such a graph or explain why none exists. If there is, determine if there is a simple graph with this exact degree sequence: either draw such a graph or explain why none exists.

a) $3, 3, 2, 2, 1$

b) $5, 5, 2, 2, 1, 1$

[+**1**] **15.** (bonus) Draw the Petersen graph.