ANOVA

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ANOVA

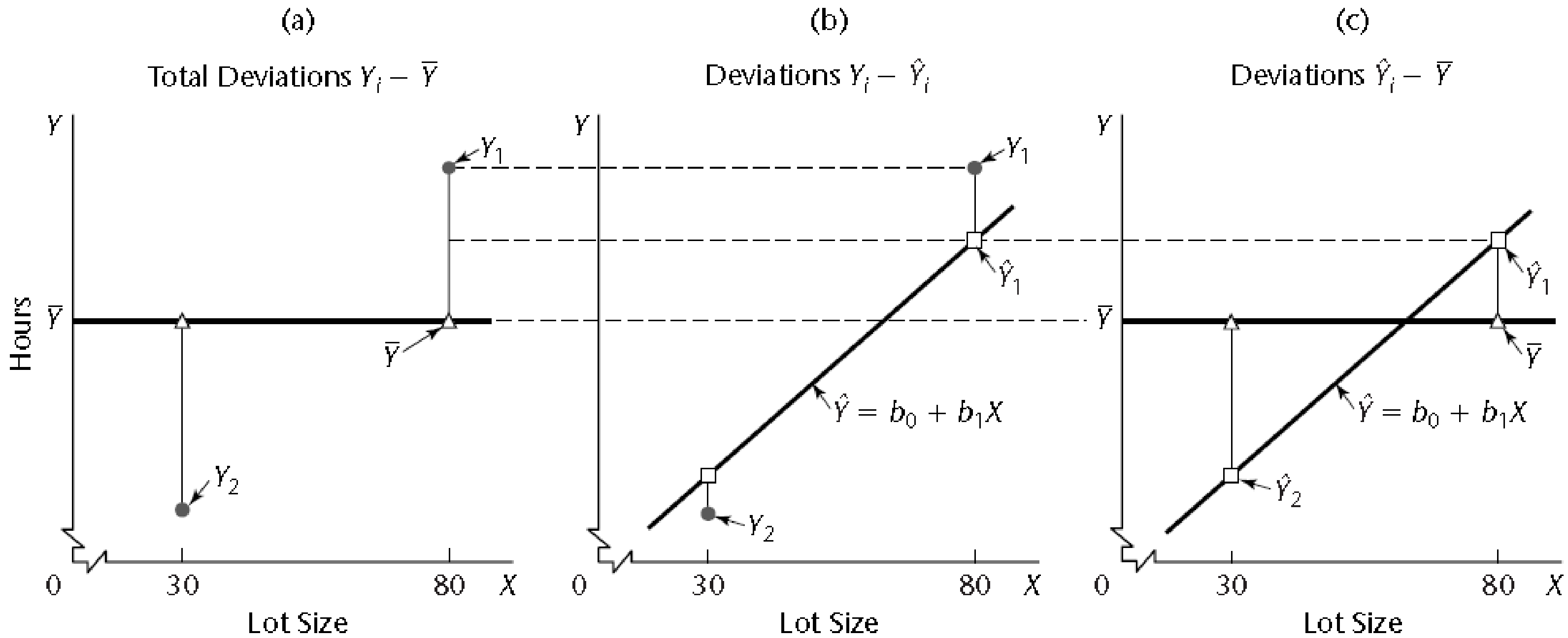
- ANOVA is nothing new but is instead a way of organizing the parts of linear regression so as to make easy inference recipes.
- Will return to ANOVA when discussing multiple regression and other types of linear statistical models.

Partitioning Total Sum of Squares

- "The ANOVA approach is based on the partitioning of sums of squares and degrees of freedom associated with the response variable Y"
- We start with the observed deviations of Y_i around the observed mean $\bar{Y}$

$$Y_i - \bar{Y}$$

Partitioning of Total Deviations



SSTO SSE SSR

Measure of Total Variation

- The measure of total variation is denoted by

$$SSTO = \sum(Y_i - \bar{Y})^2$$

- SSTO stands for *total sum of squares*
- If all Y_i's are the same, SSTO = 0
- The greater the variation of the Y_i's the greater SSTO

Variation after predictor effect

- The measure of variation of the Y_i's that is still present when the predictor variable X is taken into account is the sum of the squared deviations

$$SSE = \sum (Y_i - \hat{Y}_i)^2$$

- SSE denotes *error sum of squares*

Regression Sum of Squares

- The difference between SSTO and SSE is SSR

$$SSR = \sum(\hat{Y}_i - \bar{Y})^2$$

- SSR stands for *regression sum of squares*

Partitioning of Sum of Squares

$$\underbrace{Y_i - \bar{Y}}_{\text{Total deviation}} = \underbrace{\hat{Y}_i - \bar{Y}}_{\text{Deviation of fitted regression value around mean}} + \underbrace{Y_i - \hat{Y}_i}_{\text{Deviation around fitted regression line}}$$

Remarkable Property

- The sums of the same deviations squared has the same property!

$$(Y_i - \bar{Y})^2 = (\hat{Y}_i - \bar{Y})^2 + (Y_i - \hat{Y}_i)^2$$

or SSTO = SSR + SSE

- Proof:

Remarkable Property

- Proof: $(Y_i - \bar{Y})^2 = (\hat{Y}_i - \bar{Y})^2 + (Y_i - \hat{Y}_i)^2$

$$
\begin{aligned}
(Y_i - \bar{Y})^2 &= \sum[(\hat{Y}_i - \bar{Y}) + (Y_i - \hat{Y}_i)]^2 \\
&= \sum[(\hat{Y}_i - \bar{Y})^2 + (Y_i - \hat{Y}_i)^2 + 2(\hat{Y}_i - \bar{Y})(Y_i - \hat{Y}_i)] \\
&= \sum(\hat{Y}_i - \bar{Y})^2 + \sum(Y_i - \hat{Y}_i)^2 + 2\sum(\hat{Y}_i - \bar{Y})(Y_i - \hat{Y}_i)
\end{aligned}
$$

but

$$\sum(\hat{Y}_i - \bar{Y})(Y_i - \hat{Y}_i) = \sum \hat{Y}_i(Y_i - \hat{Y}_i) - \sum \bar{Y}(Y_i - \hat{Y}_i) = 0$$

By properties previously demonstrated

Remember: Lecture 3

- The i[th] residual is defined to be

$$e_i = Y_i - \hat{Y}_i$$

- The sum of the residuals is zero:

$$\begin{aligned}\sum_i e_i &= \sum (Y_i - b_0 - b_1 X_i) \\ &= \sum Y_i - nb_0 - b_1 \sum X_i \\ &= 0\end{aligned}$$

By first normal equation.

Remember: Lecture 3

- The sum of the weighted residuals is zero when the residual in the i^{th} trial is weighted by the fitted value of the response variable for the i^{th} trial

$$\begin{aligned}\sum_i \hat{Y}_i e_i &= \sum_i (b_0 + b_1 X_i) e_i \\ &= b_0 \sum_i e_i + b_1 \sum_i e_i X_i \\ &= 0\end{aligned}$$

By previous properties.

Breakdown of Degrees of Freedom

- SSTO
  - 1 linear constraint due to the calculation and inclusion of the mean
    - n-1 degrees of freedom
- SSE
  - 2 linear constraints arising from the estimation of β_1 and β_0
    - n-2 degrees of freedom
- SSR
  - Two degrees of freedom in the regression parameters, one is lost due to linear constraint
    - 1 degree of freedom

Mean Squares

- A sum of squares divided by its associated degrees of freedom is called a mean square
  - The *regression mean square* is

$$MSR = \frac{SSR}{1} = SSR$$

  - The *error mean square* is

$$MSE = \frac{SSE}{n-2}$$

ANOVA table for simple lin. regression

Source of Variation	SS	df	MS	E{MS}
Regression	$SSR = \sum(\hat{Y}_i - \bar{Y})^2$	1	MSR = SSR/1	$\sigma^2 + \beta_1^2 \sum(X_i - \bar{X})^2$
Error	$SSE = \sum(Y_i - \hat{Y}_i)^2$	n-2	MSE = SSE/(n-2)	σ^2
Total	$SSTO = \sum(Y_i - \bar{Y})^2$	n-1		

$$E\{MSE\} = \sigma^2$$

- We know from earlier lectures that
  - $SSE/\sigma^2 \sim \chi^2(n-2)$
- That means that $E\{SSE/\sigma^2\} = n-2$
- And thus that $E\{SSE/(n-2)\} = E\{MSE\} = \sigma^2$

$$E\{MSR\} = \sigma^2 + \beta_1^2 \sum (X_i - \bar{X})^2$$

- To begin, we take an alternative but equivalent form for SSR

$$SSR = b_1^2 \sum (X_i - \bar{X})^2$$

- And note that, by definition of variance we can write

$$\sigma^2\{b_1\} = E\{b_1^2\} - (E\{b_1\})^2$$

$$E\{MSR\} = \sigma^2 + \beta_1^2 \sum (X_i - \bar{X})^2$$

- But we know that b_1 is an unbiased estimator of β_1 so $E\{b_1\} = \beta_1$
- We also know (from previous lectures) that

$$\sigma^2\{b_1\} = \frac{\sigma^2}{\sum (X_i - \bar{X})^2}$$

- So we can rearrange terms and plug in

$$\begin{aligned} \sigma^2\{b_1\} &= E\{b_1^2\} - (E\{b_1\})^2 \\ E\{b_1^2\} &= \frac{\sigma^2}{\sum (X_i - \bar{X})^2} + \beta_1^2 \end{aligned}$$

$$E\{MSR\} = \sigma^2 + \beta_1^2 \sum (X_i - \bar{X})^2$$

- From the previous slide

$$E\{b_1^2\} \quad = \quad \frac{\sigma^2}{\sum (X_i - \bar{X})^2} + \beta_1^2$$

- Which brings us to this result

$$E\{MSR\} = E\{SSR/1\}$$
$$= E\{b_1^2\} \sum (X_i - \bar{X})^2 = \sigma^2 + \beta_1^2 \sum (X_i - \bar{X})^2$$

Comments and Intuition

- The mean of the sampling distribution of MSE is σ^2 regardless of whether X and Y are linearly related (i.e. whether $\beta_1 = 0$)
- The mean of the sampling distribution of MSR is also σ^2 when $\beta_1 = 0$.
  - When $\beta_1 = 0$ the sampling distributions of MSR and MSE tend to be the same

F Test of $\beta_1 = 0$ vs. $\beta_1 \neq 0$

- ANOVA provides a battery of useful tests. For example, ANOVA provides an easy test for
  - Two-sided test
    - $H_0 : \beta_1 = 0$
    - $H_a : \beta_1 \neq 0$
    - Test statistic

Test statistic from before

$$t^* = \frac{b_1 - 0}{s\{b_1\}}$$

ANOVA test statistic

$$F^* = \frac{MSR}{MSE}$$

Sampling distribution of F^*

- The sampling distribution of F^* when $H_0(\beta_1 = 0)$ holds can be derived starting from Cochran's theorem
- *Cochran's theorem*
  - If all *n* observations Y_i come from the same normal distribution with mean μ and variance σ^2, and SSTO is decomposed into *k* sums of squares SS_r, each with degrees of freedom df_r, then the SS_r/σ^2 terms are independent χ^2 variables with df_r degrees of freedom if

$$\sum_{r=1}^{k} df_r = n - 1$$

The F Test

- We have decomposed SSTO into two sums of squares SSR and SSE and their degrees of freedom are additive, hence, by Cochran's theorem:
  - If $\beta_1 = 0$ so that all Y_i have the same mean $\mu = \beta_0$ and the same variance σ^2, SSE/σ^2 and SSR/σ^2 are independent χ^2 variables

F^* Test Statistic

- F^* can be written as follows

$$F^* = \frac{MSR}{MSE} = \frac{\frac{SSR/\sigma^2}{1}}{\frac{SSE/\sigma^2}{n-2}}$$

- But by Cochran's theorem, we have when H_0 holds

$$F^* \sim \frac{\frac{\chi^2(1)}{1}}{\frac{\chi^2(n-2)}{n-2}}$$

F Distribution

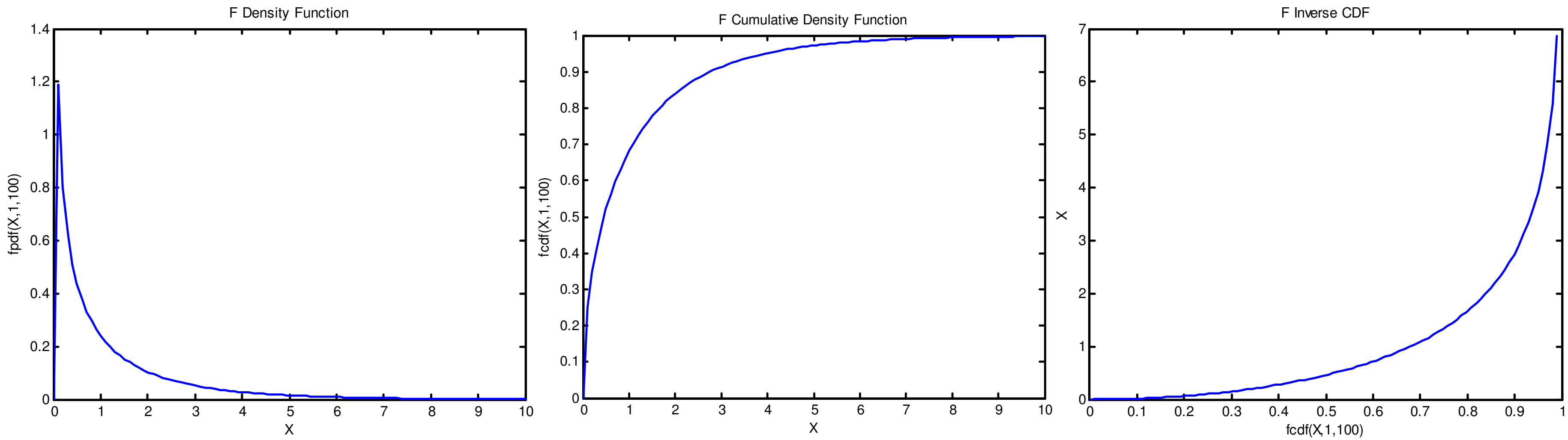
- The F distribution is the ratio of two independent χ^2 random variables.
- The test statistic F* follows the distribution
  - $F^* \sim F(1,n-2)$

Hypothesis Test Decision Rule

- Since F^* is distributed as $F(1,n-2)$ when H_0 holds, the decision rule to follow when the risk of a Type I error is to be controlled at α is:
  - If $F^* \leq F(1-\alpha; 1, n-2)$, conclude H_0
  - If $F^* > F(1-\alpha; 1, n-2)$ conclude H_a

F distribution

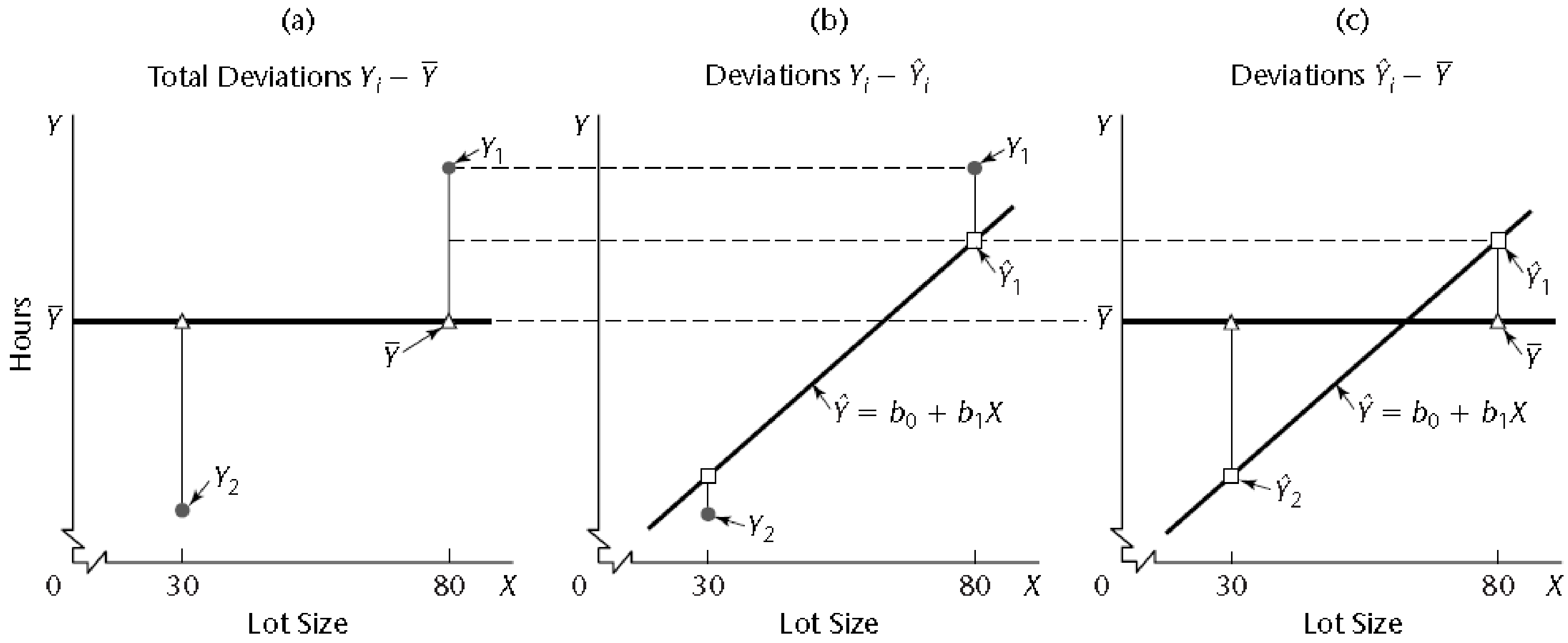
- PDF, CDF, Inverse CDF of F distribution



- Note, MSR/MSE must be big in order to reject hypothesis.

Partitioning of Total Deviations

- Does this make sense? When is MSR/MSE big?



SSTO SSE SSR

General Linear Test

- The test of $\beta_1 = 0$ versus $\beta_1 \neq 0$ is but a single example of a general test for a linear statistical models.
- The general linear test has three parts
  - Full Model
  - Reduced Model
  - Test Statistic

Full Model Fit

- The standard full simple linear regression model is first fit to the data

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$$

- Using this model the error sum of squares is obtained

$$SSE(F) = \sum[Y_i - (b_0 + b_1 X_i)]^2 = \sum(Y_i - \hat{Y}_i)^2 = SSE$$

Fit Reduced Model

- For instance, so far we have considered
  - $H_0 : \beta_1 = 0$
  - $H_a : \beta_1 \neq 0$
- The model when H_0 holds is called the *reduced* or *restricted* model. Here this results in $\beta_1 = 0$

$$Y_i = \beta_0 + \epsilon_i$$

- The SSE for the reduced model is obtained

$$SSE(R) = \sum(Y_i - b_0)^2 = \sum(Y_i - \bar{Y})^2 = SSTO$$

Test Statistic

- The idea is to compare the two error sums of squares SSE(F) and SSE(R).
- Because F has more parameters than R
  - SSE(F) $\leq$ SSE(R) always
- The relevant test statistic is

$$F^* = \frac{\frac{SSE(R)-SSE(F)}{df_R - df_F}}{\frac{SSE(F)}{df_F}}$$

  which follows the F distribution when H_0 holds.
- df_R and df_F are those associated with the reduced and full model error sumes of square respectively

R^2

- SSTO measures the variation in the observations Y_i when X is not considered
- SSE measures the variation in the Y_i after a predictor variable X is employed
- A natural measure of the effect of X in reducing variation in Y is to express the reduction in variation (SSTO-SSE = SSR) as a proportion of the total variation

$$R^2 = \frac{SSR}{SSTO} = 1 - \frac{SSE}{SSTO}$$