College of Computer & Information Science | Spring 2016
Northeastern University | 12 February 2016
CS 7880: Network Algorithms | Scribe: Tong Zhang

- Overview of the SIS Model
- Ganesh-Massoulié-Towsley: Upper Bound on Epidemic Die-Out

1 Overview of the SIS Model

We studied SIR Model/Branching Process in the previous lecture, today we will introduce SIS (Susceptible-Infective-Susceptible) model. There are three components.

2 SIS Model

- Contact network through which the infection spreads.
- Infection Rate along each edge is β.
- Recovery Rate δ.

Extent & Length of epidemic

We represent the system by a connected graph $G(V,E)$. Let $|V| = n$, and let the state at time t be represented by a vector $X(t) = (X_1(t), X_2(t), \ldots, X_n(t))^T$. X_i is defined as follows:

$$X_i(t) = \begin{cases} 1 & \text{if node } i \text{ is infected at time } t \\ 0 & \text{otherwise} \end{cases}$$

Assume that infected nodes X_i contaminate neighbors as a Poisson process with rate β and recover with a Poisson process with rate δ. This defines a continuous-time Markov process with transition rates:

$X_i :\ 0 \to 1$ at rate $\beta \sum_{(i,j)\in E} X_j$,
$X_i :\ 1 \to 0$ at rate δ.

Without loss of generality, we can assume $\delta = 1$, since it essentially corresponds to a normalization factor for the die-out time.

In other words, given $X_i(t) = 0$ we have $X_i(t + dt) = 1$ with probability $[\beta \sum_{(i,j)\in E} X_j(t)]dt$ for $dt \to 0$.

Note. X_i changes from 1 to 0 at rate r means if it takes time y to go from 1 to 0, then $Pr(y \geq t) = e^{-tr}$ (an exponential distribution with rate r).

DIE-OUT of epidemic

Let τ be the time takes for epidemic to die. Now the question we are interested in is: What is $E(\tau)$, as a function of β and G?

3 Ganesh-Massoulié-Towsley: Upper Bound on Die-Out Time

Let A be the adjacency matrix of G, $\lambda_1(A)$ be the largest eigenvalue of A.

Theorem 1. *If $\beta < \frac{1}{\lambda_1(A)}$, then $E(\tau) = O(\log n)$.*

Proof: Given $\beta\lambda_1(A) < 1$, at what t does $\sum_i X_i(t) = 0$? To approach this, we consider the function $Pr(\sum X_i(t) > 0)$ and see when this function goes to 0. Recall that,

If $X_i(t) = 1$, then

$$\begin{aligned} X_i(t+dt) &= 0 \quad \text{with probability } dt \\ X_i(t+dt) &= 1 \quad \text{with probability } 1 - dt \end{aligned}$$

If $X_i(t) = 0$, then

$$\begin{aligned} X_i(t+dt) &= 1 \quad \text{with probability } dt \cdot \beta \cdot \sum_{(i,j)\in E} X_j(t) \\ X_i(t+dt) &= 0 \quad \text{with probability } 1 - dt \cdot \beta \cdot \sum_{(i,j)\in E} X_j(t) \end{aligned}$$

The transition of X_i depends on the value of X_i, it is very hard to handle. Now we consider the continuous-time Markov process $Y = \{Y_i\}_{i\in V}$,

$Y_i(0) = X_i(0)$. For $k > 0$, the transition rates

$Y_i : k \to k+1 \quad$ at rate $\beta \sum_{(i,j)\in E} Y_j$

$Y_i : k \to k-1 \quad$ at rate Y_i

It is easy to see that $Y_i \in \{0, 1, 2, \dots\}$ (compared to $X_i \in \{0, 1\}$) and, when starting from the same initial conditions, Y_i stochastically dominates X_i. This stochastic dominance is obtained by a coupling argument in which we couple the elementary events in the probability space of X_i with those in the probability space of Y_i. We omit the details here.

By stochastic dominance we have $Pr(Y_i(t) \geq y) \geq Pr(X_i(t) \geq y)$. Therefore, we have Pr(Epidemic **not** die-out at time t)=$Pr(\sum X_i(t) > 0) \leq Pr(\sum Y_i(t) > 0)$.

We have

$$Y_i(t+dt) = \begin{cases} Y_i(t)+1 & \text{with probability } \beta \cdot \sum_{(i,j)\in E} Y_j(t) \\ Y_i(t)-1 & \text{with probability } Y_i(t)dt \\ Y_i(t) & \text{otherwise} \end{cases}$$

Note. In the above calculation we are ignoring the simultaneous occurrence of more than one event. This is because for $dt \to 0$, the associated probabilities are lower order terms (they are super-linear in dt).

Through continuous time $\{Y_i(t+dt) = Y_i(t)+1\}$ and $\{Y_i(t+dt) = Y_i(t)-1\}$ can never happen at the same time.

Since for all i we have

$$E[Y_i(t+dt) - Y_i(t)] = (\beta \sum_{(i,j)\in E} E[Y_j] - E[Y_i])dt$$

The transition rates for process Y(t) are such that

$$\frac{d(E(Y(t))}{dt} = (\beta A - I)E(Y(t))$$

where I denotes the identity matrix. Hence,

$$E(Y(t)) = e^{Mt} \cdot Y(0) \quad \text{ where } M = \beta A - I.$$

Note. $e^M = I + M + \frac{M^2}{2!} + \cdots$

Consider the following fact that if A has eigenvalue λ then $M = \beta A - I$ has eigenvalue $\beta\lambda - 1$ and e^M has eigenvalue $e^{\beta\lambda - 1}$. We obtain

$$\|E(Y(t))\|_2 \leq e^{(\beta\lambda_1 - 1)t} \cdot \|E(Y(0))\|_2.$$

Note. Cauchy-Schwarz inequality says that

$$|\langle x, y\rangle| \leq \|x\|_2 \cdot \|y\|_2$$

We obtain:

$$\sum_{i \in V} E(Y_i(t)) \leq \|E(Y(t))\|_2 \cdot \|\mathbf{1}\|_2$$

where $\mathbf{1}$ denotes the vector of ones, so $\|\mathbf{1}\|_2 = \sqrt{n}$.

Note. $Pr(Y_i > 0) \geq Pr(X_i > 0)$, which implies that $Pr(\sum Y_i > 0) \geq Pr(\sum X_i > 0)$. Moreover, by Markov Inequality it holds that:

$$\begin{aligned} Pr(\sum Y_i > 0) &= Pr(\sum Y_i \geq 1) \\ &\leq \sum E(Y_i). \end{aligned}$$

We have

$$\begin{aligned} Pr(\sum X_i(t) > 0) &\leq Pr(\sum Y_i(t) > 0) \\ &\leq \sum E(Y_i(t)) \\ &\leq \|E(Y(t))\|_2 \cdot \sqrt{n} \\ &\leq \sqrt{n} e^{(\beta\lambda_1 - 1)t} \cdot \|E(Y(0))\|_2 \\ &\leq n \cdot e^{(\beta\lambda_1 - 1)t} \end{aligned}$$

Note. $Y_i(0) = X_i(0)$, which takes values in {0,1}. So, $\|E(Y(0))\|_2 \leq \sqrt{n}$.

Say $t_1 = \frac{100 \ln n}{1 - \beta\lambda_1}$, then we have

$$Pr(\text{epidemic } \textbf{not} \text{ die-out at time } t_1) = Pr(\sum X_i(t_1) > 0) \leq \frac{1}{n^{99}}$$

We have for any $t \geq t_1$,

$$Pr(\text{epidemic } \textbf{not} \text{ die-out at time } t) \leq \frac{1}{n^{99}},$$

since the epidemic die-out is an absorbing state in the Markov process.

Therefore,

$$\begin{aligned}
E(\tau) &= \int_0^\infty Pr(\tau > t)dt \\
&= \int_0^\infty Pr(\sum X_i(t) > 0)dt \\
&\leq t_1 + \int_{t_1}^\infty Pr(\sum X_i(t) > 0)dt \\
&\leq \frac{100 \ln n}{1 - \beta\lambda_1} + \int_{t_1}^\infty \frac{1}{n^{99}} dt \\
&= O(\ln n).
\end{aligned}$$

□

Theorem 2. *If*

$$\beta > \frac{c}{\lambda_1(A) - \lambda_2(A)} \quad \textit{(where } c > 0 \textit{ is a sufficiently large constant)}$$

then $E(\tau) = \Omega(\exp(n))$.