A Spatial SIS Model in Advective Heterogeneous Environments *

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Abstract

We study the effects of diffusion and advection for a susceptible-infected-susceptible epidemic reaction-diffusion model in heterogeneous environments. The definition of the basic reproduction number $\mathcal{R}_0$ is given. If $\mathcal{R}_0 < 1$, the unique disease-free equilibrium (DFE) is globally asymptotically stable. Asymptotic behaviors of $\mathcal{R}_0$ for advection rate and mobility of the infected individuals (denoted by d_I) are established, and the existence of the endemic equilibrium when $\mathcal{R}_0 > 1$ is studied. The effects of diffusion and advection rates on the stability of the DFE are further investigated. Among other things, we find that if the habitat is a low-risk domain, there may exist one critical value for the advection rate, under which the DFE changes its stability at least twice as d_I varies from zero to infinity, while the DFE is unstable for any d_I when the advection rate is larger than the critical value. These results are in strong contrast with the case of no advection, where the DFE changes its stability at most once as d_I varies from zero to infinity.

Keywords: SIS epidemic model; reaction-diffusion-advection; spatial heterogeneity; disease-free equilibrium; endemic equilibrium

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1 Introduction

The spatial spread of diseases in heterogeneous habitats has received considerable attentions recently, as the environmental heterogeneity can be an important factor in disease

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dynamics. The following SIS (susceptible-infected-susceptible) epidemic reaction-diffusion model, which incorporated spatial heterogeneity, was proposed by Allen et al. in [3]:

$$\begin{cases} \bar{S}_t = d_S \Delta \bar{S} - \beta(x) \dfrac{\bar{S}\bar{I}}{\bar{S}+\bar{I}} + \gamma(x)\bar{I}, & x \in \Omega,\ t > 0, \\ \bar{I}_t = d_I \Delta \bar{I} + \beta(x) \dfrac{\bar{S}\bar{I}}{\bar{S}+\bar{I}} - \gamma(x)\bar{I}, & x \in \Omega,\ t > 0, \\ \dfrac{\partial \bar{S}}{\partial \nu} = \dfrac{\partial \bar{I}}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.1}$$

where $\bar{S}(x,t)$ and $\bar{I}(x,t)$ denote the density of susceptible and infected individuals in a given spatial region Ω, which is assumed to be a bounded domain in $\mathbb{R}^m$ $(m \geq 1)$ with smooth boundary $\partial\Omega$; the positive constants d_S and d_I are diffusion coefficients for the susceptible and infected populations; the positive functions $\beta(x)$ and $\gamma(x)$ are Hölder continuous on $\overline{\Omega}$ and represent the rates of disease transmission and recovery at location x, respectively. The main results of [3] concern the existence, uniqueness and asymptotic behaviors of the endemic equilibrium as the diffusion rate of the susceptible individuals approaches to zero. Allen et. al also investigated a discrete SIS-model in [2]. In [21] Peng and Liu discussed the global stability of the endemic equilibrium in some special cases. The effects of large and small diffusion rates of the susceptible and infected population on the persistence and extinction of the disease were considered in [20, 22]. Peng and Zhao [23] recently considered the same SIS reaction-diffusion model, but the rates of disease transmission and recovery are assumed to be spatially heterogeneous and temporally periodic. Ge et al. introduced a free boundary model for characterizing the spreading front of the disease in [10]. They showed that if the spreading domain is high-risk at some time, the disease will continue to spread till the whole area is infected; while if the spreading domain is low-risk, the disease may vanish or keep spreading, depending on the expanding capability and the initial number of infected individuals.

In some circumstances populations may take passive movement in certain direction, e.g., due to external environmental forces such as water flow [16, 17, 18], wind [8] and so on, which usually can be described by adding an advection term to the equation. Here, we consider a SIS epidemic reaction-diffusion model with advection. We are particularly interested in how the advection and diffusion affect the disease transmission.

1.1 SIS epidemic reaction-diffusion-advection model

In this paper, we will study the following SIS epidemic reaction-diffusion model with advection in one-dimensional space domain:

$$\begin{cases} \bar{S}_t = d_S \bar{S}_{xx} - q\bar{S}_x - \beta(x)\dfrac{\bar{S}\bar{I}}{\bar{S}+\bar{I}} + \gamma(x)\bar{I}, & 0 < x < L,\ t > 0, \\ \bar{I}_t = d_I \bar{I}_{xx} - q\bar{I}_x + \beta(x)\dfrac{\bar{S}\bar{I}}{\bar{S}+\bar{I}} - \gamma(x)\bar{I}, & 0 < x < L,\ t > 0, \\ d_S\bar{S}_x - q\bar{S} = d_I\bar{I}_x - q\bar{I} = 0, & x = 0, L,\ t > 0, \\ \bar{S}(x,0) = \bar{S}_0(x) \geq 0, \quad \bar{I}(x,0) = \bar{I}_0(x) \geq 0, & 0 < x < L, \end{cases} \tag{1.2}$$

where $\bar{S}(x,t)$ and $\bar{I}(x,t)$, respectively, represent the density of susceptible and infected individuals at time t and location x on the interval $[0,L]$; L is the size of the habitat, and we call $x = 0$ the upstream end and $x = L$ the downstream end; q is the effective speed of the current (sometimes we call q the advection speed/rate, and we remark here that q should be non-negative since $x = L$ is defined to be the downstream end). Here we impose no-flux boundary conditions at the upstream and downstream ends, respectively. It means that there is no population net flux across the boundary $x = 0$ and $x = L$. As mentioned in [3], since the term $\bar{S}\bar{I}/(\bar{S}+\bar{I})$ is a Lipschitz continuous function of $\bar{S}$ and $\bar{I}$ in the open first quadrant, its definition can be extended to the closure of the first quadrant by setting it to be zero when either $\bar{S} = 0$ or $\bar{I} = 0$. We also assume that there is a positive number of infected individuals, that is,

$$\textbf{(A)} \qquad \int_0^L \bar{I}(x,0)\mathrm{d}x > 0, \text{ with } \bar{S}(x,0) \geq 0 \text{ and } \bar{I}(x,0) \geq 0 \text{ for } x \in (0,L).$$

By the maximum principle [24], both $\bar{S}(x,t)$ and $\bar{I}(x,t)$ are positive for $x \in [0,L]$ and $t \in (0,T_{max})$, where T_{max} is the maximal existence time for solutions of (1.2). Then, by the maximum principle again, both $\bar{S}(x,t)$ and $\bar{I}(x,t)$ are bounded on $[0,L] \times (0,T_{max})$. Hence, it follows from the standard theory for semilinear parabolic systems that $T_{max} = \infty$ and system (1.2) admits a unique classical solution $(\bar{S}(x,t), \bar{I}(x,t))$ for all time [12]. Let

$$N \overset{def}{=} \int_0^L \left[\bar{S}(x,0) + \bar{I}(x,0)\right] \mathrm{d}x > 0 \tag{1.3}$$

be the total number of individuals in $(0,L)$ at $t = 0$. Summing two equations of (1.2) and integrating over $(0,L)$ gives

$$\frac{\partial}{\partial t}\int_0^L (\bar{S}+\bar{I})\mathrm{d}x = \int_0^L (d_S\bar{S}_{xx} + d_I\bar{I}_{xx})\mathrm{d}x - q\int_0^L (\bar{S}_x + \bar{I}_x)\mathrm{d}x = 0,\ t > 0. \tag{1.4}$$

Thus the total population size is constant in time, i.e.,

$$\int_0^L [\bar{S}(x,t) + \bar{I}(x,t)]\mathrm{d}x = N,\ t \geq 0. \tag{1.5}$$

From (1.5), we know that any solution $(\bar{S}(x,t), \bar{I}(x,t))$ satisfies L^1 space bound uniformly for $t \in [0,\infty)$. In fact, it can be concluded that for any fixed $q \geq 0$, $\|\bar{S}(\cdot,t)\|_{L^\infty([0,L])}$ and $\|\bar{I}(\cdot,t)\|_{L^\infty([0,L])}$ are also uniformly bounded in $[0,\infty)$, by following the argument in [1] (see also Exercise 4 of Section 3.5 in [12]). Unless otherwise stated, we assume that (A) holds and N is a fixed positive constant throughout this paper.

By adopting the same terminology as in [3], we say that x is a *low-risk* site if the local disease transmission rate $\beta(x)$ is lower than the local disease recovery rate $\gamma(x)$. A *high-risk* site is defined in a similar manner. We call that $(0,L)$ is a *low-risk domain* if $\int_0^L \beta(x)\mathrm{d}x < \int_0^L \gamma(x)\mathrm{d}x$ and a *high-risk domain* if $\int_0^L \beta(x)\mathrm{d}x > \int_0^L \gamma(x)\mathrm{d}x$. Here we note that $(0,L)$ is called a *high-risk domain* if $\int_0^L \beta(x)\mathrm{d}x \geq \int_0^L \gamma(x)\mathrm{d}x$ in [3]. For the sake of simplicity we do not consider the case $\int_0^L \beta(x)\mathrm{d}x = \int_0^L \gamma(x)\mathrm{d}x$, though some arguments still hold in this case.

1.2 The equilibrium problem

We are mainly interested in non-negative equilibrium solutions of (1.2), that is, the non-negative solutions of the following system:

$$\begin{cases} d_S\tilde{S}_{xx} - q\tilde{S}_x - \beta(x)\dfrac{\tilde{S}\tilde{I}}{\tilde{S}+\tilde{I}} + \gamma(x)\tilde{I} = 0, & 0 < x < L, \\ d_I\tilde{I}_{xx} - q\tilde{I}_x + \beta(x)\dfrac{\tilde{S}\tilde{I}}{\tilde{S}+\tilde{I}} - \gamma(x)\tilde{I} = 0, & 0 < x < L, \\ d_S\tilde{S}_x - q\tilde{S} = d_I\tilde{I}_x - q\tilde{I} = 0, & x = 0,\ L. \end{cases} \tag{1.6}$$

Here, $\tilde{S}(x)$ and $\tilde{I}(x)$ denote the density of susceptible and infected individuals, respectively, at $x \in [0,L]$. In view of (1.5), we impose the additional hypothesis

$$\int_0^L [\tilde{S}(x) + \tilde{I}(x)]\mathrm{d}x = N. \tag{1.7}$$

It is clear that only solutions $(\tilde{S}(x), \tilde{I}(x))$ satisfying $\tilde{S}(x) \geq 0$ and $\tilde{I}(x) \geq$ on $[0,L]$ are of interest. A *disease-free equilibrium* (DFE) is a solution of (1.6)-(1.7) so that $\tilde{I}(x) = 0$ for every $x \in (0,L)$; An *endemic equilibrium* (EE) of (1.6)-(1.7) is a solution in which $\tilde{I}(x) > 0$ for some $x \in (0,L)$. We denote a DFE by $(\hat{\tilde{S}}, 0)$ and an EE by $(\tilde{S}, \tilde{I})$. By direct computations and condition (1.7), we get $\hat{\tilde{S}} = qNe^{(q/d_S)x}/d_S(e^{qL/d_S} - 1)$. Thus (1.6)-(1.7) has a unique *disease-free equilibrium*, which is spatially inhomogeneous.

1.3 Statement of main results

According to the definition of the basic reproduction number in literatures [**?**, 29, 30], we define the basic reproduction number for model (1.2) as follows:

$$\mathcal{R}_0(d_I, q) = \sup_{\substack{\varphi \in H^1((0,L)) \\ \varphi \neq 0}} \left\{ \frac{\int_0^L \beta(x) e^{\frac{q}{d_I}x} \varphi^2 \mathrm{d}x}{d_I \int_0^L e^{\frac{q}{d_I}x} \varphi_x^2 \mathrm{d}x + \int_0^L \gamma(x) e^{\frac{q}{d_I}x} \varphi^2 \mathrm{d}x} \right\}.$$

From the definition of the basic reproduction number of (1.1), it can be seen that $\mathcal{R}_0$ is a smooth function of d_I and q. For the sake of convenience, we shall denote the basic reproduction number by $\mathcal{R}_0$. If the advection rate $q = 0$, we denote the basic reproduction number by $\hat{\mathcal{R}}_0$, so $\hat{\mathcal{R}}_0$ is a smooth function of d_I only. The basic reproduction number $\hat{\mathcal{R}}_0$ was introduced in [3], where it is shown that $\hat{\mathcal{R}}_0$ is a threshold value for the stability of the disease-free equilibrium: if $\hat{\mathcal{R}}_0 < 1$ then DFE is globally asymptotically stable, and if $\hat{\mathcal{R}}_0 > 1$ then the DFE is unstable. We can extend this conclusion to the model (1.2).

Theorem 1.1. *If $\mathcal{R}_0 < 1$ then the DFE is globally asymptotically stable, but if $\mathcal{R}_0 > 1$ then it is unstable.*

We now establish some qualitative properties of the basic reproduction number $\mathcal{R}_0$, in terms of d_I and q.

Theorem 1.2. *The following statements about $\mathcal{R}_0$ hold.*

(i) *Given any $q > 0$, $\mathcal{R}_0 \to \beta(L)/\gamma(L)$ as $d_I \to 0$ and $\mathcal{R}_0 \to \int_0^L \beta(x)\mathrm{d}x / \int_0^L \gamma(x)\mathrm{d}x$ as $d_I \to \infty$;*

(ii) *Given any $d_I > 0$, $\mathcal{R}_0 \to \hat{\mathcal{R}}_0$ as $q \to 0$ and $\mathcal{R}_0 \to \beta(L)/\gamma(L)$ as $q \to \infty$;*

(iii) *If $\beta(x) > (<) \gamma(x)$ on $[0, L]$, then $\mathcal{R}_0 > (<) 1$ for any $d_I > 0$ and $q > 0$.*

Part (i) shows that for any positive advection speed, the basic reproduction number $\mathcal{R}_0$ tends to the local reproduction number at the downstream end as d_I becomes arbitrarily small. This is in strong contrast with the case $q = 0$, for which $\hat{\mathcal{R}}_0 \to \max_{[0,L]} \beta(x)/\gamma(x)$ as $d_I \to 0$ (see [3]).

It is known that $\hat{\mathcal{R}}_0$ is a monotone decreasing function of d_I (see [3]), but for positive advection rate, parts (i) and (ii) suggest that the monotonicity of $\mathcal{R}_0$ with respect to d_I or q generally do not hold, as the condition at the downstream end plays a critical role in determining the dynamics of the model. From the biological point of view, since the influence of advection is from the upstream to the downstream, small diffusion or large advection tends to force the individuals to concentrate at the downstream end. Thus, when advection is strong or the diffusion is small, if the downstream end is a *low-risk* site, the disease will be eliminated; and if the downstream end is a *high-risk* site, the disease will persist.

Part (iii) of Theorem 1.2 shows that if every site of the domain is *low-risk*, the disease will be eliminated regardless of the advection speed and diffusion rate; and if every site of the domain is *high-risk*, the disease will persist for arbitrary advection speed and diffusion rate. It is natural to consider the case that the domain contains both *high-risk* sites and *low-risk* sites, i.e., the function $\beta(x) - \gamma(x)$ changes sign in the interval $(0, L)$. It turns out that this situation is much more complicated, as the answer may depend upon both locations and numbers of *high-risk* sites and *low-risk* sites. We will offer some discussions in the following theorems, which are the main analytical results of this paper.

We first study the stability of the DFE and the existence of the EE when $\beta(x) - \gamma(x)$ changes sign exactly once in $(0, L)$. It means that the domain contains only one *high-risk* sub-domain and one *low-risk* sub-domain. Moreover, from Theorem 1.2 we know that the stability of the DFE depends upon whether the downstream site is *high-risk* or *low-risk*. More precisely, we consider $\beta(x)$, $\gamma(x)$ satisfying the following assumptions:

(C1) $\beta(x)-\gamma(x)$ changes sign from negative to positive, i.e., $\beta(0)-\gamma(0) < 0 < \beta(L)-\gamma(L)$,

or

(C2) $\beta(x)-\gamma(x)$ changes sign from positive to negative, i.e., $\beta(0)-\gamma(0) > 0 > \beta(L)-\gamma(L)$.

For the biological point of view, assumption $(C1)$ implies that all *lower-risk* sites are located at the upstream and all *high-risk* sites are at the downstream. Similarly, assumption $(C2)$ implies that *high-risk* sites are distributed at the upstream and *lower-risk* sites are at the downstream.

In [3], for the model without advection, the stability of DFE is different when the habitat is a *high-risk* domain or *lower-risk* domain. We also analyze our model under these two cases. First we consider the case when the habitat is a *high-risk* domain.

Theorem 1.3. *Suppose that* $\int_0^L \beta(x)\mathrm{d}x > \int_0^L \gamma(x)\mathrm{d}x$.

(i) *If* $(C1)$ *holds, then the DFE is unstable for any* $q > 0$ *and* $d_I > 0$;

(ii) *If* $(C2)$ *holds, then there exists a unique curve*

$$\Gamma_1 = \{(d_I, \rho_1(d_I)) : \mathcal{R}_0(d_I, \rho_1(d_I)) = 1, d_I \in (0, \infty)\}$$

in $d_I - q$ *plane such that for every* $d_I > 0$ *the DFE is unstable for* $0 < q < \rho_1(d_I)$ *and the DFE is globally asymptotically stable for* $q > \rho_1(d_I)$. *Moreover, the function*

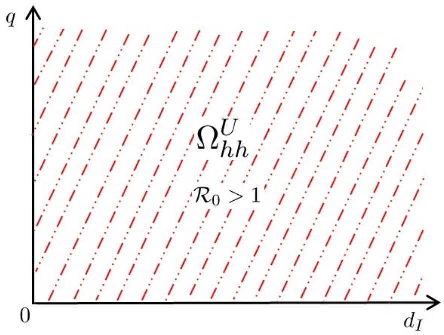

Figure 1: Illustration of the parameter regions of (d_I, q) in Theorem 1.3 part (i): If $\beta(x), \gamma(x)$ satisfy $\int_0^L \beta(x)\mathrm{d}x > \int_0^L \gamma(x)\mathrm{d}x$ and $\beta(L) > \gamma(L)$, then the DFE is unstable in the first quadrant.

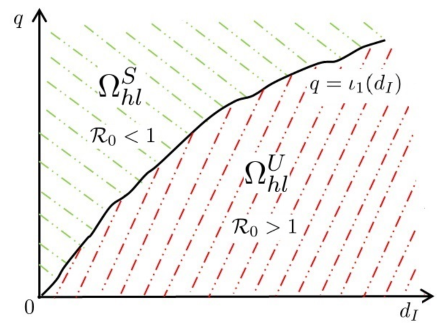

Figure 2: Illustration of the parameter regions of (d_I, q) in Theorem 1.3 part (ii): If $\beta(x), \gamma(x)$ satisfy $\int_0^L \beta(x)\mathrm{d}x > \int_0^L \gamma(x)\mathrm{d}x$ and $\beta(L) < \gamma(L)$, then the DFE is stable in Ω^S_{hl} and unstable in Ω^U_{hl}.

$\rho_1 : (0,\infty) \to (0,\infty)$ *satisfies*

$$\lim_{d_I \to 0+} \rho_1(d_I) = 0, \qquad \lim_{d_I \to \infty} \frac{\rho_1(d_I)}{d_I} = \theta_1,$$

where θ_1 *is the unique positive solution of*

$$\int_0^L [\beta(x) - \gamma(x)] e^{\theta_1 x} \mathrm{d}x = 0.$$

For different values of $d_I, q > 0$, the stability of the DFE will be different under suitable assumptions about $\beta(x)$ and $\gamma(x)$. To be more precise, we give the following definitions:

Definition 1.1. *Denote the regions on the $d_I - q$ plane identified in Theorem* 1.3 *as follows:*

$$\Omega^S_{hh} = \left\{ (d_I, q) : \mathcal{R}_0 < 1, \ \int_0^L \beta(x)\mathrm{d}x > \int_0^L \gamma(x)\mathrm{d}x, \ \beta(L) > \gamma(L) \right\},$$
$$\Omega^U_{hh} = \left\{ (d_I, q) : \mathcal{R}_0 > 1, \ \int_0^L \beta(x)\mathrm{d}x > \int_0^L \gamma(x)\mathrm{d}x, \ \beta(L) > \gamma(L) \right\},$$
$$\Omega^S_{hl} = \left\{ (d_I, q) : \mathcal{R}_0 < 1, \ \int_0^L \beta(x)\mathrm{d}x > \int_0^L \gamma(x)\mathrm{d}x, \ \beta(L) < \gamma(L) \right\},$$
$$\Omega^U_{hl} = \left\{ (d_I, q) : \mathcal{R}_0 > 1, \ \int_0^L \beta(x)\mathrm{d}x > \int_0^L \gamma(x)\mathrm{d}x, \ \beta(L) < \gamma(L) \right\}.$$

Remark 1.1. See Figures 1 and 2 for graphical illustrations of the stable and unstable regions of the DFE in Theorem 1.3. For part (i), $\Omega_{hh}^S = \emptyset$ and $\Omega_{hh}^U = \mathbb{R}^+ \times \mathbb{R}^+$, i.e., the unstable region of the DFE is the first quadrant in $d_I - q$ plane in this case. For part (ii), $\Omega_{hl}^S = \{(d_I, q) : d_I > 0, q > \rho_1(d_I)\}$ and $\Omega_{hh}^U = \{(d_I, q) : d_I > 0, 0 < q < \rho_1(d_I)\}$. The regions Ω_{hl}^S and Ω_{hl}^U are separated by the curve $q = \rho_1(d_I)$, where $q = \rho_1(d_I)$ is determined in Theorem 1.3. Some qualitative behaviors of $q = \rho_1(d_I)$ are given in Theorem 1.3.

In [3] it is shown that for the model without advection, DFE is always unstable when the domain is a *high-risk* domain, which means that the disease always persist for any d_I. Part (i) of Theorem 1.3 implies that if the habitat is a *high-risk* domain and the downstream end is a *high-risk* site, then the disease persists for arbitrary advection rate. From the biological point of view, for the model with advection, the advection causes the individuals to concentrate at the downstream end which is a *high-risk* site (we can understand it as a "bad" site), thus disease always persist in this case. For part (ii), we see that for any fixed advection rate, the DFE is stable for small d_I and is unstable for large d_I. In particular, the DFE changes its stability at least once as d_I varies from 0 to ∞. Biologically this indicates that since the downstream end is a *low-risk* site (similarly, we can understand it as a "good" site), the advection transports the individuals to a favorable location and thus it can help eliminate the disease. Hence, the disease will persist if the advection rate q is small relative to d_I, and the disease will be eliminated if the advection rate is large relative to d_I.

Next we give the precise description of the stability of the DFE when the habitat is a *low-risk* domain.

Theorem 1.4. *Suppose that $\int_0^L \beta(x)\mathrm{d}x < \int_0^L \gamma(x)\mathrm{d}x$, then there exists a constant $d_I^* > 0$ such that the following statements hold, where d_I^* is the unique positive root of the equation $\hat{\mathcal{R}}_0(d_I) = 1$:*

(i) *If $(C1)$ holds, then*

- *for $d_I \in (0, d_I^*]$, the DFE is unstable for any $q > 0$;*
- *for $d_I \in (d_I^*, \infty)$, there exists a curve*

$$\Gamma_2 = \{(d_I, \rho_2(d_I)) : \mathcal{R}_0(d_I, \rho_2(d_I)) = 1, d_I \in (d_I^*, \infty)\}$$

in $d_I - q$ plane such that the DFE is globally asymptotically stable for $0 < q < \rho_2(d_I)$ and the DFE is unstable for $q > \rho_2(d_I)$. Moreover, the function $\rho_2 : (d_I^*, \infty) \to (0, \infty)$ is a monotone increasing function of d_I and satisfies

$$\lim_{d_I \to d_I^*+} \rho_2(d_I) = 0, \qquad \lim_{d_I \to \infty} \frac{\rho_2(d_I)}{d_I} = \theta_2,$$

where θ_2 is the unique positive solution of

$$\int_0^L [\beta(x) - \gamma(x)] e^{\theta_2 x} \mathrm{d}x = 0.$$

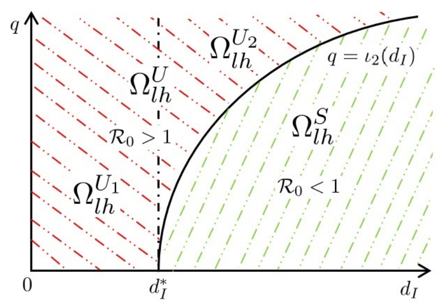

Figure 3: Illustration of the parameter regions of (d_I, q) in Theorem 1.4 part (i): If $\beta(x), \gamma(x)$ satisfy $\int_0^L \beta(x)\mathrm{d}x < \int_0^L \gamma(x)\mathrm{d}x$ and $\beta(L) > \gamma(L)$, then the DFE is stable in Ω_{lh}^S and unstable in Ω_{lh}^U.

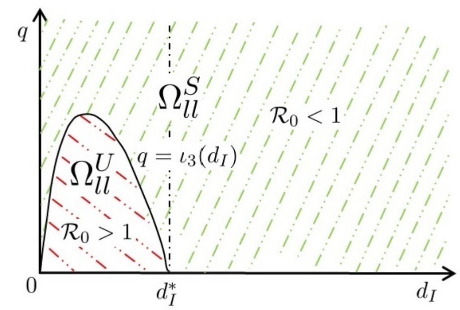

Figure 4: Illustration of the parameter regions of (d_I, q) in Theorem 1.4 part (ii): If $\beta(x), \gamma(x)$ satisfy $\int_0^L \beta(x)\mathrm{d}x < \int_0^L \gamma(x)\mathrm{d}x$ and $\beta(L) < \gamma(L)$, then the DFE is stable in Ω_{ll}^S and unstable in Ω_{ll}^U.

(ii) *If* $(C2)$ *holds, then*

- *for* $d_I \in (0, d_I^*)$, *there exists a curve*

$$\Gamma_3 = \{(d_I, \rho_3(d_I)) : \mathcal{R}_0(d_I, \rho_3(d_I)) = 1, d_I \in (0, d_I^*)\}$$

in $d_I - q$ *plane such that the DFE is unstable for* $0 < q < \rho_3(d_I)$ *and the DFE is globally asymptotically stable for* $q > \rho_3(d_I)$. *Moreover, the function* $\rho_3 : (0, d_I^*) \to (0, \infty)$ *satisfies*

$$\lim_{d_I \to 0+} \rho_3(d_I) = 0, \qquad \lim_{d_I \to d_I^*-} \rho_3(d_I) = 0;$$

- *for* $d_I \in [d_I^*, \infty)$, *the DFE is globally asymptotically stable for any* $q > 0$.

Similar to Definition 1.1, we give the following definition.

Definition 1.2. *Denote the regions on the* $d_I - q$ *plane identified in Theorem* 1.4 *as follows:*

$$\Omega_{lh}^S = \left\{(d_I, q) : \mathcal{R}_0 < 1, \int_0^L \beta(x)\mathrm{d}x < \int_0^L \gamma(x)\mathrm{d}x, \ \beta(L) > \gamma(L)\right\},$$

$$\Omega_{lh}^U = \left\{(d_I, q) : \mathcal{R}_0 > 1, \int_0^L \beta(x)\mathrm{d}x < \int_0^L \gamma(x)\mathrm{d}x, \ \beta(L) > \gamma(L)\right\},$$

$$\Omega_{ll}^S = \left\{(d_I, q) : \mathcal{R}_0 < 1, \int_0^L \beta(x)\mathrm{d}x < \int_0^L \gamma(x)\mathrm{d}x, \ \beta(L) < \gamma(L)\right\},$$

$$\Omega_{ll}^U = \left\{(d_I, q) : \mathcal{R}_0 > 1, \int_0^L \beta(x)\mathrm{d}x < \int_0^L \gamma(x)\mathrm{d}x, \ \beta(L) < \gamma(L)\right\}.$$

Remark 1.2. See Figures 3 and 4 for graphical illustrations of the stable and unstable regions of the DFE in Theorem 1.4. For part (i), $\Omega_{lh}^S = \{(d_I, q) : d_I > d_I^*, 0 < q < \rho_2(d_I)\}$ and $\Omega_{lh}^U = \Omega_{lh}^{U_1} \cup \Omega_{lh}^{U_2}$, where $\Omega_{lh}^{U_1} = \{(d_I, q) : 0 < d_I < d_I^*, q > 0\}$ and $\Omega_{lh}^{U_2} = \{(d_I, q) : d_I = d_I^*, q > 0\} \cup \{(d_I, q) : d_I > d_I^*, q > \rho_2(d_I)\}$. The regions Ω_{lh}^S and Ω_{lh}^U are separated by the curve $q = \rho_2(d_I)$, where $q = \rho_2(d_I)$ is determined in Theorem 1.4 and some qualitative behaviors of $q = \rho_2(d_I)$ are given in Theorem 1.4. For part (ii), $\Omega_{ll}^S = \{(d_I, q) : 0 < d_I < d_I^*, q > \rho_3(d_I)\} \cup \{(d_I, q) : d_I \geq d_I^*, q > 0\}$ and $\Omega_{ll}^U = \{(d_I, q) : 0 < d_I < d_I^*, 0 < q < \rho_3(d_I)\}$. The regions Ω_{ll}^S and Ω_{ll}^U are separated by the curve $q = \rho_3(d_I)$, where $q = \rho_3(d_I)$ is determined in Theorem 1.4 and some qualitative behaviors of $q = \rho_3(d_I)$ are given in Theorem 1.4.

For the model without advection, it is shown in [3] that when the domain is a *low-risk* domain, the DFE is stable if the mobility of infected individuals lies above a threshold value d_I^* and is unstable if the mobility of infected individuals lies below d_I^*. Part (i) of Theorem 1.4 implies that similar threshold value (denoted as d_I^{**}) exists for the model with advection, if the habitat is a *low-risk* domain and the downstream end is a *high-risk* site. In Theorem 1.4, the function $\rho_2(d_I)$ is a monotone increasing function of d_I so that d_I^{**} is uniquely determined by $\rho_2(d_I^{**}) = q$. Biologically, in the model with advection, since the downstream end is a "bad" site, the advection is beneficial to the persistence of the disease. Thus increasing advection will increase the threshold value for d_I at which the DFE changes the stability, i.e., d_I^{**} is an increasing function of q.

An interesting finding is part (ii) of Theorem 1.4. Set $\hat{q} = \max_{[0,d_I^*]} \rho_3(d_I)$. If $q > \hat{q}$, then the DFE is always stable for any $d_I > 0$. It means that there is a maximum advection speed $\hat{q}$ such that if the advection is very strong (i.e., if q is larger than $\hat{q}$), the disease will always be eliminated, regardless of the diffusion rate for infected individuals. Interestingly, if $0 < q < \hat{q}$, then the stability of the DFE changes at least twice: the DFE is stable for both small and large d_I and it is unstable for some intermediate values of d_I. When d_I is very small, advection will transport individuals to the downstream end, which is a "good" site in this case, so the disease will not persist. If d_I is very large, the disease can not persist as the habitat is a "low-risk" domain. However, for intermediate d_I, in this case the disease will persist since the advection rate q is relatively small with respect to d_I, i.e., d_I neither large or small so that the dynamics of the model is neither dominated solely by diffusion nor advection.

The following result establishes the existence of endemic equilibrium for a class of transmission and recovery rates:

Theorem 1.5. *Assume that* $\beta(x) - \gamma(x)$ *changes sign once in* $(0, L)$*. If* $\mathcal{R}_0 > 1$*, then problem* (1.6)-(1.7) *has at least an EE.*

In Theorems 1.3 and 1.4, we studied the stability of the DFE when the habitat contains exactly one *high-risk* sub-domain and one *lower-risk* sub-domain, i.e., $\beta(x) - \gamma(x)$ changes sign once in $(0, L)$. Next we consider the stability of the DFE when $\beta(x) - \gamma(x)$ changes sign twice in $(0, L)$ and illustrate that Theorems 1.3 and 1.4 may fail for such transmission and recovery rates.

Theorem 1.6. *Assume that $\beta(x) - \gamma(x)$ change signs twice in $(0, L)$.*

(i) *If $\int_0^L \beta(x)\mathrm{d}x > \int_0^L \gamma(x)\mathrm{d}x$ and $\beta(L) < \gamma(L)$, then there exists some positive constant Λ, independent of d_I and q, such that for every $d_I > \Lambda$, there exists a constant $Q > 0$ (dependent on d_I) such that $\mathcal{R}_0 > 1$ for $0 < q < Q$ and $\mathcal{R}_0 < 1$ for $q > Q$;*

(ii) *If $\int_0^L \beta(x)\mathrm{d}x < \int_0^L \gamma(x)\mathrm{d}x$ and $\beta(L) > \gamma(L)$, then there exists some positive constant $\Lambda > d_I^*$, independent of d_I and q, such that for every $d_I > \Lambda$, there exists a constant $Q > 0$ (dependent on d_I) such that $\mathcal{R}_0 < 1$ for $0 < q < Q$ and $\mathcal{R}_0 > 1$ for $q > Q$;*

(iii) *If $\int_0^L \beta(x)\mathrm{d}x > \int_0^L \gamma(x)\mathrm{d}x$ and $\beta(L) > \gamma(L)$, then there exists a constant $\Lambda > 0$ independent of d_I and q such that for every $d_I > \Lambda$, either $\mathcal{R}_0 > 1$ for any $q > 0$, or there exists a constant $\hat{Q} > 0$ independent of d_I such that $\mathcal{R}_0 > 1$ for $q \neq \hat{Q}$ and $\mathcal{R}_0 = 1$ for $q = \hat{Q}$, or there exist two constants $Q_2 > Q_1 > 0$ (both dependent on d_I) such that $\mathcal{R}_0 > 1$ for $q \in (0, Q_1) \cup (Q_2, \infty)$ and $\mathcal{R}_0 < 1$ for $q \in (Q_1, Q_2)$;*

(iv) *If $\int_0^L \beta(x)\mathrm{d}x < \int_0^L \gamma(x)\mathrm{d}x$ and $\beta(L) < \gamma(L)$, then there exists a constant $\Lambda > d_I^*$ independent of d_I and q such that for every $d_I > \Lambda$, either $\mathcal{R}_0 < 1$ for any $q > 0$, or there exists a constant $\hat{Q} > 0$ independent of d_I such that $\mathcal{R}_0 < 1$ for $q \neq \hat{Q}$ and $\mathcal{R}_0 = 1$ for $q = \hat{Q}$, or there exist two constants $Q_2 > Q_1 > 0$ (both dependent on d_I) such that $\mathcal{R}_0 < 1$ for $q \in (0, Q_1) \cup (Q_2, \infty)$ and $\mathcal{R}_0 > 1$ for $q \in (Q_1, Q_2)$.*

Part (i) is qualitatively similar to that of part (ii) of Theorem 1.3, when d_I is large enough. Part (ii) is qualitatively similar to that of part (i) of Theorem 1.4, when d_I is large enough. The results of parts (iii) and (iv) are noteworthy. For part (iii), if d_I is large enough, there are three cases:

Case 1. The DFE is unstable for any $q > 0$. This result is similar to that of Theorem 1.3 part (i).

Case 2. There exists a curve (in fact, it approaches a line with positive slope as $d_I \to \infty$) such that the DFE is unstable for any (d_I, q) except those on the curve.

Case 3. There exist two functions $q = \rho_4(d_I)$ and $q = \rho_5(d_I)$ such that the DFE is stable for $\rho_4(d_I) < q < \rho_5(d_I)$ and is unstable for $0 < q < \rho_4(d_I)$ and $q > \rho_5(d_I)$. Moreover, $q = \rho_4(d_I)$ and $q = \rho_5(d_I)$ satisfy

$$\lim_{d_I \to \infty} \frac{\rho_4(d_I)}{d_I} = \alpha_1, \quad \lim_{d_I \to \infty} \frac{\rho_5(d_I)}{d_I} = \alpha_2,$$

where $\alpha_1 < \alpha_2$ are the positive roots of

$$\int_0^L [\beta(x) - \gamma(x)] e^{\alpha_1 x} \mathrm{d}x = \int_0^L [\beta(x) - \gamma(x)] e^{\alpha_2 x} \mathrm{d}x = 0.$$

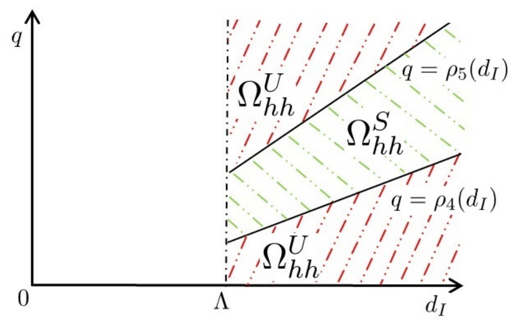

Figure 5: Illustration of the parameter regions of (d_I, q) in Theorem 1.6 part (iii): If $\beta(x), \gamma(x)$ satisfy $\int_0^L \beta(x)\mathrm{d}x > \int_0^L \gamma(x)\mathrm{d}x$ and $\beta(L) > \gamma(L)$, then the DFE is stable in Ω_{hh}^S and unstable in Ω_{hh}^U.

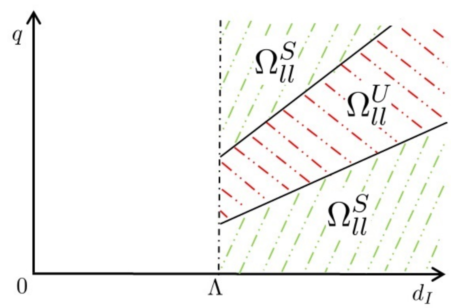

Figure 6: Illustration of the parameter regions of (d_I, q) in Theorem 1.6 part (iv): If $\beta(x), \gamma(x)$ satisfy $\int_0^L \beta(x)\mathrm{d}x < \int_0^L \gamma(x)\mathrm{d}x$ and $\beta(L) < \gamma(L)$, then the DFE is stable in Ω_{ll}^S and unstable in Ω_{ll}^U.

This last case is different from part (i) of Theorem 1.3: For each fixed $d_I > \Lambda$, the DFE changes the stability exactly twice, from unstable to stable to unstable, as q varies from 0 to ∞.

Remark 1.3. If $\beta(x) - \gamma(x)$ changes sign twice, we can only determine the stability of the DFE when the infected individuals diffusion rate d_I is large enough. It is unclear how the DFE changes its stability as d_I varies from 0 to ∞.

The rest of the paper is organized as follows. In Section 2, we consider the stability of the DFE and discuss some properties of the basic reproduction number in terms of the advection speed and the mobility of infected individuals. In Section 3, when the habitat changes from *high-risk sites* to *lower-risk sites* (or from *lower-risk sites* to *high-risk sites*), we study how the stability of the DFE depends precisely on the advection speed and the mobility of infected individual. The existence of the *endemic equilibrium* is established, by using the bifurcation theory and Leray-Schauder degree theory. Section 4 is devoted to the stability analysis of the DFE when the habitat contains several *high-risk* sub-domains and *lower-risk* sub-domains.

2 Qualitative properties of $\mathcal{R}_0$ and stability of the DFE

By the definition of the basic reproduction number $\mathcal{R}_0$, there exists some positive function $\Phi(x) \in C^2([0,L])$ such that

$$\begin{cases} -d_I\Phi_{xx} + q\Phi_x + \gamma(x)\Phi = \dfrac{1}{\mathcal{R}_0}\beta(x)\Phi, & 0 < x < L, \\ d_I\Phi_x(0) - q\Phi(0) = 0, \;\; d_I\Phi_x(L) - q\Phi(L) = 0. \end{cases} \tag{2.1}$$

Set $\varphi(x) = e^{-(q/d_I)x}\Phi(x)$, then $\varphi(x)$ satisfies

$$\begin{cases} -d_I\varphi_{xx} - q\varphi_x + \gamma(x)\varphi = \dfrac{1}{\mathcal{R}_0}\beta(x)\varphi, & 0 < x < L, \\ \varphi_x(0) = \varphi_x(L) = 0. \end{cases} \tag{2.2}$$

When $q = 0$, we denote the basic reproduction number $\mathcal{R}_0$ by $\hat{\mathcal{R}}_0$. The basic reproduction number $\hat{\mathcal{R}}_0$ was introduced in [3], where the following statements are established:

Lemma 2.1. *Suppose that $\beta(x) - \gamma(x)$ changes sign in $(0, L)$.*

- (i) *$\hat{\mathcal{R}}_0$ is a monotone decreasing function of d_I with $\hat{\mathcal{R}}_0 \to \max\{\beta(x)/\gamma(x) : x \in [0,L]\}$ as $d_I \to 0$ and $\hat{\mathcal{R}}_0 \to \int_0^L \beta(x)\mathrm{d}x / \int_0^L \gamma(x)\mathrm{d}x$ as $d_I \to \infty$;*
- (ii) *If $\int_0^L \beta(x)\mathrm{d}x \geq \int_0^L \gamma(x)\mathrm{d}x$, then $\hat{\mathcal{R}}_0 > 1$ for all $d_I > 0$;*
- (iii) *If $\int_0^L \beta(x)\mathrm{d}x < \int_0^L \gamma(x)\mathrm{d}x$, then there exists some threshold value $d_I^* \in (0,\infty)$ such that $\hat{\mathcal{R}}_0 > 1$ for $d_I < d_I^*$ and $\hat{\mathcal{R}}_0 < 1$ for $d_I > d_I^*$.*

Remark 2.1. *From the characterizations of d_I^* in [4], the threshold value d_I^* can be defined by*

$$d_I^* = \sup\left\{ \frac{\int_0^L (\beta-\gamma)\phi^2 \mathrm{d}x}{\int_0^L |\nabla\phi|^2 \mathrm{d}x} : \phi \in H^1((0,L)) \text{ and } \int_0^L (\beta-\gamma)\phi^2\mathrm{d}x > 0 \right\}. \tag{2.3}$$

It can be shown that if $\int_0^L \beta(x)\mathrm{d}x < \int_0^L \gamma(x)\mathrm{d}x$ then d_I^* is positive and finite; If $\int_0^L \beta(x)\mathrm{d}x \leq \int_0^L \gamma(x)\mathrm{d}x$ then $d_I^* = \infty$.

In the following we discuss some qualitative properties of $\mathcal{R}_0$ and the stability of the DFE. Theorem 1.1 is a consequence of Lemmas 2.3 and 2.4, Theorem 1.2 follows from Lemma 2.5, Corollary 2.6 and Lemma 2.7.

To study the stability of the DFE, we need to consider an eigenvalue problem associated with (1.2). We linearize (1.2) around the DFE to obtain

$$\begin{cases} \bar{\xi}_t = d_S\bar{\xi}_{xx} - q\bar{\xi}_x - [\beta(x) - \gamma(x)]\bar{\eta}, & 0 < x < L, \; t > 0, \\ \bar{\eta}_t = d_I\bar{\eta}_{xx} - q\bar{\eta}_x + [\beta(x) - \gamma(x)]\bar{\eta}, & 0 < x < L, \; t > 0. \end{cases}$$

Here $\bar{\xi}(x,t) = \bar{S}(x,t) - \hat{\bar{S}}(x,t)$ and $\bar{\eta}(x,t) = \bar{I}(x,t)$. Let $\bar{\xi}(x,t) = e^{-\lambda t}\xi(x)$ and $\bar{\eta}(x,t) = e^{-\lambda t}\eta(x)$ be the solution of the linear system. We then derive an eigenvalue problem

$$\begin{cases} d_S\xi_{xx} - q\xi_x - [\beta(x) - \gamma(x)]\eta + \lambda\xi = 0, & 0 < x < L, \\ d_I\eta_{xx} - q\eta_x + [\beta(x) - \gamma(x)]\eta + \lambda\eta = 0, & 0 < x < L, \end{cases} \tag{2.4}$$

with boundary conditions

$$\begin{cases} d_S\xi_x(0) - q\xi(0) = 0, \ \ d_S\xi_x(L) - q\xi(L) = 0, \\ d_I\eta_x(0) - q\eta(0) = 0, \ \ d_I\eta_x(L) - q\eta(L) = 0. \end{cases} \tag{2.5}$$

In view of (1.7), we impose an additional condition

$$\int_0^L [\xi(x) + \eta(x)]\mathrm{d}x = 0. \tag{2.6}$$

In fact, we only need to consider the eigenvalue problem

$$\begin{cases} d_I\eta_{xx} - q\eta_x + [\beta(x) - \gamma(x)]\eta + \lambda\eta = 0, & 0 < x < L, \\ d_I\eta_x(0) - q\eta(0) = 0, \ \ d_I\eta_x(L) - q\eta(L) = 0. \end{cases} \tag{2.7}$$

Set $\zeta(x) = e^{-(q/d_I)x}\eta(x)$, then $\zeta(x)$ satisfies

$$\begin{cases} d_I\zeta_{xx} + q\zeta_x + [\beta(x) - \gamma(x)]\zeta + \lambda\zeta = 0, & 0 < x < L, \\ \zeta_x(0) = \zeta_x(L) = 0. \end{cases} \tag{2.8}$$

It is well known that all eigenvalues are real, and the smallest eigenvalue, denote by $\lambda_1(d_I, q)$, is simple, and its corresponding eigenfunction ϕ_1 can be chosen positive [28].

First, we give the relation between $\mathcal{R}_0$ and $\lambda_1(d_I, q)$.

Lemma 2.2. *For any $d_I > 0$ and $q > 0$, we have $\mathcal{R}_0 > 1$ when $\lambda_1(d_I, q) < 0$, $\mathcal{R}_0 = 1$ when $\lambda_1(d_I, q) = 0$, and $\mathcal{R}_0 < 1$ when $\lambda_1(d_I, q) > 0$.*

Proof. Recall that $(\lambda_1(d_I, q), \phi_1)$ is the principal eigen-pair, i.e.,

$$\begin{cases} -d_I(\phi_1)_{xx} - q(\phi_1)_x + [\gamma(x) - \beta(x)]\phi_1 = \lambda_1(d_I, q)\phi_1, & 0 < x < L, \\ (\phi_1)_x(0) = (\phi_1)_x(L) = 0. \end{cases} \tag{2.9}$$

We multiply (2.1) by ϕ_1 and (2.9) by Φ, integrate by parts in $(0, L)$, and subtract the resulting equations to obtain

$$\lambda_1(d_I, q)\int_0^L \Phi(x)\phi_1(x)\mathrm{d}x = \left(\frac{1}{\mathcal{R}_0} - 1\right)\int_0^L \beta(x)\Phi(x)\phi_1(x)\mathrm{d}x.$$

Since $\int_0^L \Phi(x)\phi_1(x)\mathrm{d}x$ and $\int_0^L \beta(x)\Phi(x)\phi_1(x)\mathrm{d}x$ are both positive, we conclude that $\mathcal{R}_0 > 1$ when $\lambda_1(d_I, q) < 0$, $\mathcal{R}_0 = 1$ when $\lambda_1(d_I, q) = 0$, and $\mathcal{R}_0 < 1$ when $\lambda_1(d_I, q) > 0$. □

We remark that the positive function ϕ_1 satisfies equation (2.2) when $\mathcal{R}_0 = 1$, i.e., ϕ_1 denotes the corresponding eigenfunction of the basic reproduction number $\mathcal{R}_0 = 1$. The following lemma shows that the stability of the DFE relies on the magnitude of $\mathcal{R}_0$.

Lemma 2.3. *If $\mathcal{R}_0 < 1$, then the DFE is stable, but if $\mathcal{R}_0 > 1$ then it is unstable.*

Proof. **1.** Suppose first that $\mathcal{R}_0 < 1$. We will show that the DFE is linearly stable. Suppose the conclusion is false, then we can find (λ, ξ, η) which is a solution of (2.4)-(2.5) with the condition (2.6), with at least one of ξ and η not identical zero, and that $\mathrm{Re}(\lambda) \le 0$. Suppose that $\eta \equiv 0$, then $\xi \not\equiv 0$ on $[0, L]$. Furthermore, from (2.4) with boundary condition (2.5), we have

$$\begin{cases} d_S \xi_{xx} - q\xi_x + \lambda \xi = 0, & 0 < x < L, \\ d_S \xi_x(0) - q\xi(0) = 0, \ d_S \xi(L) - q\xi(L) = 0. \end{cases}$$

It is easy to see that λ is real and nonnegative, and therefore $\lambda = 0$. We find that $\xi = \xi_0 e^{(q/d_I)x}$, where ξ_0 is some constant to be determined later. By (2.6), $\xi_0 = 0$, i.e., $\xi \equiv 0$ on $[0, L]$. This is a contradiction. Then we conclude that $\eta \not\equiv 0$ on $[0, L]$. From (2.7), λ must be real and $\lambda \le 0$. Since $\lambda_1(d_I, q)$ is the principal eigenvalue, then $\lambda_1(d_I, q) \le \lambda \le 0$. Lemma 2.2 implies that $\mathcal{R}_0 \ge 1$, which is a contradiction. Then we conclude that if (λ, ξ, η) is a solution of (2.4)-(2.5), with at least one of ξ and η not identical zero on $[0, L]$, then $\mathrm{Re}(\lambda) > 0$. This proves the linear stability of the DFE.

2. Suppose that $\mathcal{R}_0 > 1$. We claim that the DFE is linearly unstable. In fact, $(\lambda_1(d_I, q), \phi_1)$ is the principal eigen-pair of (2.8). Then $(\lambda_1(d_I, q), e^{(q/d_I)x}\phi_1)$ satisfies the second equation of (2.4). Lemma 2.2 implies that $\lambda_1(d_I, q) < 0$. Consider the first equation of (2.4) with $(\lambda, \eta) = (\lambda_1(d_I, q), e^{(q/d_I)x}\phi_1)$, i.e.,

$$\begin{cases} d_S \xi_{xx} - q\xi_x + \lambda_1(d_I, q)\xi = (\beta(x) - \gamma(x)) e^{(q/d_I)x}\phi_1, & 0 < x < L, \\ d_S \xi_x(0) - q\xi(0) = 0, \ \ d_S \xi_x(L) - q\xi(L) = 0. \end{cases}$$

We see that there is a unique function ξ_1 satisfying the above equation and the condition $\int_0^L (\xi_1 + e^{(q/d_I)x}\phi_1)\mathrm{d}x = 0$. Thus, (2.4)-(2.5) has a solution $(\lambda_1(d_I, q), \xi_1, e^{(q/d_I)x}\phi_1)$ satisfying $\lambda_1(d_I, q) < 0$ and $e^{(q/d_I)x}\phi_1 > 0$ in $(0, L)$. Hence, the DFE is linearly unstable. □

Next we show that if $\mathcal{R}_0 < 1$ then the DFE is globally asymptotically stable.

Lemma 2.4. *If $\mathcal{R}_0 < 1$, then $(\bar{S}, \bar{I}) \to (\hat{\bar{S}}, 0)$ in $C([0, L])$ as $t \to \infty$.*

Proof. Suppose that $\mathcal{R}_0 < 1$. By the equation of $\bar{I}$ in (1.2), we have

$$\frac{\partial \bar{I}}{\partial t} \le d_I \bar{I}_{xx} - q\bar{I}_x + [\beta(x) - \gamma(x)]\bar{I}, \ \ x \in (0, L), \ \ t > 0.$$

Set $u(x,t) = Me^{-\lambda_1(d_I,q)t}e^{(q/d_I)x}\phi_1$ where $(\lambda_1(d_I,q),\phi_1)$ is the principal eigen-pair, $\lambda_1(d_I,q) > 0$ by Lemma 2.2, $\phi_1 > 0$ on $[0,L]$. M is chosen so large that $\bar{I}(x,0) \le u(x,0)$ for every $x \in (0,L)$. Hence, $u(x,t)$ satisfies

$$\begin{cases} u_t = d_I u_{xx} - qu_x + [\beta(x)-\gamma(x)]u, & x \in (0,L),\ t>0, \\ d_I u_x(0,t) - qu(0,t) = 0,\ d_I u_x(L,t) - qu(L,t) = 0, & t>0. \end{cases}$$

By the comparison principle, $\bar{I}(x,t) \le u(x,t)$ for every $x \in (0,L)$ and $t \ge 0$. Since $u(x,t) \to 0$ as $t \to \infty$ for every $x \in (0,L)$, we also have that $\bar{I}(x,t) \to 0$ as $t \to \infty$ for every $x \in (0,L)$.

Finally we show that $\bar{S}$ tends to $\hat{\hat{S}}$ as $t \to \infty$. Rewrite the first equation of (1.2) as

$$\bar{S}_t - d_S\bar{S}_{xx} + q\bar{S}_x = \left[\gamma(x) - \beta(x)\frac{\bar{S}}{\bar{S}+\bar{I}}\right]\bar{I},\ \ x \in (0,L),\ t>0.$$

By the continuity of $\beta(x)$ and $\gamma(x)$ on $[0,L]$, together with the above argument about $\bar{I}(x,t)$, we have

$$\left|\bar{S}_t - d_S\bar{S}_{xx} + q\bar{S}_x\right| \le C_1 e^{-\lambda_1(d_I,q)t},\ \ x \in (0,L),\ t>0,$$

for some positive constant C_1. Since the right-hand side tends to 0 exponentially, it follows that $\bar{S}(x,t)$ tends to some positive function $\check{\check{S}}$ as $t \to \infty$, where $\check{\check{S}}$ depends only on x and satisfies $\int_0^L \check{\check{S}}(x)\mathrm{d}x = N$. Thus, $\check{\check{S}} = \hat{\hat{S}}$. □

We remark that the global asymptotic stability of the DFE when $\mathcal{R}_0 < 1$ implies that there is no EE in this case.

To prove Theorem 1.2, we first establish the following asymptotic behavior $\mathcal{R}_0$ for d_I and q.

Lemma 2.5. *If $q/d_I \to \infty$ and $q^2/d_I \to \infty$, then $\mathcal{R}_0 \to \beta(L)/\gamma(L)$.*

Proof. Set $w(x) = e^{-(q/d_I)Ax}\Phi(x)$, where A is some constant which will be chosen differently for different purposes. Recall that $(\mathcal{R}_0, \Phi(x))$ satisfies (2.1), then w satisfies

$$\begin{cases} -d_I w_{xx} + q(1-2A)w_x + \left[\dfrac{q^2}{d_I}A(1-A) + \gamma(x)\right]w = \dfrac{1}{\mathcal{R}_0}\beta(x)w, & 0<x<L, \\ d_I w_x(0) = q(1-A)w(0),\ \ d_I w_x(L) = q(1-A)w(L). \end{cases} \tag{2.10}$$

Set $A = 1 + C_1 d_I/q^2$, where C_1 is some positive constant to be chosen later. Then w satisfies

$$\begin{cases} -d_I w_{xx} - q\left(1 + 2\dfrac{C_1 d_I}{q^2}\right)w_x + \left[-C_1\left(1+\dfrac{C_1 d_I}{q^2}\right) + \gamma(x) - \dfrac{1}{\mathcal{R}_0}\beta(x)\right]w = 0, & 0<x<L, \\ d_I w_x(0) = -\dfrac{C_1 d_I}{q}w(0),\ \ d_I w_x(L) = -\dfrac{C_1 d_I}{q}w(L). \end{cases} \tag{2.11}$$

Let $x_* \in [0, L]$ such that $w(x_*) = \min_{x\in[0,L]} w(x)$. Since $w_x(0) < 0$, then $x_* \neq 0$. If $x_* \in (0, L)$, then $w_{xx}(x_*) \geq 0$ and $w_x(x_*) = 0$. By (2.11) we have

$$-C_1 \left(1 + \frac{C_1 d_I}{q^2}\right) + \gamma(x_*) - \frac{1}{\mathcal{R}_0}\beta(x_*) \geq 0,$$

which is impossible for any small d_I/q^2 if we choose $C_1 = \gamma(x_1) = \max_{x\in[0,L]} \gamma(x)$ for some x_1. Therefore, $x_* = L$; i.e., $w(x) \geq w(L)$ for any $x \in [0, L]$. Hence,

$$\frac{\Phi(x)}{\Phi(L)} \geq e^{-\frac{q}{d_I}\left(1+\frac{C_1 d_I}{q^2}\right)(L-x)}. \tag{2.12}$$

Next, we choose $A = 1 - C_2 d_I/q^2$, where C_2 is some positive constant to be chosen later. By (2.10), w satisfies

$$\begin{cases} -d_I w_{xx} - q\left(1 - 2\frac{C_2 d_I}{q^2}\right) w_x + \left[C_2\left(1 - \frac{C_2 d_I}{q^2}\right) + \gamma(x) - \frac{1}{\mathcal{R}_0}\beta(x)\right] w = 0, & 0 < x < L, \\ d_I w_x(0) = \frac{C_2 d_I}{q} w(0), \ \ d_I w_x(L) = \frac{C_2 d_I}{q} w(L). \end{cases} \tag{2.13}$$

Let $x^* \in [0, L]$ such that $w(x^*) = \max_{x\in[0,L]} w(x)$. Since $w_x(0) > 0$, then $x^* \neq 0$. If $x^* \in (0, L)$, then $w_{xx}(x^*) \leq 0$ and $w_x(x^*) = 0$. By (2.13) we have

$$C_2\left(1 - \frac{C_2 d_I}{q^2}\right) + \gamma(x^*) - \frac{1}{\mathcal{R}_0}\beta(x^*) \leq 0. \tag{2.14}$$

Set $\beta(x_2) = \min_{x\in[0,L]} \beta(x)$ and $\beta(x_3) = \max_{x\in[0,L]} \beta(x)$, choose $C_2 = 2\beta(x_3)\gamma(x_1)/\beta(x_2)$ and $d_I/q^2 < \beta(x_2)/4\gamma(x_1)\beta(x_3)$, we can get

$$\begin{aligned} C_2\left(1 - \frac{C_2 d_I}{q^2}\right) + \gamma(x^*) - \frac{1}{\mathcal{R}_0}\beta(x^*) &\geq C_2\left(1 - \frac{C_2 d_I}{q^2}\right) + \gamma(x^*) - \frac{\gamma(x_1)}{\beta(x_2)}\beta(x^*) \\ &\geq C_2\left(1 - \frac{C_2 d_I}{q^2}\right) + \gamma(x^*) - \frac{\gamma(x_1)}{\beta(x_2)}\beta(x_3) \\ &\geq C_2\left(1 - \frac{C_2 d_I}{q^2}\right) - \frac{\gamma(x_1)}{\beta(x_2)}\beta(x_3) \\ &> 0, \end{aligned}$$

which contradicts (2.14). Therefore, $x^* = L$; i.e., $w(x) \leq w(L)$ for any $x \in [0, L]$. Hence,

$$\frac{\Phi(x)}{\Phi(L)} \leq e^{-\frac{q}{d_I}\left(1-\frac{C_2 d_I}{q^2}\right)(L-x)}. \tag{2.15}$$

Integrating (2.1) in $(0, L)$ and dividing the result by $\Phi(L)$, we have

$$\int_0^L \gamma(x)\frac{\Phi(x)}{\Phi(L)}\mathrm{d}x = \frac{1}{\mathcal{R}_0}\int_0^L \beta(x)\frac{\Phi(x)}{\Phi(L)}\mathrm{d}x. \tag{2.16}$$

Set $y = q(L-x)/d_I$, then Φ satisfies

$$e^{-\left(1+\frac{C_1 d_I}{q^2}\right)y} \le \frac{\Phi(L - d_I y/q)}{\Phi(L)} \le e^{-\left(1-\frac{C_2 d_I}{q^2}\right)y}. \tag{2.17}$$

We can rewrite (2.16) as

$$\int_0^{qL/d_I} \gamma(L - d_I y/q)\frac{\Phi(L - d_I y/q)}{\Phi(L)}\mathrm{d}y = \frac{1}{\mathcal{R}_0}\int_0^{qL/d_I} \beta(L - d_I y/q)\frac{\Phi(L - d_I y/q)}{\Phi(L)}\mathrm{d}y. \tag{2.18}$$

By (2.17), we can apply the Lebesgue dominant convergence theorem and pass to the limit in (2.18) to obtain

$$\begin{aligned}
\lim_{\substack{q/d_I\to\infty\\ q^2/d_I\to\infty}} \mathcal{R}_0 &= \frac{\lim\limits_{q/d_I\to\infty,\ q^2/d_I\to\infty} \int_0^{qL/d_I} \beta(L - d_I y/q)\frac{\Phi(L-d_I y/q)}{\Phi(L)}\mathrm{d}y}{\lim\limits_{q/d_I\to\infty,\ q^2/d_I\to\infty} \int_0^{qL/d_I} \gamma(L - d_I y/q)\frac{\Phi(L-d_I y/q)}{\Phi(L)}\mathrm{d}y} \\
&= \frac{\int_0^\infty \beta(L)e^{-y}\mathrm{d}y}{\int_0^\infty \gamma(L)e^{-y}\mathrm{d}y} \\
&= \frac{\beta(L)}{\gamma(L)}.
\end{aligned}$$

This completes the proof. □

Remark 2.2. We refer to [5] and [6] for some recent works on the asymptotic behaviors of the principal eigenvalue as advection rate tends to infinity or the diffusion coefficient tends to zero. Part of Lemma 2.5, e.g., the case $d_I \to \infty$ and $q/d_I \to \infty$, is not covered by [5] and [6].

By Lemma 2.5, we have the following result:

Corollary 2.6. *The following statements about $\mathcal{R}_0$ hold.*

(i) *Given any $d_I > 0$, $\mathcal{R}_0 \to \hat{\mathcal{R}}_0$ as $q \to 0$;*

(ii) *Given any $d_I > 0$, $\mathcal{R}_0 \to \beta(L)/\gamma(L)$ as $q \to \infty$;*

(iii) *Given any $q > 0$, $\mathcal{R}_0 \to \beta(L)/\gamma(L)$ as $d_I \to 0$;*

(iv) *Given any* $q>0$, $\mathcal{R}_0 \to \int_0^L \beta(x)\mathrm{d}x / \int_0^L \gamma(x)\mathrm{d}x$ *as* $d_I \to \infty$.

Proof. We only need to prove part (iv) as (i) is obvious and (ii) and (iii) follow from Lemma 2.5. By the definition of the basic reproduction number $\mathcal{R}_0$, we can obtain

$$\frac{1}{\mathcal{R}_0} = \inf_{\substack{\varphi \in H^1((0,L)) \\ \varphi \neq 0}} \left\{ \frac{d_I \int_0^L e^{\frac{q}{d_I}x} \varphi_x^2 \mathrm{d}x + \int_0^L \gamma(x) e^{\frac{q}{d_I}x} \varphi^2 \mathrm{d}x}{\int_0^L \beta(x) e^{\frac{q}{d_I}x} \varphi^2 \mathrm{d}x} \right\} \leq \frac{\int_0^L \gamma(x) e^{\frac{q}{d_I}x} \mathrm{d}x}{\int_0^L \beta(x) e^{\frac{q}{d_I}x} \mathrm{d}x} \leq \frac{\max_{x\in[0,L]} \gamma(x)}{\min_{x\in[0,L]} \beta(x)},$$

where the first inequality is obtained by setting $\varphi \equiv 1$. Thus, $1/\mathcal{R}_0$ is uniformly bounded for $d_I > 0$, passing to a subsequence if necessary, it has a finite limit $1/\bar{\mathcal{R}}_0$ as $d_I \to \infty$. By standard elliptic regularity and the Sobolev embedding theorem [11], the function Φ in (2.1) is uniformly bounded in $C^2([0,L])$ for all $d_I \geq 1$. Therefore, dividing both sides of (2.1) by d_I and passing to some subsequence if necessary, we can get $\Phi \to \bar{\Phi}$ in $C([0,L])$ as $d_I \to \infty$ for some positive constant $\bar{\Phi}$. We multiply (2.1) by $e^{-(q/d_I)x}$ and integrate by parts over $(0,L)$ to obtain

$$\int_0^L \gamma(x)\Phi(x) e^{-\frac{q}{d_I}x} \mathrm{d}x = \frac{1}{\mathcal{R}_0} \int_0^L \beta(x)\Phi(x) e^{-\frac{q}{d_I}x} \mathrm{d}x.$$

Letting $d_I \to \infty$ we obtain $\bar{\mathcal{R}}_0 = \int_0^L \beta(x)\mathrm{d}x / \int_0^L \gamma(x)\mathrm{d}x$. ☐

Lemma 2.7. *The following statements about* $\mathcal{R}_0$ *hold.*

(i) *If* $\beta(x) > \gamma(x)$ *on* $[0,L]$*, then* $\mathcal{R}_0 > 1$ *for any* $d_I > 0$ *and* $q > 0$*;*

(ii) *If* $\beta(x) < \gamma(x)$ *on* $[0,L]$*, then* $\mathcal{R}_0 < 1$ *for any* $d_I > 0$ *and* $q > 0$*.*

Proof. (i) By the definition of $\mathcal{R}_0$ and the condition (i), we have

$$\mathcal{R}_0 = \sup_{\substack{\varphi \in H^1((0,L)) \\ \varphi \neq 0}} \left\{ \frac{\int_0^L \beta(x) e^{\frac{q}{d_I}x} \varphi^2 \mathrm{d}x}{d_I \int_0^L e^{\frac{q}{d_I}x} \varphi_x^2 \mathrm{d}x + \int_0^L \gamma(x) e^{\frac{q}{d_I}x} \varphi^2 \mathrm{d}x} \right\} \geq \frac{\int_0^L e^{\frac{q}{d_I}x} \beta(x) \mathrm{d}x}{\int_0^L e^{\frac{q}{d_I}x} \gamma(x) \mathrm{d}x} > 1.$$

The first inequality is obtained by $\varphi \equiv 1$.

(ii) We subtract both sides of (2.2) by $\beta(x)\varphi$, multiply by $e^{(q/d_I)x}\varphi$ and integrate by parts over $(0,L)$ to obtain

$$d_I \int_0^L e^{\frac{q}{d_I}x} \varphi_x^2 \mathrm{d}x + \int_0^L [\gamma(x) - \beta(x)] e^{\frac{q}{d_I}x} \varphi^2 \mathrm{d}x = \left(\frac{1}{\mathcal{R}_0} - 1 \right) \int_0^L \beta(x) e^{\frac{q}{d_I}x} \varphi^2 \mathrm{d}x.$$

Since $\beta(x) < \gamma(x)$ on $[0,L]$, we have

$$\left(\frac{1}{\mathcal{R}_0} - 1 \right) \int_0^L \beta(x) e^{\frac{q}{d_I}x} \varphi^2 \mathrm{d}x \geq \int_0^L [\gamma(x) - \beta(x)] e^{\frac{q}{d_I}x} \varphi^2 \mathrm{d}x > 0,$$

which implies that $\mathcal{R}_0 < 1$. ☐

3 Further properties of $\mathcal{R}_0$: $\beta(x) - \gamma(x)$ changing sign once

In this section, we study the stability of the DFE and the existence of the EE when $\beta(x) - \gamma(x)$ changes sign exactly once in $(0, L)$ and $\beta(x)$, $\gamma(x)$ satisfy the assumption $(C1)$ or $(C2)$.

The following result will be frequently used in this section.

Lemma 3.1. *Let ϕ_1 be a positive eigenfunction corresponding to $\mathcal{R}_0 = 1$. If $\beta(x) - \gamma(x)$ changes sign once in $(0, L)$ and assumption $(C1)$ (or $(C2)$) holds, then $(\phi_1)_x > 0$ (or < 0) in $(0, L)$.*

Proof. We only consider the case that $\beta(x) - \gamma(x)$ changes sign once in $(0, L)$ and $(C1)$ holds, i.e., $\beta(0) - \gamma(0) < 0 < \beta(L) - \gamma(L)$. Note that ϕ_1 satisfies

$$\begin{cases} -d_I(\phi_1)_{xx} - q(\phi_1)_x + \gamma(x)\phi_1 = \beta(x)\phi_1, & 0 < x < L, \\ (\phi_1)_x(0) = (\phi_1)_x(L) = 0. \end{cases} \tag{3.1}$$

Multiplying (3.1) by $e^{(q/d_I)x}$, we rewrite the equation as

$$-d_I(e^{(q/d_I)x}(\phi_1)_x)_x = [\beta(x) - \gamma(x)]e^{(q/d_I)x}\phi_1.$$

By assumption $(C1)$, there exists some $x_0 \in (0, L)$ such that $\beta(x) - \gamma(x) < 0$ in $(0, x_0)$ and $\beta(x) - \gamma(x) > 0$ in (x_0, L). Thus, $(e^{(q/d_I)x}(\phi_1)_x)_x > 0$ in $(0, x_0)$ and $(e^{(q/d_I)x}(\phi_1)_x)_x < 0$ in (x_0, L). Therefore, $e^{(q/d_I)x}(\phi_1)_x$ is strictly increasing in $(0, x_0)$ and strictly decreasing in (x_0, L). According to the boundary condition $(\phi_1)_x(0) = (\phi_1)_x(L) = 0$, we get $e^{(q/d_I)x}(\phi_1)_x > 0$ in $(0, L)$. Thus, $(\phi_1)_x > 0$ in $(0, L)$. □

In subsection 3.1, we study the stability of the DFE in terms of d_I and q. Subsection 3.2 is devoted to the existence of the EE.

3.1 The stability of the DFE

In this subsection, we study the stability of the DFE and our main goal is to establish Theorems 1.3 and 1.4. Theorem 1.3 is a consequence of Lemmas 3.4 and 3.5, and Theorem 1.4 follows from Lemmas 3.6 and 3.7.

In order to consider the stability of the DFE in terms of the diffusion rate d_I and the advection rate q, we need to study the asymptotic behavior of $q(d_I)$ for sufficiently large d_I, where d_I and $q(d_I)$ satisfy $\mathcal{R}_0(d_I, q(d_I)) = 1$. To this end, we first consider some auxiliary function F.

For any continuous function $m(x)$ on $[0, L]$, define

$$F(\eta) = \int_0^L e^{\eta x} m(x) \mathrm{d}x, \quad 0 \le \eta < \infty.$$

Lemma 3.2. *Assume that $m(x)$ is continuous on $[0, L]$. If $m(L) > (<)\, 0$, then there exists some positive constant M such that $F(\eta) > (<)\, 0$ for any $\eta > M$.*

Proof. Since

$$\lim_{\eta \to +\infty} \eta e^{-\eta L} F(\eta) = \lim_{\eta \to +\infty} \int_0^L \eta e^{\eta(x-L)} m(x) \mathrm{d}x = m(L) > (<)\, 0,$$

we can see that there exists some positive constant M such that $F(\eta) > (<)\, 0$ for $\eta > M$. $\square$

Lemma 3.3. *Suppose that $m(x)$ changes sign once in $(0, L)$. The following statements hold:*

(i) *If $m(x)$ satisfies $m(L) > 0$ and $\int_0^L m(x)\mathrm{d}x > 0$, then $F(\eta) > 0$ for any $\eta > 0$;*

(ii) *If $m(x)$ satisfies $m(L) < 0$ and $\int_0^L m(x)\mathrm{d}x < 0$, then $F(\eta) < 0$ for any $\eta > 0$;*

(iii) *If $m(x)$ satisfies $m(L) > 0$ and $\int_0^L m(x)\mathrm{d}x < 0$, then $F(\eta)$ has a unique positive root, denoted by η_1, for $\eta \in (0, +\infty)$ and $F'(\eta_1) > 0$;*

(iv) *If $m(x)$ satisfies $m(L) < 0$ and $\int_0^L m(x)\mathrm{d}x > 0$, then $F(\eta)$ has a unique positive root, denoted by η_1, for $\eta \in (0, +\infty)$ and $F'(\eta_1) < 0$.*

Proof. It suffices to prove parts (i) and (iii). If $m(x)$ changes sign once in $(0, L)$ and $m(L) > 0$, i.e., there exists $x_1 \in (0, L)$ such that $m(x_1) = 0$, then $m(x)(x - x_1) > 0$ for $x \in (0, L)$ and $x \ne x_1$. By direct computation we obtain

$$[e^{-x_1 \eta} F(\eta)]' = e^{-x_1 \eta}[F'(\eta) - x_1 F(\eta)] = \int_0^L e^{\eta(x - x_1)} m(x)(x - x_1) \mathrm{d}x > 0. \tag{3.2}$$

Here the prime notation denotes differentiation by η.

Since $F(0) = \int_0^L m(x)\mathrm{d}x > 0$ and $e^{-x_1\eta} F(\eta)$ is strictly increasing in $\eta \in (0, \infty)$, $F(\eta) > 0$ for any $\eta \ge 0$. Thus (i) of Lemma 3.3 holds.

If $m(x)$ satisfies $\int_0^L m(x)\mathrm{d}x < 0$, by $m(L) > 0$ and Lemma 3.2, then $F(\eta)$ has at least a positive root for $\eta \in (0, +\infty)$. As $e^{-x_1\eta} F(\eta)$ is strictly increasing in $\eta \in (0, \infty)$, $F(\eta) = 0$ has a unique positive root, denoted as η_1. By equation (3.2), $F'(\eta_1) > x_1 F(\eta_1) = 0$. This completes the proof of (iii). $\square$

Remark 3.1. In parts (iii) and (iv) of Lemma 3.4, the monotonicity of $F(\eta)$ for $\eta \in (0, \infty)$ generally do not hold. For part (iii), we can show that $F(\eta)$ is strictly increasing in (η_1, ∞). Similarly in part (iv), $F(\eta)$ is strictly decreasing for $\eta \in (\eta_1, \infty)$.

Lemma 3.4. *Suppose that $\beta(x) - \gamma(x)$ changes sign once in $(0, L)$ and $\int_0^L \beta(x)\mathrm{d}x > \int_0^L \gamma(x)\mathrm{d}x$.*

(i) *If $\beta(x)$ and $\gamma(x)$ satisfy (C1), then $\mathcal{R}_0 > 1$ for any $d_I > 0$ and $q > 0$;*

(ii) *If $\beta(x)$ and $\gamma(x)$ satisfy (C2), then for every $d_I > 0$, there exists a unique $\bar{q} = \bar{q}(d_I)$ such that $\mathcal{R}_0 > 1$ for $0 < q < \bar{q}$ and $\mathcal{R}_0 < 1$ for $q > \bar{q}$.*

Proof. (i) We subtract both sides of (2.2) by $\beta(x)\varphi$, multiply by $e^{(q/d_I)x}/\varphi$ and integrate by parts over $(0, L)$ to obtain

$$d_I \int_0^L \frac{e^{\frac{q}{d_I}x}\varphi_x^2}{\varphi^2}\mathrm{d}x + \int_0^L [\beta(x) - \gamma(x)]e^{\frac{q}{d_I}x}\mathrm{d}x = \left(1 - \frac{1}{\mathcal{R}_0}\right)\int_0^L \beta(x)e^{\frac{q}{d_I}x}\mathrm{d}x.$$

By Lemma 3.3 with $m(x) = \beta(x) - \gamma(x)$, we see that $\int_0^L [\beta(x) - \gamma(x)]e^{\frac{q}{d_I}x}\mathrm{d}x > 0$ for any $d_I > 0$ and $q > 0$. We can thus conclude that

$$\left(1 - \frac{1}{\mathcal{R}_0}\right)\int_0^L \beta(x)e^{\frac{q}{d_I}x}\mathrm{d}x \geq \int_0^L [\beta(x) - \gamma(x)]e^{\frac{q}{d_I}x}\mathrm{d}x > 0,$$

which implies that $\mathcal{R}_0 > 1$ for any $d_I > 0$ and $q > 0$.

(ii) First we compute $\partial\mathcal{R}_0/\partial q$. One differentiates both sides of (2.2) with respect to q to get

$$\begin{cases} -d_I\varphi'_{xx} - \varphi_x - q\varphi'_x + \gamma(x)\varphi' = -\dfrac{\mathcal{R}'_0}{\mathcal{R}_0^2}\beta(x)\varphi + \dfrac{1}{\mathcal{R}_0}\beta(x)\varphi', & 0 < x < L, \\ \varphi'_x(0) = \varphi'_x(L) = 0. \end{cases} \tag{3.3}$$

Here the prime notation denotes differentiation with respect to q. Multiplying (3.3) by $e^{(q/d_I)x}\varphi$ and integrating the resulting equation in $(0, L)$, we get

$$\begin{aligned} &d_I \int_0^L e^{\frac{q}{d_I}x}\varphi'_x\varphi_x\mathrm{d}x - \int_0^L e^{\frac{q}{d_I}x}\varphi_x\varphi\mathrm{d}x + \int_0^L \gamma(x)e^{\frac{q}{d_I}x}\varphi'\varphi\mathrm{d}x \\ &= -\frac{\mathcal{R}'_0}{\mathcal{R}_0^2}\int_0^L \beta(x)e^{\frac{q}{d_I}x}\varphi^2\mathrm{d}x + \frac{1}{\mathcal{R}_0}\int_0^L \beta(x)e^{\frac{q}{d_I}x}\varphi'\varphi\mathrm{d}x. \end{aligned} \tag{3.4}$$

Multiplying (2.2) by $e^{(q/d_I)x}\varphi'$ and integrating the resulting equation in $(0, L)$, we obtain

$$d_I \int_0^L e^{\frac{q}{d_I}x}\varphi_x\varphi'_x\mathrm{d}x + \int_0^L \gamma(x)e^{\frac{q}{d_I}x}\varphi\varphi'\mathrm{d}x = \frac{1}{\mathcal{R}_0}\int_0^L \beta(x)e^{\frac{q}{d_I}x}\varphi\varphi'\mathrm{d}x. \tag{3.5}$$

Subtracting (3.4) and (3.5), we obtain

$$\frac{\partial \mathcal{R}_0}{\partial q} = \frac{\mathcal{R}_0^2 \int_0^L e^{\frac{q}{d_I}x} \varphi_x \varphi \mathrm{d}x}{\int_0^L \beta(x) e^{\frac{q}{d_I}x} \varphi^2 \mathrm{d}x}. \tag{3.6}$$

By Corollary 2.6 we have

$$\lim_{q\to\infty} \mathcal{R}_0 = \frac{\beta(L)}{\gamma(L)} < 1.$$

On the other hand, we have $\lim_{q\to 0} \mathcal{R}_0 = \hat{\mathcal{R}}_0 > 1$ for any d_I. Then, there must exist at least some $\bar{q}$ such that $\mathcal{R}_0(\bar{q}) = 1$. We only need to show that for any $\bar{q}$ satisfying $\mathcal{R}_0(\bar{q}) = 1$, $\mathcal{R}_0'(\bar{q}) < 0$. From (3.6) and Lemma 3.1, we see that for any $\bar{q} > 0$ satisfying $\mathcal{R}_0(\bar{q}) = 1$,

$$\mathcal{R}_0'(\bar{q}) = \frac{\int_0^L e^{\frac{\bar{q}}{d_I}x} (\phi_1)_x \phi_1 \mathrm{d}x}{\int_0^L \beta(x) e^{\frac{\bar{q}}{d_I}x} (\phi_1)^2 \mathrm{d}x} < 0.$$

This implies that $\bar{q}$ is the unique point satisfying $\mathcal{R}_0(\bar{q}) = 1$. □

Remark 3.2. In Lemma 3.4, we can not obtain the monotonicity of $\mathcal{R}_0$ with respect to q. We can only determine the sign of the derivative for $\mathcal{R}_0$ at $\bar{q}$. If $\beta(x) - \gamma(x)$ is strictly monotone, see [13] for related results.

From part (ii) of Lemma 3.4, we see that there exists a function $q = \rho_1(d_I)$ such that $\mathcal{R}_0(d_I, \rho_1(d_I)) = 1$. We determine the asymptotic profile of $\rho_1(d_I)$ in the following result.

Lemma 3.5. *Suppose that $\beta(x) - \gamma(x)$ changes sign once in $(0, L)$ and $\int_0^L \beta(x)\mathrm{d}x > \int_0^L \gamma(x)\mathrm{d}x$. If $\beta(x)$, $\gamma(x)$ satisfy (C2), then there exists a function $\rho_1 : (0,\infty) \to (0,\infty)$ such that $\mathcal{R}_0(d_I, \rho_1(d_I)) = 1$. Moveover, ρ_1 satisfies*

$$\lim_{d_I\to 0} \rho_1(d_I) = 0, \ \lim_{d_I\to\infty} \frac{\rho_1(d_I)}{d_I} = \theta_1,$$

where θ_1 is the unique solution of

$$\int_0^L [\beta(x) - \gamma(x)] e^{\theta_1 x} \mathrm{d}x = 0.$$

Proof. **1.** First, we consider the limit of $\rho_1(d_I)/d_I$ as $d_I \to \infty$. Suppose that $\rho_1(d_I)/d_I \to \infty$ as $d_I \to \infty$. By Lemma 2.5 and assumption (C2) we get

$$\lim_{\substack{\rho_1(d_I)\to\infty \\ \rho_1(d_I)/d_I\to\infty}} \mathcal{R}_0(d_I, \rho_1(d_I)) = \frac{\beta(L)}{\gamma(L)} < 1,$$

which contradicts $\mathcal{R}_0(d_I, \rho_1(d_I)) = 1$.

Next, we claim that $\rho_1(d_I)/d_I \to \theta_1$ as $d_I \to \infty$, where θ_1 is the unique positive root of $\int_0^L e^{\theta_1 x}[\beta(x) - \gamma(x)]\mathrm{d}x = 0$. By the above argument, $\rho_1(d_I)/d_I$ is bounded for large d_I. Passing to a subsequence if necessary, as $d_I \to \infty$, we assume that $\rho_1(d_I)/d_I \to \theta_*$ for some non-negative number θ_*. Let $\tilde{\varphi}$ denote an eigenfunction of the eigenvalue $\mathcal{R}_0(d_I, \rho_1(d_I)) = 1$, uniquely determined by $\max_{[0,L]} \tilde{\varphi} = 1$. Hence, $\tilde{\varphi}$ satisfies

$$\begin{cases} -d_I\big(e^{\frac{\rho_1(d_I)}{d_I}x}\tilde{\varphi}_x\big)_x + [\gamma(x) - \beta(x)]e^{\frac{\rho_1(d_I)}{d_I}x}\tilde{\varphi} = 0, & 0 < x < L, \\ \tilde{\varphi}_x(0) = \tilde{\varphi}_x(L) = 0. \end{cases} \tag{3.7}$$

Integrating (3.7) in $(0, L)$, we have

$$\int_0^L e^{\frac{\rho_1(d_I)}{d_I}x}[\beta(x) - \gamma(x)]\tilde{\varphi}\mathrm{d}x = 0.$$

By standard regularity and passing to a subsequence if necessary, as $d_I \to \infty$, we may assume that $\tilde{\varphi} \to 1$ in $C([0, L])$. Passing to the limit by letting $d_I \to \infty$, we have

$$\int_0^L e^{\theta_* x}[\beta(x) - \gamma(x)]\mathrm{d}x = 0.$$

By Lemma 3.3 with $m(x) = \beta(x) - \gamma(x)$, we know $F(\eta)$ has a unique positive root, i.e., $\theta_* = \theta_1$.

2. Suppose that there exists some positive constant q^* such that $q = \rho_1(d_I) \to q^*$ as $d_I \to 0$. By Lemma 2.5, we conclude

$$\lim_{\substack{\rho_1(d_I) \to q^* \\ \rho_1(d_I)/d_I \to \infty}} \mathcal{R}_0(d_I, \rho_1(d_I)) = \frac{\beta(L)}{\gamma(L)} < 1,$$

which contradicts $\mathcal{R}_0(d_I, \rho_1(d_I)) = 1$. Suppose that $q = \rho_1(d_I) \to \infty$ as $d_I \to 0$, we can reach the contradiction similarly. Thus, it must be the case that $\lim_{d_I \to 0} \rho_1(d_I) = 0$. $\square$

Next we consider the case that $\beta(x)$, $\gamma(x)$ satisfy $\int_0^L \beta(x)\mathrm{d}x < \int_0^L \gamma(x)\mathrm{d}x$. With slight modification to the proof of Lemma 3.4, we can prove:

Lemma 3.6. *Suppose that $\beta(x) - \gamma(x)$ changes sign once in $(0, L)$ and $\int_0^L \beta(x)\mathrm{d}x < \int_0^L \gamma(x)\mathrm{d}x$, then there exists some constant $d_I^* > 0$ such that the following statements hold, where d_I^* is the unique positive root of the equation $\hat{\mathcal{R}}_0(d_I) = 1$.*

(i) *If $\beta(x)$ and $\gamma(x)$ satisfy $(C1)$, then*

- *for $d_I \in (0, d_I^*]$, $\mathcal{R}_0 > 1$ for any $q > 0$;*

- *for $d_I \in (d_I^*, +\infty)$, there exists a unique $\bar{q} = \bar{q}(d_I)$ such that $\mathcal{R}_0 < 1$ for $0 < q < \bar{q}$ and $\mathcal{R}_0 > 1$ for $q > \bar{q}$.*

(ii) *If $\beta(x)$ and $\gamma(x)$ satisfy $(C2)$, then*

- *for $d_I \in (0, d_I^*)$, there exists a unique $\bar{q} = \bar{q}(d_I)$ such that $\mathcal{R}_0 > 1$ for $0 < q < \bar{q}$ and $\mathcal{R}_0 < 1$ for $q > \bar{q}$;*
- *for $d_I \in [d_I^*, +\infty)$, then $\mathcal{R}_0 < 1$ for any $q > 0$.*

Proof. We only prove part (i) as the proof of (ii) is similar. Since $\beta(x)$ and $\gamma(x)$ satisfy $(C1)$, using the same proof of part (ii) in Lemma 3.4, we know that if there exists $\bar{q} > 0$ such that $\mathcal{R}_0(\bar{q}) = 1$, then $\bar{q}$ is unique, and $\mathcal{R}_0'(\bar{q}) > 0$. Thus, for $d_I \in (d_I^*, +\infty)$, the proof is completed. For $d_I \in (0, d_I^*]$, from Lemma 2.1 we know that $\lim_{q\to 0} \mathcal{R}_0 = \hat{\mathcal{R}}_0 \geq 1$. By Corollary 2.6 and condition (C1), $\lim_{q\to\infty} \mathcal{R}_0 = \beta(L)/\gamma(L) > 1$. It implies that there does not exist $q^* > 0$ such that $\mathcal{R}_0 = 1$. Thus, $\mathcal{R}_0 > 1$ for any $q > 0$. □

Remark 3.3. For $d_I = d_I^*$, we know that $q = \bar{q}(d_I^*) = 0$ and $\mathcal{R}_0(d_I^*, 0) = \hat{\mathcal{R}}_0(d_I^*) = 1$. More precisely, for $d_I = d_I^*$, if $\beta(x)$ and $\gamma(x)$ satisfy $(C1)$, then $\mathcal{R}_0 > 1$ for $q > \bar{q}(d_I^*) = 0$; if $\beta(x)$ and $\gamma(x)$ satisfy $(C2)$, then $\mathcal{R}_0 < 1$ for $q > \bar{q}(d_I^*) = 0$.

Similar to Lemma 3.5, we determine the asymptotic behavior of $\rho_2(d_I)$ and $\rho_3(d_I)$.

Lemma 3.7. *Suppose that $\beta(x) - \gamma(x)$ changes sign once in $(0, L)$ and $\int_0^L \beta(x)\mathrm{d}x < \int_0^L \gamma(x)\mathrm{d}x$, then there exists a constant $d_I^* > 0$ such that the following statements hold, where d_I^* is the unique positive root of the equation $\hat{\mathcal{R}}_0(d_I) = 1$.*

(i) *If $\beta(x)$ and $\gamma(x)$ satisfy $(C1)$, then there exists a function $\rho_2 : (d_I^*, \infty) \to (0, \infty)$ such that $\mathcal{R}_0(d_I, \rho_2(d_I)) = 1$. Moveover, ρ_2 is a monotone increasing function of d_I and satisfies*

$$\lim_{d_I\to d_I^*+} \rho_2(d_I) = 0, \quad \lim_{d_I\to\infty} \frac{\rho_2(d_I)}{d_I} = \theta_2,$$

where θ_2 is the unique solution of

$$\int_0^L [\beta(x) - \gamma(x)]e^{\theta_2 x}\mathrm{d}x = 0.$$

(ii) *If $\beta(x)$ and $\gamma(x)$ satisfy $(C2)$, there exists a function $\rho_3 : (0, d_I^*) \to (0, \infty)$ such that $\mathcal{R}_0(d_I, \rho_3(d_I)) = 1$. Moreover, ρ_3 satisfies*

$$\lim_{d_I\to 0+} \rho_3(d_I) = 0, \quad \lim_{d_I\to d_I^*-} \rho_3(d_I) = 0.$$

Proof. (i) To prove the monotone property of $\rho_2(d_I)$ about d_I, we claim that if $\beta(x)-\gamma(x)$ changes sign once in $(0,L)$ and $(C1)$ holds, then for any d_I satisfying $\mathcal{R}_0(d_I)=1$, $\mathcal{R}_0'(d_I)<0$. Here the prime notation denotes differentiation by d_I.

We differentiate both sides of (2.2) with respect to d_I to get

$$\begin{cases}-\varphi_{xx}-d_I\varphi_{xx}'-q\varphi_x'+\gamma(x)\varphi'=-\dfrac{\mathcal{R}_0'}{\mathcal{R}_0^2}\beta(x)\varphi+\dfrac{1}{\mathcal{R}_0}\beta(x)\varphi', \quad & 0<x<L,\\ \varphi_x'(0)=\varphi_x'(L)=0. \end{cases} \tag{3.8}$$

Multiplying (3.8) by $e^{(q/d_I)x}\varphi$ and integrating the resulting equation in $(0,L)$, we get

$$\begin{aligned}&-\int_0^L e^{\frac{q}{d_I}x}\varphi_{xx}\varphi\mathrm{d}x+d_I\int_0^L e^{\frac{q}{d_I}x}\varphi_x'\varphi_x\mathrm{d}x+\int_0^L\gamma(x)e^{\frac{q}{d_I}x}\varphi'\varphi\mathrm{d}x\\ &=-\frac{\mathcal{R}_0'}{\mathcal{R}_0^2}\int_0^L\beta(x)e^{\frac{q}{d_I}x}\varphi^2\mathrm{d}x+\frac{1}{\mathcal{R}_0}\int_0^L\beta(x)e^{\frac{q}{d_I}x}\varphi'\varphi\mathrm{d}x.\end{aligned} \tag{3.9}$$

Multiplying (2.2) by $e^{(q/d_I)x}\varphi'$ and integrating the resulting equation in $(0,L)$, we obtain

$$d_I\int_0^L e^{\frac{q}{d_I}x}\varphi_x'\varphi_x\mathrm{d}x+\int_0^L\gamma(x)e^{\frac{q}{d_I}x}\varphi'\varphi\mathrm{d}x=\frac{1}{\mathcal{R}_0}\int_0^L\beta(x)e^{\frac{q}{d_I}x}\varphi'\varphi\mathrm{d}x. \tag{3.10}$$

Subtracting (3.9) and (3.10), we obtain

$$\frac{\partial\mathcal{R}_0}{\partial d_I}=\frac{\mathcal{R}_0^2\int_0^L e^{\frac{q}{d_I}x}\varphi_{xx}\varphi\mathrm{d}x}{\int_0^L\beta(x)e^{\frac{q}{d_I}x}\varphi^2\mathrm{d}x}=-\frac{\mathcal{R}_0^2\int_0^L e^{\frac{q}{d_I}x}\varphi_x^2\mathrm{d}x}{\int_0^L\beta(x)e^{\frac{q}{d_I}x}\varphi^2\mathrm{d}x}-\frac{q\mathcal{R}_0^2\int_0^L e^{\frac{q}{d_I}x}\varphi_x\varphi\mathrm{d}x}{d_I\int_0^L\beta(x)e^{\frac{q}{d_I}x}\varphi^2\mathrm{d}x}. \tag{3.11}$$

By Lemma 3.1, we see that for any d_I satisfying $\mathcal{R}_0(d_I)=1$, $(\phi_1)_x>0$, which implies that

$$\mathcal{R}_0'(d_I)=-\frac{\mathcal{R}_0^2\int_0^L e^{\frac{q}{d_I}x}[(\phi_1)_x]^2\mathrm{d}x}{\int_0^L\beta(x)e^{\frac{q}{d_I}x}\phi_1^2\mathrm{d}x}-\frac{q\mathcal{R}_0^2\int_0^L e^{\frac{q}{d_I}x}(\phi_1)_x\phi_1\mathrm{d}x}{d_I\int_0^L\beta(x)e^{\frac{q}{d_I}x}\phi_1^2\mathrm{d}x}<0.$$

This proves our assertion.

For any $(d_I,\rho_2(d_I))$ satisfying $\mathcal{R}_0(d_I,\rho_2(d_I))=1$, we differentiate $\mathcal{R}_0(d_I,\rho_2(d_I))=1$ with respect to d_I to get

$$\frac{\partial\mathcal{R}_0}{\partial q}\cdot\rho_2'(d_I)+\frac{\partial\mathcal{R}_0}{\partial d_I}=0,$$

where the prime notation denotes differentiation by d_I. By part (i) of Lemma 3.6 and the above claim, $\partial\mathcal{R}_0/\partial q>0$ and $\partial\mathcal{R}_0/\partial d_I<0$ for $(d_I,\rho_2(d_I))$ satisfying $\mathcal{R}_0(d_I,\rho_2(d_I))=1$. Thus $\rho_2'(d_I)>0$.

Next we prove $\lim_{d_I\to d_I^*+}\rho_2(d_I)=0$. Suppose that there exists $q^*>0$ such that $q=\rho_2(d_I)\to q^*$ as $d_I\to d_I^*+$. Then there exists a positive function $\phi^*(x)\in C^2([0,L])$ such that

$$\begin{cases}-d_I^*\phi_{xx}^*-q^*\phi_x^*+\gamma(x)\phi^*=\beta(x)\phi^*, \quad & 0<x<L,\\ \phi_x^*(0)=\phi_x^*(L)=0.\end{cases} \tag{3.12}$$

Since d_I^* is the unique positive root of the equation $\hat{\mathcal{R}}_0 = 1$, then there exists a positive function $\hat{\phi}(x) \in C^2([0, L])$ such that

$$\begin{cases} -d_I^* \hat{\phi}_{xx} + \gamma(x)\hat{\phi} = \beta(x)\hat{\phi}, & 0 < x < L, \\ \hat{\phi}_x(0) = \hat{\phi}_x(L) = 0. \end{cases} \tag{3.13}$$

Multiplying (3.12) by $\hat{\phi}$, (3.13) by ϕ^*, integrating over $(0, L)$ and subtracting, we obtain

$$q^* \int_0^L \phi_x^* \hat{\phi} \mathrm{d}x = 0.$$

From Lemma 3.1, we know that $\phi_x^* > 0$ in $(0, L)$. Since ϕ_x^* and $\hat{\phi}$ are all positive functions, then $q^* = 0$. Hence, $\lim_{d_I \to d_I^*+} \rho_2(d_I) = 0$.

By a similar argument as in the proof of Lemma 3.5, we can prove

$$\lim_{d_I \to \infty} \frac{\rho_2(d_I)}{d_I} = \theta_2,$$

where θ_2 is the unique solution of

$$\int_0^L [\beta(x) - \gamma(x)] e^{\theta_2 x} \mathrm{d}x = 0.$$

(ii) By the above argument and the proof of Lemma 3.5, the proof is complete. □

3.2 The endemic equilibrium

In this subsection, we apply the bifurcation analysis and degree theory to study the existence of *endemic equilibrium* when the *disease-free equilibrium* is unstable. Theorem 1.5 is a consequence of Lemmas 3.8 and 3.10.

In subsection 3.2.1, we will prove the problem (1.6)-(1.7) has at least an EE for $(d_I, q) \in \Omega_{hl}^U \cup \Omega_{lh}^{U_2} \cup \Omega_{ll}^U$ and the EE exists for $(d_I, q) \in \Omega_{hh}^U \cup \Omega_{lh}^{U_1}$ in subsection 3.2.2.

3.2.1 Bifurcation from the *disease-free equilibrium*

In this subsection we prove the existence of *endemic equilibrium* of (1.6)-(1.7) using bifurcation theory. By the transformation $\tilde{S} = e^{\frac{q}{d_S}x} S$, $\tilde{I} = e^{\frac{q}{d_I}x} I$ we can convert the steady states system (1.6) to

$$\begin{cases} d_S S_{xx} + qS_x - \beta(x)\dfrac{e^{\frac{q}{d_I}x}SI}{e^{\frac{q}{d_S}x}S + e^{\frac{q}{d_I}x}I} + \gamma(x)e^{\left(\frac{q}{d_I}-\frac{q}{d_S}\right)x}I = 0, & 0 < x < L, \\ d_I I_{xx} + qI_x + \beta(x)\dfrac{e^{\frac{q}{d_S}x}SI}{e^{\frac{q}{d_S}x}S + e^{\frac{q}{d_I}x}I} - \gamma(x)I = 0, & 0 < x < L, \\ S_x(0) = S_x(L) = 0,\ I_x(0) = I_x(L) = 0, \\ \int_0^L [e^{\frac{q}{d_S}x}S + e^{\frac{q}{d_I}x}I]\mathrm{d}x = N, \end{cases} \tag{3.14}$$

where the constant N is the total population size. We shall study (3.14) instead of (1.6)-(1.7) since the structure of the solution set of (3.14) is the same as that of (1.6)-(1.7). For (3.14), the unique *disease-free equilibrium* is denoted by $(\hat{S}, 0) = (qN/d_S(e^{qL/d_S} - 1), 0)$. We will apply the local and global bifurcation theorems to consider a branch of positive solutions of (3.14) bifurcating from the branch of semi-trivial solutions given by

$$\Gamma_S := \{(q, (\hat{S}, 0)) : 0 < q < \infty\}.$$

We now set up the abstract framework for our bifurcation analysis. Fix $d_S, d_I > 0$ and let q be the bifurcation parameter. For $p > 1$, set

$$X = \left\{u \in W^{2,p}((0, L)) : u_x(0) = u_x(L) = 0\right\}, \quad Y = L^p((0, L)),$$

and define the set of positive solutions of (3.14) to be

$$O = \{(q, (S, I)) \in \mathbb{R}^+ \times X \times X : q > 0, S > 0, I > 0,\ (q, (S, I))\textit{satisfies } (3.14)\}.$$

We have the following result about the bifurcation from the *disease-free equilibrium*.

Lemma 3.8. *Suppose that $d_S, d_I > 0$ and $\beta(x) - \gamma(x)$ changes sign once in $(0, L)$, then*

1. *$q_* > 0$ is a bifurcation point for the positive solutions of (3.14) from the semi-trivial branch Γ_S if and only if q_* satisfies $\mathcal{R}_0(d_I, q_*) = 1$. More precisely,*
   - (I) *when $\int_0^L \beta(x)\mathrm{d}x > \int_0^L \gamma(x)\mathrm{d}x$, for any $d_I > 0$, such q_* exists uniquely if and only if $\beta(x)$ and $\gamma(x)$ satisfy condition $(C2)$;*
   - (II) *when $\int_0^L \beta(x)\mathrm{d}x < \int_0^L \gamma(x)\mathrm{d}x$, such q_* exists uniquely if and only if either assumption $(C1)$ holds and $d > d_I^*$ or assumption $(C2)$ holds and $0 < d < d_I^*$, where d_I^* is the unique positive root of $\hat{\mathcal{R}}_0 = 1$.*
2. *All positive solutions of (3.14) near $(q_*, (\hat{S}, 0)) \in \mathbb{R} \times X \times X$ can be parameterized as*

$$\Gamma = \{(q(\tau), (\hat{S} + S_1(\tau), I_1(\tau))) : \tau \in [0, \delta)\} \tag{3.15}$$

for some $\delta > 0$, $(q(\tau), (\hat{S} + S_1(\tau), I_1(\tau)))$ is a smooth curve with respect to τ and satisfies $q(0) = q_*$, $S_1(0) = I_1(0) = 0$.

3. *There exists a connected component Σ of $\overline{O}$ such that $\Gamma \subseteq \Sigma$, and the following statements about Σ hold:*

   Case (I) $\int_0^L \beta(x)\mathrm{d}x > \int_0^L \gamma(x)\mathrm{d}x$. *If $(C2)$ holds, then for Σ, the projection of Σ to the q-axis satisfies $Proj_q\Sigma = [0, q_*]$ and the connected component Σ connects to $(0, (S_*, I_*))$, where (S_*, I_*) is an endemic equilibrium of (3.14) when $q = 0$.*

   Case (II) $\int_0^L \beta(x)\mathrm{d}x < \int_0^L \gamma(x)\mathrm{d}x$.

   (i) *If $(C1)$ holds and $d_I > d_I^*$, then for Σ, the projection of Σ to the q-axis satisfies $Proj_q\Sigma = [q_*, \infty)$, and (3.14) has no positive solution for $0 < q < q_*$.*

   (ii) *If $(C2)$ holds and $0 < d_I < d_I^*$, then for Σ, $Proj_q\Sigma = [0, q_*]$ and the connected component Σ connects to $(0, (S_*, I_*))$, where (S_*, I_*) is an endemic equilibrium of (3.14) when $q = 0$.*

Proof. **1.** We define a mapping $F : \mathbb{R}^+ \times X \times X \to Y \times Y \times \mathbb{R}$ by

$$F(q,(S,I)) = \begin{pmatrix} d_S S_{xx} + qS_x - \beta(x)\dfrac{e^{\frac{q}{d_I}x}SI}{e^{\frac{q}{d_S}x}S + e^{\frac{q}{d_I}x}I} + \gamma(x)e^{\left(\frac{q}{d_I}-\frac{q}{d_S}\right)x}I \\ d_I I_{xx} + qI_x + \beta(x)\dfrac{e^{\frac{q}{d_S}x}SI}{e^{\frac{q}{d_S}x}S + e^{\frac{q}{d_I}x}I} - \gamma(x)I \\ \int_0^L [e^{\frac{q}{d_S}x}S + e^{\frac{q}{d_I}x}I]\mathrm{d}x - N \end{pmatrix}.$$

Then the pair (S, I) satisfies $F(q,(S,I)) = 0$ if and only if (S, I) is a solution of (3.14). Note that $F(q,(\hat{S},0)) = 0$ for any $q > 0$. The Fréchet derivatives of F at $(\hat{S}, 0)$ are given by

$$D_{(S,I)}F(q,(\hat{S},0)) \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = \begin{pmatrix} d_S\Phi_{xx} + q\Phi_x + [\gamma(x) - \beta(x)]e^{\left(\frac{q}{d_I}-\frac{q}{d_S}\right)x}\Psi \\ d_I\Psi_{xx} + q\Psi_x + [\beta(x) - \gamma(x)]\Psi \\ \int_0^L [e^{\frac{q}{d_S}x}\Phi + e^{\frac{q}{d_I}x}\Psi]\mathrm{d}x \end{pmatrix},$$

$$D_{q,(S,I)}F(q,(\hat{S},0)) \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = \begin{pmatrix} \Phi_x + \left(\frac{x}{d_I} - \frac{x}{d_S}\right)[\gamma(x) - \beta(x)]e^{\left(\frac{q}{d_I}-\frac{q}{d_S}\right)x}\Psi \\ \Psi_x \\ \int_0^L [\frac{x}{d_S}e^{\frac{q}{d_S}x}\Phi + \frac{x}{d_I}e^{\frac{q}{d_I}x}\Psi]\mathrm{d}x \end{pmatrix},$$

$$D_{(S,I)(S,I)}F(q,\hat{S},0) \begin{bmatrix} \Phi \\ \Psi \end{bmatrix}^2 = \begin{pmatrix} \frac{2}{\hat{S}}\beta(x)e^{2\left(\frac{q}{d_I}-\frac{q}{d_S}\right)x}\Psi^2 \\ -\frac{2}{\hat{S}}\beta(x)e^{\left(\frac{q}{d_I}-\frac{q}{d_S}\right)x}\Psi^2 \\ 0 \end{pmatrix}.$$

We find that $(q_*, (\hat{S}, 0))$ is a degenerate solution of (3.14) if

$$\begin{cases} d_S\Phi_{xx} + q\Phi_x + [\gamma(x) - \beta(x)]e^{\left(\frac{q}{d_I} - \frac{q}{d_S}\right)x}\Psi = 0, & 0 < x < L, \\ d_I\Psi_{xx} + q\Psi_x + [\beta(x) - \gamma(x)]\Psi = 0, & 0 < x < L, \\ \displaystyle\int_0^L \left[e^{\frac{q}{d_S}x}\Phi + e^{\frac{q}{d_I}x}\Psi\right] \mathrm{d}x = 0, & \\ \Phi_x(0) = \Phi_x(L) = 0, \;\; \Psi_x(0) = \Psi_x(L) = 0 & \end{cases} \tag{3.16}$$

has a nontrivial solution (Φ_1, ϕ_1). The second equation of (3.16) has a positive solution ϕ_1 only when $q = q_*$ satisfies $\mathcal{R}_0(d_I, q_*) = 1$. Actually, ϕ_1 is an eigenfunction corresponding to the eigenvalue $\mathcal{R}_0(d_I, q_*) = 1$ and Φ_1 satisfies

$$\begin{cases} d_S(\Phi_1)_{xx} + q(\Phi_1)_x + [\gamma(x) - \beta(x)]e^{\left(\frac{q}{d_I} - \frac{q}{d_S}\right)x}\phi_1 = 0, & 0 < x < L, \\ \displaystyle\int_0^L \left[e^{\frac{q}{d_S}x}\Phi_1 + e^{\frac{q}{d_I}x}\phi_1\right] \mathrm{d}x = 0, & \\ (\Phi_1)_x(0) = (\Phi_1)_x(L) = 0. & \end{cases} \tag{3.17}$$

It can be seen that Φ_1 is uniquely determined by ϕ_1 in (3.17). Hence, $q = q_*$ is the only possible bifurcation point along Γ_S where positive solutions of (3.14) bifurcates, and such q_* exists if and only if $\mathcal{R}_0 = 1$. By Lemmas 3.4 and 3.6, the necessary and sufficient conditions for the occurrence of bifurcation are established.

2. At $(q, (S, I)) = (q_*, (\hat{S}, 0))$, the kernel $\mathrm{Ker}(D_{(S,I)}F(q_*, (\hat{S}, 0))) = \mathrm{span}\{(\Phi_1, \phi_1)\}$, where (Φ_1, ϕ_1) satisfy (3.16) with $q = q_*$. From the above discussion we know that (Φ_1, ϕ_1) is unique (up to a multiple of constant). The range of $D_{(S,I)}F(q_*, (\hat{S}, 0))$ is given by

$$\mathrm{Range}(D_{(S,I)}F(q_*, (\hat{S}, 0))) = \left\{(f, g, k) \in Y \times Y \times \mathbb{R} : \int_0^L g\phi_1 e^{\frac{q}{d_I}x}\mathrm{d}x = 0\right\},$$

which is of co-dimension one. Since $(\phi_1)_x$ does not change sign in $(0, L)$ from Lemma 3.1, then $\int_0^L (\phi_1)_x\phi_1 e^{\frac{q}{d_I}x}\mathrm{d}x \neq 0$. Hence, we can get

$$D_{q,(S,I)}F(q_*, (\hat{S}, 0))[(\Phi_1, \phi_1)] \notin \mathrm{Range}(D_{(S,I)}F(q_*, (\hat{S}, 0))).$$

Consequently we can apply the local bifurcation theorem in [7] to F at $(q_*, (\hat{S}, 0))$ and conclude that the set of positive solutions to (3.14) is a smooth curve

$$\Gamma = \{(q(\tau), (\hat{S} + S_1(\tau), I_1(\tau))) : \tau \in [0, \delta)\},$$

such that $q(0) = q_*$, $S_1(\tau) = \tau\hat{S} + o(|\tau|)$, $I_1(\tau) = o(|\tau|)$. Moreover, $q'(0)$ can be calculated (see, e.g., [9, 26]):

$$q'(0) = -\frac{\langle l, D_{(S,I)(S,I)}F(q_*, (\hat{S}, 0))[\Phi_1, \phi_1]^2\rangle}{2\langle l, D_{q,(S,I)}F(q_*, (\hat{S}, 0))[\Phi_1, \phi_1]\rangle} = \frac{\int_0^L \beta(x)e^{\left(\frac{2q}{d_I} - \frac{q}{d_S}\right)x}\phi_1^3\mathrm{d}x}{\hat{S}\int_0^L e^{\frac{q}{d_I}x}\phi_1(\phi_1)_x\mathrm{d}x},$$

where l is the linear functional on $Y \times Y \times \mathbb{R}$ defined by $\langle l, [f, g, k] \rangle = \int_0^L g\phi_1 e^{\frac{q}{d_I}x} \mathrm{d}x$.

3. For the proof of part 3, the existence of the connected component Σ follows from the global bifurcation theorem in [27] (see also [25]), and it is known that Σ is either unbounded, or connects to another $(q, (\hat{S}, 0))$, or Σ connects to another point on the boundary of O.

Case (I): $\int_0^L \beta(x)\mathrm{d}x > \int_0^L \gamma(x)\mathrm{d}x$ and $(C2)$ holds. From the above proof of part 2 and Lemma 3.1, we know that there exists a unique q_* such that the local bifurcation occurs at $(q_*, (\hat{S}, 0))$ and $q'(0) < 0$, i.e., the bifurcation direction is subcritical. It means that (3.14) has a positive solution when $q_* - \delta < q < q_*$ for some small $\delta > 0$. By Lemma 3.4, $\mathcal{R}_0 > 1$ when $q_* - \delta < q < q_*$ for some small $\delta > 0$. According to Lemma 2.4, (3.14) has no positive solution when $\mathcal{R}_0 < 1$, then (3.14) has no positive solution when $q > q_*$, i.e., the projection of Σ to the q-axis $Proj_q\Sigma \subset [0, q_*]$. Since the positive solutions have a uniform L^∞ bound for $0 \le q \le q^*$, then Σ must be bounded in $\overline{O}$. It can be seen that the third option must happen here. Therefore Σ must connect to $(0, (S_*, I_*))$, where (S_*, I_*) is the unique *endemic equilibrium* of (3.14) when $q = 0$. Thus, $0 \in Proj_q\Sigma$. This complete the proof of Case (I).

Case (II): $\int_0^L \beta(x)\mathrm{d}x < \int_0^L \gamma(x)\mathrm{d}x$. If $(C1)$ holds and $d_I > d_I^*$, from the above bifurcation analysis and Lemma 3.1, there exists a unique bifurcation point q_* such that $q'(0) > 0$, i.e., the bifurcation direction is supercritical. That is, (3.14) has a positive solution when $q_* < q < q_* + \delta$ for some small $\delta > 0$. Hence $\mathcal{R}_0 > 1$ when $q_* < q < q_* + \delta$ for some small $\delta > 0$ by Lemma 3.6. According to Lemma 2.4, (3.14) has no positive solution when $\mathcal{R}_0 < 1$, then (3.14) has no positive solution when $0 < q < q_*$. Hence the first option must happen here. Assume that there exists $q^* > q_*$ such that $Proj_q\Sigma = [q_*, q^*)$, then it is a contradiction as all positive solutions have an uniform L^∞ bound for $q = q^*$. Therefore, the projection of Σ to the q-axis $Proj_q\Sigma = [q_*, \infty)$. If $(C2)$ holds and $0 < d_I < d_I^*$, the proof is similar to that of case (I). □

Here we note that when $\int_0^L \beta(x)\mathrm{d}x < \int_0^L \gamma(x)\mathrm{d}x$, if condition $(C1)$ holds and $d_I = d_I^*$, $q_*(d_I^*) = 0$ is the unique bifurcation point. For the connected component Σ, the projection of Σ to the q-axis satisfies $Proj_q\Sigma = [0, \infty)$, i.e., (3.14) has at least one positive solution for any $q > 0$ in this case.

From Lemma 3.8, we see that the EE of (1.6)-(1.7) exists for $(d_I, q) \in \Omega_{hl}^U \cup \Omega_{lh}^{U_2} \cup \Omega_{ll}^U$.

3.2.2 The Leray-Schauder degree argument

In this subsection, we will apply the degree theory to prove the existence of the *endemic equilibrium* when the *disease-free equilibrium* is unstable. First we give *a priori* estimate of the positive solutions of (3.14).

Lemma 3.9. *For any $\varepsilon > 0$, there exist two positive constants $\overline{C}$ and $\underline{C}$, depending on $d_I, \varepsilon, \|\beta\|_\infty, \|\gamma\|_\infty$ and N such that for any $\varepsilon \le d_S \le 1/\varepsilon$, $0 \le q \le 1/\varepsilon$, if $\mathcal{R}_0 \ne 1$, then any* *positive solution of* (3.14) *satisfies*

$$\underline{C} \le S(x), I(x) \le \overline{C}, \text{ for any } x \in [0, L]. \tag{3.18}$$

Proof. In the introduction we have discussed that S and I have the upper bound $\overline{C}$, $\overline{C}$ depending on $d_I, \varepsilon, \|\beta\|_\infty, \|\gamma\|_\infty$ and N. We only need to prove that S and I have positive lower bounds.

We claim that $\max_{[0,L]} I(x)$ is bounded below by some positive constant. To establish this assertion, we argue by contradiction. Suppose that there exist $(S_i(x), I_i(x))$ are positive solutions of (3.14) such that

$$\max_{x\in[0,L]} I_i(x) \to 0, \text{ as } i \to \infty,$$

and $(S_i(x), I_i(x))$ satisfies

$$\begin{cases} d_{S,i}(S_i)_{xx} + q_i(S_i)_x - \beta(x)\dfrac{e^{\frac{q_i}{d_I}x}S_iI_i}{e^{\frac{q_i}{d_{S,i}}x}S_i + e^{\frac{q_i}{d_I}x}I_i} + \gamma(x)e^{\left(\frac{q_i}{d_I}-\frac{q_i}{d_{S,i}}\right)x}I_i = 0, & 0 < x < L,\\ d_I(I_i)_{xx} + q_i(I_i)_x + \beta(x)\dfrac{e^{\frac{q_i}{d_{S,i}}x}S_iI_i}{e^{\frac{q_i}{d_{S,i}}x}S_i + e^{\frac{q_i}{d_I}x}I_i} - \gamma(x)I_i = 0, & 0 < x < L,\\ (S_i)_x(0) = (S_i)_x(L) = 0, \ (I_i)_x(0) = (I_i)_x(L) = 0,\\ \displaystyle\int_0^L [e^{\frac{q_i}{d_{S,i}}x}S_i + e^{\frac{q_i}{d_I}x}I_i]\mathrm{d}x = N, \end{cases}$$

where $\varepsilon \le d_{S,i} \le 1/\varepsilon$ and $0 \le q_i \le 1/\varepsilon$. Passing to a subsequence if necessary, we assume $d_{S,i} \to d_S > 0$ and $q_i \to q \ge 0$. Since $\|I_i\|_\infty$ are uniformly bounded, let $\tilde{I}_i = I_i/\|I_i\|_\infty$, then $\tilde{I}_i$ satisfies

$$\begin{cases} d_I(\tilde{I}_i)_{xx} + q_i(\tilde{I}_i)_x + \beta(x)\tilde{I}_i\dfrac{e^{\frac{q_i}{d_{S,i}}x}S_i}{e^{\frac{q_i}{d_{S,i}}x}S_i + e^{\frac{q_i}{d_I}x}I_i} - \gamma(x)\tilde{I}_i = 0, & 0 < x < L,\\ (\tilde{I}_i)_x(0) = (\tilde{I}_i)_x(L) = 0. \end{cases}$$

By standard regularity and the Sobolev embedding theorem [11], passing to a subsequence if necessary, we know that $I_i \to 0$ and $\tilde{I}_i \to I^*$ in $C^1([0, L])$, where $I^* > 0$ and $\|I^*\|_\infty = 1$. From $\int_0^L [e^{\frac{q_i}{d_{S,i}}x}S_i + e^{\frac{q_i}{d_I}x}I_i]\mathrm{d}x = N$ and $I_i \to 0$ in $C^1([0, L])$, by the equation of S_i we can get $S_i \to \hat{S} > 0$ in $C^1([0, L])$. Thus it is easy to see that I^* satisfies

$$\begin{cases} d_I I^*_{xx} + qI^*_x + [\beta(x) - \gamma(x)]I^* = 0, & 0 < x < L,\\ I^*_x(0) = I^*_x(L) = 0. \end{cases}$$

From $I^* > 0$, then 0 is the principal eigenvalue, which contradicts the assumption $\mathcal{R}_0 \ne 1$ for any $d_I > 0$ and $0 \le q \le 1/\varepsilon$. Thus there exists some positive constant $\underline{C}$ such that

$\max_{[0,L]} I(x) \geq \underline{C}$. By the Harnack inequality (e.g., modifying the argument in [14]), there exists some positive constant C^*, depending on $d_I, \varepsilon, \|\beta\|_\infty, \|\gamma\|_\infty$ and N such that

$$\max_{x\in[0,L]} I(x) \leq C^* \min_{x\in[0,L]} I(x).$$

It implies that $I(x)$ has uniform positive lower bound.

Next we prove that $S(x)$ has a uniform positive lower bound. Set $S(x_0) = \min_{[0,L]} S$. Applying the minimum principle in [15], we obtain

$$\beta(x_0)\frac{e^{\frac{q}{d_I}x_0}S(x_0)}{e^{\frac{q}{d_S}x_0}S(x_0) + e^{\frac{q}{d_I}x_0}I(x_0)} - \gamma(x_0)e^{\left(\frac{q}{d_I}-\frac{q}{d_S}\right)x_0} \geq 0, \tag{3.19}$$

which implies that

$$\beta(x_0)\frac{S(x_0)}{I(x_0)} \geq \beta(x_0)\frac{e^{\frac{q}{d_I}x_0}S(x_0)}{e^{\frac{q}{d_S}x_0}S(x_0) + e^{\frac{q}{d_I}x_0}I(x_0)} \geq \gamma(x_0)e^{\left(\frac{q}{d_I}-\frac{q}{d_S}\right)x_0}.$$

Thus we deduce that

$$S(x_0) \geq \frac{\gamma(x_0)e^{\left(\frac{q}{d_I}-\frac{q}{d_S}\right)x_0}}{\beta(x_0)}I(x_0) \geq C\min_{x\in[0,L]} I.$$

Then the proof is complete. □

Lemma 3.10. *Assume that $\beta(x) - \gamma(x)$ changes sign once in $(0, L)$, then (3.14) has at least an endemic equilibrium, provided that one of the following conditions holds:*

- (i) *$d_I > 0$, $q > 0$, $\int_0^L \beta(x)\mathrm{d}x > \int_0^L \gamma(x)\mathrm{d}x$ and (C1) holds;*
- (ii) *$0 < d_I < d_I^*$, $q > 0$, $\int_0^L \beta(x)\mathrm{d}x < \int_0^L \gamma(x)\mathrm{d}x$ and (C1) holds.*

Proof. For any nonnegative pair $(f, g) \in C([0, L]) \times C([0, L])$, we can extend the range of f, g properly such that the Lipschitz continuous function $fg/(e^{(\tau qx/d_S)}f + e^{(\tau qx/d_I)}g)$ be defined for $f, g \in \mathbb{R}$ and $\tau \in [0, 1]$. We can define a compact operator family from $[0, 1] \times C([0, L]) \times C([0, L])$ to $C([0, L]) \times C([0, L])$ as follows:

$$\begin{cases} (\tau d_S + (1-\tau)d_I)u_{xx} + \tau q u_x + \gamma(x)e^{\left(\frac{\tau q}{d_I}-\frac{\tau q}{d_S}\right)x}v = \beta(x)\dfrac{e^{\frac{\tau q}{d_I}x}fg}{e^{\frac{\tau q}{d_S}x}f + e^{\frac{\tau q}{d_I}x}g}, & 0 < x < L, \\ d_I v_{xx} + \tau q v_x - \gamma(x)v = -\beta(x)\dfrac{e^{\frac{\tau q}{d_S}x}fg}{e^{\frac{\tau q}{d_S}x}f + e^{\frac{\tau q}{d_I}x}g}, & 0 < x < L, \\ u_x(0) = u_x(L) = 0,\ v_x(0) = v_x(L) = 0, \\ \displaystyle\int_0^L \left[e^{\frac{\tau qx}{\tau d_S + (1-\tau)d_I}}u + e^{\frac{\tau qx}{d_I}}v\right]\mathrm{d}x = N. \end{cases} \tag{3.20}$$

In fact, for any $\tau \in [0,1]$ and $(f,g) \in C([0,L]) \times C([0,L])$, since the operator $d_I \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \tau q \frac{\mathrm{d}}{\mathrm{d}x} - \gamma(x)$ is invertible, v is uniquely determined by the second equation of (3.20). From the first and last equations of (3.20), u is also uniquely determined. Thus, we can define $\mathcal{G}_\tau(f,g) := (u,v)$.

By conditions (i) and (ii), from Figures 1 and 3 we see that $(d_I, q) \in \Omega_{hh}^U \cup \Omega_{lh}^{U_1}$, where Ω_{hh}^U and $\Omega_{lh}^{U_1}$ are defined in Definitions 1.1 and 1.2, respectively. That is, $\mathcal{R}_{0,\tau} > 1$ for any $\tau \in [0,1]$, where $\mathcal{R}_{0,\tau}$ is defined by

$$\mathcal{R}_{0,\tau} = \sup_{\substack{\varphi \in H^1((0,L)) \\ \varphi \neq 0}} \left\{ \frac{\int_0^L \beta(x) e^{\frac{\tau q}{d_I} x} \varphi^2 \mathrm{d}x}{d_I \int_0^L e^{\frac{\tau q}{d_I} x} \varphi_x^2 \mathrm{d}x + \int_0^L \gamma(x) e^{\frac{\tau q}{d_I} x} \varphi^2 \mathrm{d}x} \right\}.$$

By Lemma 3.9, for any $\tau \in [0,1]$, we know that there exist two positive constants $\overline{C} = \overline{C}(d_S, d_I, q, \|\beta\|_\infty, \|\gamma\|_\infty, N)$ and $\underline{C} = \underline{C}(d_S, d_I, q, \|\beta\|_\infty, \|\gamma\|_\infty, N)$ such that any positive solution (u,v) of (3.20) satisfies $\underline{C} \leq u, v \leq \overline{C}$.

We define

$$D = \left\{ (u,v) \in C([0,L]) \times C([0,L]) : \underline{C}/2 \leq u, v \leq 2\overline{C} \right\}.$$

Then for any $\tau \in [0,1]$ and $(S,I) \in \partial D$, we know that$(S,I) \neq \mathcal{G}(\tau, (S,I))$. As a result, the Leray-Schauder degree $\deg(\mathbf{I} - \mathcal{G}(\tau, (\cdot,\cdot)), D, 0)$ is well defined, and it is independent of τ, where $\mathbf{I}$ is the identity map. Furthermore, (S,I) solves (3.14) if and only if (S,I) satisfies $(S,I) = \mathcal{G}(1,(S,I))$. It is easy to see that $(S,I) \in D$ satisfies $(\mathbf{I} - \mathcal{G}(0,(\cdot,\cdot)))(S,I) = 0$ implies that (S,I) is a positive solution of

$$\begin{cases} d_I S_{xx} - \beta(x) \dfrac{SI}{S+I} + \gamma(x) I = 0, & 0 < x < L, \\ d_I I_{xx} + \beta(x) \dfrac{SI}{S+I} - \gamma(x) I = 0, & 0 < x < L, \\ S_x(0) = S_x(L) = 0, \ I_x(0) = I_x(L) = 0, \\ \displaystyle\int_0^L [S+I] \mathrm{d}x = N. \end{cases} \tag{3.21}$$

From [3], the problem has a unique positive solution when the basic reproduction number $\hat{\mathcal{R}}_0 > 1$, denoted by (S_*, I_*), and (S_*, I_*) satisfying $S_* + I_* = N/L$. We need to determine the linear stability of (S_*, I_*). Linearizing equation (3.21) around (S_*, I_*), we get the following linearized system:

$$\begin{cases} -d_I \Phi_{xx} + \beta(x) \dfrac{I_*^2}{(S_*+I_*)^2} \Phi + \beta(x) \dfrac{S_*^2}{(S_*+I_*)^2} \Psi - \gamma(x) \Psi = \mu \Phi, & 0 < x < L, \\ -d_I \Psi_{xx} - \beta(x) \dfrac{S_*^2}{(S_*+I_*)^2} \Psi + \gamma(x) \Psi - \beta(x) \dfrac{I_*^2}{(S_*+I_*)^2} \Phi = \mu \Psi, & 0 < x < L, \\ \Phi_x(0) = \Phi_x(L) = 0, \ \Psi_x(0) = \Psi_x(L) = 0, \\ \displaystyle\int_0^L [\Phi + \Psi] \mathrm{d}x = 0. \end{cases} \tag{3.22}$$

Adding the first two equations of (3.22) and integrating the resulting equation in $(0, L)$, combining with the boundary condition, we obtain $\Phi = -\Psi$. Then we can convert the first equation of (3.22) to

$$-d_I \Phi_{xx} + \left(2\frac{L}{N}\beta(x)I_* + \gamma(x) - \beta(x)\right)\Phi = \mu\Phi.$$

Since I_* is a positive solution of (3.21), then $-d_I \frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2\frac{L}{N}\beta(x)I_* + \gamma(x) - \beta(x)$ is a positive operator and we can obtain $\mu > 0$. Thus the unique positive solution (S_*, I_*) is linearly stable. Hence, by the well-known Leray-Schauder degree index formula (see, e.g., theorem 2.8.1 in [19]), we get

$$\deg(\mathbf{I} - \mathcal{G}(0, (\cdot, \cdot)), D, 0) = 1.$$

Therefore, by the homotopy invariance of the Leray-Schauder degree, we get

$$\deg(\mathbf{I} - \mathcal{G}(1, (\cdot, \cdot)), D, 0) = \deg(\mathbf{I} - \mathcal{G}(0, (\cdot, \cdot)), D, 0),$$

for $(d_I, q) \in \Omega^U_{hh} \cup \Omega^{U_1}_{lh}$. Therefore, we know that $\deg(\mathbf{I} - \mathcal{G}(1, (\cdot, \cdot)), D, 0) = 1$. By the properties of the degree, when $(d_I, q) \in \Omega^U_{hh} \cup \Omega^{U_1}_{lh}$, $\mathcal{G}(1, (\cdot, \cdot))$ has a fixed point in D, i.e., (3.14) has at least one positive solution. □

By Lemma 3.10, the EE of (1.6)-(1.7) exists for $(d_I, q) \in \Omega^U_{hh} \cup \Omega^{U_1}_{lh}$.

4 Properties of $\mathcal{R}_0$: $\beta(x) - \gamma(x)$ changing sign twice

The goal of this section is to establish Theorem 1.6. Here using the same definition in Section 3, for any given continuous function $m(x)$ on $[0, L]$, define

$$F(\eta) = \int_0^L e^{\eta x} m(x)\mathrm{d}x, \quad 0 \le \eta < \infty.$$

To study the stability of the DFE in terms of the diffusion rate d_I and the advection rate q, we need consider the results about the positive roots of the auxiliary function F. We have the following result:

Lemma 4.1. *Let $m(x)$ change sign twice for $x \in [0, L]$, i.e., there exist $0 < x_1 < x_2 < L$ such that $m(x_1) = m(x_2) = 0$. The following statements about F hold.*

(i) *If $m(x)$ satisfies $m(L) < 0$ and $\int_0^L m(x)\mathrm{d}x > 0$, then $F(\eta)$ has a unique positive root η_1 for $\eta \in (0, +\infty)$ and $F'(\eta_1) < 0$;*

(ii) *If $m(x)$ satisfies $m(L) > 0$ and $\int_0^L m(x)\mathrm{d}x < 0$, then $F(\eta)$ has a unique positive root η_1 for $\eta \in (0, +\infty)$ and $F'(\eta_1) > 0$;*

(iii) *If* $m(x)$ *satisfies* $m(L) > 0$ *and* $\int_0^L m(x)\mathrm{d}x > 0$, *then* $F(\eta)$ *has at most two positive roots for* $\eta \in (0, +\infty)$*;*

(iv) *If* $m(x)$ *satisfies* $m(L) < 0$ *and* $\int_0^L m(x)\mathrm{d}x < 0$, *then* $F(\eta)$ *has at most two positive roots for* $\eta \in (0, +\infty)$*.*

Proof. We only need to prove (i) and (iii). Parts (ii) and (iv) can be established in the same way.

1. Set $G_1(\eta) := e^{-x_2\eta}[x_1F(\eta) - F'(\eta)]$. Here the prime notation denotes differentiation with respect to η. By the condition in (i), we know that $m(x) > 0$ for $x \in (x_1, x_2)$ and $m(x) < 0$ for $x \in (0, x_1) \cup (x_2, L)$, i.e., $m(x)(x - x_1)(x - x_2) < 0$ for $x \in (0, L)$ and $x \neq x_i\,(i = 1, 2)$. Thus, for any $\eta > 0$, one can get

$$\begin{aligned} G_1'(\eta) &= -e^{-x_2\eta}[F''(\eta) - (x_1 + x_2)F'(\eta) + x_1x_2F(\eta)] \\ &= -\int_0^L e^{\eta(x-x_2)}m(x)(x - x_1)(x - x_2)\mathrm{d}x \\ &> 0, \end{aligned}$$

i.e., $G_1(\eta)$ is a strictly increasing function for $\eta \in (0, \infty)$. By condition (i) and Lemma 3.2, we know that there exists at least a positive root of F. Let η_1 denote the smallest positive one, then $F'(\eta_1) \le 0$. There are two cases, $F'(\eta_1) = 0$ and $F'(\eta_1) < 0$.

Assume that $F'(\eta_1) = 0$, since $F''(\eta) - (x_1 + x_2)F'(\eta) + x_1x_2F(\eta) = \int_0^L e^{\eta x}m(x)(x - x_1)(x - x_2)\mathrm{d}x < 0$, then we have $F''(\eta_1) - (x_1 + x_2)F'(\eta_1) + x_1x_2F(\eta_1) = F''(\eta_1) < 0$. It implies that F attains a strict local maximum at η_1. This is a contradiction. Hence $F'(\eta_1) < 0$. Next we claim that η_1 is the unique positive root of F. Suppose that there exists another positive root $\eta_2 > \eta_1$ such that $F(\eta_2) = 0$. Since $F(\eta_1) = 0$ and $F'(\eta_1) < 0$, then $F(\eta) < 0$ in (η_1, η_2), i.e., $F'(\eta_2) \ge 0$. By the definition of $G_1(\eta)$, we know $G_1(\eta_1) = -e^{-x_2\eta_1}F'(\eta_1) > 0$ and $G_1(\eta_2) = -e^{-x_2\eta_2}F'(\eta_2) \le 0$, this reaches the contradiction as $G_1(\eta)$ is strictly increasing. This completes the proof of (i).

2. For part (iii), by condition and Lemma 3.2, it is known that either $F > 0$ for any $\eta \ge 0$ or F has positive roots in $(0, \infty)$. Set $G_2(\eta) = e^{-x_2\eta}[F'(\eta) - x_1F(\eta)]$ and η_1 is the first zero of F. Similar to the proof of part (i), we know G_2 is strictly monotone increasing in $(0, +\infty)$ and $F'(\eta_1) \le 0$. There are two cases, $F'(\eta_1) = 0$ and $F'(\eta_1) < 0$.

Case 1: If $F'(\eta_1) = 0$, then we claim that η_1 is the unique positive root of F. Since $F''(\eta) - (x_1 + x_2)F'(\eta) + x_1x_2F(\eta) = \int_0^L e^{\eta x}m(x)(x - x_1)(x - x_2)\mathrm{d}x > 0$, then we have $F''(\eta_1) - (x_1 + x_2)F'(\eta_1) + x_1x_2F(\eta_1) = F''(\eta_1) > 0$. It implies that F attains a strict local minimum at η_1. Next we claim that there is no other root of F. If not, there exist $\eta_2 > \eta_1 > 0$ such that $F(\eta_2) = 0$ and $F(\eta) > 0$ in (η_1, η_2), which means that $F'(\eta_2) \le 0$. Thus $G_2(\eta_1) = e^{-x_2\eta_1}[F'(\eta_1) - x_1F(\eta_1)] = 0 \ge e^{-x_2\eta_2}F'(\eta_2) = e^{-x_2\eta_2}[F'(\eta_2) - x_1F(\eta_2)] =$

$G_2(\eta_2)$. This contradicts the fact that G_2 is strictly increasing. Therefore, F has a unique positive root η_1, i.e., $F \geq 0$ for any $\eta \in (0, +\infty)$ in this case.

Case 2: If $F'(\eta_1) < 0$, then we claim that there exists a unique $\eta_2 > \eta_1$ such that $F(\eta_2) = 0$. Moreover, $F'(\eta_2) > 0$.

Since $F'(\eta_1) < 0$, by Lemma 3.2, F has another positive root, denoted by η_2. We have $F(\eta) < 0$ in (η_1, η_2) and $F'(\eta_2) \geq 0$. If $F'(\eta_2) = 0$, then

$$F''(\eta_2) = F''(\eta_2) - (x_1 + x_2)F'(\eta_2) + x_1 x_2 F(\eta_2) = \int_0^L e^{\eta_2 x} m(x)(x - x_1)(x - x_2)\mathrm{d}x > 0,$$

which implies that F attains a strict local minimum at η_2. This is a contraction. Hence, $F'(\eta_2) > 0$. Next we claim that there is no the positive root of F for $\eta > \eta_2$. Suppose that there exist $\eta_3 > \eta_2$ such that $F(\eta_3) = 0$ and $F(\eta) > 0$ in (η_2, η_3). Then $F'(\eta_3) \leq 0$. Thus $G_2(\eta_2) = e^{-x_2\eta_2} F'(\eta_2) > 0 \geq e^{-x_2\eta_3} F'(\eta_3) = e^{-x_2\eta_3}[F'(\eta_3) - x_1 F(\eta_3)] = G_2(\eta_3)$. This contradicts the fact that G_2 is strictly increasing. Thus F has exactly two positive roots in this case. This completes the proof of (iii). □

Remark 4.1. For part (iii) (or (iv)), we can further show that $F(\eta) > (<)\ 0$ for any $\eta > 0$ with the additional condition $\int_0^L m(x)(x - x_1)\mathrm{d}x > (<)\ 0$. This implies that the first alternative in part (iii) can occur.

Proof the Theorem 1.6. We only need to prove (i) and (iii). Parts (ii) and (iv) can be established in the same way.

1. For part (i), similarly as in the proofs of Lemmas 3.5 and 3.6, we know that there exists some positive constant Λ, independent of d_I and q such that for each $d_I > \Lambda$, there exists some $\tilde{q} = \tilde{q}(d_I)$ such $\mathcal{R}_0(d_I, \tilde{q}) = 1$, and $\tilde{q}/d_I \to \eta_0$ as $d_I \to \infty$, where η_0 is the unique positive root of $F(\eta) = 0$.

Next we claim that if d_I is sufficiently large, for any $\tilde{q}$ satisfying $\mathcal{R}_0(d_I, \tilde{q}) = 1$, we have

$$\frac{\partial \mathcal{R}_0}{\partial q}(d_I, \tilde{q}) < 0.$$

To establish our assertion, let $\tilde{\varphi}$ denote an eigenfunction of the eigenvalue $\mathcal{R}_0(d_I, \tilde{q}) = 1$, uniquely determined by $\max_{[0,L]} \tilde{\varphi} = 1$. Hence, $\tilde{\varphi}$ satisfies

$$\begin{cases} -d_I\left(e^{\frac{\tilde{q}}{d_I}x}\tilde{\varphi}_x\right)_x + [\gamma(x) - \beta(x)]e^{\frac{\tilde{q}}{d_I}x}\tilde{\varphi} = 0, & 0 < x < L, \\ \tilde{\varphi}_x(0) = \tilde{\varphi}_x(L) = 0. \end{cases} \tag{4.1}$$

By (3.6), we have

$$\frac{\partial \mathcal{R}_0}{\partial q}(d_I, \tilde{q}) = \frac{\mathcal{R}_0^2 \int_0^L e^{\frac{\tilde{q}}{d_I}x}\tilde{\varphi}_x\tilde{\varphi}\mathrm{d}x}{\int_0^L \beta(x)e^{\frac{\tilde{q}}{d_I}x}\tilde{\varphi}^2\mathrm{d}x}.$$

Multiplying (4.1) by $\int_0^x \tilde{\varphi}(s)\mathrm{d}s$ and integrating in $(0, L)$, we obtain

$$d_I \int_0^L e^{\frac{\tilde{q}}{d_I}x} \tilde{\varphi}_x \tilde{\varphi} \mathrm{d}x + \int_0^L [\gamma(x) - \beta(x)] e^{\frac{\tilde{q}}{d_I}x} \tilde{\varphi} \left(\int_0^x \tilde{\varphi}(s)\mathrm{d}s \right) \mathrm{d}x = 0.$$

Thus,

$$d_I \frac{\partial \mathcal{R}_0}{\partial q}(d_I, \tilde{q}) = \frac{\int_0^L [\beta(x) - \gamma(x)] e^{\frac{\tilde{q}}{d_I}x} \tilde{\varphi} \left(\int_0^x \tilde{\varphi}(s)\mathrm{d}s \right) \mathrm{d}x}{\int_0^L \beta(x) e^{\frac{\tilde{q}}{d_I}x} \tilde{\varphi}^2 \mathrm{d}x}.$$

As $d_I \to \infty$, $\tilde{q}/d_I \to \eta_0$ and $\tilde{\varphi} \to 1$, we have

$$\lim_{d_I \to \infty} d_I \frac{\partial \mathcal{R}_0}{\partial q}(d_I, \tilde{q}) = \frac{\int_0^L x e^{\eta_0 x} [\beta(x) - \gamma(x)] \mathrm{d}x}{\int_0^L \beta(x) e^{\eta_0 x} \mathrm{d}x}.$$

By part (i) of Lemma 4.1,

$$\int_0^L x e^{\eta_0 x} [\beta(x) - \gamma(x)] \mathrm{d}x = F'(\eta_0) < 0.$$

Therefore, there exists some constant $Q > 0$ (dependent on d_I) such that $\mathcal{R}_0 > 1$ for $0 < q < Q$ and $\mathcal{R}_0 < 1$ for $q > Q$.

2. For part (iii), from part (iii) of Lemma 4.1, we know that there are three cases for F:

Case 1. $F(\eta) > 0$ for any $\eta > 0$.

Case 2. $F(\eta)$ has a unique positive root η_1 for $\eta \in (0, +\infty)$ and $F'(\eta_1) = 0$.

Case 3. $F(\eta)$ has two positive roots η_1 and η_2 ($\eta_1 < \eta_2$) for $\eta \in (0, +\infty)$ and $F'(\eta_1) < 0$, $F'(\eta_2) > 0$.

For Case 1, we can show that there exists some positive constant Λ independent of d_I and q such that for every $d_I > \Lambda$, $\mathcal{R}_0 > 1$ for any $q > 0$.

For Case 2, the proof of this case is exactly the same as the above proof of part (i). We can obtain that there exists some positive constant Λ independent of d_I and q such that for every $d_I > \Lambda$, there exists some $\tilde{q} = \tilde{q}(d_I)$ such $\mathcal{R}_0(d_I, \tilde{q}) = 1$, and $\tilde{q}/d_I \to \eta_0$ as $d_I \to \infty$, where η_0 is the unique positive root of $F(\eta) = 0$. Moreover, we can obtain that if d_I is sufficiently large, for any $\tilde{q}$ satisfying $\mathcal{R}_0(d_I, \tilde{q}) = 1$, we have $\partial \mathcal{R}_0(d_I, \tilde{q})/\partial q = 0$. Thus, we know that there exists some positive constant Λ independent of d_I and q such that for every $d_I > \Lambda$, there exists a constant $Q > 0$ (dependent on d_I) such that $\mathcal{R}_0 > 1$ for $q \in (0, Q) \cup (Q, \infty)$ and $\mathcal{R}_0 = 1$ for $q = Q$.

For Case 3, we need to modify the above proof of part (i). Following the same way as in proof of part (i), for each $d_I > 0$, there exist $\tilde{q}_1 = \tilde{q}_1(d_I)$ and $\tilde{q}_2 = \tilde{q}_2(d_I)$ ($\tilde{q}_1 < \tilde{q}_2$)

such that $\mathcal{R}_0(d_I, \tilde{q}_i) = 1$ $(i = 1, 2)$, and $\tilde{q}_1/d_I \to \eta_1$, $\tilde{q}_2/d_I \to \eta_2$ as $d_I \to \infty$. And if d_I is sufficiently large, then

$$\frac{\partial \mathcal{R}_0}{\partial q}(d_I, \tilde{q}_1) < 0, \quad \frac{\partial \mathcal{R}_0}{\partial q}(d_I, \tilde{q}_2) > 0.$$

Therefore, there exist two constants $Q_2 > Q_1 > 0$ (both dependent on d_I) such that $\mathcal{R}_0 > 1$ for $q \in (0, Q_1) \cup (Q_2, \infty)$ and $\mathcal{R}_0 < 1$ for $q \in (Q_1, Q_2)$. The proof is complete. □

We end this section with an example to show that the third option in part (iii) of Theorem 1.6 can happen.

Example 4.2. *Consider the system*

$$\begin{cases} d_S\tilde{S}_{xx} - q\tilde{S}_x - (x + \sin x + \tau)\dfrac{\tilde{S}\tilde{I}}{\tilde{S} + \tilde{I}} + x\tilde{I} = 0, & 0 < x < 2\pi, \\ d_I\tilde{I}_{xx} - q\tilde{I}_x + (x + \sin x + \tau)\dfrac{\tilde{S}\tilde{I}}{\tilde{S} + \tilde{I}} - x\tilde{I} = 0, & 0 < x < 2\pi, \\ d_S\tilde{S}_x(0) - q\tilde{S}(0) = 0, \ \ d_S\tilde{S}_x(2\pi) - q\tilde{S}(2\pi) = 0, \\ d_I\tilde{I}_x(0) - q\tilde{I}(0) = 0, \ \ d_I\tilde{I}_x(2\pi) - q\tilde{I}(2\pi) = 0, \end{cases} \tag{4.2}$$

where d_S, d_I, q are all positive constants, $0 < \tau < 1/2$ is a constant. From (4.2), we know that $\beta(x) = x + \sin x + \tau$ and $\gamma(x) = x$. It is easy to see that $\beta(x) - \gamma(x) = \sin x + \tau$ changes sign twice in $(0, 2\pi)$, $\beta(2\pi) - \gamma(2\pi) = \tau > 0$ and $\int_0^{2\pi}[\beta(x) - \gamma(x)]\mathrm{d}x = 2\tau\pi > 0$.

For $\tau > 0$, by direct computation we get

$$\begin{aligned} F(\eta) = \int_0^{2\pi} e^{\eta x}(\sin x + \tau)\mathrm{d}x &= \int_0^{2\pi} e^{\eta x}\sin x\mathrm{d}x + \tau\int_0^{2\pi} e^{\eta x}\mathrm{d}x \\ &= \frac{1 - e^{2\pi\eta}}{1 + \eta^2} + \frac{\tau\left(e^{2\pi\eta} - 1\right)}{\eta} \\ &= \frac{\left(e^{2\pi\eta} - 1\right)}{\eta}\left(\tau - \frac{\eta}{1 + \eta^2}\right). \end{aligned}$$

Then there exist two positive roots $\eta_{1,2} = (1 \mp \sqrt{1 - 4\tau^2})/2\tau$ such that $F(\eta_1) = F(\eta_2) = 0$. More precisely, from Lemma 4.1, we know that $F(\eta) < 0$ for $\eta \in (\eta_1, \eta_2)$ and $F(\eta) > 0$ as $\eta \in [0, \eta_1) \cup (\eta_2, \infty)$. By Theorem 1.6, there exists some constant $\Lambda > 0$ independent of d_I and q such that for every $d_I > \Lambda$, there exist at most two constant $Q_2 > Q_1 > 0$ (both dependent on d_I) such that $\mathcal{R}_0 > 1$ for $q \in (0, Q_1) \cup (Q_2, \infty)$ and $\mathcal{R}_0 < 1$ for $q \in (Q_1, Q_2)$. It implies that the disease persists when the advection rate is suitably small or large, and the disease is eliminated for intermediate advection rate.

References

[1] Nicholas D. Alikakos. An application of the invariance principle to reaction-diffusion equations. *J. Differential Equations*, 33(2):201–225, 1979.

[2] L. J. S. Allen, B. M. Bolker, Y. Lou, and A. L. Nevai. Asymptotic profiles of the steady states for an SIS epidemic patch model. *SIAM J. Appl. Math.*, 67(5):1283–1309, 2007.

[3] L. J. S. Allen, B. M. Bolker, Y. Lou, and A. L. Nevai. Asymptotic profiles of the steady states for an SIS epidemic reaction-diffusion model. *Discrete Contin. Dyn. Syst.*, 21(1):1–20, 2008.

[4] Robert Stephen Cantrell and Chris Cosner. *Spatial ecology via reaction-diffusion equations*. Wiley Series in Mathematical and Computational Biology. John Wiley & Sons Ltd., Chichester, 2003.

[5] Xinfu Chen and Yuan Lou. Principal eigenvalue and eigenfunctions of an elliptic operator with large advection and its application to a competition model. *Indiana Univ. Math. J.*, 57(2):627–658, 2008.

[6] Xinfu Chen and Yuan Lou. Effects of diffusion and advection on the smallest eigenvalue of an elliptic operator and their applications. *Indiana Univ. Math. J.*, 61(1):45–80, 2012.

[7] Michael G. Crandall and Paul H. Rabinowitz. Bifurcation from simple eigenvalues. *J. Functional Analysis*, 8:321–340, 1971.

[8] Karin A. Dahmen, David R. Nelson, and Nadav M. Shnerb. Life and death near a windy oasis. *J. Math. Biol.*, 41(1):1–23, 2000.

[9] Yihong Du and Junping Shi. Allee effect and bistability in a spatially heterogeneous predator-prey model. *Trans. Amer. Math. Soc.*, 359(9):4557–4593, 2007.

[10] Jing Ge, Kwang Ik Kim, Zhigui Lin, and Huaiping Zhu. A SIS reaction–diffusion–advection model in a low-risk and high-risk domain. *J. Differential Equations*, 259(10):5486–5509, 2015.

[11] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.

[12] Daniel Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1981.

[13] Sze-Bi Hsu and Yuan Lou. Single phytoplankton species growth with light and advection in a water column. *SIAM J. Appl. Math.*, 70(8):2942–2974, 2010.

[14] C.-S. Lin, W.-M. Ni, and I. Takagi. Large amplitude stationary solutions to a chemotaxis system. *J. Differential Equations*, 72(1):1–27, 1988.

[15] Yuan Lou and Wei-Ming Ni. Diffusion, self-diffusion and cross-diffusion. *J. Differential Equations*, 131(1):79–131, 1996.

[16] Frithjof Lutscher, Mark A. Lewis, and Edward McCauley. Effects of heterogeneity on spread and persistence in rivers. *Bull. Math. Biol.*, 68(8):2129–2160, 2006.

[17] Frithjof Lutscher, Edward McCauley, and Mark A Lewis. Spatial patterns and coexistence mechanisms in systems with unidirectional flow. *Theoretical Population Biology*, 71(3):267–277, 2007.

[18] Frithjof Lutscher, Elizaveta Pachepsky, and Mark A. Lewis. The effect of dispersal patterns on stream populations. *SIAM Rev.*, 47(4):749–772, 2005.

[19] Louis Nirenberg. *Topics in nonlinear functional analysis.* New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2001.

[20] Rui Peng. Asymptotic profiles of the positive steady state for an SIS epidemic reaction-diffusion model. I. *J. Differential Equations*, 247(4):1096–1119, 2009.

[21] Rui Peng and Shengqiang Liu. Global stability of the steady states of an sis epidemic reactioncdiffusion model. *Nonlinear Anal. TMA*, 71(4):239–247, 2009.

[22] Rui Peng and Fengqi Yi. Asymptotic profile of the positive steady state for an SIS epidemic reaction-diffusion model: effects of epidemic risk and population movement. *Phys. D*, 259:8–25, 2013.

[23] Rui Peng and Xiao-Qiang Zhao. A reaction-diffusion SIS epidemic model in a time-periodic environment. *Nonlinearity*, 25(5):1451–1471, 2012.

[24] Murray H. Protter and Hans F. Weinberger. *Maximum principles in differential equations.* Springer-Verlag, New York, 1984.

[25] Paul H. Rabinowitz. Some global results for nonlinear eigenvalue problems. *J. Functional Analysis*, 7:487–513, 1971.

[26] Junping Shi. Persistence and bifurcation of degenerate solutions. *J. Funct. Anal.*, 169(2):494–531, 1999.

[27] Junping Shi and Xuefeng Wang. On global bifurcation for quasilinear elliptic systems on bounded domains. *J. Differential Equations*, 246(7):2788–2812, 2009.

[28] Hal L. Smith. *Monotone dynamical systems*, volume 41 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1995.

[29] P. van den Driessche and James Watmough. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. *Math. Biosci.*, 180:29–48, 2002.

[30] Wendi Wang and Xiao-Qiang Zhao. Basic reproduction numbers for reaction-diffusion epidemic models. *SIAM J. Appl. Dyn. Syst.*, 11(4):1652–1673, 2012.