Math 220: Differential Calculus of One Variable Functions
Northwestern University, Lecture Notes

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These are notes which provide a basic summary of each lecture for Math 220, "Differential Calculus of One Variable Functions", taught by the author at Northwestern University. The book used as a reference is the 2nd edition of *Essential Calculus: Early Transcendentals* by Stewart. Watch out for typos! Comments and suggestions are welcome.

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Lecture 1: Review of Functions

Differential calculus is the study of change. More concretely, it is concerned with studying the rate at which one quantity changes in response to a change in a different but related quantity. This idea is fundamental to numerous applications, and manifests itself in, for example: the rate at which the price of a product changes in response to a change in cost of production; the rate at which a chemical dissolves in a liquid over a period of time; the rate at which glucose enters or leaves the bloodstream; and the rate at which the distance a car travels changes with respect to time. Calculus provides a single mathematical framework from which to study these and many other phenomena. It is truly no exaggeration to say that the development of calculus is one of the pinnacle achievements of human thought.

Functions. The core mathematical concept which calculus is concerned with is that of the *derivative* of a *function*. We'll introduce derivatives later on, but for now we focus on a review of functions and their properties. This is all material with which we will assume familiarity.

A *function* is an operation which takes an input and produces an output. For our purposes, a function will take a number as input and will produce a number as an output. For instance, when we write something like

$$f(x) = x^2,$$

we are referring to a function we call f which takes a number x as input and outputs the value $f(x) = x^2$, which we say is the value of f at x, or the result of evaluating f at x. For instance, the result of evaluating f at 2 is $f(2) = 2^2 = 4$, and the result of evaluating f at 3 is $f(3) = 3^2 = 9$; so, this function takes 2 as an input and outputs 4, and outputs 9 for the input 3.

Example. The area of the region which a circle of radius r encloses can be described by a function whose input is the radius r. Call this function A, so that $A(r) = \pi r^2$ is the resulting area for a given radius r. For instance, a unit circle has area $A(1) = \pi$, and a circle of radius 2 has area $A(2) = 4\pi$.

Now say that g is the function which measures the difference in the area between one circle and that of the circle with radius 1 larger. So, $g(r)$ should describe the difference between the area of a circle of radius r and one of radius $r + 1$. We can express this using the previous area function as

$$g(r) = A(r + 1) - A(r).$$

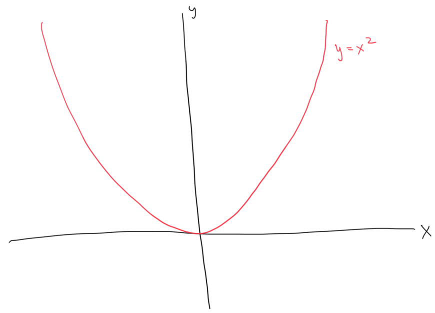
Using the explicit formula for $A(r)$ and doing some algebra gives:

$$g(r) = \pi(r + 1)^2 - \pi r^2 = \pi(r^2 + 2r + 1) - \pi r^2 = 2\pi r + \pi.$$

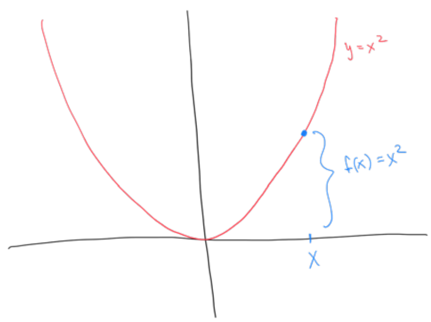
The point here is simply to observe how we can use an already-studied function, such as A, to construct a new function such as g in this case. Understanding how different functions might relate to one another will be a crucial concept going forward.

Graphs. Functions can visualized via their *graphs*, which are curves drawn in the xy-plane which keep track of the values the functions in question. Concretely, the graph of a function f consists of the points (x, y) where the y-coordinate is precisely the value of the function $f(x)$ at the corresponding x-coordinate. In other words, it is the curve defined by the equation $y = f(x)$.

For instance, the function $f(x) = x^2$ from above has a graph which looks like:

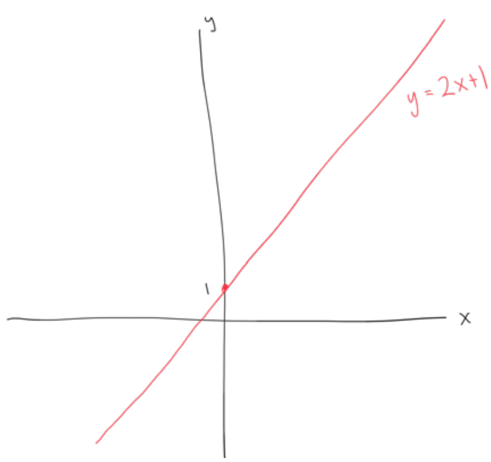


This shape is called a *parabola*, and is defined by the equation $y = x^2$. Visually, it represents the value of f at x as the *height* of the corresponding point on the graph:



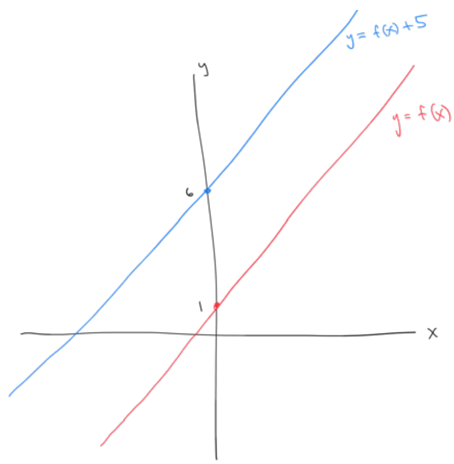
In general, when looking at the graph of a function, the horizontal axis keeps track of the inputs and the vertical axis (or "height") keeps track of the outputs.

Linear example. The function $f(x) = 2x+1$ is an example of a *linear* function, so named because its graph looks like a straight line:



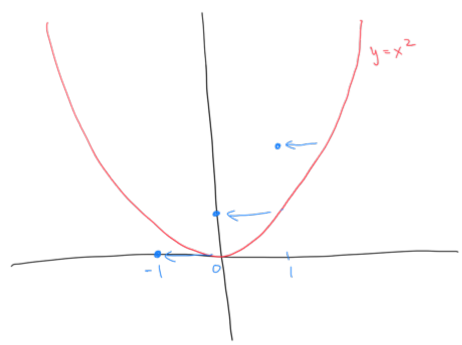
This line has *slope* 2, which means that increasing the value of x by 1 causes the value of y to increase by 2. Slope is a measure of how steep a line is.

Modifications. Say that $f(x) = 2x + 1$ is the same linear function as above, and consider now the function g defined by $g(x) = f(x) + 5$. The question is: how does the graph of g relate to that of f? Note that the values of g are obtained by increasing the values of f by 5, which visually in terms of graphs means that the heights on the graph of f should be increased by 5 to obtain the graph of g. So, the graph of g is still a line, in fact parallel to the graph of f, only shifted up 5:

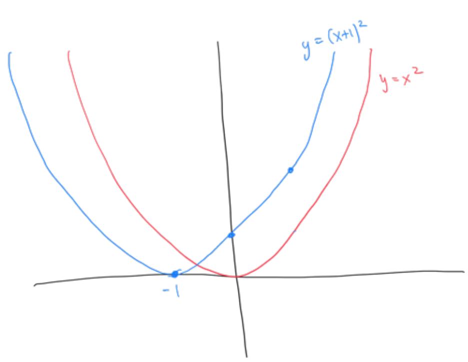


There is nothing special about the fact that we are dealing with lines heres: no matter what type of graph f had, the graph of $g(x) = f(x) + 5$ would still maintain the same shape, only shifted up by 5.

Suppose now that f is the function $f(x) = x^2$ we saw previously. Define h to be the function $h(x) = f(x+1) = (x+1)^2$. Again we want to know how the graph of h compares to that of f. Note that in this case we are modifying *inputs* instead of outputs like we did above. At, say, $x = -1$, the value of h should be the value f has at $x = -1 + 1 = 0$, meaning that the height at $x = -1$ on the graph of h should be the height on the graph of f at $x = 0$. Similarly, the height at $x = 0$ on the graph of h should be the height at $x = 0 + 1 = 1$ on the graph of f, and the height at $x = 1$ on the graph of h should be the height at $x = 2$ on the graph of f:



Visually, this has the effect of shifting the graph of f to the *left*.

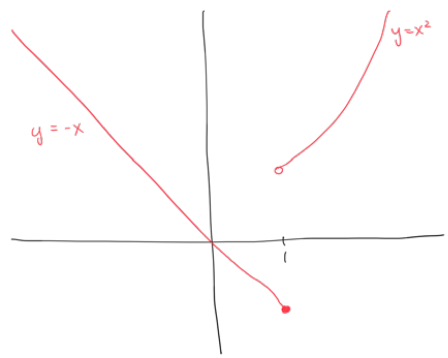


The idea is that for the the graph of h, we don't have to move as far to the right to obtain certain heights as we did for f since we are inputting $x+1$ (which occurs to the right of x) and not simply x. In general, regardless of function f is, the function $h(x) = f(x+1)$ would still have the same shape as the graph of f, only shifted to the left by 1.

Piecewise example. Consider the function f defined by

$$f(x) = \begin{cases} x^2 & \text{if } x > 1 \\ -x & \text{if } x \leq 1 \end{cases}.$$

This is often called a *piecewise* function, since the expression for the outputs it gives depends on the "piece" the value of x comes from. So, in this case the graph should match the parabola $y = x^2$ we saw previously, but only for values of x which are larger than 1. For values of x which are less than or equal to 1, the graph of f should match up with the line $y = -x$:

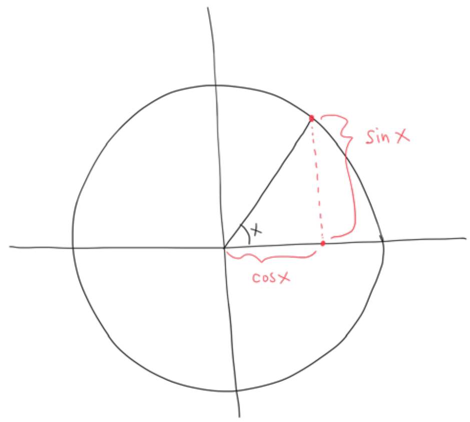


We use the standard convention that an "open" dot means that at that point, the function value is NOT the one given by the open dot, whereas a "full" dot indicates that that height IS the value of the function at that point.

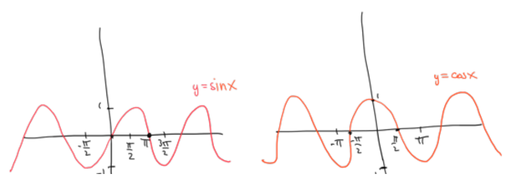
Domain. Consider the function $f(x) = \sqrt{4-x}$. The new observation is that the formula defining this function might fail to make sense for certain values of x. For instance, for $x = 10$, $\sqrt{4-10} = \sqrt{-6}$ is undefined. Thus, we often talk about the *domain* of a function, which is the collection of inputs for which the function is actually defined. In this case, in order for $\sqrt{4-x}$ to make sense, we need the term under the square root to be nonnegative: $4 - x \geq 0$. (Otherwise we would be trying to take the square root of a negative number.)

The requirement $4 - x \geq 0$ is the same as $4 \geq x$, which can rephrase as saying that x belongs to the *interval* $(-\infty, 4]$. This notation denotes the collection of all numbers less than or equal to 4, where the "equal to" part comes from the use of the *closed* bracket at 4. By contrast, $(-\infty, 4)$ denotes the collection of all numbers which are strictly *less* than 4. We thus say that the domain of $f(x) = \sqrt{4-x}$ is the interval $(-\infty, 4]$.

Trigonometric functions. We'll see many types of functions in this course. In particular, we'll talk about *exponential* and *logarithmic* functions later on. For now, the other important type of function with which we should be familiar are *trigonometric functions*; specifically, $\sin x$ and $\cos x$. Recall that these values can be defined by lookin at a unit circle: if x denotes the angle a line segment makes with the positive x-axis, then $\cos x$ is the x-coordinate (horizontal length) of the corresponding point on the unit circle, while $\sin x$ is the y-coordinate (vertical height) of that point:



You should be familiar with the fact that we can measure angles both in terms of *degrees* and *radians*. (The radian measure of an angle indicates how much circumference of the circle has been used up by that angle.) For most aspects of calculus, we will actually only care about radians. Then the graphs of $y = \sin x$ and $y = \cos x$ are as follows:



Lecture 2: Limits of Functions

Warm-Up 1. Suppose f and g are the functions defined by $f(x) = \frac{x^2-4}{x-2}$ and $g(x) = x + 2$. Are f and g the same function? The point is that it is possible to simplify the expression for f by factoring the numerator, thereby obtaining the expression for g:

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x + 2.$$

So, this seems to say that $f(x)$ and $g(x)$ are always the same, so f and g should be different expressions for the same function. However, this misses the fact that f is not technically defined at $x = 2$, since this value for x gives an undefined fraction with 0 in the denominator, while g *is* defined at $x = 2$.

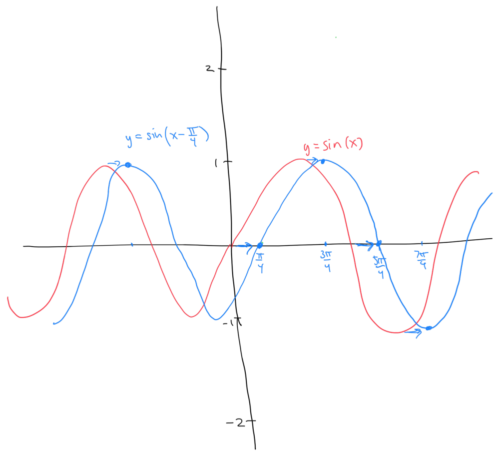
The point is that f and g are not the same function since their *domains* are different. The domain of g consists of all numbers, where as the domain of f consists of all numbers *expect* 2. What is true is that $f(x) = g(x)$ for all $x \neq 2$, so that f and g give the same values on the collection of numbers excluding 2. But, nonetheless, f and g are not technically the same function.

Now, suppose we define the function h in a piecewise-manner by:

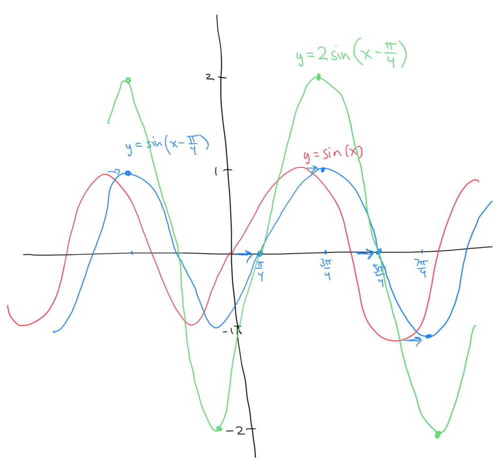
$$h(x) = \begin{cases} \frac{x^2-4}{x-2} & x \neq 2 \\ 4 & x = 2. \end{cases}$$

Then, for $x \neq 2$, h gives the value $h(x) = \frac{x^2-4}{x-2} = x + 2$, while for $x = 2$ h gives the value $h(2) = 4 = 2 + 2 = g(2)$. Thus, this function h *is* the same as the function $g(x) = x + 2$, only that we've written it in a more elaborate way.

Warm-Up 2. We draw the graph of $f(x) = 2\sin(x - \frac{\pi}{4})$. The idea is to build up from the simpler graph of $y = \sin x$ by focusing on how the subtraction of $\frac{\pi}{4}$ and then the multiplication by 2 affect the graph. Start with the graph of $y = \sin x$. Changing the input to be $x - \frac{\pi}{4}$ instead of just x has the effect of shifting the graph to the *right* by $\frac{\pi}{4}$. Indeed, the height of $y = \sin(x - \frac{\pi}{4})$ at a point x is actually that of the normal sine function at the point $x - \frac{\pi}{4}$ *to the left* of x, so we take this height and draw it *further to the right* at x:



Now, in $f(x) = 2\sin(x - \frac{\pi}{4})$ we take the values $y = \sin(x - \frac{\pi}{4})$ and *double* them. This has the effect of doubling vertical distance to the x-axis, so the graph of $y = f(x)$ looks like:



Again, note how this was pieced together from the ground up starting with $y = \sin x$.

One final comment: we can also say that f is obtained as a result of the *composition* of two other functions. If $g(x) = 2\sin x$ and $h(x) = x - \frac{\pi}{4}$, then $f(x) = 2\sin(x - \frac{\pi}{4})$ is obtained by first applying h to x, and then applying g to the result:

$$f(x) = g(h(x)).$$

The right side is called the *composition* of g and h. Such compositions will play a key role later on when discussing what's called the *chain rule*.

Limits. The notion of a *limit* is a central one in all of calculus, and describes the phenomena where quantities can *approach* a certain value. We use the notation

$$\lim_{x \to a} f(x)$$

to denote what's called the "*limit of f as x approaches a*", and concretely equals the number (if it exists) which the values $f(x)$ approach as x gets closer and closer to a. Graphically this limit equals the height which the heights $f(x)$ approach as the input x-coordinate approach a.

One key point to emphasize is that the value $f(a)$ of f *at* a, or whether $f(a)$ is even defined, plays no role in this first definition; it only matters how f behaves *near* and close to a. Now, we'll see that quite often the value of a limit might indeed agree with $f(a)$, but this reflects a certain nice property a function can have, namely that it be *continuous* at a. This is something we'll come to later this week, but for now we are really only focused on understanding the values of f near but not at a.

This intuitive idea as to what a limit is will be enough for our purposes, but we should point out that a completely precise definition of "limit" can be given. You can find this in the book, and is often referred to as the "ϵ-δ" (pronounced epsilon-delta) definition. We will not look at this precise definition in this class, but is the type of thing you would study in detail in a theoretical calculus class such as *Real Analysis*, which is Math 320 at Northwestern. Even though we don't study this precise definition in this class, I claim it is actually quite intuitive once you unwind the various symbols used, and I'd be happy to discuss it in office hours if you're interested. The development of this formal definition is what motivated much of mathematics in the 18th and 19th centuries.

Example 1. Let's jump into an example which will illustrate various ways of thinking about limits. We are interested in the following:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}.$$

Now, going by our intuitive definition above, we are looking for the number which the values $f(x) = \frac{x^2-4}{x-2}$ approach as x gets closer and closer to 2. If nothing else, we can work out some of these values and try to guess at what they are approaching. The table

x	1.8	1.9	1.99	2.001	2.1
$f(x)$	3.8	3.9	3.99	4.001	4.1

encodes various values of $f(x) = \frac{x^2-4}{x-2}$ for x close to 2. The takeaway is that it seems the values $f(x)$ approaches 4 as x approaches 2: at $x = 1.8, 1.9, 1.99$ respectively, we have $f(x) = 3.8, 3.9, 3.99$ respectively approaching 4, and similarly if we approach 2 from the "right" with 2.1 and 2.001, the corresponding function values 4.1 and 4.001 seem to be getting closer to 4. So, we guess based on this alone that

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4.$$

Now, in this case we can be more precise. Recall that in this limit we are really only interested in the values of $f(x) = \frac{x^2-4}{x-2}$ for x "close" to 2. In particular, we will never consider $x = 2$, and we have seen before that for $x \neq 2$ we actually know that $f(x) = \frac{x^2-4}{x-2}$ is the same as $x + 2$. This means that the following should be true:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2).$$

The point is that $\frac{x^2-4}{x-2}$ and $x+2$ are give literally the *same* values for the values of x in question (i.e. those close to but not equal to 2), so they should certainly give the same limit value. (We'll see this graphically in a bit.) So, we are now reduced to determining the value of

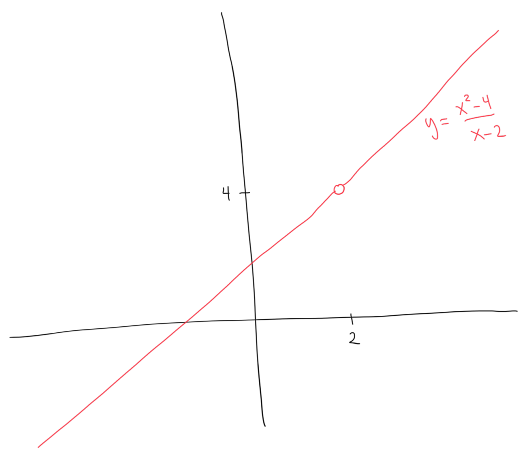
$$\lim_{x\to 2}(x+2).$$

But this is more straightforward: as x gets closer and closer 2, $x+2$ gets closer and closer to $2+2=4$. Thus

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}(x+2)=4,$$

as we had guessed previously by plugging in some numbers close to 2. This idea of finding a simpler expression for a given function, one that works at least close to the point we are approaching, will be an important one for computing more general types of limits, as we'll see next time.

Finally, let's interpret all of this graphically. The graph of $f(x)=\frac{x^2-4}{x-2}$ looks like that of $y=x+2$, except that our graph should not be defined at $x=2$:



Again, this is because we already saw that $\frac{x^2-4}{x-2}$ has the same as $x+2$ for $x\neq 2$, and that $\frac{x^2-4}{x-2}$ is undefined at $x=2$. Visually, if you imagine points on the x-axis getting closer and closer to 2, we are wanting to know what the value the corresponding heights are approaching, which we can see to be 4 as expected.

Example 2. This next example starts to give a sense of how limiting procedures might show up in practice. Suppose a car begins to move and that the distance it has traveled after t seconds is given by the function $d(t)=3t^2-t$, measured in say feet. We are interested in knowing the speed at which the car is moving at $t=1$ seconds, or more precisely what is usually called the *instantaneous* speed at $t=1$. Later once we discuss the concept of a *derivative* we'll see that this is something we can compute fairly easily, but for now the best we can do is *approximate* this speed. What we *can* compute using the information given is the *average* speed the car has traveled over a given time interval, and we make the definition that *instantaneous* speed is obtained as a limit of average speed as the time interval in question shrinks:

$$\text{instantaneous speed} = \text{limit (average speed)}.$$

Now, the average speed the car has traveled from $t=1$ seconds to $t=1+h$ seconds, where h is some number indicating the length of time elapsed, is given by

$$\text{average speed}=\frac{d(1+h)-d(1)}{(1+h)-1}=\frac{d(1+h)-d(1)}{h}.$$

In other words, the average speed over a time interval is the ratio of the distance traveled over that time interval to the length of the time interval, i.e. how much time has elapsed. We want to know the value these average speeds approach as h approaches 0, which corresponds to a shrinking time interval around $t = 1$ seconds. Here are some values:

t	1	1.001	1.01	1.1	2
$d(t)$	2	2.005003	2.0501	2.53	10.

For instance, the average speed from $t = 1$ to $t = 2$ seconds is

$$\frac{d(2) - d(1)}{2 - 1} = \frac{10 - 2}{1} = 8.$$

This number isn't so important for the question we're trying to answer since 2 isn't so close to 1. The average speed from $t = 1$ to $t = 1.1$ seconds (which corresponds to $h = 0.1$) is:

$$\frac{d(1.1) - d(1)}{1.1 - 1} = \frac{0.53}{0.1} = 5.3.$$

The average speed from $t = 1$ to $t = 1.01$ seconds (corresponding to $h = 0.01$) is:

$$\frac{d(1.01) - d(1)}{1.01 - 1} = \frac{0.0501}{0.01} = 5.01,$$

and the average speed from $t = 1$ to $t = 1.001$ seconds (which corresponds to $h = 0.001$) is:

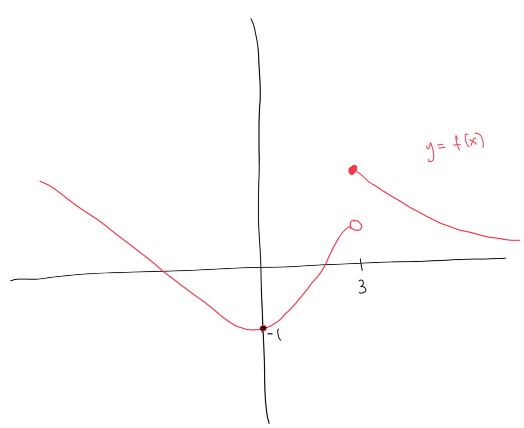
$$\frac{d(1.001) - d(1)}{1.001 - 1} = \frac{0.005003}{0.001} = 5.003.$$

The point is that these average speeds seem to be approaching 5 as h gets smaller, i.e. as the time elapsed past $t = 1$ seconds shrinks. So, we make the guess that the instantaneous speed at $t = 1$ is

$$\lim_{h \to 0} \begin{pmatrix} \text{average speed} \\ \text{from } t = 1 \text{ to } t = 1 + h \end{pmatrix} = 5.$$

Later we will see that this is not just a guess, but is in fact true.

Example 3. Finally we look at one more graphical example. Consider the function f whose graph is given by



Based on the graph, we can say for instance that

$$\lim_{x\to 0} f(x) = -1,$$

since the vertical heights of points on the graph are approaching -1 as the input x approaches 0. Now, what about $\lim_{x\to 3} f(x)$? This limit actually *does not exist*, since it is NOT true that the height $f(x)$ approaches a single, specific value as x approaches 3. In fact, the behavior of f is different to the left of $x = 3$ than to the right. What we *can* say is that as we approach 3 from the *left*, the value of $f(x)$ gets closer to 2, while as we approach 3 from the *right*, the value of $f(x)$ approaches 4. We use *one-sided* limits to describe this phenomena:

$$\lim_{x\to 3^-} f(x) = 2$$

denotes the limit as we approach x from the left, and

$$\lim_{x\to 3^+} f(x) = 4$$

denotes the limit as we approach x from the right. The fact that these are not the same is what says that there is no single, unique value which $f(x)$ approaches in general as x approaches 3, so

$$\lim_{x\to 3} f(x) \text{ does not exist.}$$

Lecture 3: More on Limits

Warm-Up 1. We will estimate the value of

$$\lim_{x\to 0} \frac{\sin x}{x}.$$

This is actually a very important limit we will see a few times throughout the quarter since it explains much of the behavior of various trig functions. Note that the given function $f(x) = \frac{\sin x}{x}$ is not defined at 0, which is not an issue here since, after all, a limit only cares about the values of a function *near* the point being approached.

We record a few values below, which can be found with the help of a calculator:

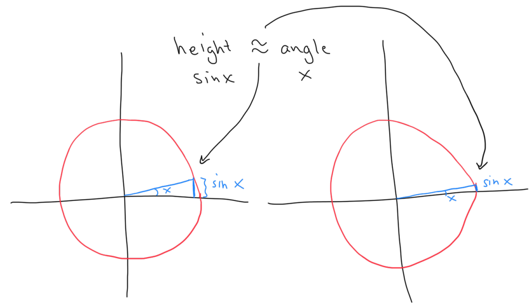
x	-0.1	-0.01	0.001	0.01	0.1
$\frac{\sin x}{x}$	0.998	0.99998	0.9999998	0.99998	0.998

Based on these values it seems that $\frac{\sin x}{x}$ is getting closer and closer to 1 as x gets closer and closer to 0, so our guess is that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

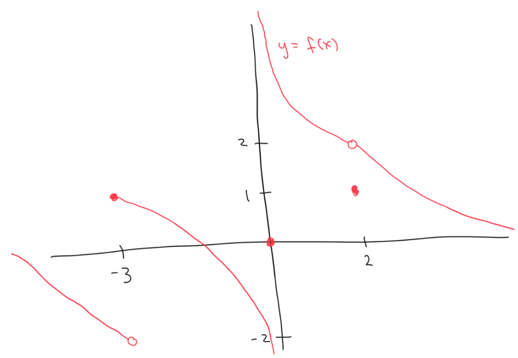
This is in fact true (!), but justifying it precisely takes some work and requires looking at the geometry of sectors of circles and triangles. This is worked out in the book, so you can take a look there if interested. For our purposes, this is a limit you should have ingrained in your minds.

But, we can give some nice geometric intuition for why you should expect this limit to be 1. Saying that this limit is 1 is saying that the fraction $\frac{\sin x}{x}$ gets closer and closer to 1, which means that the numerator and denominator should in fact be getting closer and closer to each other! Saying that limit is 1 amounts to the claim that for x that are really close to 0, the value of $\sin x$ is approximately the "same" as the value of x, or more precisely that the value of $\sin x$ is close to the value of x, and it gets even closer as x approaches 0. Looking at what x and $\sin x$ look like on a unit circle:



shows that this makes sense: the small height $\sin x$ drawn has pretty close to the same value as the small angle x. Of course, for larger angles this is not the case, but in this limit we only care about angles close to 0.

Warm-Up 2. Consider the function f whose graph is drawn below:



We want to determine the points a for which $\lim\limits_{x\to a} f(x)$ does not exist. First, $a = 0$ is such a point. As we approach 0, the value (i.e. heights) of the function either get larger and larger (from the right side) or more and more negative (from the left side), so these values do not approach a specific, unique number. Thus ${}_{x\to 0}f(x)$ does not exist.

Next, $a = -3$ is also a point at which the limit does not exist. The issue is here is that while the values of f do get closer and closer to specific numbers regardless of whether we approach from the left or right, the specific numbers we get are different depending on how we approach -3. What we can say is that the one-sided limits

$$\lim_{x\to 3^-} f(x) = -2 \text{ and } \lim_{x\to 3^+} f(x) = 1$$

exist, but the fact that these are different means that

$$\lim_{x\to -3} f(x)$$

does not exist since the values of $f(x)$ do not approach a *unique* number as x approaches -3.

Everywhere else the limit does exist. In particular, note that

$$\lim_{x\to 2} = 2$$

exists even though the value of f at 2 is not 2, it is $f(2) = 1$. The value of f at 2 plays no role in this limit however, since we only care about the values of f *near* 2, and these values are indeed getting closer and closer to 2.

Computing limits. Now we move to computing limits explicitly. The simplest types are ones where we can just literally evaluate the function in question at the point we are approaching. For instance,

$$\lim_{x\to 1}(x^2+2x-5) = -2,$$

which is what we get when we evaluate $f(x) = x^2+2x-5$ at $x = 1$. This works in this case because we can just keep track of what is happening to each term in the expression for f, one at a time: as $x \to 1$ (notation for "x approaches 1"), x^2 will get closer and closer to 1^2, $2x$ will get closer and closer to $2(1)$, and -5 remains -5 throughout since it doesn't change as x changes. Thus, what we are really saying is that

$$\lim_{x\to 1}(x^2+2x-5) = 1^2+2(1)-5 = -2.$$

This will work for any *polynomial* function such as $f(x) = x^2+2x-5$, which is a function whose expression involves only positive powers of x and constants.

This idea also works for other functions, in general for functions where the values we obtain "make sense" in that we don't run into any undefined expressions. For instance, consider

$$\lim_{x\to 2}\sqrt{\frac{4x^2-10}{x+2}}.$$

Now, to be sure, the function given by this square root is not defined for all x, since the term under the square root might be negative, but it is at least defined for all x which are close to 2. Tracking what happens to each term gives that as $x \to 2$: $4x^2$ approaches $4(2)^2 = 16$, -10 remains -10, x in the denominator approaches 2, and $+2$ in the denominator remains. Thus

$$\lim_{x\to 2}\sqrt{\frac{4x^2-10}{x+2}} = \sqrt{\frac{4(2)^2-10}{2+2}} = \sqrt{\frac{6}{4}}.$$

Check the book for some "limit laws" along these lines. Again, as long as you don't run into any undefined expression, this type of computation will be valid.

But not all things are as nice as this. In general, our strategy for computing for now will be to find an *alternate* expression for our function, which will be valid for the types of values we are considering. All of the examples which follow use this technique, but the specific way in which it is applied differs. There is no set way to know what to do in each scenario, and it is only through practice that these things become simpler to grasp.

Example 1. Consider the limit

$$\lim_{x\to 2}\frac{|x-2|}{x-2}.$$

Certainly the numerator gets closer to 0 as $x \to 2$, and so does the denominator. However, in this case this is not enough to go on, since it does not make sense to say that this fraction approaches $\frac{0}{0}$, and undefined expression. We really have to consider the behavior of the *entire* fraction, and not simply the numerator and denominator separately.

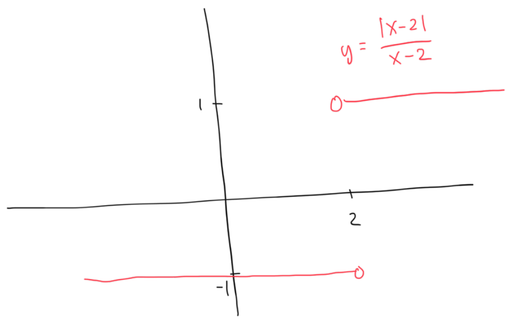
The goal is to find an alternate expression for $\frac{|x-2|}{x-2}$. But this is not so bad in this case, since we can be pretty explicit about what $|x-2|$ actually equals: if $x-2$ is positive, then $|x-2| = x-2$ since taking the absolute value of a positive number produces that number itself; while if $x-2$ is negative, then $|x-2| = -(x-2)$ since when $x-2$ is negative, $-(x-2)$ will in fact be positive. Thus, we can give the precise value of the function in question, depending on whether $x-2$ is positive (which occurs when $2 < x$) or negative (which occurs when $x < 2$):

$$\frac{|x-2|}{x-2} = \begin{cases} \frac{x-2}{x-2} & x > 2 \\ \frac{-(x-2)}{x-2} & x < 2. \end{cases}$$

Of course, we can simplify these further:

$$\frac{|x-2|}{x-2} = \begin{cases} 1 & x > 2 \\ -1 & x < 2. \end{cases}.$$

The point is that the at-first-glance complicated looking expression $\frac{|x-2|}{x-2}$ ends up having a pretty simple alternate description, 1 or -1, depending on the value of x we consider. The graph of the function $f(x) = \frac{|x-2|}{x-2}$ thus looks like:



and so

$$\lim_{x \to 2} \frac{|x-2|}{x-2}$$

does not exist since the values approach different numbers from the left of 2 versus the right of 2.

Example 2. Take the function $d(t) = 3t^2 - t$ we used in an example last time, giving the distance a car has traveled after t seconds. We argued last time that the instantaneous speed of the car at $t = 1$ is given by the limit

$$\lim_{h \to 0} \frac{d(1+h) - d(1)}{h}$$

of approximating average speeds as our time interval shrinks around $t = 1$; in other words, as we move from $t = 1 + h$ seconds to $t = 1$ second. Using the given expression for d, we have

$$d(1+h) = 3(1+h)^2 - (1+h) \text{ and } d(1) = 3(1)^2 - 1 = 2,$$

so this limit becomes

$$\lim_{h \to 0} \frac{[3(1+h)^2 - (1+h)] - 2}{h}$$

which we will now compute. First, we can simplify the expression for the numerator:

$$3(1+h)^2 - (1+h) - 2 = 3(1 + 2h + h^2) - 1 - h - 2 = 5h + 3h^2.$$

Our limit is now:

$$\lim_{h \to 0} \frac{5h + 3h^2}{h}.$$

We can simplify one step further by factoring h out of the numerator:

$$5h + 3h^2 = h(5 + 3h),$$

so we get

$$\lim_{h\to 0} \frac{h(5+3h)}{h} = \lim_{h\to 0}(5+3h).$$

The point of all this is that we have found an alternate expression $5+3h$ for our original function which is valid for the values of $h \neq 0$ we are considering:

$$\frac{3(1+h)^2 - (1+h) - 2}{h} = 5+3h \text{ for } h \neq 0.$$

The limit of our original function should be the same as the limit of this alternate expression, so

$$\lim_{h\to 0} \frac{[3(1+h)^2 - (1+h)] - 2}{h} = \lim_{h\to 0}(5+3h) = 5+3(0) = 5.$$

Note that this was a value we guessed last time by plugging in values of h close to 0, or equivalently values of $1+h$ close to 1.

Example 3. Now we compute

$$\lim_{x\to 2} \frac{\sqrt{4x+1}-3}{x-2}.$$

This function is undefined at $x = 2$, so our goal again is to find an alternate expression valid for $x \neq 2$. Now, it would make things convenient to not have the square root term in the numerator, simply because in general expressions without square roots are simply than ones with square roots. So, we look for a way to rewrite this function in a way which avoids having a square root in the numerator. This we can do by multiplying the entire fraction by

$$\frac{\sqrt{4x+1}+3}{\sqrt{4x+1}+3}.$$

In other words, we multiply by 1—so that the value of the fraction does not change—only 1 written in a funny way as $\sqrt{4x+1}+3$ divided by itself. This gives:

$$\frac{(\sqrt{4x+1}-3)}{(x-2)} \frac{(\sqrt{4x+1}+3)}{(\sqrt{4x+1}+3)} = \frac{4x+1-9}{(x-2)(\sqrt{4x+1}+3)} = \frac{4x-8}{(x-2)(\sqrt{4x+1}+3)}.$$

Now, to be clear, the numerator comes from the following product:

$$\begin{aligned}(\sqrt{4x+1}-3)(\sqrt{4x+1}+3) &= \sqrt{4x-1}^2 + 3\sqrt{4x+1} - 3\sqrt{4x+1} - 9 \\ &= (4x-1) - 9,\end{aligned}$$

which follows the $(a-b)(a+b) = a^2 - b^2$ identity. The point is that the two terms we get which still involve $\sqrt{4x+1}$ when multiplying everything out actually cancel each other out, so no square root terms remain. Seeing that this was going to happen is precisely why we multiplied numerator and denominator by $\sqrt{4x+1}+3$ at the start, again motivated by the identity $(a-b)(a+b) = a^2 - b^2$.

Next, we can simplify a bit more as follows:

$$\frac{4x-8}{(x-2)(\sqrt{4x+1}+3)} = \frac{4(x-2)}{(x-2)(\sqrt{4x+1}+3)} = \frac{4}{\sqrt{4x+1}+3}.$$

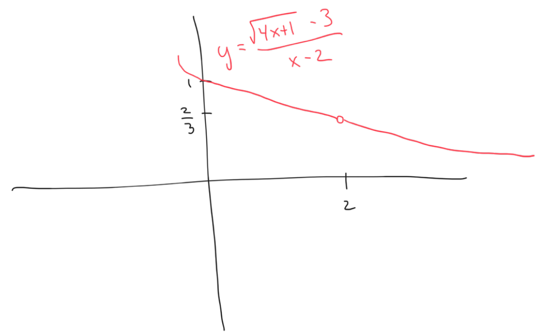
The point again is that we have found an alternate expression for our original function which is valid for $x \neq 2$:

$$\frac{\sqrt{4x+1}-3}{x-2} = \frac{4}{\sqrt{4x+1}+3} \text{ for } x \neq 2.$$

Thus we can now finish our limit computation:

$$\lim_{x\to 2}\frac{\sqrt{4x+1}-3}{x-2} = \lim_{x\to 2}\frac{4}{\sqrt{4x+1}+3} = \frac{4}{\sqrt{4(2)+1}+3} = \frac{4}{6} = \frac{2}{3}.$$

The graph of this function looks like the following (which you can see using a computer):



which shows that it makes sense for the limit to be $\frac{2}{3}$.

Example 4. Finally we consider

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x^2+x}\right).$$

First, as a side remark, note that neither

$$\lim_{x\to 0}\frac{1}{x} \text{ nor } \lim_{x\to 0}\frac{1}{x^2+x}$$

exist. Indeed, the issue is that as x gets closer and closer to 0, the denominators of these fractions also approach 0 while the numerator remains constant at 1, and this causes the fractions to get larger and larger and larger. (More precisely, when approaching 0 from the left, the fractions get more and more negative.) Hence these fractions do not approach a specific value, so the limits do not exist. (Later we will talk about it means for a limit to "equal infinity", which is really just a way of saying that a limit does not exist for a specific reason dealing with infinity. This is not the type of limit we're talking about just yet.)

This is worth pointing out because you might be tempted to break the given limit into two pieces and say that:

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x^2+x}\right) \stackrel{?}{=} \lim_{x\to 0}\frac{1}{x} - \lim_{x\to 0}\frac{1}{x^2+x}.$$

Since the limits on the right do not exist, you might thus say that our given limit does not exist either. But this is wrong:

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x^2+x}\right) \neq \lim_{x\to 0}\frac{1}{x} - \lim_{x\to 0}\frac{1}{x^2+x}.$$

The issue is that "limit law"

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a}f(x) + \lim_{x\to a}g(x)$$

only applies when both limits on the right exist *independently* on their own. That is not true in our case, so such a "splitting" of our limit into two pieces is not valid.

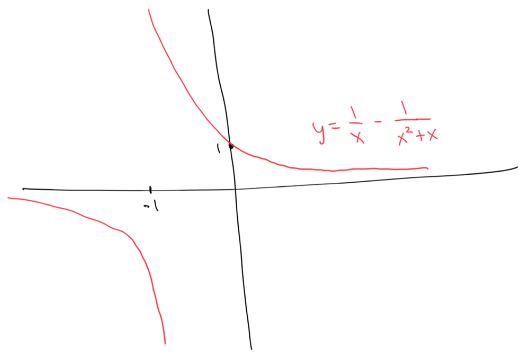
Instead we find, as usual, an alternate expression for our given function. We combine our fractions into one via:

$$\frac{1}{x} - \frac{1}{x(x+1)} = \frac{x+1}{x(x+1)} - \frac{1}{x(x+1)} = \frac{x}{x(x+1)} = \frac{1}{x+1}.$$

Thus, $\frac{1}{x+1}$ is an alternate expression for our function which is valid for $x \neq 0$, so:

$$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x^2+x} \right) = \lim_{x\to 0} \frac{1}{x+1} = \frac{1}{0+1} = 1.$$

Using a computer to draw the graph of $f(x) = \frac{1}{x} - \frac{1}{x^2+x}$ verifies that this limit should indeed be 1:



Note that this function has what's called an *asymptote* at $x = -1$, which is a concept we'll discuss later on.

Lecture 4: Continuous Functions

Warm-Up. We compute the following limits:

$$\lim_{x\to 0} \frac{\sqrt{3+x} - \sqrt{3}}{x} \qquad \lim_{x\to 3^+} \frac{x^2+2x-15}{|x-3|} \qquad \lim_{x\to 3^-} \frac{x^2+2x-15}{|x-3|}$$

The strategy in each case is to find an alternate expression for the function given which is valid for the values of x being considered.

First, we can rewrite the function in the first limit by multiplying numerator and denominator by $\sqrt{3+x} + \sqrt{3}$, which will have the effect of eliminating the square root terms in the numerator:

$$\frac{(\sqrt{3+x} - \sqrt{3})}{x} \frac{(\sqrt{3+x} + \sqrt{3})}{(\sqrt{3+x} + \sqrt{3})} = \frac{3+x-3}{x(\sqrt{3+x} - \sqrt{3})} = \frac{x}{x(\sqrt{3+x} + \sqrt{3})} = \frac{1}{\sqrt{3+x} + \sqrt{3}}.$$

The point is that this final expression gives the same value as the original function for $x \neq 0$. Thus

$$\lim_{x\to 0} \frac{\sqrt{3+x} - \sqrt{3}}{x} = \lim_{x\to 0} \frac{1}{\sqrt{3+x} + \sqrt{3}} = \frac{1}{\sqrt{3+0} + \sqrt{3}} = \frac{1}{2\sqrt{3}}.$$

For the second and third limits, we can factor the given numerator into $(x-3)(x+5)$. After doing so, the point is that we explicitly say what the denominator $|x-3|$ equals depending on whether $x-3$ is positive or negative. For $x-3>0$, so $x>3$, $|x-3| = x-3$ since the absolute value of a positive number is that number itself, while for $x-3<0$, so $x<3$, $|x-3| = -(x-3)$

since when taking absolute value we can turn a negative number into a positive one by multiplying by -1. We get:

$$\text{for } x > 3, \qquad \frac{x^2+2x-15}{|x-3|} = \frac{(x-3)(x+5)}{x-3} = x+5$$

$$\text{for } x < 3, \qquad \frac{x^2+2x-15}{|x-3|} = \frac{(x-3)(x+5)}{-(x-3)} = -(x+5).$$

Thus:

$$\lim_{x\to 3^+} \frac{x^2+2x-15}{|x-3|} = \lim_{x\to 3^+} (x+5) = 3+5 = 8, \text{ and}$$

$$\lim_{x\to 3^-} \frac{x^2+2x-15}{|x-3|} = \lim_{x\to 3^-} -(x+5) = -(3+5) = -8.$$

As a consequence, note that $\lim_{x\to 3} \frac{x^2+2x-15}{|x-3|}$ does not exist since approaching 3 from the right vs the left gives a different value.

Squeeze Theorem. Now we consider:

$$\lim_{x\to 0} x^2 \cos\left(\frac{x^2-100}{x}\right).$$

In this case our technique of finding an alternate expression for the given function isn't so useful, since there is no simpler way to rewrite the cosine part. To be clear, we can certainly rewrite the expression being plugged into cosine in the following way:

$$\cos\left(\frac{x^2-100}{x}\right) = \cos\left(x - \frac{100}{x}\right),$$

but after this there is no way to simplify or rewrite further. We also can't simply plug in $x = 0$ since our function is not defined at $x = 0$. So, we need something else.

The key idea in this case is that we have further information about the cosine function, namely that it always gives values between -1 and 1:

$$-1 \leq \cos(\text{whatever}) \leq 1.$$

In our case, we can say for sure that:

$$-1 \leq \cos\left(\frac{x^2-100}{x}\right) \leq 1 \text{ for any } x \neq 0.$$

Multiplying through by x^2, which is never negative so the inequalities don't change directions, gives:

$$-x^2 \leq x^2 \cos\left(\frac{x^2-100}{x}\right) \leq x^2 \text{ for any } x \neq 0.$$

Alternatively, starting with our original function, we are making use of the $\cos \leq 1$ inequality to say

$$x^2 \cos\left(\frac{x^2-100}{x}\right) \leq x^2 \cdot 1,$$

since keeping x^2 the same but replacing the cosine part by something larger gives a larger value overall, and the $-1 \leq \cos$ inequality to say

$$x^2(-1) \leq x^2 \cos\left(\frac{x^2 - 100}{x}\right)$$

since replacing the cosine term by something smaller while leaving x^2 unchanged gives a smaller value overall.

Either way, we arrive at

$$-x^2 \leq x^2 \cos\left(\frac{x^2 - 100}{x}\right) \leq x^2.$$

Now the punchline: the left side $-x^2$ approaches 0 as $x \to 0$, and so does the right side x^2. Thus, the term in the middle in "squeezed" between two terms, each of which are getting closer and closer to 0, so the conclusion is that this term in the middle must also be getting closer and closer to 0:

$$\lim_{x \to 0} x^2 \cos\left(\frac{x^2 - 100}{x}\right) = 0.$$

The precise reason why this holds is known as the *Squeeze Theorem*, which says exactly that given functions satisfying

$$g(x) \leq f(x) \leq h(x),$$

if the "outer" functions $g(x)$ and $h(x)$ have the same limit as $x \to a$, then so does the function $f(x)$ in the middle. This gives us a new way to compute limits which doesn't involve finding alternate expressions for functions, but instead uses the ability to compare functions via inequalties.

Here is another example. Consider:

$$\lim_{x \to 0}[5x^4 \cos(\tfrac{1}{x}) + 2x^2 \sin(\tfrac{1}{x}) + 4].$$

Each of the sine and cosine terms gives values between -1 and 1, so we have the following comparisons:

$$-5x^4 - 2x^2 + 4 \leq 5x^4 \cos(\tfrac{1}{x}) + 2x^2 \sin(\tfrac{1}{x}) + 4 \leq 5x^4 + 2x^2 + 4.$$

Again this comes from "bounding" the sine and cosine terms but larger and smaller values while keeping everything else unchanged. The function on the left has limit 4 as $x \to 0$, and so does the function on the right. Thus the Squeeze Theorem guarantees that

$$\lim_{x \to 0}[5x^4 \cos(\tfrac{1}{x}) + 2x^2 \sin(\tfrac{1}{x}) + 4] = 4$$

as well.

Continuity. Say we wanted to compute the following limit:

$$\lim_{x \to 0} \cos\left(\frac{\sqrt{3+x} - \sqrt{3}}{x}\right).$$

Ideally we would like to be able to say something like:

$$\lim_{x \to 0} \cos\left(\frac{\sqrt{3+x} - \sqrt{3}}{x}\right) = \cos(\text{whatever the term inside approaches}).$$

That is, we saw in the Warm-Up that

$$\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3}}{x}=\frac{1}{2\sqrt{3}},$$

so why shouldn't we just be able to make use of this in the limit we are now considering to say that:

$$\lim_{x\to 0}\cos\left(\frac{\sqrt{3+x}-\sqrt{3}}{x}\right)=\cos\left(\frac{1}{2\sqrt{3}}\right).$$

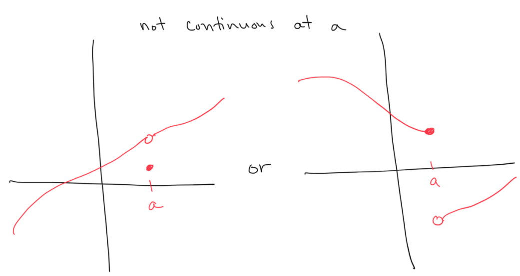
This computation does in fact work, but it depends on an important property of the cosine function, namely that it is *continuous*.

We say that a function f is *continuous at a* if the following is true:

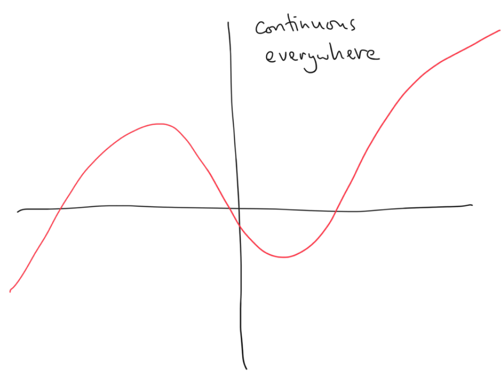
$$\lim_{x\to a} f(x)=f(a).$$

In other words, continuous functions are *precisely* the ones for which evaluating a limit *does* amount to plugging in the value we are approaching. This isn't true in general, but it is true for continuous functions by definition. Intuitively, the idea is that the value of a continuous function at a point can be fully determined solely by its behavior *near* that point; or in other words, the value of $f(a)$ can be guessed exactly from the nearby values used in $\lim_{x\to a} f(x)$.

Visually, a function will fail to be continuous when we have something like the following:



That is, a non-continuous function could have a "hole" at a point or a "jump". A continuous function then, by contrast, is one whose graph has no such holes or jumps:



Apart from making some limits simpler to compute, continuous functions have other nice properties as we'll see.

Example. We determine the value of c which makes the following function continuous at 2:

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & x \neq 2 \\ c & x = 2 \end{cases}$$

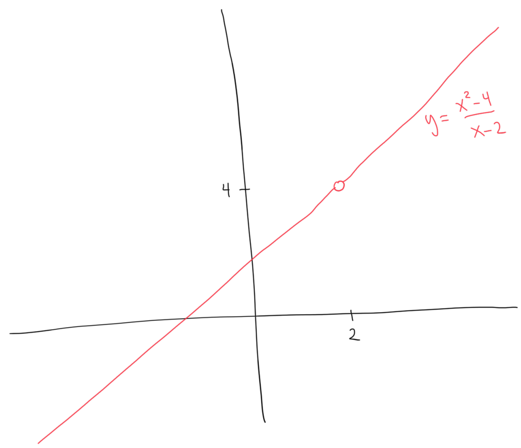
i.e., by the definition of continuity, this is asking for the value of c which makes it true that

$$\lim_{x\to 2} f(x) = f(2).$$

In our case, $f(2) = c$ is precisely the value we are trying to determine, so this question amounts to computing the required limit, and setting c equal to whatever its value is. This is a computation we've done before:

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4,$$

so we know that we must take $c = 4$. It is only this value of c which makes this function continuous at 2. Visually, the graph of this function for $x \neq 2$ looks the same as the line $y = x + 2$



and we are saying that if we want to be able to fill in the "hole" at $x = 2$ and obtain something continuous, we must set $f(2)$ to be 4.

Exchanging functions and limits. Let us rewrite the condition needed for f to be continuous at a in the following way:

$$\lim_{x\to a} f(x) = f(a) = f\left(\lim_{x\to a} x\right).$$

All we did was rewrite a as $\lim\limits_{x\to a} x$. The point is that this suggests for a continuous function it is true that we can essentially *exchange* the limit operation with the function itself, or in other words "bring" the limit "inside" the function:

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right).$$

Now, there are some assumptions needed in order for this to work out, namely that f must be continuous at whatever value $\lim\limits_{x\to a} g(x)$ has, but it reflects a crucial property of continuous function. Indeed, this makes limits involving continuous functions often simpler to compute.

For instance, going back to a previous example, this is precisely why

$$\lim_{x\to 0} \cos\left(\frac{\sqrt{3+x}-\sqrt{3}}{x}\right) = \cos\left(\frac{1}{2\sqrt{3}}\right).$$

To be clear, since the cosine function is continuous everywhere, we can exchange the cosine function with the limit operation:

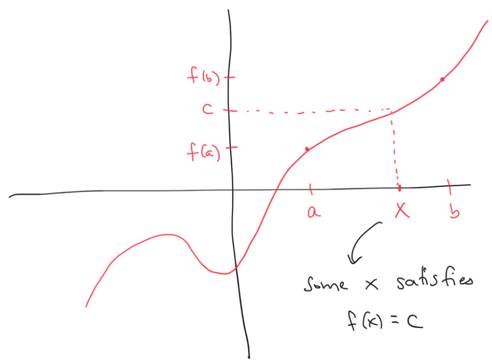
$$\lim_{x\to 0} \cos\left(\frac{\sqrt{3+x}-\sqrt{3}}{x}\right) = \cos\left(\lim_{x\to 0} \frac{\sqrt{3+x}-\sqrt{3}}{x}\right).$$

We are left computing the limit inside cosine, which again we did in the Warm-Up.

Many functions you know and love are continuous: sine, cosine, polynomial functions involving powers of x, square roots, absolute value, etc. For instance, here is another instance of exchanging function and limit: the square root function is continuous, so

$$\lim_{x\to 3} \sqrt{\frac{\sqrt{3+x}-\sqrt{3}}{x}} = \sqrt{\lim_{x\to 3}\left(\frac{\sqrt{3+x}-\sqrt{3}}{x}\right)} = \sqrt{\frac{1}{2\sqrt{3}}}.$$

Intermediate Value Theorem. Apart from being able to exchange functions and limits, we finish with one other key property of continuous functions. Consider the following picture of the graph of a continuous function:

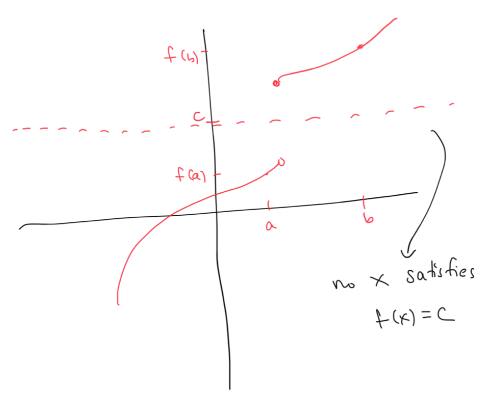


The key observation is that the number c lies between two known values of the function:

$$f(a) < c < f(b).$$

Since f is continuous, the *Intermediate Value Theorem* guarantees that there is then *a* value of x (between a and b) we can plug into the function in order to obtain c: some x between a and b satisfies $f(x) = c$. That is, any number which is "intermediate" between two known values of a continuous function must itself be attained as an actual value. This is clear in the picture, where we can draw a value of x which gives c as the output.

By contrast, this is not necessarily true for functions which aren't continuous, say:



In this case, even though c is between $f(a)$ and $f(b)$, there is no x we can plug into this function in order to obtain c. Of course, this function is not continuous so there is no reason to expect that such an x would exist. Think of the Intermediate Value Theorem as giving yet another reason why the graph of a continuous function cannot have any "jumps".

Example. Here is a typical application of the Intermediate Value Theorem. We want to argue that there has to be some number x which satisfies the equation

$$\cos x = x^3.$$

Now, at the end of this, we will not be able to say that this number x actually is; all that we will be able to say is that there *is* some x which works.

Consider the function $f(x) = \cos x - x^3$. Note first that

$$f(0) = \cos 0 - 0^3 = 1.$$

Now, $\cos 1$ is definitely smaller than 1, which we can see without having to use a calculator to compute $\cos 1$ by thinking about a right triangle with hypotenuse 1: the length of each side is smaller than the length of the hypotenuse, so $\cos 1 < 1$. This means that

$$f(1) = \cos 1 - 1^3 < 0.$$

The point is that we now know 0 is intermediate between $f(0) = 1$ and $f(1)$:

$$f(1) < 0 < f(0).$$

Since $f(x) = \cos x - x^3$ is continuous, because $\cos x$ and x^3 are each continuous and subtracting continuous functions still gives something continuous, the Intermediate Value Theorem then says that there must be some x between 0 and 1 which satisfies

$$f(x) = \cos x - x^3 = 0, \text{ or equivalently } \cos x = x^3.$$

So, we know that such a number has to exist without computing it explicitly, all due to continuity.

This type of reasoning is key to finding approximate solutions to such equations. The idea is to narrow down the search for a number x satisfying $\cos x = x^3$ using the Intermediate Value Theorem: in this case, we know there has to be such a number between 0 and 1. A computer can then narrow down the search further, eventually finding if nothing else an approximate solution.

Lecture 5: Derivatives

Warm-Up 1. We determine the value of a which makes the function f defined by

$$f(x) = \begin{cases} ax - x^3 + a & x \leq 4 \\ \frac{x^2-16}{x-4} - a & x > 4 \end{cases}$$

continuous on all of $(-\infty, \infty)$. To be clear, to say that f is continuous on all of $(-\infty, \infty)$ means that it is continuous at each point x in the interval $(-\infty, \infty)$.

First, no matter what a is, we claim this function is for sure continuous at any $c < 4$. The point is that at and *anywhere near* such a point, the function f has values given by $f(x) = ax - x^3 + a$. So, the behavior of f at and near any $c < 4$ is the same as that of $g(x) = ax - x^2 + a$, and

$g(x) = ax - x^2 + a$ is certainly continuous because it is a polynomial function. Another way of saying this is that since near any $c < 4$, the value of f is the same as that of g, we have

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \lim_{x\to c}(ax - x^2 + a) = ac - c^2 + a = f(c).$$

Again, all that matters in questions of continuous is how the function behaves at an "near" the point in question, and at $c < 4$, that behavior is the same as that of the continuous function $g(x) = ax - x^2 + a$.

For the same reason, this function is definitely continuous at any $c > 4$ regardless of the value of a. Again, at and near any $c > 4$, this function behaves in the same way as

$$h(x) = \frac{x^2 - 16}{x - 4} - a,$$

and this latter function is continuous at any $c > 4$. Indeed, both $x^2 - 16$ and $x - 4$ are continuous expressions, and the quotient of any continuous expressions is also continuous as long as the denominator is nonzero, which is the case for $c > 4$. So, the given function f is equal to a continuous function at and near any $c > 4$, so f is also continuous at any $c > 4$.

Thus we only really have to determine what happens at $x = 4$ itself, which is where we will get some restrictions on a. In order to be continuous at 4 we need the following to be true:

$$\lim_{x\to 4} f(x) = f(4).$$

To be clear, this saying two things: first that the limit on the left actually exists, and second that it equals $f(4)$. To see whether or not this limit exists, we consider left- and right-sided limits:

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (ax - x^3 + a) = 5a - 64,$$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \left(\frac{x^2 - 16}{x - 4} - a\right) = \lim_{x\to 4^+} (x + 4 - a) = 8 - a.$$

Thus in order for $\lim_{x\to 4} f(x)$ to even exist requires that

$$5a - 64 = 8 - a, \text{ so } a = 12.$$

It is only for this value of a that the limit we need to equal $f(4)$ exists, so it is only for this value of a that f has the possibility of being continuous at 4.

When $a = 12$, we thus have

$$\lim_{x\to 4} f(x) = -4,$$

which comes from either $5a - 64$ or $8 - a$ when $a = 12$. In order for f to be continuous at 4 this would have to equal

$$f(4) = 4a - 4^3 + a = 4(12) - 64 + 12 = -4$$

when we take into account that a is 12. This agrees with the limit above, so we now know that f is continuous at 4 when $a = 12$. Thus f is continuous at all points of $(-\infty, \infty)$ only for $a = 12$.

Warm-Up 2. We justify the fact that there is a nonzero number x satisfying the equality

$$\sin x = x^2 - x.$$

Note that $x = 0$ for sure satisfies this since $\sin 0 = 0 = 0^2 - 0$, but we are claiming here is there is some *other* value of x which satisfies this equation as well. The point is that we can interpret this as an "intermediate value property", for instance by saying that the continuous function $f(x) = \sin x - x^2 + x$ should have 0 as a value "intermediate" between two known values. If so, then the Intermediate Value Theorem will say that there is indeed something we can plug in to this function to give 0 as an actual value.

To be clear, the function $f(x) = \sin x - x^2 + x$ is continuous because is the sum of three continuous functions: $\sin x, -x^2$, and x. Now, we have:

$$f(1) = \sin 1 - 1^2 + 1 = \sin 1 > 0$$

since on the unit circle the point occurring at an angle of 1 radian has positive height. Also:

$$f(2) = \sin 2 - 2^2 + 2 = \sin 2 - 2 < 0$$

since regardless of what $\sin 2$ is, it for sure is smaller than or equal to 1. Thus we have

$$f(2) < 0 < f(1)$$

so the Intermediate Value Theorem guarantees that there is some number x between 1 and 2 satisfying

$$f(x) = \sin x - x^2 + x = 0, \text{ which is the same as } \sin x = x^2 - x.$$

This is what we set out to justify. The fact that this x value is nonzero becomes simply from knowing that it must be between 1 and 2.

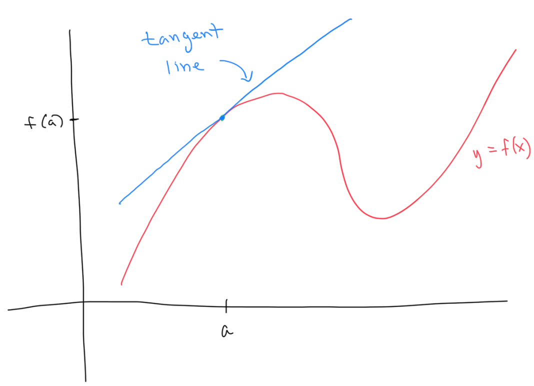
Now, we are done with the problem, but we can do even better. So far we know there is some x between 1 and 2 satisfying $\sin x = x^2 - x$. But we can also compute the following using a calculator:

$$f(1.7) \approx -0.198 < 0 < f(1.5) \approx 0.247$$

so the Intermediate Value Theorem guarantees that in fact there is a number x between 1.5 and 1.7 which satisfies $\sin x = x^2 - x$. The point is that we have narrowed down the range of the types we numbers we are looking for, from $(1, 2)$ to $(1.5, 1.7)$. This type of "narrowing down" of possibilities is useful in instances where you care about getting good approximations to numbers satisfying some given equation. (Or at least, this is something which people who come up with algorithms which actually find such approximate value would care about.)

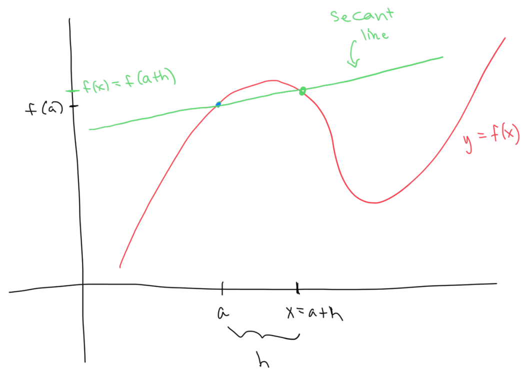
Tangent slopes. We now move towards the fundamental notion of this course, and indeed one of the key notions in all of calculus: the notion of a *derivative*. (The second key notion, that of an *integral*, is what Math 224 covers.) Before giving a definition, we first provide motivation in terms of slopes of tangent lines.

Consider the following function and point $x = a$:



We are interested in finding the slope of what's called the *tangent line* to the graph of f at $x = a$. Intuitively, the tangent line at a is the line which "best approximates" the graph near a, or the line which lies "flattest" against the graph at a. (What it means to "best approximate" the graph is something we'll discuss later on.)

We can obtain this slope via slopes of "approximating" lines. Take a point $x = a+h$ a bit away from a:

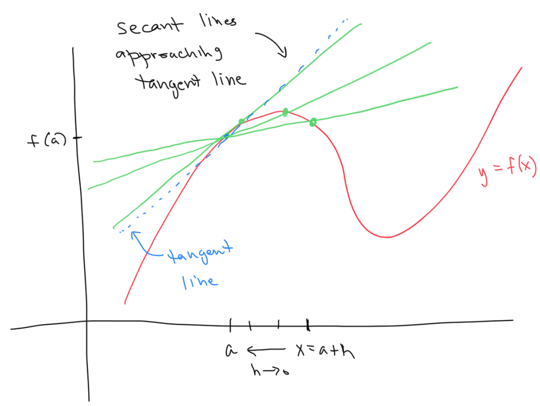


The line which passes through the points $(a, f(a))$ and $(a+h, f(a+h)) = (x, f(x))$ on the graph is called a *secant* line, and its slope is given by

$$\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(x) - f(a)}{x - a}.$$

In both of these expressions, we are taking the difference in y-coordinates divided by the difference in x-coordinates, which is precisely what the slope of the line passing through these two points should be. The only difference is that on the right we are calling this second point x, whereas on the left we write it as $a+h$ for some $h \neq 0$. (Any x can be expressed as $x = a+h$ for an appropriate choice of h.) Note that the denominator in the first fraction simplifies to h, which is the difference between a and $x = a+h$.

Now, imagine what happens as the point $x = a+h$ gets closer and closer to a, or equivalently as h gets closer and closer to 0:



The point is that as we take this "limit", the secant lines are themselves getting closer and closer to the tangent line we want, and so the slope of the tangent line should be precisely the limit of the slopes of the secant lines. This limit is what we call the "derivative of f at a", which we now define.

Derivatives. The *derivative* of f at a is the limit (if it exists) given by

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \text{ or equivalently } \lim_{x\to a}\frac{f(x)-f(a)}{x-a}.$$

Again, these give the same value, with the only difference between that on the left we set x to be $x=a+h$ and interpret $a+h=x\to a$ as $h\to 0$ instead. (The first version is usually simpler to work with in practice, so it's the one we mainly use.) If this limit exists, we denote it by $f'(a)$ (pronounced "f prime of a") and we say that f is *differentiable* at a. If this limit does not exist, then we say that f is *not differentiable* at a. Geometrically, $f'(a)$ gives the slope of the tangent line to the graph of f at a. (To be fully precise, the tangent line is usually *defined* to be the line whose slope is precisely $f'(a)$. The reasoning we gave above when coming up with the limit definition of a derivative shows that this is a good definition to make.)

For now we will focus on this tangent slope interpretation of a derivative, but let us mention the other main interpretation. The point is that the quantity

$$\frac{f(a+h)-f(a)}{h} \quad \text{or} \quad \frac{f(x)-f(a)}{x-a}$$

gives what's called the *average rate of change* of f between a and $x=a+h$. When we take the limit in the definition of a derivative, we are thus seeing what happens to this average rate of change as the second "sampling" point $x=a+h$ gets closer and closer to a. The resulting value, if it exists, gives what's called the *instantaneous rate of change* of f at a:

$$f'(a) = \text{instantaneous rate of change of } f \text{ at } a.$$

This tells us the rate by which the value of f is changing *instantly at* the point a itself, or the rate at which f is "increasing" or "decreasing" at a. This is probably the more important interpretation of derivatives, since not all functions will have easy-to-visualize graphs so tangent lines are not always simple to picture.

Example 1. We find the equation of the tangent line to the graph of $f(x)=3x^2-x$ at the point $(1,2)$. The slope of the line we want is precisely $f'(1)$, where the 1 comes from the x-coordinate of the point $(1,2)$ on the graph we're looking at. We compute this derivative using the limit definition:

$$\begin{aligned}
f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}\\
&= \lim_{h\to 0}\frac{[3(1+h)^2-(1+h)]-2}{h}\\
&= \lim_{h\to 0}\frac{3(1+2h+h^2)-1-h-2}{h}\\
&= \lim_{h\to 0}\frac{3h^2+5h}{h}\\
&= \lim_{h\mathbf{t}00}(3h+5)\\
&= 5.
\end{aligned}$$

So the tangent line we want has slope 5. We can now use either of the standard equations for a line:

$$y=mx+b \text{ or } y-y_0=m(x-x_0)$$

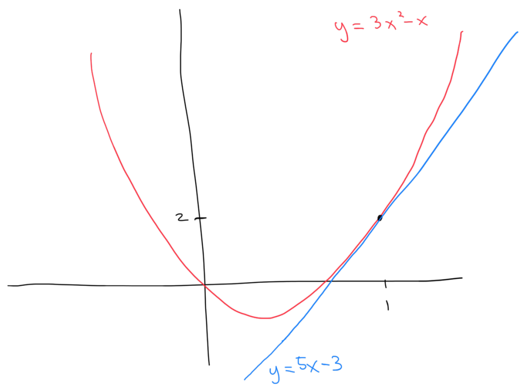
where m denotes the slope. In the first case, we have $y = 5x + b$ for some b, which can be found using the fact that the line should pass through $(1, 2)$

$$2 = 5(1) + b, \text{ so } b = -3.$$

In the second case, (x_0, y_0) denotes a point on the line, such as $(1, 2)$, so we get:

$$y - 2 = 5(x - 1).$$

Either way, after simplifying, we get that the equation of the tangent line at $(1, 2)$ is $y = 5x - 3$:



Note that it makes sense that $f'(1)$ should be positive since the function is *increasing* at $x = 1$, meaning it has a positive rate of change at this point.

Example 2. We find the slope of the tangent line to the curve $y = -2\sqrt{x}$ at the point with x-coordinate 2. Viewing this curve as the graph of the function $f(x) = -2\sqrt{x}$, we are thus wanting to compute $f'(2)$. Again we use the limit definition:

$$\begin{aligned} f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0} \frac{-2\sqrt{2+h} - (-2\sqrt{2})}{h} \\ &= \lim_{h \to 0} \frac{-2\sqrt{2+h} + 2\sqrt{2}}{h}. \end{aligned}$$

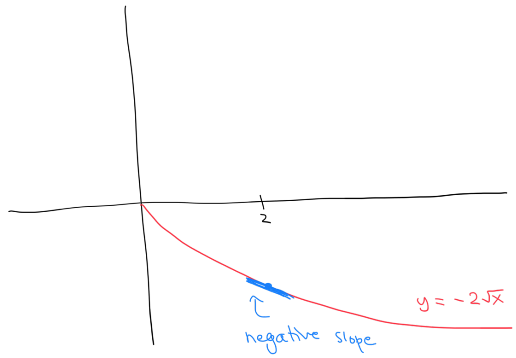
To find an alternate expression for this function, we multiply numerator and denominator by $-2\sqrt{2+h} - 2\sqrt{2}$, which will have the effect of getting rid of the square roots in the numerator:

$$\frac{(-2\sqrt{2+h} + 2\sqrt{2})}{h} \frac{(-2\sqrt{2+h} - 2\sqrt{2})}{(-2\sqrt{2+h} - 2\sqrt{2})} = \frac{4(2+h) - 4(2)}{h(-2\sqrt{2+h} - 2\sqrt{2})} = \frac{4}{-2\sqrt{2+h} - 2\sqrt{2}}.$$

Thus, back to our limit, we get:

$$f'(2) = \lim_{h \to 0} \frac{-2\sqrt{2+h} + 2\sqrt{2}}{h} = \lim_{h \to 0} \frac{4}{-2\sqrt{2+h} - 2\sqrt{2}} = \frac{4}{-2\sqrt{2} - 2\sqrt{2}} = -\frac{1}{\sqrt{2}}.$$

So, the slope of the tangent line to the curve $y = -2\sqrt{x}$ at $x = 2$ is $-\frac{1}{\sqrt{2}}$. Note it makes sense that this is negative since $y = -2\sqrt{x}$ is decreasing at $x = 2$, so it should have a negative rate of change:



Lecture 6: More on Derivatives

Warm-Up 1. Say that the height (in feet) of a flying bird at t seconds is given by the function $d(t) = t^3 - 6t^2 + 10t$. We determine the speed at which the bird is flying at $t = 1$ second. The point is that "speed" is precisely defined to be the rate of change of distance with respect to time, so the speed of the bird is given by the rate of change of $d(t)$, i.e. the derivative of $d(t)$ at $t = 1$. This uses the "instantaneous rate of change" interpretation of a derivative.

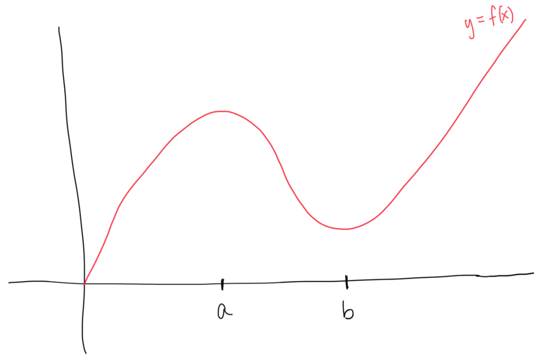
So, we compute $d'(1)$ using the limit definition:

$$\begin{aligned}
d'(2) &= \lim_{h\to 0} \frac{d(1+h) - d(1)}{h} \\
&= \lim_{h\to 0} \frac{[(1+h)^3 - 6(1+h)^2 + 10(1+h)] - 5}{h} \\
&= \lim_{h\to 0} \frac{1 + 3h^2 + 3h + h^3 - 6(1 + 2h + h^2) + 10 + 10h - 5}{h} \\
&= \lim_{h\to 0} \frac{h^3 - 3h^2 + h}{h} \\
&= \lim_{h\to 0} (h^2 - 3h + 1) \\
&= 1.
\end{aligned}$$

The speed of the bird at $t = 1$ second is thus 1 ft/sec. Since this is positive, the bird is actually moving up (i.e. increasing in height), instead of down (i.e. decreasing in height) as would occur when the derivative is negative.

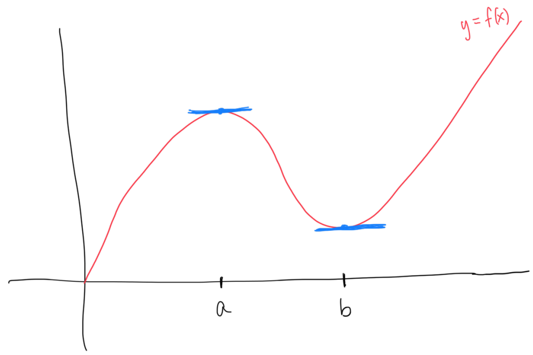
Note that this limit involved a fair amount of algebra, which takes time. We will see starting next time that we can develop ways of computing derivatives which *avoid* having to compute a limit every single time. The fact that these limit computations can get faily tedious should make us appreciate the techniques we'll talk about soon enough.

Warm-Up 2. Consider the graph of the following function f:



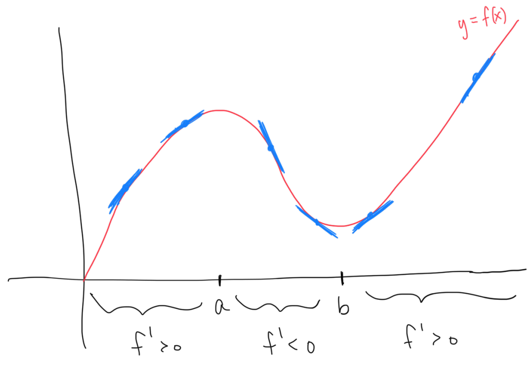
This is actually roughly what the graph of the function in the first Warm-Up looks like, so the graph which traces out the height of the bird. We determine the points at which the derivative of this function is zero, the points at which it is positive, and the points at which it is negative.

First, if we interpret $f'(x)$ as the slope of the tangent line at x, then points where the derivative is zero should be points where the tangent line is horizontal, so at the points labeled $x = a$ and $x = b$ in the picture:



These are also the points at which the rate of change of the function is zero, using the other interpretation of derivatives. Indeed, at the point $x = a$ the function switches from increasing to decreasing, so the *instantaneous* rate of change *at* $x = a$ itself is zero. Similarly, at $x = b$ the function switches from decreasing to increasing, and the rate of change at $x = b$ itself is again zero. In terms of the bird flying, these are points at which the bird is "momentarily at rest" since at these instants it is changing direction; in other words, these are points at which the bird has speed equal to 0.

Now, between a and b the function is decreasing in value, meaning the rate of change at these points should be negative and hence at these points the derivative is negative. We can also see this by noting that the tangent lines at these points have negative slope:



To the left of $x = a$ *and* to the right of $x = b$, the function has a positive rate of change since it is increasing, or equivalently the tangent lines at these points have positive slope. Thus the derivative at these points is positive. (Some various slopes are indicated by the blue line segments in the pictures above.) To summarize, we have $f'(a) = 0$ and $f'(b) = 0$, $f'(x) < 0$ for $a < x < b$, and $f'(x) > 0$ for $x < a$ and $x > b$.

Nonexistence example. Consider the function f defined by

$$f(x) = \begin{cases} |x-2| & x < 4 \\ 6 & x \geq 4. \end{cases}$$

We ask for the points at which the derivative of f does *not* exist, or in other words the values of a for which the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

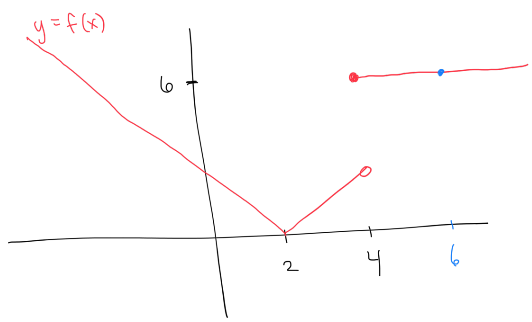
does not exist. First, we claim that $a = 2$ is such a value. Indeed, we have:

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{|(2+h) - 2| - |2-2|}{h} = \lim_{h \to 0} \frac{|h|}{h}.$$

To be clear, we used the expression $f(x) = |x-2|$ for f for values of x *close to* 2 in the second equality. But the resulting limit does not exist since its value depends on how we approach 0:

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1 \text{ and } \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0} \frac{-h}{h} = -1.$$

Thus the limit defining $f'(2)$ does not exist, so $f'(2)$ does not exist. (We say that f is *not differentiable* at 2.) Geometrically, this is reflected in the fact that the graph of f has a "sharp point" or "corner" at $x = 2$:



This is one type of behavior which can cause a derivative, or a tangent line, to fail to exist.

Now, the derivative of f also fails to exist at $a = 4$. One way to see this is to again work with the limit definition:

$$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{f(4+h) - 6}{h}.$$

If we approach 0 from the right, we get (since $4 + h > 4$ in this case):

$$\lim_{h \to 0^+} \frac{f(4+h) - 6}{h} = \lim_{h \to 0^+} \frac{6-6}{h} = \lim_{h \to 0^+} 0 = 0.$$

However, from the left we get (using $4 + h < 4$ in this case):

$$\lim_{h \to 0^-} \frac{f(4+h) - 6}{h} = \lim_{h \to 0^-} \frac{|(4+h) - 2| - 6}{h} = \lim_{h \to 0^-} \frac{|2+h| - 6}{h} = \lim_{h \to 0^-} \frac{h-4}{h},$$

which does not exist. (This expression simplifies to $1 - \frac{4}{h}$, and the problem is that the $\frac{4}{h}$ does not approach a specific number as $h \to 0$.) Since this left-sided limit does not exist,

$$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$$

does not exist either, so f is not differentiable at 4.

However, we didn't actually need to go through all this: the fact that f is not even *continuous* at 4 (as we can see from the graph) is alone enough reason to guarantee it cannot be differentiable at 4. The point is that if a function is indeed differentiable at a point, it must have been continuous there to start with; so, if it is not continuous there, the derivative cannot exist there. You can check the book for a "proof" of this fact if interested, but intuitively it says that a tangent line has no hope of existing if our function isn't even continuous. Now, to be sure, just knowing that a function *is* continuous at a point does NOT mean it is differentiable there; for instance, this same function is continuous at 2 and yet not differentiable at 2. So, the reasoning only works one way: not continuous for sure means not differentiable, but continuous does not necessarily mean differentiable. Again, differentiable isn't just asking about whether the graph has a "hole" or "jump", but rather whether it has a "sharp" point.

At all other points, apart from 2 and 4, this function is indeed differentiable. For instance, at and near $x = 6$ the function has the constant value 6, so it experiences zero rate of change at this point. Said another way, the tangent line is horizontal at $x = 6$. (In fact, the tangent line at $x = 6$ is the same horizontal line $y = 6$ as the one giving the graph of f itself at this point.) So, $f'(6) = 0$ and the same is true at all $x > 4$.

Derivatives as functions. By varying the point at which are taking considering the derivative of f, we can think of this derivative as a function itself:

$$f'(x) = \text{the derivative of } f \text{ evaluated at } x$$

where x is a variable which varies. We often denote this function by f' without making reference to the point at which it is being evaluated. Alternate notations are

$$\frac{df}{dx}, \text{ or } \frac{d}{dx}(f).$$

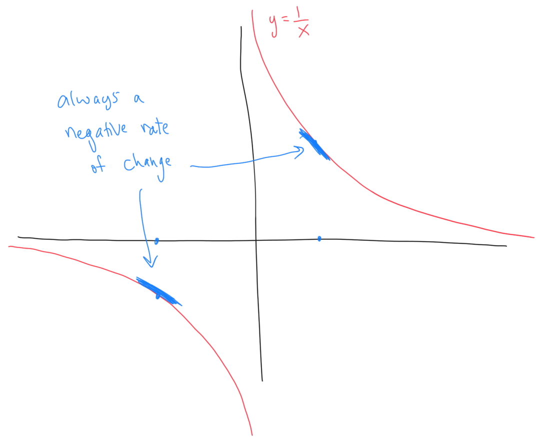
In the second notation, $\frac{d}{dx}$ stands for "the operation of taking derivative", so the second notation means to apply this operation to f, thereby resulting in the derivative of f. The first notation $\frac{df}{dx}$ is one we'll come back to later when we give some intuitive meaning to "df" and "dx" themselves. The point is that in *some* sense, a derivative can be interpreted as a fraction of "infinitesimal" quantities, but again we'll elaborate later.

Example. We compute the derivative of the function $f(x) = \frac{1}{x}$. The derivative of f at an arbitrary point x is:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h \to 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{x-(x+h)}{hx(x+h)} \\
&= \lim_{h\to 0} \frac{-h}{hx(x+h)} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} \\
&= -\frac{1}{x^2}.
\end{aligned}
$$

Thus the derivative of the function $f(x) = \frac{1}{x}$ is the function $f'(x) = -\frac{1}{x^2}$. Note that this derivative always has a negative value, since x^2 is always positive. This makes sense since $f(x) = \frac{1}{x}$ is always decreasing:



Thus, at any $x \neq 0$, the rate of change of $f(x) = \frac{1}{x}$ should indeed be negative, as $f'(x) = -\frac{1}{x^2}$ is. (Or, equivalently, the tangent line to the graph of $y = \frac{1}{x}$ at any $x \neq 0$ should have negative slope.)

Second derivatives. Continuing with the example above, imagine that we forget about the original f and consider only the function $g(x) = -\frac{1}{x^2}$. We can ask about *its* derivative, which turns out to be

$$g'(x) = \frac{2}{x^3}.$$

(We'll see why this is the derivative of $y = -\frac{1}{x^2}$ next time.) This function, namely the derivative *of the derivative of* f, is what we call the *second derivative* of f and we denote it by f'':

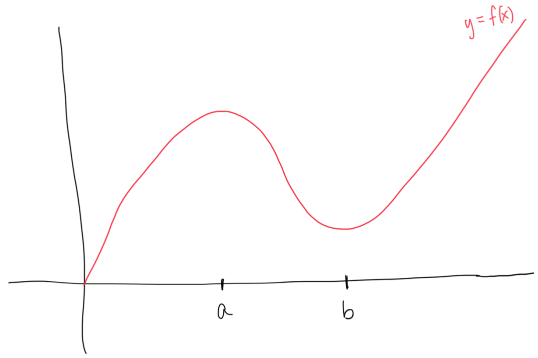
$$f'' = \text{derivative of } f'.$$

The expression f'' is pronounced "f double prime". We will now often refer to f' as being the *first* derivative of f, so the number of "primes" (i.e. apostrophes) present indicates how many derivatives we're taking. In this example then, we have the original function, its first derivative, and its second derivative as:

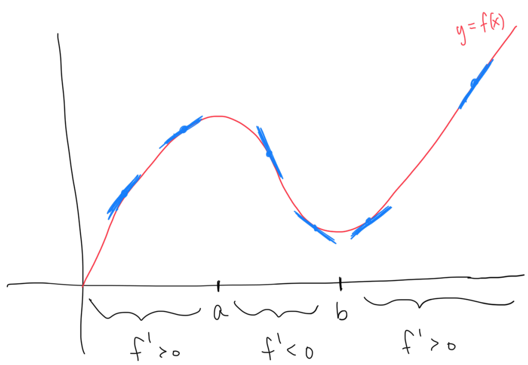
$$f(x) = \frac{1}{x} \qquad f'(x) = -\frac{1}{x^2} \qquad f''(x) = \frac{2}{x^3}.$$

We could keep going, and talk about the third, fourth, and fifth derivatives of f, and so on.

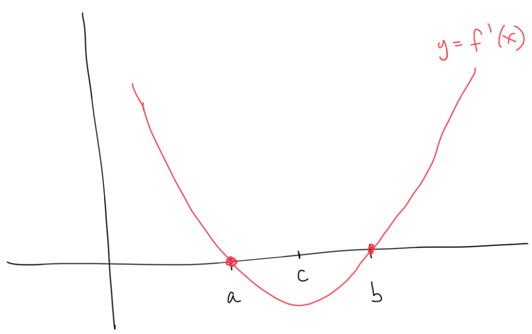
Example. Consider the function f from Warm-Up 2:



We now first give a sketch of the graph of f'. The function $y = f'(x)$ keeps track of the rates of change (or tangent slopes) in the picture above. Recall we worked out previously that this function, f', should be zero at $x = a$ and $x = b$, negative between a and b, and positive to the left of a and to the right of b:

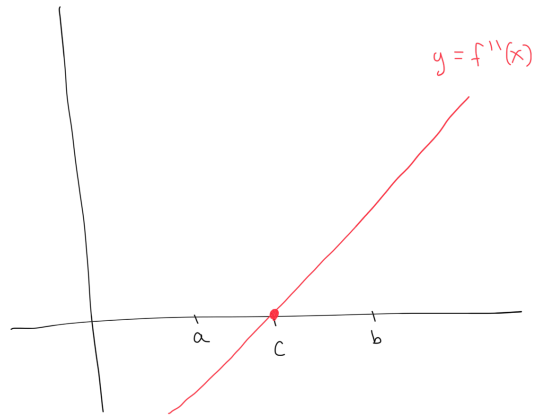


Thus, the graph of f' should roughly look like:



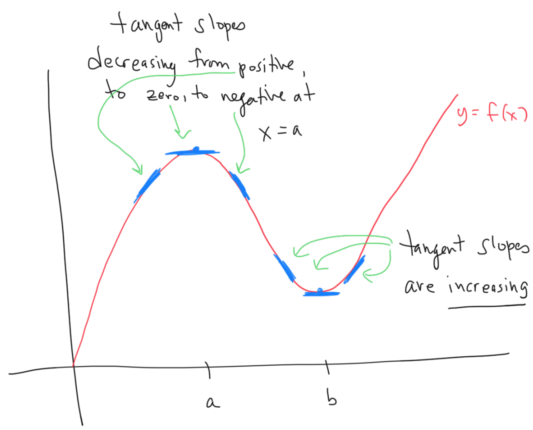
Indeed, this new graph has the value 0 at a and b, reflecting $f'(a) = 0 = f'(b)$, it has negative values between a and b since f' is negative for these values, and it has positive values for $x < a$ and $x > b$ as expected from above. Note that f' is zero at both a and b and should be negative between, its graph must have the type of behavior drawn above where it hits the x-axis at a, dips below the x-axis and then comes back up to hit zero as $x = b$ again. The exactly shape might not be possible to narrow down precisely, but this is a rough sketch.

Now, let us take the derivative of *this* new function f', and draw the graph of $y = f''(x)$, namely the second derivative of f. This new function $(f')'$ (i.e. the derivative of f') should have the value zero at the point $x = c$ drawn above since at this point f' as zero rate of change, or zero tangent slope. So, the graph of $y = f''(x)$ should hit zero at c. To the left of $x = c$ the function f' is decreasing in value, so its derivative f'' should have negative values for these points, while for $x >$ the function f' is increasing in value, so $f'' = (f')'$ should have positive values for these points. Thus we roughly get:



Note it makes sense that $f''(a)$ should be negative and that $f''(b)$ should be positive as this graph of f'' indicates: going back to the graph of f', f' is indeed decreasing at $x = a$, so the derivative of f' at a should be negative, and f' is increasing at $x = b$, so $(f')'$ evaluated at b should be positive.

Finally, we see how to interpret this second derivative geometrically in terms of the original graph of f. The key is to notice NOT how f itself is changing in the graph of f, but rather how the *tangent slopes* are changing:



After all, f'' indeed measures the rate of f', so the rate of change of the *tangent slopes* of f. At $x = a$ for instance the tangent slope is zero: $f'(a) = 0$. However, the left of a the slope is positive while to the right it is negative. This means that the tangent slopes themselves are getting *smaller* in value, or decreasing, as we pass through a. Thus, the rate of change of these tangent slopes, as measured by the derivative of f', should be negative:

$$f''(a) < 0 \text{ since } f' \text{ is } \textit{decreasing} \text{ at } a.$$

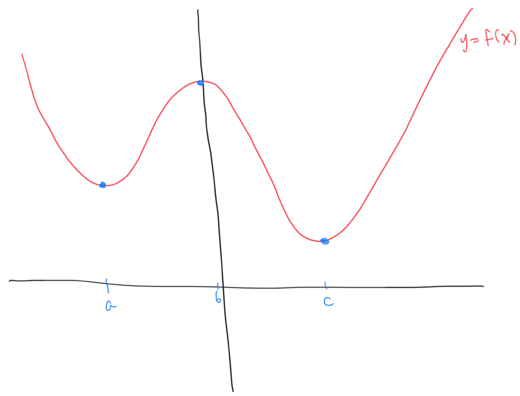
At $x = b$ on the other hand, the tangent slopes are *increasing* since they move from negative to the left of b, to zero at b, to positive to the right of b. Thus, f' is increasing at b, so its rate of change should be positive:

$$f''(b) > 0 \text{ since } f' \text{ is } \textit{increasing} \text{ at } b.$$

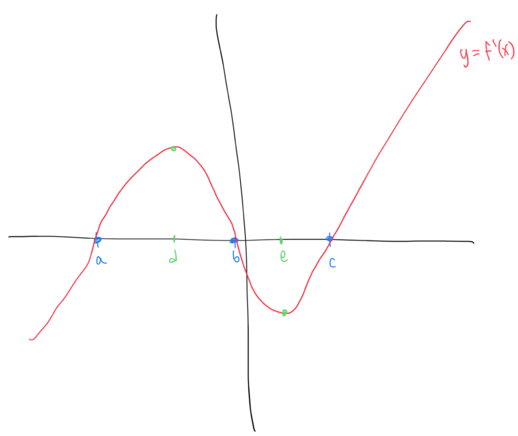
The point is that it is possible to interpret second derivatives geometrically in terms of the original function via this "rate of change of tangent slopes" approach. Later we'll use this idea to draw more accurate graphs of functions, where the second derivative will tell us what's called the *concavity* of the graph, which in a sense measures how the graph "turns". To finish with one possible practical interpretation of second derivatives, imagine that $y = f(x)$ is again giving the path a bird takes when flying. Then the first derivative f' measures the rate of change of its height, which we interpret as its speed. Now the second derivative f'' measures the rate of change of f', or in other words the rate at which *speed* is changing, which is what we normally call *acceleration*. Positive acceleration (i.e. second derivative) would mean that the bird is speeding up, while negative second derivative means the bird is slowing down.

Lecture 7: Differentiation Rules

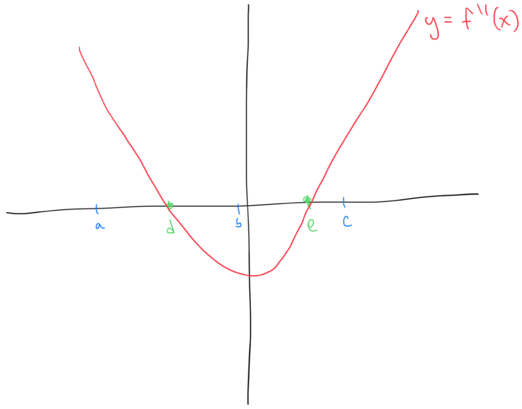
Warm-Up. Consider the graph of the function f drawn below:



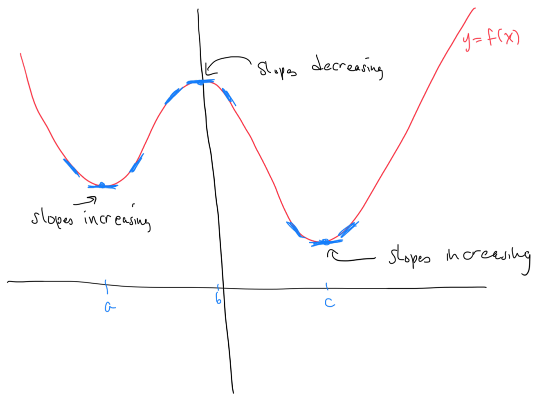
We draw rough sketches of the graph of f' and of the graph of f''. First, the points labeled a, b, c above are the points at which f' is zero, since these are the points at which the tangent line to the graph of f is horizontal. To the left of a, f is decreasing so its rate of change f' is negative; between a and b, f is increasing so its rate of change f' is positive, so the graph of f' between these points must be above the x-axis while hitting the x-axis at $x = a$ and b; and so on, continuing in this way we get that the graph of f' roughly looks like:



Now, the points labeled d, e above are those where f' is zero since the tangent line to the graph of f' is horizontal at these points, so the graph of $(f')' = f''$ should hit the x-axis at these points. By comparing increasing vs decreasing as we did before we get that the graph of f'' roughly looks like:



Finally, we point out that this graph of f'' actually makes sense if we interpret it back in terms of the graph of f. For instance, $f''(a) > 0$ based on the graph of f''. If we consider the graph of f at this point, we see that tangent slopes to the left, at, and to the right of a look like:



These slopes are thus negative to the left of a, zero at a, and positive to the right of a, which is to say that these slopes are *increasing* at a. Thus, the function f' which measures these slopes should be increasing at a, so its rate of change $(f')' = f''$ should indeed be positive at a. The same phenomena occurs at $x = c$. At $x = b$, the tangent slopes are *decreasing* since they move from positive, to zero, to negative, so the rate of change of f' at b, which is $f''(b)$, should indeed be negative.

Derivative of x^n. We compute the derivative of $f(x) = x^n$, where n is a positive integer. According to the limit definition, we need to evaluate:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}.$$

For this we need an expression for

$$(x+h)^n = \underbrace{(x+h)(x+h)(x+h)\cdots(x+h)}_{n \text{ times}}.$$

If we multiply all of this out, using distributive properties over and over again, one term we get is x^n, which comes from taking the "x" of each factor (of which there are n) and multiplying them together. Next, we get a bunch of terms which looks like $x^{n-1}h$. These come from taking an "h"

from one factor and then "x" from all the rest. For instance, if we take the h from the first factor and then x from the remaining $n-1$ factors we have

$$h \cdot \underbrace{x \cdot x \cdots x}_{n-1 \text{ times}} = x^{n-1}h.$$

But we can also take h from the second factor and then x from the rest:

$$x \cdot h \cdot \underbrace{x \cdot x \cdots x}_{n-2 \text{ times}} = x^{n-1}h.$$

In total, there are n terms which look like $x^{n-1}h$, coming from n possible factors from the "h" can be chosen, so these all together contribute a $nx^{n-1}h$ to the expression for $(x+h)^n$.

All the remaining terms which show up when multiplying out $(x+h)^n$ involve powers of h which are at least 2, meaning there will be terms with h^2 in them, or h^3, or h^4, all the way to h^n. Thus $(x+h)^n$ looks like

$$(x+h)^n = x^n + nx^{n-1}h + (\text{terms with } h \text{ to a power larger than or equal to 2}).$$

Thus we compute:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{[x^n + nx^{n-1}h + (\text{terms with } h \text{ to a power larger than or equal to 2})] - x^n}{h} \\
&= \lim_{h\to 0} \frac{nx^{n-1}h + (\text{terms with } h \text{ to a power larger than or equal to 2})}{h} \\
&= \lim_{h\to 0} [nx^{n-1} + (\text{terms with } h \text{ to a power larger than or equal to 1})].
\end{aligned}$$

This last line comes from the fact that dividing all the terms with h to at least a second power will produce terms with h to at least the first power. But then when taking the limit all of these terms approach 0 since they still have an h in them, so we are left with:

$$\lim_{h\to 0} [nx^{n-1} + (\text{terms with } h \text{ to a power larger than or equal to 1})] = nx^{n-1}.$$

Thus we have computed that

$$\text{for } f(x) = x^n, \;\; f'(x) = nx^{n-1}.$$

Note the pattern: the exponent comes down and then we subtract 1 from the exponent.

So, for instance:

$$\frac{d}{dx}(x^6) = 6x^5, \text{ and } \frac{d}{dx}(x^{20}) = 20x^{19}.$$

It turns out that the same rule applies to other powers of x, even negative and fractional powers:

$$\frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}}, \text{ and } \frac{d}{dx}(x^{-2}) = -2x^{-3}.$$

Basic derivatives. Differentiation rules are what allow us to compute derivatives without having to resort to a tedious limit computation every, single time. Just as in the x^n case, all of these rules can be justified via the limit approach, but from now on we take these results for granted.

Here are the first basic differentiation rules to know:

- $\frac{d}{dx}(\text{constant}) = (\text{constant})' = 0$. This says that the derivative of a constant function is zero, which makes sense since a constant function does not change in value at all, so it should have zero rate of change everywhere.
- $\frac{d}{dx}(x^n) = (x^n)' = nx^{n-1}$. This is the derivative we computed above in the case where n is a positive integer, but it actually works no matter what n is. The pattern is that "the exponent comes down and then we subtract one from the exponent."
- $\frac{d}{dx}(f(x)+g(x)) = (f(x)+g(x))' = f'(x)+g'(x)$. This says that "the derivative of a sum is the sum of derivatives." It comes from the intuitive fact that to obtain the rate at which $f+g$ is changing, we should simply take the rate at which f changes and add the rate at which g changes.
- $\frac{d}{dx}(cf(x)) = (cf(x))' = cf'(x)$. This says that constants can be "pulled outside" of derivative computations. The idea is that since the values of cf are the values of f scaled by c, the rate at which cf changes should just be the rate at which f changes scaled by c.
- $\frac{d}{dx}(\sin x) = (\sin x)' = \cos x$. This and the next derivative say that sine and cosine *almost* get exchanged when taking derivatives, except for the negative which shows up in the derivative of cosine below. This is actually something you'll justify via the limit definition on a written homework problem.
- $\frac{d}{dx}(\cos x) = (\cos x)' = -\sin x$. Again, note the negative which shows up here, in contrast to the derivative of $\sin x$ above.

Example. The derivative of $f(x) = 3x^5$ is

$$3(\text{derivative of } x^5) = 3 \cdot 5x^4 = 15x^4$$

using the "pull the constant out" rule and the derivative of x^n.

The derivative of $f(x) = 4\sqrt{x} + 12\sin x$ is

$$(\text{derivative of } 4\sqrt{x}) + (\text{derivative of } 12\sin x)$$

using the "derivative of sum is the sum of derivative property". The first term above, using the fact that $\sqrt{x} = x^{1/2}$ is then

$$4(\text{derivative of } x^{1/2}) = 4 \cdot \frac{1}{2}x^{-1/2} = \frac{2}{x^{1/2}} = \frac{2}{\sqrt{x}}.$$

The derivative of the $12\sin x$ portion is

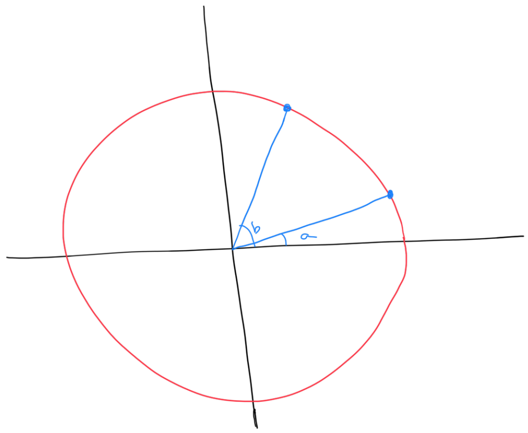
$$12(\text{derivative of } \sin x) = 12\cos x.$$

Thus all together we have:

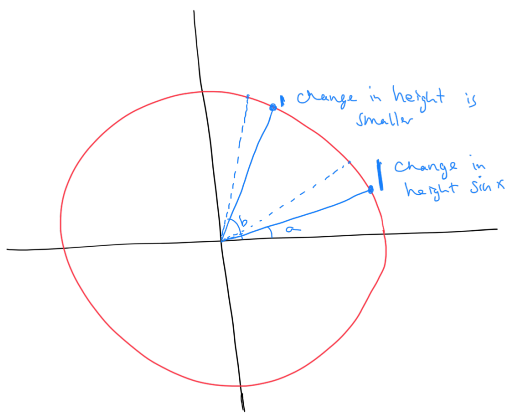
$$\text{for } f(x) = 4\sqrt{x} + 12\sin x,\ f'(x) = \frac{2}{\sqrt{x}} + 12\cos x.$$

Again, each term here comes from differentiating each portion separately.

Intuition for sine and cosine. Let us give some intuition for why it makes sense that the derivative of $\sin x$ should be $\cos x$ and why the derivative of $\cos x$ should be $-\sin x$. (Again, you'll justify this more carefully on a homework problem.) Take a unit circle with two angles a and b:

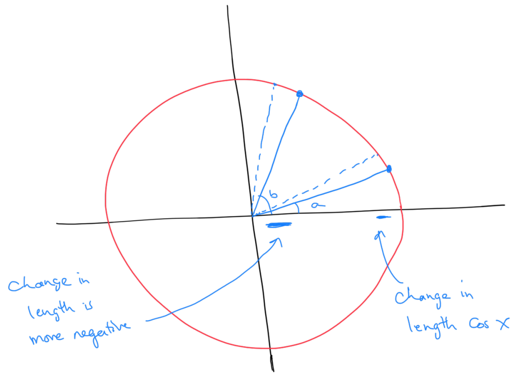


The function $f(x) = \sin x$ gives the vertical height of the corresponding points on the circle. As the angle increases, this height increases as well so $f'(a)$ and $f'(b)$ should be positive. But notice that the amount by which this height changes at a is actually *larger* than the amount by which it changes at b:



So, $f'(a)$ should be positive and larger than $f'(b)$, which should also be positive. The idea is that as $\sin x$ increases (in the first quadrant), its derivative should actually *decrease*. The function $\cos x$ indeed has this property: as $\sin x$ increases in the first quadrant, $\cos x$ decreases. Thus, it seems plausible that the derivative of $\sin x$ should be related to $\cos x$, and the fact is that this is exactly correct: the rate at which the height $\sin x$ is changing at a is precisely the horizontal length $\cos a$ of this point.

In a similar way, consider now the horizontal lengths corresponding to a and b, which are measured by the function $g(x) = \cos x$. Now as the angle increases by a small amount, these horizontal lengths are *decreasing*, so the rate of change of $\cos x$ (i.e. its derivative) should be negative. Moreover, the rate at which $\cos x$ decreases seems to *more negative* at b than at a, so $g'(b)$ should be more negative than $g'(a)$:



The function $-\sin x$ behaves in a similar way (the negative is there since we already said g' should be negative in the first quadrant), so again it seems plausible that the derivative of $\cos x$ should be related to $-\sin x$. The fact is that the rate at which the length $\cos x$ is changing at a point with angle x is exactly given by the negative of the vertical height at that point, which is $-\sin x$.

Example. We determine the values of a and b for which the parabola $y = ax^2 + bx$ has tangent line at the point $(1, 1)$ given by $y = 3x - 2$. First, in order for this parabola to pass through $(1, 1)$ we need it to be true that

$$1 = a(1)^2 + b(1) = a + b,$$

which gives one condition on a and b. Now, the tangent line at $(1, 1)$ should be $y = 3x - 2$, which has slope 3. Thus it should be true that the derivative $f'(1)$ of $y = f(x) = ax^2 + bx$ at $x = 1$ should equal 3. This derivative is

$$f'(x) = 2ax + b, \text{ so } f'(1) = 2a + b = 3$$

is a condition on a and b. So, we need to find a and b which satisfy both equations

$$a + b = 1 \text{ and } 2a + b = 3.$$

The first gives $b = 1 - a$, and plugging this into the second gives

$$2a + (1 - a) = 3, \text{ so } a = 2.$$

Then $b = 1 - a = 1 - 2 = -1$. Thus we get that $y = 2x^2 - x$ is the specific parabola for which the tangent line at $(1, 1)$ is $y = 3x - 2$.

Product Rule. We now continue to build up to more elaborate functions. The *product rule* is what tells us how to take the derivative of the product $f(x)g(x)$ of two functions. It says that:

$$\frac{d}{dx}(f(x)g(x)) = (f(x)g(x))' = f(x)g'(x) + f'(x)g(x).$$

Each term comes from taking the derivative of one factor and leaving the other alone, so we get two terms depending on which factor we are taking the derivative of. For instance, the derivative of $f(x) = x \sin x$ is

$$f'(x) = (\text{derivative of } x) \sin x + x(\text{derivative of } \sin x) = \sin x + x \cos x.$$

This is a totally non-obvious fact, so you should not be kicking yourself if you can't see why you should expect this to be the correct answer. It is correct because it can be derived from the limit definition, which you can see in the book if interested, but it is not meant to be "easy to see". Certainly, it makes sense that the derivative of $f(x)g(x)$ should *somehow* be related to the derivatives of f and g individually: the rate of change $f(x)g(x)$ should depend on both the rate of change of $f(x)$ and of $g(x)$ since both of these changes have an effect on how $f(x)g(x)$ changes. But, beyond making sense that $(f(x)g(x))'$ should relate to $f'(x)$ and $g'(x)$ somehow, the exact formula given by the product rule takes some getting used to.

(And now a brief historical aside. Actually, for a long time people *did* make the guess that the derivative of $f(x)g(x)$ should be $f'(x)g'(x)$. However, this wasn't giving answers which agreed with what people were seeing in scientific experiments, so people understood it needed to be fixed somehow. Based experimental results, someone finally guessed that the answer should actually be $f'(x)g(x) + f(x)g'(x)$, and it still took another hundred years or so afterwards for Newton and Leibniz—the two people credited with inventing calculus—to prove that this was actually true using the precise definition of "derivative" as a limit, which wasn't a thing people used before. Before Newton and Leibniz people only thought about derivatives as rates of change in an intuitive sense, and indeed perhaps the main contribution of Newton and Leibniz *was* to actually give precise definitions for these concepts.)

Example. We compute the derivative of

$$f(x) = 3x^2 \sin x - x^{1/3} \cos x.$$

The derivative of the first term is:

$$(\text{derivative of } 3x^2) \sin x + 3x^2 (\text{derivative of } \sin x) = 6x \sin x + 3x^2 \cos x.$$

The derivative of $-x^{1/3} \cos x$ is:

$$(\text{derivative of } -x^{1/3}) \cos x - x^{1/3} (\text{derivative of } \cos x) = -\frac{1}{3} x^{-2/3} \cos x - x^{1/3}(-\sin x).$$

Thus altogether we get

$$f'(x) = 6x \sin x + 3x^2 \cos x - \frac{1}{3x^{2/3}} + x^{1/3} \sin x.$$

Another example. We determine the points at which the tangent line to the graph of $f(x) = \sin x \cos x$ is horizontal, which should be the points at which the derivative $f'(x)$ is zero. Using the product rule, this derivative is

$$f'(x) = (\text{derivative of } \sin x)(\cos x) + (\sin x)(\text{derivative of } \cos x) = (\cos x)(\cos x) + (\sin x)(-\sin x).$$

Thus $f'(x) = \cos^2 x - \sin^2 x$, where we use the standard notations $\cos^2 x = (\cos x)^2$ and similarly for $\sin^2 x$. We want this to be zero:

$$\cos^2 x - \sin^2 x = 0, \text{ so } \cos^2 x = \sin^2 x.$$

Taking square roots gives that either $\cos x = \sin x$ or $\cos x = -\sin x$. The values of x which satisfy $\cos x = \sin x$ are

$$\frac{\pi}{4}, \frac{5\pi}{4}, \text{ and either of these plus a multiple of } 2\pi,$$

and the values which satisfy $\cos x = -\sin x$ are

$$\frac{3\pi}{4}, \frac{7\pi}{4}, \text{ and either of these plus a multiple of } 2\pi.$$

Hence the points at which the tangent line is horizontal are those with x-coordinates of the form $\frac{\pi}{4} + 2\pi n$ or $\frac{5\pi}{4} + 2\pi n$, and those with x-coordinates of the form $\frac{3\pi}{4} + 2\pi n$ or $\frac{7\pi}{4} + 2\pi n$.

Quotient Rule. The derivative of a fraction $\frac{f(x)}{g(x)}$ is given by:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

As in the product rule, each term in the numerator comes from taking the derivative of one piece of $\frac{f(x)}{g(x)}$ while leaving the other the same, only that here we have an extra negative which shows up. One way to remember this is the following jingle, which *to this day* is how I still compute such derivatives myself: call the numerator "HI" and the denominator "LO", then the quotient rule is "LO D-HI minus HI D-LO over LO LO", where "D" means to take the derivative.

Derivative of $\tan x$. Here is a classical use of the quotient rule. We compute the derivative of $f(x) = \tan x = \frac{\sin x}{\cos x}$. According to the quotient rule, this is:

$$f'(x) = \frac{\cos x(\text{derivative of } \sin x) - \sin x(\text{derivative of } \cos x)}{(\cos x)(\cos x)} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x},$$

which simplifies to $\frac{1}{\cos^2 x}$ using the fact that $\cos^2 x + \sin^2 x = 1$ in the numerator. This is normally written

$$f'(x) = \sec^2 x,$$

where the secant function $\sec x$ is the reciprocal of $\cos x$: $\sec x = \frac{1}{\cos x}$.

Lecture 8: Chain Rule

Warm-Up 1. We compute the derivative of

$$f(x) = 3x^2 \tan x + \frac{\cos x}{1+\sqrt{x}}.$$

The first term requires the product rule:

$$\frac{d}{dx}(3x^2 \tan x) = (\text{derivative of } 3x^2)\tan x + 3x^2(\text{derivative of } \tan x) = 6x\tan x + 3x^2\sec^2 x.$$

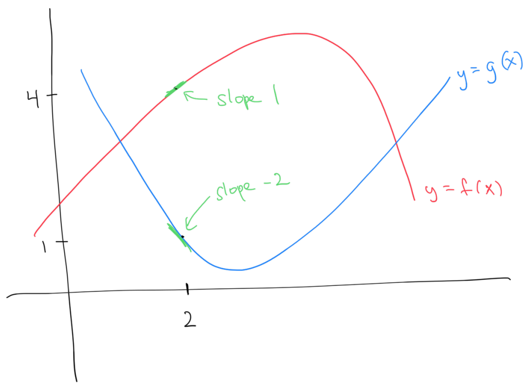
The second requires the quotient rule:

$$\frac{d}{dx}\left(\frac{\cos x}{1+x^{1/2}}\right) = \frac{(1+x^{1/2})\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(1+x^{1/2})}{(1+\sqrt{x})^2} = \frac{(1+x^{1/2})(-\sin x) - \cos x(\frac{1}{2}x^{-1/2})}{(1+\sqrt{x})^2}.$$

Putting it all together gives:

$$f'(x) = 6x\tan x + 3x^2\sec^2 x + \frac{-(1+\sqrt{x})\sin x - \frac{\cos x}{2\sqrt{x}}}{(1+\sqrt{x})^2}.$$

Warm-Up 2. Consider the functions f and g whose graphs are drawn below:



We determine the derivatives of $h(x) = f(x)g(x)$ and $r(x) = \frac{f(x)}{g(x)}$ at $x = 2$. First we have:

$$h'(x) = f'(x)g(x) + f(x)g'(x), \text{ so } h'(2) = f'(2)g(2) + f(2)g'(2) = 1(1) + 4(-2) = -7$$

where we use the drawn slopes to determine $f'(2)$ and $g'(2)$. Next we have:

$$r'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}, \text{ so } r'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{g(2)^2} = \frac{1(1) - 4(-2)}{1^2} = 9.$$

These points in this problems was that we don't have explicit expressions for f and g, but nonetheless we can determine the information we need from the given graphs.

Chain Rule. The final step on our quest to build up to more elaborate functions is a procedure for computing the derivative of a *composition* of functions $f(g(x))$. After this, we will be able to pretty much compute any derivative thrown at us, because most functions we will encounter are built up out of composing, multiplying, dividing, adding, subtracting, and scaling simpler functions.

The *chain rule* states that the derivative of $f(g(x))$ is:

$$(f(g(x))' = f'(g(x))g'(x).$$

So, the derivative of a composition $f(g(x))$ is the product of individual derivatives, only with the derivative of the "outermost" function (f in this case) being evaluated at the "inner" function (g in this case). The key in such computations is in figuring out what these "outer" and "inner" functions are, and this is something which becomes easier with practice. Never forget, however, that the outer function is not being evaluated at x, but rather at the value $g(x)$ of the inner function.

The chain rule too can be justified using the limit definition of derivatives, but here is some intuition. We want to know the rate at which $f(g(x))$ changes. Changing the value of x by a small amount definitely changes the value of g, so the rate we want should depend on $g'(x)$ somehow. But this change in the value of $g(x)$ *in turn* cases a change in the value of $f(g(x))$ since $g(x)$ is the *input* into f. Thus the rate of change f should also play a role, meaning that both $f'(g(x))$ and $g'(x)$ should show up in the derivative of $f(g(x))$.

Example 1. We compute the derivative of $h(x) = \sin(x^2)$. Just for the sake of matching the notation in the chain rule, this function is $h(x) = f(g(x))$ where $f(x) = \sin x$ and $g(x) = x^2$; i.e. x^2 is the function being plugged into $\sin x$, so here $\sin x$ is the "outer" function and x^2 the "inner" function. We have $f'(x) = \cos x$ and $g'(x) = 2x$, so the chain rule gives:

$$h'(x) = f'(g(x))g'(x) = f'(x^2)g'(x) = \cos(x^2)(2x) = 2x\cos(x^2).$$

This is the answer, but it is somewhat cumbersome to write out what $f(x)$ and $g(x)$ are every single time we want to use the chain rule. So, let us instead formulate this approach in the following way. We want to differentiate $\sin(x^2)$. We focus on the outermost function first, which is sin. The derivative of sin is cos, and for now we leave whatever these are being evaluated at *alone*:

$$\frac{d}{dx}(\sin(\text{inside})) = \cos(\text{inside})(\text{other stuff to come}).$$

We then multiply by the derivative of the "inside" expression:

$$\frac{d}{dx}(\sin(\text{inside})) = \cos(\text{inside})(\text{derivative of inside}).$$

So, we do get:

$$\frac{d}{dx}(\underbrace{\sin}_{\text{outside}}(\underbrace{x^2}_{\text{inside}})) = \underbrace{\cos}_{\text{derivative of outside}}(\underbrace{x^2}_{\text{inside}})\underbrace{2x}_{\text{derivative of inside}} = 2x\cos(x^2).$$

Instead, suppose we had the function $r(x) = \sin^2 x = (\sin x)^2$. The outside function is the squaring function, so for this derivative the exponent 2 comes down and then we subtract one from the exponent, all while the inside function $\sin x$ is left as is, and then we multiply by the derivative of the inside function:

$$\frac{d}{dx}(\text{inside})^2 = 2(\text{inside})^1(\text{derivative of inside}).$$

This gives:

$$\frac{d}{dx}(\underbrace{\sin x}_{\text{inside}})^2 = \underbrace{2}_{\text{derivative of outside}}(\underbrace{\sin x}_{\text{inside}})^1\underbrace{\cos x}_{\text{derivative of inside}} = 2\sin x\cos x.$$

Example 2. We find the derivative of the function $f(x) = 3(1 + x\cos x)^4$. For the "outside" derivative of the fourth power function, we bring the 4 down and subtract 1 from the exponent, then multiply by the derivative of the inner function:

$$\frac{d}{dx}(3(\text{inside}^4)) = 12(\text{inside})^3(\text{derivative of inside}).$$

So in our case we get:

$$f'(x) = 12(\underbrace{1 + x\cos x}_{\text{inside}})^3(\underbrace{\cos x - x\sin x}_{\text{derivative of inside}}).$$

To be clear, the derivative of the "inner' $1 + x\cos x$ is computed using the product rule.

Example 3. Let us put some different techniques together and compute the derivative of

$$f(x) = \frac{x\cos(x^2) + (1 + x^3)^{3/2}}{1 + x}.$$

First we have a quotient rule:

$$f'(x) = \frac{(1 + x)(x\cos(x^2) + (1 + x^3)^{3/2})' - (x\cos(x^2) + (1 + x^3)^{3/2})(1 + x)'}{(1 + x)^2}$$

where the primes $'$ denote still to-be-computed derivatives. Next for $(x\cos(x^2)+(1+x^3)^{3/2})'$ we use a product rule on the first term and a chain rule on the second:

$$(x\cos(x^2)+(1+x^3)^{3/2})' = \cos(x^2)+x(\cos(x^2))'+\frac{3}{2}(1+x^3)^{1/2}(3x^2).$$

The middle term on the right requires another chain rule:

$$(\cos(x^2))' = -\sin(x^2)(2x),$$

so

$$(x\cos(x^2)+(1+x^3)^{3/2})' = \cos(x^2)+x(-2x\sin(x^2))+\frac{3}{2}(1+x^3)^{1/2}(3x^2).$$

Putting it all together and simplifying a bit gives:

$$f'(x) = \frac{(1+x)[\cos(x^2)-2x^2\sin(x^2)+\frac{9}{2}x^2\sqrt{1+x^3}]-[x\cos(x^2)+\sqrt{(1+x^3)^3}]}{(1+x)^2}.$$

This is a messy looking expression, but the point is that we found it simply by following our differentiation rules, differentiating one piece at a time. Sure it may be tedious at times, but the process of computing derivatives should just be a mechanical one where we just grind it out, even when working with complicated looking expressions.

Final example. In this final example we work with a function which is not explicitly given to use, but about which we have enough information to compute the required derivative. Suppose g and h are functions satisfying

$$g'(3)=5,\ g'(1)=-2,\ h(1)=3,\ \text{and } h'(1)=4.$$

We find the derivative of $f(x)=g(x^2h(x))$ at $x=1$. Again, this is no different than anything before, except that we do not know g and h explicitly but are only given some information about them. First, the chain rule gives:

$$f'(x) = g'(x^2h(x))(\text{derivative of } x^2h(x)) = g'(x^2h(x))(2xh(x)+x^2h'(x)).$$

To be clear, when computing the derivative of the "outside" function g, the "inner" $x^2h(x)$ at which it is being evaluated remains as is, and then we multiply by the derivative of this inner function, which is computed using the product rule. Evaluating at $x=1$ gives:

$$f'(1) = g'(h(1))(2h(1)+h'(1)) = g'(3)(2\cdot 3+4) = 5(10) = 50.$$

Note that the given information $g'(1)=2$ was not needed; it was included as a red herring because without recalling the exact form of the chain rule you might randomly guess that the g' term should be evaluated at $x=1$ since, after all, that is what we are plugging into f' in the end, but the point is that according to the chain rule this g' term should be evaluated not at x but rather at the output of $h(x)$, which in this case is $h(1)=3$.

Lecture 9: Implicit Differentiation

Warm-Up 1. We find the points at which the tangent line to the curve $y=\sqrt{3+x^2}$ is parallel to $x-2y=1$. This line has slope $\frac{1}{2}$, so we are looking for points where the value of y' is $\frac{1}{2}$. We compute this derivative using the chain rule:

$$y=(3+x^2)^{1/2},\ \text{so } y'=\frac{1}{2}(3+x^2)^{-1/2}(2x)=\frac{x}{\sqrt{3+x^2}}.$$

So we want the value(s) of x satisfying

$$\frac{x}{\sqrt{3+x^2}} = \frac{1}{2}.$$

Clearing denominators gives

$$2x = \sqrt{3+x^2}.$$

Now we square both sides to get:

$$4x^2 = 3 + x^2.$$

There is a subtle point here that this squaring might introduce extraneous values of x which don't actually describe the type of point we want. We'll come back to this in a second, and is something I missed when first working through this in class until a fellow classmate pointed it out.

Solving for x gives $3x^2 = 3$, so $x = \pm 1$. However, if we go back to the condition

$$2x = \sqrt{3+x^2}$$

right before the squaring, we see that in fact x must be positive since $2x$ is set to equal a positive square root. Thus $x = -1$ isn't actually a point we want to consider since the slope at this point is actually $-\frac{1}{2}$. This was the extraneous solution introduced when squaring: $x = -1$ definitely satisfies $4x^2 = 3 + x^2$, but not the $2x = \sqrt{3+x^2}$ condition. So, we only get one point at which the tangent line has slope $\frac{1}{2}$, namely the one with x-coordinate $x = 1$. The y-coordinate here is $y = \sqrt{3+1} = 2$, so $(1, 2)$ is the point we want.

Warm-Up 2. We find the derivative of

$$f(x) = \sin(\cos(\sin(1 + x^2))).$$

Clearly this should be a chain rule application since we are plugging one function into another, but actually this will make use of *multiple* chain rules, one for each time we are plugging one function into another: $1 + x^2$ plugged into $\sin x$, which is then plugged into $\cos x$, which is then plugged into $\sin x$. The goal is to unwind it all, computing the derivative starting with the "outermost" function and working our way "inward".

First we get:

$$f'(x) = \cos(\cos(\sin(1 + x^2))(\text{derivative of inside})$$

where "inside" is $\cos(\sin(1 + x^2))$. Next, the derivative of this "inside" is

$$\sin(\sin(1 + x^2))(\text{derivative of new inside})$$

where "new inside" is $\sin(1 + x^2)$. The derivative of "new inside" is

$$\cos(1 + x^2)(\text{derivative of } 1 + x^2) = \cos(1 + x^2)(2x).$$

Putting it all together gives:

$$f'(x) = \cos(\cos(\sin(1 + x^2)))\sin(\sin(1 + x^2))\cos(1 + x^2)2x.$$

This looks a bit involved, but the point is that each term comes a chain rule application, where again we work out way from the inward starting from the "outside".

Implicitly defined functions. So, we should now be able to compute the derivative of most any function, at least if we are given an explicit formula for it. However, not all functions one might encounter are as nice as this. Often it turns out that we cannot (easily) find an explicit expression for a function, and the best we can do is to define it *implicitly.* This means that we can find a constraining equation the function must satisfy, which essentially characterizes it although not in explicit form.

For instance, a unit circle has equation $x^2 + y^2 = 1$, and so we can take this to implicitly define y as a function of x. Now, in this case, it is actually possible to find an *explicit* formula for y has a function of x by solving for y, or to be more precise we get two functions $y = \sqrt{1+x^2}$ and $y = -\sqrt{1+x^2}$ for the top and bottom halves of the circle. But if we instead had an equation like

$$x \sin y + y \sin x = 1,$$

we would not be able to solve for y explicitly. Nonetheless, we would still like to be able to compute the derivative of such a function using only its implicit definition, and the point is that this often does work out. Even in the case of the circle, it turns out that computing $\frac{dy}{dx}$, which gives the slope of the tangent line at different points, using the $x^2 + y^2 = 1$ equation alone is faster than using the explicit expression for y.

The technique of implicit differentiation is what allows us to compute derivatives of functions in this way even if we don't have an explicit formula for the function in question. We'll see the technique in action in the examples below, and the key point is to differentiate both sides of the equation implicitly defining our function and then "solve" for the derivative we want.

Example 1. Consider the curve defined by the equation

$$\sqrt{xy} = 1 + x^2 y.$$

We interpret this implicitly defining y as a function of x, and we would like to find the derivative y' of this function. To do so, we differentiate both sides of the given equation. The point is that the equation says that the function $\sqrt{xy}$ is supposed to equal the function $1 + x^2y$, and if two functions are equal then their derivatives should be as well. In other words, differentiating both sides of this equality will maintain the equality.

The derivative of the right side is

$$0 + (x^2y)' = 2xy + x^2y',$$

which comes from the product rule: recall that y is a function, so x^2y is a product of two functions and so its derivative is $(x^2)'y + x^2(y')$. The key is not to forget the y' term.

The derivative of the left side $(xy)^{1/2}$ is

$$\frac{1}{2}(xy)^{-1/2}(y + xy'),$$

which again uses the product rule on the xy term. Thus after differentiating both sides of $\sqrt{xy} = 1 + x^2y$ we get the equality

$$\frac{y + xy'}{2\sqrt{xy}} = 2xy + x^2y'.$$

This new equation contains y', which is precisely what we are trying to find. We then use this equation to *solve* for y' directly. Multiplying through by the denominator gives

$$y + xy' = 2\sqrt{xy}(2xy + x^2y'), \text{ or } y + xy' = 2xy\sqrt{xy} + 2x^2y'\sqrt{xy}.$$

We group together all the y' terms on one side and the non-y' terms on the other:

$$xy' - 2x^2y'\sqrt{xy} = 2xy\sqrt{xy} - y.$$

We can factor the common y' out of the terms on the left:

$$(x - 2x^2\sqrt{xy})y' = 2xy\sqrt{xy} - y,$$

and finally isolate y' by dividing by $x - 2x^2\sqrt{xy}$:

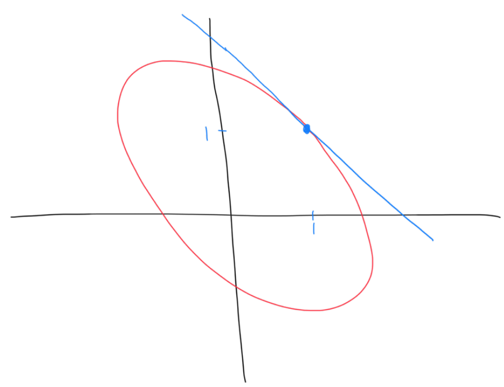
$$y' = \frac{2xy\sqrt{xy} - y}{x - 2x^2\sqrt{xy}}.$$

This is the desired derivative! Again, we did not find explicitly what y was, but we were able to find its derivative anyway. The only difference between this expression and the type of derivative expressions we've computed up until now is that in this one the expression for the derivative y' involves the unknown function y itself. Nonetheless, this type of expression will usually be good enough for what we need to do. For instance, the slope of the tangent line at a point (a, b) on this curve is

$$y' = \frac{2ab\sqrt{ab} - b}{a - 2a^2\sqrt{ab}},$$

which comes from setting $x = a$ and $y = b$.

Example 2. Consider the curve given by the equation $x^2 + xy + y^2 = 3$. We want to find the equation of the tangent line to the curve at the point $(1, 1)$, which does lie on the curve in question since $x = 1, y = 1$ satisfy the given equation. This curve is actually an ellipse, so the tangent line we want looks like:



As before, the function y giving the y-coordinate of a point on this curve in terms of the x-coordinate is only implicitly determined by the given equation and we do not have an explicit formula for y, so in order to compute $y' = \frac{dy}{dx}$ we use implicit differentiation.

We first differentiate both sides of $x^2 + xy + y^2 = 3$. Differentiating the right side certainly gives 0. Now, differentiating the terms on the left gives

$$2x + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2).$$

For the middle term, we recall that y is still denoting a function, so we use the product rule since we are taking the derivative of the function x times the function y:

$$\frac{d}{dx}(xy) = y + xy'.$$

For the third term, which is the function y being squared, we need the chain rule:

$$\frac{d}{dx}(y^2) = 2yy'.$$

To be clear, y is the "inner" function here and the squaring function is the "outer" one, so the derivative of the "outer" function evaluated at the "inner" one is the $2y$ term, and the y' comes from the derivative of the "inner" function. So, overall we end up with:

$$2x + y + xy' + 2yy' = 0$$

after differentiating both sides of $x^2 + xy + y^2 = 3$. Our goal is to find y', which gives slopes of tangent lines along the curve, so we group all y' terms on one side:

$$xy' + 2yy' = -2x - y,$$

factor out the common y' on the left:

$$(x + 2y)y' = -2x - y,$$

and finally divide by $x + 2y$:

$$y' = \frac{-2x - y}{x + 2y}.$$

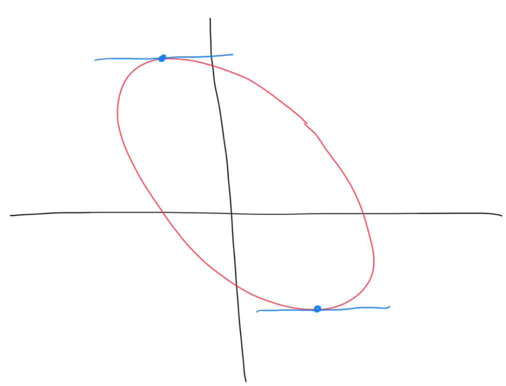
This thus gives the slope of the tangent line to the ellipse at a point (x, y). So, at $(1, 1)$, the slope of the tangent line is:

$$y' = \frac{-2 - 1}{1 + 2} = -1.$$

Note in the picture above that it indeed makes sense the tangent line we want should have negative slope. The equation of the actual tangent line, taking into account that it should pass through $(1, 1)$, is then:

$$y - 1 = -1(x - 1), \text{ or } y = -x + 2.$$

Now, going back to the picture of the ellipse, there seem to be two points where the tangent line should be horizontal:



We now find these points. Recall the slope formula we derived above:

$$y' = \frac{-2x - y}{x + 2y}.$$

In order to have a horizontal tangent line, this derivative should be 0:

$$\frac{-2x-y}{x+2y}=0.$$

Multiplying through by the denominator on the left gives:

$$-2x-y=0, \text{ so } y=-2x.$$

This says that the points (x, y) at which the tangent line is horizontal, it should be true that $y=-2x$. In order to find these actual points, we must finally make use of the fact that the points (x, y) we want should also satisfy the equation of the ellipse:

$$x^2+xy+y^2=3.$$

Since $y=-2x$, this final equation becomes

$$x^2+x(-2x)+(-2x)^2=3, sox^2-2x^2+4x^2=3 \text{ and thus } x^2=1.$$

Thus $x=\pm 1$ are the x-coordinates of the points at which the tangent line is horizontal. To find the y-coordinates we use the equation of the ellipse:

$$\text{for } x=1, 1+y+y^2=3 \text{ or } 0=y^2+y-2=(y+2)(y-1), \text{ and}$$

$$\text{for } x=-1, 1-y+y^2=3 \text{ or } 0=y^2-y-2=(y-2)(y+1).$$

So, for $x=1$, we get that either $y=-2$ or $y=1$, but of these only $y=1$ satisfies $y=-2x$ with $x=1$, so this point is $(1,-2)$. For $x=-1$ we get that either $y=2$ or $y=-1$, but here only $y=2$ satisfies $y=-2x$ with $x=-1$, so we get the point $(-1,2)$. Thus the two points at which the tangent line is horizontal are $(1,-2)$ and $(-1,2)$.

Lecture 10: Related Rates

Warm-Up 1. Suppose a function $f(x)$ satisfies $f(\frac{\pi}{4})=\frac{\pi}{3}$ and

$$x\sin f(x)+f(x)\sin x=1.$$

We wish to compute $f'(\frac{\pi}{4})$.

Differentiating both sides of the equation implicitly defining f gives:

$$\sin f(x)+x(\cos f(x))f'(x)+f'(x)\sin x+f(x)\cos x=0.$$

To be clear, the first two terms come from the product rule applied to $x\sin f(x)$, and the third and fourth terms are the result of the product rule applied to $f(x)\sin x$. Also, when differentiating $\sin f(x)$, we use the chain rule since $f(x)$ is being plugged into the sine function:

$$\frac{d}{dx}(\sin f(x))=(\cos f(x))(\text{derivative of } f(x)).$$

In the resulting equation we group all the $f'(x)$ terms on one side and the rest of the terms on the other side:

$$x(\cos f(x))f'(x)+f'(x)\sin x=-\sin f(x)-f(x)\cos x, \text{ or } f'(x)[x\cos f(x)+\sin x]=-\sin f(x)-f(x)\cos x.$$

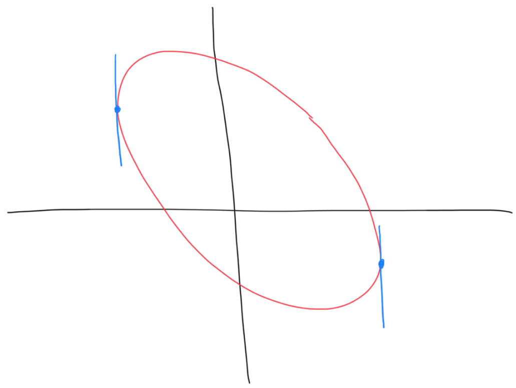
Thus the derivative of f is:

$$f'(x) = \frac{-\sin f(x) - f(x)\cos x}{x\cos f(x) + \sin x}.$$

Evaluating at $\frac{\pi}{4}$ gives:

$$f'(\frac{\pi}{4}) = \frac{-\sin f(\frac{\pi}{4}) - f(\frac{\pi}{4})\cos\frac{\pi}{4}}{\frac{\pi}{4}\cos f(\frac{\pi}{4}) + \sin\frac{\pi}{4}} = \frac{-\frac{3}{2} - \frac{\pi}{3}\frac{\sqrt{2}}{2}}{\frac{\pi}{4}\frac{1}{2} + \frac{\sqrt{2}}{2}}.$$

Warm-Up 2. Let us return to the ellipse we considered last time with equation $x^2 + xy + y^2 = 3$:



Based on the picture, there appear to be two points at which the tangent line is actually *vertical*. The question is how can we get detect a vertical tangent line using the derivative y' we found last time? A vertical line has undefined slope, or we sometimes say it has "infinite" slope. In order to have undefined slope, the expression we found for y' last time:

$$y' = \frac{-2x - y}{x + 2y}$$

should be undefined, which occurs when the *denominator* is zero and the numerator is not. Thus, vertical tangent lines occur when $x + 2y = 0$, so $x = -2y$. To find the corresponding points we use the ellipse equation with $x = -2y$:

$$(-2y)^2 + (-2y)y + y^2 = 3, \text{ so } 3y^2 = 3 \text{ and thus } y = \pm 1.$$

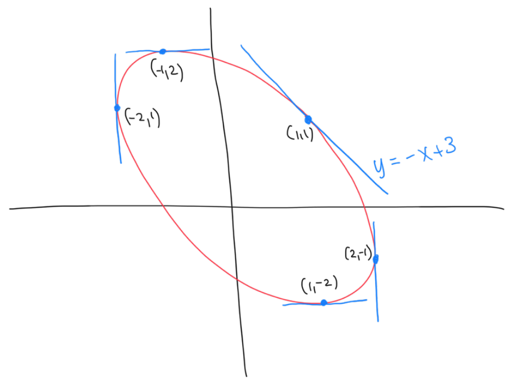
With $y = 1$ in the ellipse equation we get

$$x^2 + x + 1 = 3, \text{ so } (x+2)(x-1) = x^2 + x - 2 = 0$$

and with $y = -1$ we get

$$x^2 - x + 1 = 3, \text{ so } (x-2)(x+1) = x^2 - x - 2 = 0.$$

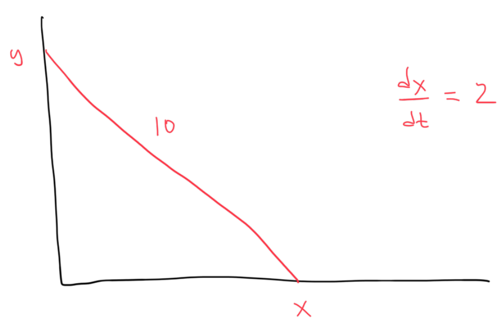
So, for $y = 1$ we have $x = -2$ or $x = 1$, but only $x = -2$ satisfies $x = -2y$; for $y = -1$ we have $x = 2$ or $x = -1$, but only $x = 2$ satisfies $x = -2y$. Thus we conclude that the points at which the vertical tangent lines occur are $(-2, 1)$ and $(2, -1$. Here then are all the points and tangent lines we've found:



Hinting at something we'll do later on, suppose we weren't given the picture of the ellipse in the first place; how could we then actually draw it? One approach is to use the derivatives we've computed to find the points at which the tangent lines are horizontal or vertical. From these points, and a picture of what the horizontal and vertical tangent lines look like, the idea is that it should be possible to come up with a rough sketch of the ellipse itself. After all, the horizontal and vertical tangents tell us the points at which the ellipse is "turning", which goes a long way towards telling us what the curve actually looks like. We'll push this idea further in a few weeks.

Related rates. In many situations, there are different quantities we are interested in, related by some type of equation. By differentiating this equation we can obtain an equation relating the *rates of change* for these quantities, and possibly use information about one rate of change to determine information about the other. Such *related rates* scenarios are a key application of derivatives.

Example 1. Suppose a 10 ft ladder is leaning against a wall:



If we pull the base of the ladder further away from the wall, say at a rate of 2 ft/sec, we want to determine the rate at which the height the ladder is at changes when the base is 6 ft away from the ball. We label the distance of the base from the wall by x, so we know that $\frac{dx}{dt} = 2$ since the ladder is being pulled away at a rate of 2 ft/sec. If we label the height by y, we want to thus find $\frac{dy}{dt}$ when $x = 6$.

The key observation is that since the ladder is 10 ft long, we must have

$$x^2 + y^2 = 10^2 = 100$$

by the Pythagorean Theorem. Implicitly differentiating both sides with respect to the time t, recalling that x and y are *both* functions depending on t, we get:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

This equation relates the two rates $\frac{dx}{dt}$ and $\frac{dy}{dt}$. By isolating $\frac{dy}{dt}$, we get

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

We have $\frac{dx}{dt} = 2$, and also $y = 8$ when $x = 6$ since these values must satisfy $x^2 + y^2 = 100$. Thus the height is changing at a rate of

$$\frac{dy}{dt} = -\frac{6}{8}(2) = -\frac{3}{2}\text{ft/sec}.$$

Note it makes sense that this should be negative since the ladder is slipping *down* the wall.

Example 2. Suppose we drain water from a cylindrical tank of radius 3 cm at a rate of 2 cm^3. We want to know that rate at which the height of the water in the tank is changing. If V denotes the volume of water and h the height, we have the relation

$$V = \pi r^2 h = 4\pi h$$

since the radius is $r = 2$ in our case. Taking derivatives of both sides gives:

$$\frac{dV}{dt} = 4\pi\frac{dh}{dt}.$$

We have $\frac{dV}{dt} = -2$ (negative since water is being *drained* so the volume is decreasing), so

$$-2 = 4\pi\frac{dh}{dt}, \text{ and thus } \frac{dh}{dt} = -\frac{1}{2\pi}.$$

Thus the height is decreasing at a rate of $\frac{1}{2\pi}$ cm/sec.

Example 3. Say we have a candle in the shape of a cylinder, which is melting. If the volume is changing at a rate of -1 and the radius is changing at a rate of 1, we want to know the rate at which the height is changing when the radius and height are both 1. If r denotes the radius and h the height, we have that the volume V is

$$V = \pi r^2 h.$$

Differentiating (in this case V, r, and h are all functions depending on time t) gives:

$$\frac{dV}{dt} = 2\pi r\frac{dr}{dt}h + \pi r^2\frac{dh}{dt},$$

where we use the product and chain rules in order to compute the derivative of $\pi r^2 h$. In our case, we have $\frac{dV}{dt} = -1$, $\frac{dr}{dt} = 1$, and $r = h = 1$, so:

$$-1 = 2\pi(1)(1)(1) + \pi(1^2)\frac{dh}{dt}, \text{ and thus } \frac{dh}{dt} = \frac{-1-2\pi}{\pi}$$

is the rate at which the height is changing. (Note it makes sense since is negative, since the height is decreasing, while the radius is increasing since the melting wax is making the width larger.)

Example 4. Suppose the length ℓ and width w of a rectangle are both changing as time passes, say at rates of $\frac{d\ell}{dt} = 8$ cm/sec and $\frac{dw}{dt} = 3$ cm/sec. We want to know the rate at which the area is changing when the length is 20 cm and the width is 3 cm. The area is given by

$$A = \ell w.$$

Differentiating both sides gives

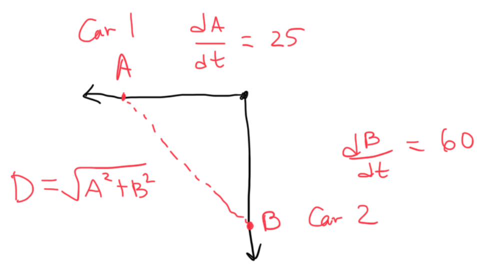
$$\frac{dA}{dt} = \frac{d\ell}{dt}w + \ell\frac{dw}{dt}.$$

Plugging in the given values gives

$$\frac{dA}{dt} = 8(3) + 20(3) = 84,$$

so the area is changing at a rate of 84 cm^2.

Example 5. Finally, suppose two cars take off from the same point, one moving west at a speed of 25, and the other moving south at a speed of 60:



We want to know the rate at which the distance between the cars is increasing after 2 hours. The distance between the cars is

$$D = \sqrt{A^2 + B^2}.$$

Differentiating gives

$$\frac{dD}{dt} = \frac{1}{2\sqrt{A^2+B^2}}\left((2A\frac{dA}{dt} + 2B\frac{2B}{dt}\right),$$

where we use multiple chain rules on the right side. After 2 hours, the first car has traveled a distance $A = 50$, while the second has traveled a distance of $B = 120$. Since $\frac{dA}{dt} = 25$ and $\frac{dB}{dt} = 60$, we thus get:

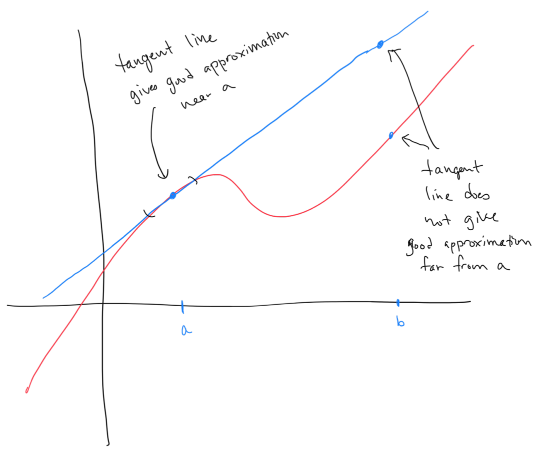
$$fracdDdt = \frac{100(25) + 240(60)}{2\sqrt{50^2 + 120^2}}$$

as the rate at which the distance between the cars is changing.

Lecture 11: Linear Approximation

Approximating functions. So far we've seen the point of view that derivatives describes rates of change, or slopes. Certainly being able to find a rate of change is important in many applications, but finding slopes is not really that important in other fields. Rather, one of the most important uses of derivatives in practice comes from using them to compute *approximations*.

Consider the graph of a function and its tangent line at a given point:



The point is that *near* $x = a$ itself, the tangent line provides a pretty good approximation to the actual graph. Certainly, the tangent line is not a very good approximation at points far away from a, but here we are only talking about the behavior near a. So, we can use the function which gives this tangent line to approximate the actual values of the given function near a. This tangent line function is:

$$L(x) = f(a) + f'(a)(x - a),$$

and we call it the *linear (or tangent line) approximation* to f at a, or the *linearization* of f at a. (The expression for this function comes from using the point-slope form of the equation of a line

$$y - y_0 = m(x - x_0)$$

which in this case should have slope $m = f'(a)$ and should pass through $(x_0, y_0) = (a, f(a))$.) The point is that for x "close" to a, we have $f(x) \approx L(x)$, where the notation $\approx$ means "is approximately equal to to". The "closer" x is to a, the better an approximation this is.

Here is some more intuition. Consider the following expression for $f'(a)$:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

This says that x gets closer to a, the value of $\frac{f(x)-f(a)}{x-a}$ should be getting closer to $f'(a)$, so

$$\frac{f(x) - f(a)}{x - a} \approx f'(a) \text{ for } x \text{ close to } a.$$

Rearranging gives

$$f(x) - f(a) \approx f'(a)(x - a) \text{ for } x \text{ close to } a,$$

and finally

$$f(x) \approx f(a) + f'(a)(x - a) \text{ for } x \text{ close to } a.$$

The right hand side here is precisely the linear approximation $L(x)$, and so we conclude that the value of $L(x)$ is close to the value of $f(x)$ for x close to a.

Example 1. Consider $f(x) = x^4 + 3x^2$ at $a = -1$. The linear approximation to f at -1 is

$$L(x) = f(-1) + f'(-1)(x - (-1)) = f(-1) + f'(-1)(x + 1).$$

In this case, we have $f(-1) = 4$ and

$$f'(x) = 4x^3 + 6x, \text{ so } f'(-1) = -10.$$

Thus the linear approximation is

$$L(x) = 4 - 10(x + 1),$$

which is just the function giving the tangent line to the graph of f at $x = -1$.

Example 2. Now we compute the linearization of $f(x) = \sin x$ at $a = \frac{\pi}{6}$. This looks like

$$L(x) = f(\tfrac{\pi}{6}) + f'(\tfrac{\pi}{6})(x - \tfrac{\pi}{6}).$$

We have $f(\frac{\pi}{6}) = \sin\frac{\pi}{6} = \frac{1}{2}$. Also:

$$f'(x) = \cos x, \text{ so } f'(\tfrac{\pi}{6}) = \cos\tfrac{\pi}{6} = \tfrac{\sqrt{3}}{2}.$$

Thus the linearization is

$$L(x) = \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\left(x - \tfrac{\pi}{6}\right).$$

Now let us use this to actually approximate some value of sine, say $\sin(\frac{\pi}{6} + 0.1)$. The input $\frac{\pi}{6} + 0.1$ is pretty close to the value $\frac{\pi}{6}$ at which we computed the linearization, so we expect that $L(x)$ should give a pretty approximation to $f(x)$ for this value of $x = \frac{\pi}{6} + 0.1$. We have:

$$L(\tfrac{\pi}{6} + 0.1) = \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}([\tfrac{\pi}{6} + 0.1] - \tfrac{\pi}{6}) = \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}(0.1).$$

To be clear, the 0.1 represents how close $\frac{\pi}{6} + 0.1$ is to the point $\frac{\pi}{6}$ at which we computed the linearization. The value given the linearization is about 0.5866 (using a calculator), and the actual value of $\sin(\frac{\pi}{6} + 0.1)$ is about 0.5839, so we see that the value given by linear approximation is indeed fairly close to the actual value.

Example 2. Let us approximate the value of $\sqrt[5]{32.01}$. The point is that we can do this by comparing this value to that $\sqrt[5]{32} = 2$. That is, we approximate the value we want using the linear approximation to the function $f(x) = \sqrt[5]{x}$ at 32. (We use 32 since it is a number close to 32.01 for which the value of $f(32)$ is easier to determine.) Thus we use the approximation:

$$f(32.01) \approx f(32) + f'(32)(32.01 - 32).$$

We have $f(32) = \sqrt[5]{32} = 2$, and:

$$f'(x) = \frac{1}{5}x^{-4/5}, \text{ so } f'(32) = \frac{1}{5(32)^{4/5}} = \frac{1}{5 \cdot 2^4} = \frac{1}{5 \cdot 16}.$$

Thus

$$\sqrt[5]{32.01} \approx \sqrt[5]{32} + \frac{1}{5 \cdot 16}(0.1) = 2 + \frac{1}{800}.$$

This approximate value is thus $2 + \frac{1}{800} = 2.00125$, while the actual value of $\sqrt[5]{32.01}$ is about 2.000125, so fairly close. We could also approximate, say, $\sqrt[5]{33}$ by plugging in $x = 33$ into the linear approximation:

$$\sqrt[5]{33} \approx f(32) + f'(32)(33 - 32) = 2 + \frac{1}{5 \cdot 16}(1) = 2 + \frac{1}{80} = 2.0125$$

which is again pretty close to the actual value of about 2.0123. Of course, if we try to use to approximate $\sqrt[5]{40}$, the value we get won't be so accurate since 40 is not "close" to 32.

Higher order approximations. One objection to what we did above is that we still had to use a calculator to see how good our approximations actually were, or for instance to work out what $\frac{1}{2}+\frac{\sqrt{3}}{2}(0.1)$ actually looked like as a decimal. So, if are going to use a calculator anyway, what is the point of finding linear approximations? But actually, the *real* question here is: how does my calculator or computer program *know* that $\sin(\frac{\pi}{6}+0.1)$ should be about 0.5839? I guarantee you that this value of sine is not one which is programmed specifically into the software your calculator runs, so what actually goes on behind the scenes?

The answer, and the point of it all, is that what actually *is* programmed into your calculator is the expression for the linear approximation $L(x)$ itself, or more precisely, an expression for an even better approximation! Here is what the software running your calculator or computer actually does: take the input you provide it, look up a linear or other type of approximation for the function you're using, and then use this approximation to compute the number which should be displayed. *This* gets at the heart of one of the core uses of derivatives in practice—to develop methods for approximating quantities! Indeed, often times in applications you don't even have an explicit formula for a function $f(x)$ itself, but the idea is that as long as you can determine information about one or more of its derivatives based on whatever data you have available, you can use these derivatives to find quite accurate approximations to the unknown $f(x)$.

In this course we are only discussing linear approximations, or what are also called *first-order* approximations. In Math 224 you'll learn about "higher-order" approximations, which give even better approximations to functions! For instance, the second-order approximation to $f(x)$ near $x=a$ is given by

$$f(a)+f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2,$$

where now we've included a second derivative term. You might recognize the first two terms here as giving the first-order (or linear) approximation we've been looking at, which we are now modifying by adding a "second-order" term. It turns out that this gives an even better approximation to $f(x)$ than the linear approximation does. Next you can construct "third-order" approximations which include a third derivative term, and so on. This is the topic known as *approximation by Taylor polynomials*, which again you'll see a lot of Math 224. Even better, you'll see ways to controlling how good an approximation these "Taylor polynomial" expressions actually are; for instance, if you wanted to approximate $f(x)$ to up to 5 decimal digits of accuracy, you can actually work out which "order" Taylor polynomial you would need to achieve that. Awesome stuff!

Differentials. Let us return to the approximation given by tangent lines:

$$f(x)\approx f(a)+f'(a)(x-a).$$

Rearrange terms to get

$$\underbrace{f(x)-f(a)}_{\text{change in outputs}} \approx f'(a) \underbrace{(x-a)}_{\text{change in inputs}}.$$

The left side gives the *actual* change in the value of f when we input x versus a, and this says that the right side gives a way to approximate this change using only the derivative at a and the change $x-a$ in inputs. Indeed, intuitively the idea is that the derivative $f'(a)$ tells us how to convert a "change in inputs" $x-a$ into a "change in outputs" $f(x)-f(a)$: the change in outputs is well-approximated by multiplying the change in inputs by the derivative, at least when the change in inputs is small.

The right side of the expression above is called a *differential*, or more precisely the *differential* of a function $f(x)$ is the expression

$$df = f'(x)\,dx.$$

We interpret "dx" as representing a "small change" in x, and this differential expression tells us the corresponding "change in f" df. Again, it is the derivative $f'(x)$ which relates these two changes to one another. So, we can use such differentials df to approximate *actual* changes in the values of f. Note that if we were to treat df and dx as numbers, we could "divide" by dx to get

$$\frac{df}{dx} = f'(x),$$

which is where the $\frac{df}{dx}$ notation for a derivative comes from! The idea is that we are taking a "small infinitesimal change in f" divided by a "small infinitesimal change in x", and such a fraction should indeed describe some kind of "infinitesimal slope". (The word *infinitesimal* is not being defined precisely, but take it to mean "incredibly incredibly incredibly small".)

Example. Suppose we measure the radius of a disk to be 24 cm. Then the measured area (using πr^2) would be $\pi(24)^2 = 576\pi$ cm^2. But, perhaps our measuring tool was not completely accurate, so that we can only guarantee the measured radius is accurate to within 0.2 cm. The question we ask is how accurate is the measured area is, or more precisely, by how much could the measured area differ from the actual area?

We can estimate this using differentials. The differential of the area function $A = \pi r^2$ is

$$dA = 2\pi r\,dr,$$

where $2\pi r$ comes from the derivative $\frac{dA}{dr}$ of A with respect to r. The "error" in measuring the radius is at most $dr = 0.2$, since this describes the maximum amount by which the measure radius could be from the actual radius. With this possible change in r, we thus estimate the corresponding change in A to be:

$$dA = 2\pi(24)(0.2) = 9.6\pi.$$

That is, the maximal error in computing the area this way is 9.6π cm^2, meaning that the area value of 576π we computed before is only accurate to within 9.6π, so the actual area would fall between $576\pi - 9.6\pi$ and $576\pi + 9.6\pi$.

Final example. Say we measure the side length of a cube to be 30 cm, but that we can only guarantee this is accurate to within 0.1 cm. What is the maximal error in the measured volume of 30^3 cm^3? The volume of the cube is given by $V = x^3$ where x denotes the length of one of the sides, so we can estimate the possible change in V using the differential

$$dV = \frac{dV}{dx}\,dx = 3x^2\,dx.$$

With $x = 30$ and $dx = 0.1$, this gives $dV = 3(30)^2(0.1) = 180$ cm^3 as an estimate for the maximal error in the volume obtained via this measurement.

Lecture 12: Limits Involving Infinity

Warm-Up 1. We approximate the value of $\sqrt[4]{258}$, using the fact that $4^4 = 256$. The point is that $\sqrt[4]{256} = 4$, so 256 is a point close to 258 at which the value fourth root function. So, we use a linear approximation to $f(x) = \sqrt[4]{x}$ at the point 256:

$$f(x) \approx f(256) + f'(256)(x - 256) \text{ for } x \text{ close to } 256.$$

We have $f(256) = \sqrt[4]{256} = 4$ and

$$f'(x) = \frac{1}{4}x^{-3/4}, \text{ so } f'(256) = \frac{1}{4(256)^{3/4}} = \frac{1}{4(4)^3} = \frac{1}{256}.$$

Thus

$$\sqrt[4]{258} \approx f(256) + f'(256)(258 - 256) = 4 + \frac{1}{256}(2).$$

For comparison, this approximate value is $4 + \frac{1}{128} = 4.0078125$, which is pretty close to the actual value of $\sqrt[4]{258} = 4.0077897$.

The actual change (Δf in the book's notation) in the value of f at 256 versus 258 is

$$\Delta f = f(258) - f(256) = \sqrt[4]{258} - \sqrt[4]{256} = 0.0077897,$$

which itself is approximated by the differential $df = f'(x)\,dx = \frac{1}{4x^{3/4}}\,dx$ at $x = 256$ when the change in x is $dx = 2$:

$$df = \frac{1}{4(256)^{3/4}}(2) = \frac{2}{256} = \frac{1}{128} = .0078125.$$

Warm-Up 2. Suppose we measure the radius of a cylinder of height 4 in to be 12 in, but that this measurement is only accurate to within a maximal error of 0.2 in. What is the corresponding maximal error in the measured volume? The volume of a cylinder of radius r and height 4 is

$$V = 4\pi r^2, \text{ so the differential } dV = 8\pi r\,dr$$

gives the possible error in the volume. (To be clear, we differentiated V here with respect to r.) Given a maximal error of $dr = 0.2$ in the measured radius, we get:

$$dV = 8\pi(12)(0.2)$$

as the maximal error in V, which is about 60.32 in^3. The volume with the measured radius of 12 would be $V = 4\pi(12)^2$, or about 1809.56 in^3, so the point is that the *actual* volume should fall between $1809.56 - 60.32$ and $1809.56 + 60.32$.

Vertical asymptotes. Now we go back to limits, this time considering two types of limits which can involve infinity: namely ones which "equal" infinity, or ones where we take x "approaching" infinity. First, consider

$$\lim_{x\to 0^+} \frac{1}{x}.$$

We have said before that this limit does not exist, since the values of $\frac{1}{x}$ do not approach a *specific concrete* number as the values of x get closer and closer to zero from the right. But, in this case, this limit fails to exist more precisely because the values of $\frac{1}{x}$ get larger and larger, or more and more positive, x gets closer to 0 from the right. This is contrast to a limit like

$$\lim_{x\to 0} \sin(\tfrac{1}{x}),$$

which also fails to exist but in this case because the values of $\sin(\frac{1}{x})$ "oscillate" wildly and do not approach any one specific value, as opposed to having values that are getting more and more positive.

We use the notation

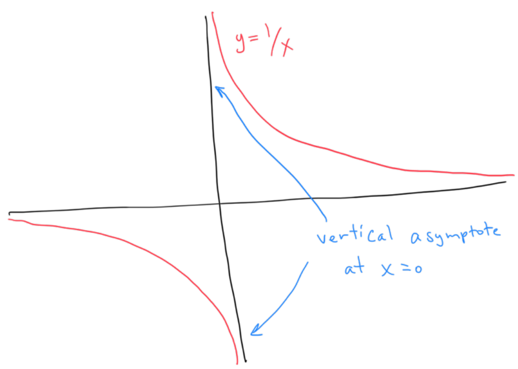
$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$

to indicate that the values of $\frac{1}{x}$ get more and more and more positive without restriction as x gets closer and closer to 0 from the right. To be absolute clear: this limit still DOES NOT EXIST! We are not saying that this limit "exists" and equals the number "infinity", all we are saying is that this limit fails to exist for the specific reason that the values of the function in question get more and more positive. Similarly, we write

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

to indicate that the values of $\frac{1}{x}$ get more and more *negative* without restriction as x gets closer to 0 from the left. Again, this limit still does NOT exist, but it fails to exist for the specific reason that its values get more and more negative. We are NEVER treating ∞ as a number, only as a symbol used to indicate a certain type of behavior.

Graphically, this says that the vertical line at $x = 0$ is a *vertical asymptote* of the function $f(x) = \frac{1}{x}$, since as x approaches 0 the graph of $y = \frac{1}{x}$ approaches this vertical line, either in the positive infinite or negative infinite direction:



In general, limits which "equal" ∞ or $-\infty$ indicate a vertical asymptote. (Again, treat the word "equal" here loosely: we are NOT saying that these limits literally exist and have infinite value.)

Example. We determine the following limits:

$$\lim_{x\to -3^-} \frac{x+2}{x+3} \qquad \lim_{x\to -3^+} \frac{x+2}{x+3} \qquad \lim_{x\to -3} \frac{x+2}{(x+3)^2}$$

If nothing else, we can compute a few values of, say, the function in the first two limits at values of x close to -3:

x	-3.01	-3.001	-3.0001	-2.9999	-2.999	-2.99
$\frac{x+2}{x+3}$	101	1001	10001	-9999	-999	-99

The point is that as $x \to 3^-$, it seems that values of the function in question are getting more and more positive, while as $x \to 3^+$, it seems that the values are getting more and more negative. Thus we might guess that

$$\lim_{x\to -3^-} \frac{x+2}{x+3} = \infty \quad \text{and} \quad \lim_{x\to -3^+} \frac{x+2}{x+3} = -\infty$$

Now, we verify this more precisely as follows. The key is determine the behavior of the different portions of $\frac{x+2}{x+3}$ for the values of x being considered. For instance, the numerator approaches -1 as x approaches -3 from the left. The denominator, however, approaches 0. Thus the fraction itself starts to behave more and more like an expression of the form

$$\frac{-1}{\text{number getting close to zero}}.$$

This is precisely the type of behavior which would indicate some type of infinite or negative infinite limit, since a fraction with a nonzero numerator but denominators getting smaller and smaller will itself be getting larger and larger in either the positive or negative directions. This is contrast to some of the types of limits we saw before, where having denominator and numerator which are *both* approaching 0 requires more care since no general conclusion can be made about fractions behaving like

$$\frac{\text{number getting close to 0}}{\text{another number getting close to 0}}.$$

The difference now is that only the *denominator* is approaching 0.

So, the first two limits we are considering should be some type of infinity. But now we can say more: in the first case, where x approaches -3 from the left, the denominator $x+3$ itself is always negative, so our fraction is behaving like a fraction which looks like

$$\frac{-1}{\text{negative number getting close to zero}}.$$

But this fraction is itself positive since numerator and denominator are both negative, so this fraction should be getting larger and larger in the *positive* direction:

$$\lim_{x\to -3^-} \frac{x+2}{x+3} = \infty$$

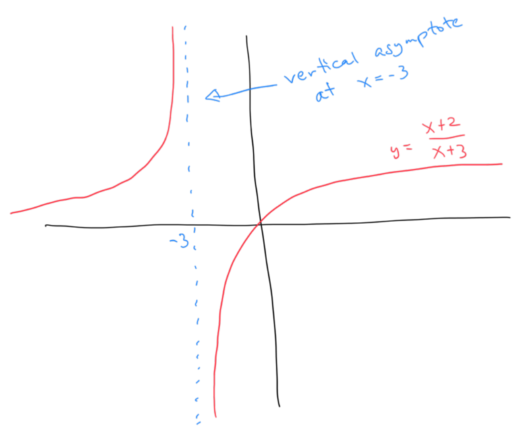
Instead, if we approach -3 from the right, $x+3$ is always positive so our limit behaves like

$$\frac{-1}{\text{positive number getting close to zero}}.$$

Such a fraction is negative, so in this case we get fractions getting more and more negative:

$$\lim_{x\to -3^+} \frac{x+2}{x+3} = -\infty.$$

The function $f(x) = \frac{x+2}{x+3}$ should thus have a vertical asymptote at $x = -3$, with the graph approaching this line in the positive direction from the left and the negative direction from the right:



The point is that a vertical asymptote the graph should be getting more and more vertical itself, but never actually exactly vertical.

Now, for

$$\lim_{x \to -3} \frac{x+2}{(x+3)^2},$$

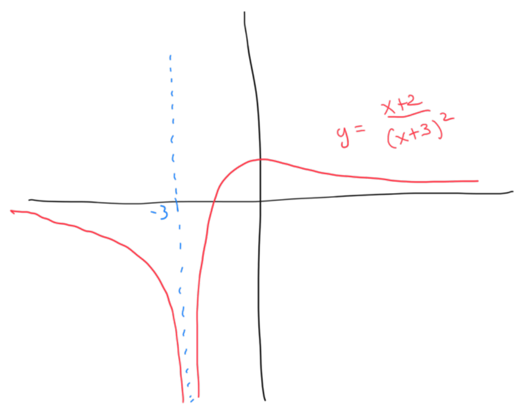
in this case it does not matter how we approach -3 since the denominator is always positive due to the squaring effect. Certainly, $x+3$ itself can be negative or positive depending on whether x is to the left or right of 3, but $(x+3)^2$ is always positive. Thus this limit is one which behaves like

$$\frac{-1}{\text{positive number getting close to zero}}$$

for $x \to -3^-$ and $x \to -3^+$, so

$$\lim_{x \to -3} \frac{x+2}{(x+3)^2} = -\infty.$$

A picture of the graph of $y = \frac{x+2}{(x+3)^2}$ verifies this vertical asymptote $x = -3$ as well, where now from either side the graph approaches the vertical line $x = -3$ in the negative direction:



Horizontal asymptotes. Going back to $f(x) = \frac{1}{x}$, we can ask now what happens to $f(x)$ as the values of x themselves get larger and larger, say more and more positive. In this case, the fraction $\frac{1}{x}$ gets smaller and smaller, and in fact gets closer and closer to 0. We express this as giving the value of following limit:

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

Thus, when we take a limit where x *goes to* ∞, we are wanting to know what value the function approaches as x gets more and more positive. Now, to be clear, the following expression is NONSENSE:

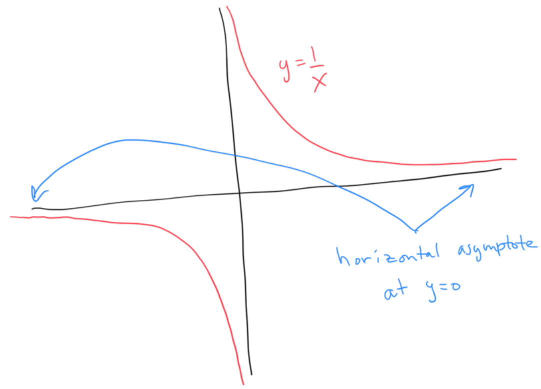
$$\text{NONSENSE}: \lim_{x \to \infty} \frac{1}{x} = \frac{1}{\infty} = 0.$$

It makes no sense to "plug in ∞" since ∞ is a not a number, merely a symbol we use to indicate a certain type of behavior. The reason why this limit equals 0 is NOT because "1 divided by ∞ is zero", but rather because the values of $\frac{1}{x}$ get closer and closer to 0 as x gets larger and larger.

Similarly, we can consider x getting more and more negative:

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

since again the fraction $\frac{1}{x}$ (which is negative in this case) is getting smaller and smaller as x gets more and more negative. Graphically, the values of these two limits is reflected in the fact that the graph of $y = \frac{1}{x}$ approaches the horizontal line $y = 0$ both in the "positive infinite" direction and the "negative infinite" direction:



We call the line $y = 0$ a *horizontal asymptote* of the function $f(x) = \frac{1}{x}$, and in general horizontal asymptotes arise by considering limits where $x \to \infty$ or $x \to -\infty$.

Example. We compute the following limit:

$$\lim_{x\to\infty} \frac{3x^3 - 2x + 2}{4x^3 - x^2}.$$

First, here is nonsensical answer, using the idea that the numerator and denominator individually each get larger and larger:

$$\text{NONSENSE} : \lim_{x\to\infty} \frac{3x^3 - 2x = 2}{4x^3 - x^2} = \frac{\infty}{\infty} = 1$$

Nope! It makes no sense to treat ∞ as if it were a number, so certainly "dividing" ∞ by itself makes no sense. Instead, we have to approach this computation in a different way, but finding an alternate way of rewriting the values of our function.

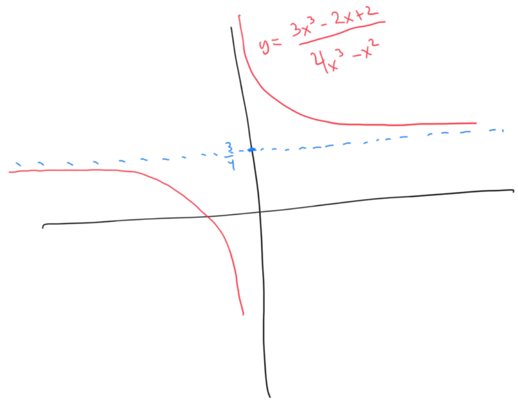
Note what happens if we multiply numerator and denominator by $\frac{1}{x^3}$:

$$\frac{(3x^3 - 2x + 2)}{(4x^3 - x^2)} \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \frac{\frac{3x^3}{x^3} - \frac{2x}{x^3} + \frac{2}{x^3}}{\frac{4x^3}{x^3} - \frac{x^2}{x^3}} = \frac{3 - \frac{2}{x^2} + \frac{2}{x^3}}{4 - \frac{1}{x}}.$$

This doesn't change the value of the limit we want at all since we are just rewriting our function in an alternate way, and the point is that in this new expression it is more straightforward to analyze each piece one-at-a-time: the $\frac{2}{x^2}$, $\frac{2}{x^3}$, and $\frac{1}{x}$ terms all each approach 0 since in each case we have a fraction with a nonzero numerator and a denominator getting larger and larger, so:

$$\lim_{x\to\infty} \frac{3x^3 - 2x + 2}{4x^3 - x^2} = \lim_{x\to\infty} \frac{3 - \frac{2}{x^2} + \frac{2}{x^3}}{4 - \frac{1}{x}} = \frac{3 - 0 + 0}{4 - 0} = \frac{3}{4}.$$

Thus, the function here has a horizontal asymptote at $y = \frac{3}{4}$, a horizontal line which the graph approaches as x gets more and more positive:



Actually, we get the same thing look at $x \to -\infty$ in this case, since even for x getting more and more negative, the fractions we get with powers of x in the denominator still get closer and closer to 0, so

$$\lim_{x\to-\infty} \frac{3x^3-2x+2}{4x^3-x^2} = \lim_{x\to\infty} \frac{3-\frac{2}{x^2}+\frac{2}{x^3}}{4-\frac{1}{x}} = \frac{3-0+0}{4-0} = \frac{3}{4}.$$

Thus the graph of our function also approaches the horizontal asymptote at $y = \frac{3}{4}$ in the negative direction as well, as drawn above.

Final observation. Notice what would happen if for instance we only had a $3x^2$ terms in the numerator in the example above as opposed to $3x^3$. Dividing by x^3 in this case gives the following computation:

$$\lim_{x\to\infty} \frac{3x^2-2x+2}{4x^3-x^2} = \lim_{x\to\infty} \frac{\frac{3}{x}-\frac{2}{x^2}+\frac{2}{x^3}}{4-\frac{1}{x}} = \frac{0-0+0}{4-0} = 0,$$

so here this function has a horizontal asymptote at $y = 0$. The difference here is that each term in the numerator ends up approaching 0, while the denominator does not. This is a general pattern: for fractions of polynomials, when considering a limit where $x \to \pm\infty$, if the larger power of x on top is smaller than that below, the limit is zero; when the largest powers are the same, the limit is some nonzero number, and when the power on top is larger than below, the limit will be $\pm\infty$. The strategy for computing these types of limits is to divide by the *largest* power of x which occurs.

Lecture 13: Exponential Functions

Warm-Up 1. We determine the vertical and horizontal asymptotes of the function

$$f(x) = \frac{3x^2+4x-2}{x^2-2x+1}.$$

Vertical asymptotes arise when the values of the function get more and more positive or more and more negative, which for a fraction happens when the denominator approaches 0 but the numerator does not. In this case the denominator can be written as $(x-1)^2$, so it is zero when $x = 1$. Since the numerator is not 0 at $x = 1$, this should give a vertical asymptote.

To verify this we consider the following limit:

$$\lim_{x\to1} \frac{3x^2+4x-2}{x^2-2x+1} = \lim_{x\to1} \frac{3x^2+4x-2}{(x-1)^2}.$$

The denominator approaches 0 along *positive* numbers, which $(x-1)^2$ is never negative. The numerator approaches 5, so the fraction overall behaves more and more like the expression

$$\frac{5}{\text{positive numbers getting closer to 0}}.$$

Such expressions get more and more positive without restriction, so

$$\lim_{x\to 1} \frac{3x^2+4x-2}{x^2-2x+1} = \infty$$

and $f(x)$ indeed has a vertical asymptote at $x=1$.

In this case we did not have to consider $\lim_{x\to 1^-}$ and $\lim_{x\to 1^+}$ separately since either way the denominator $(x-1)^2$ only gives positive numbers getting closer and closer to zero. If we instead were considering the function

$$g(x) = \frac{3x^2+4x-2}{x-1},$$

then $x-1$ would give positive numbers approaching zero when $x \to 1^+$, but negative numbers approaching zero when $x \to 1^-$. In this case the limit as $x \to 1^-$ would be $-\infty$.

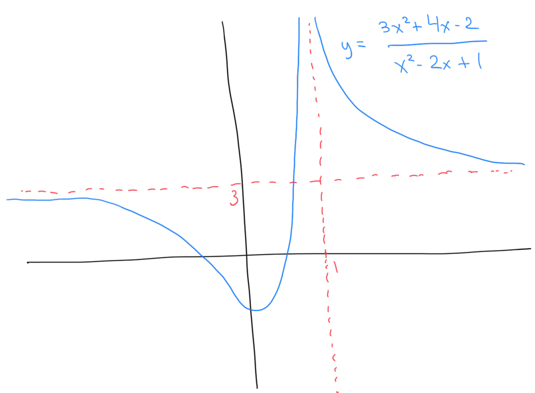
To determine horizontal asymptotes we consider the limits as $x \to \infty$ and $x \to -\infty$. To compute either one we must rewrite the given function in an alternate way since the limit of the numerator and denominator separately are each ∞. We can do by dividing numerator and denominator each by the largest power of x which occurs. So:

$$\frac{3x^2+4x-2}{x^2-2x+1}\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right) = \frac{3+\frac{4}{x}-\frac{2}{x^2}}{1-\frac{2}{x}+\frac{1}{x^2}}.$$

As $x\to\infty$, or $x \to -\infty$, all the terms with a power of x in the denominator approach 0, so we get both:

$$\lim_{x\to\infty} \frac{3x^2+4x-2}{x^2-2x+1} = \lim_{x\to\infty} \frac{3+\frac{4}{x}-\frac{2}{x^2}}{1-\frac{2}{x}+\frac{1}{x^2}} = \frac{3}{1} = 3 \quad \text{and} \quad \lim_{x\to-\infty} \frac{3+\frac{4}{x}-\frac{2}{x^2}}{1-\frac{2}{x}+\frac{1}{x^2}} = 3.$$

Thus the given function has a horizontal asymptote at $y=3$, which the graph of f approaches in both the positive and negative directions. The graph of f ends up looking like



where we can see both the vertical and horizontal asymptotes we found.

Warm-Up 2. We compute the following limit:

$$\lim_{x\to\infty}(\sqrt{9x^2+x}-3x).$$

We cannot simply take the limit of each term separately since both terms approach infinity, so we must rewrite this expression. Thinking back to previous limits we've looked at, we can do this by multiplying numerator and denominator by the same expression only with $+3x$ instead of $-3x$:

$$(\sqrt{9x^2+x}-3x)\frac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x}=\frac{(9x^2+x)-9x^2}{\sqrt{9x^2+x}+3x}=\frac{x}{\sqrt{9x^2+x}+3x}.$$

So, so far we have

$$\lim_{x\to\infty}(\sqrt{9x^2+x}-3x)=\lim_{x\to\infty}\frac{x}{\sqrt{9x^2+x}+3x}.$$

Now, to compute this limit we want to again try the idea of dividing everything through by the largest power of x, but first we have to deal with the terms in the square root. We can rewrite this square root as

$$\sqrt{9x^2+x}=\sqrt{x^2\left(9+\frac{1}{x^2}\right)}$$

with the goal of taking out an x term. But we should be careful: the value of $\sqrt{x^2}$ depends on whether x is positive or negative, since $\sqrt{x^2}=x$ only works if x is not negative. This is fine in our case since we are taking $x\to\infty$ and not $x\to-\infty$. Thus we get

$$\sqrt{9x^2+x}=\sqrt{x^2\left(9+\frac{1}{x^2}\right)}=x\sqrt{9+\frac{1}{x^2}}\text{ for }x>0.$$

Thus we can rewrite our limit once more as

$$\lim_{x\to\infty}\frac{x}{x\sqrt{9+\frac{1}{x^2}}+3x}.$$

Now we can divide everything through by the largest power of x, which is x, to get:

$$\lim_{x\to\infty}\frac{x}{x\sqrt{9+\frac{1}{x^2}}+3x}=\lim_{x\to\infty}\frac{1}{\sqrt{9+\frac{1}{x^2}}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{6}.$$

Hence $f(x)=\sqrt{9x^2+x}-3x$ has a horizontal asypmtote at $y=\frac{1}{6}$ in the positive direction.

Exponentials. Now we work towards defining new types of functions. To start, we consider what are called *exponential* functions, which are the functions we get by taking powers a *fixed* positive number:

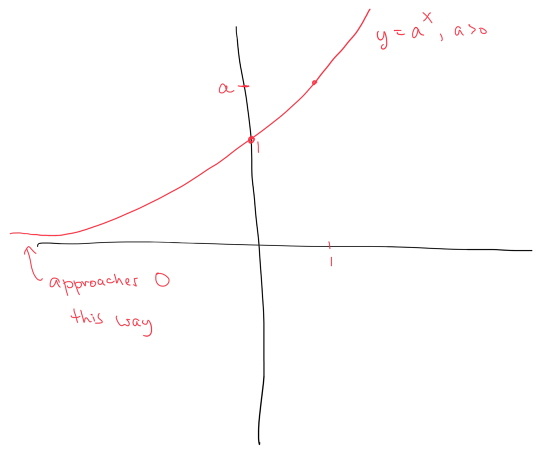
$$f(x)=a^x\text{ for some fixed number }a>0.$$

For instance, the exponential function $f(x)=2^x$ computes various powers of 2. The point here is that *exponent* is the thing which is varying and depending on the variable x, and not the number we are taking powers of x. (As opposed to something like x^2, where the exponent doesn't change but the number of which we are taking powers does.)

Exponential have various special properties which make them worthy of study, in particular the most important exponential function of them all, e^x, which we'll introduce shortly. For instance, we have identities like

$$2^x 2^{3x} = 2^{x+3x}$$

by the ways in which powers behave. As usual, we interpret a^x when x is negative as taking a reciprocal, so for instance 2^{-3} denotes $\frac{1}{2^3}$, and when x is a fraction we interpret the exponential as some type of root, such as $2^{1/2} = \sqrt{2}$. The graph of an exponential function $f(x) = a^x$ for $a > 0$ in general looks like



which makes sense: the values of a^x get more and more positive as x gets more and more positive, while the values of $a^x = \frac{1}{a^{-x}}$ get closer and closer to zero as x gets more and more negative, or equivalently as $-x$ gets more and more positive so that a^{-x} in the denominator gets more and more positive as well.

Example. We compute the following limit

$$\lim_{x\to\infty} (3 \cdot 4^{-x} + 4).$$

The key is that as $x \to \infty$, $4^{-x} = \frac{1}{4^x}$ approaches 0 since the denominator gets larger and larger. Thus

$$\lim_{x\to\infty} (3 \cdot 4^{-x} + 4) = 3 \cdot 0 + 4 = 4.$$

So, the function $f(x) = 3 \cdot 4^{-x} + 4$ has a horizontal asymptote at $y = 4$ in the positive direction.

If instead we look at $x \to -\infty$, then $-x$ actually gets more and more positive, so 4^{-x} in this case gets larger and larger. Thus

$$\lim_{x\to-\infty} (3 \cdot 4^{-x} + 4) = \infty,$$

so $f(x) = 3 \cdot 4^{-x} + 4$ does not have a horizontal asymptote in the *negative* direction.

The number e. Among all exponential functions, the one we care about most is

$$f(x) = e^x$$

where e denotes a specific number called *Euler's number*. One possible definition of e is that it is the value of the following limit:

$$e = \lim_{x\to 0} (1 + x)^{1/x}.$$

As a decimal, the value of e is about $e \approx 2.71828$, but of course the decimal expansion of e actually has infinitely many terms.

Now, why should we care about the number found by computing the limit above? Why is it special? Indeed, the actual value of e doesn't matter all that much to us—what *does* matter is the fact that the function

$$f(x) = e^x$$

obtained by taking powers of this number e is (essentially) the *only* function which equals its own derivative! This, and this alone, is why e matters at all; if we tried to describe all possible functions satisfying $f'(x) = f(x)$, that is functions which equal their own derivative, we find that they are all directly related to e^x. We'll talk more about this next time.

Example. We determine the following limit:

$$\lim_{x \to 1^+} e^{\frac{6}{1-x^2}}.$$

As $x \to 1^+$, the expression $1 - x^2$ approaches 0 among *negative* numbers, since $1 - x^2 < 0$ if $x > 1$. Thus the fraction

$$\frac{6}{1-x^2} \quad \text{behaves like} \quad \frac{6}{\text{negative numbers approaching } 0},$$

so this fraction gets more and more negative, i.e. goes off to $-\infty$. Thus, in the limit we want, we are taking powers of e where the exponent gets more and more negative:

$$e^{\text{numbers getting more and more negative}}.$$

This limit should thus be 0, since e^a does approach 0 as a gets more and more negative:

$$\lim_{x \to 1^+} e^{\frac{6}{1-x^2}} = 0.$$

If we instead considered $x \to 1^-$, then $1 - x^2$ approaches 0 using *positive* numbers, so the exponent $\frac{6}{1-x^2}$ in this case goes to ∞. Thus the resulting limit

$$\lim_{x \to 1^-} e^{\frac{6}{1-x^2}} = \infty$$

is ∞ since we are raising e to powers which get more and more positive.

Final example. Finally we compute

$$\lim_{x \to \infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}.$$

Both the numerator and denominator actually themselves go to ∞ (since the e^{3x} terms get larger but the e^{-3x} terms approach 0, so in order to compute this limit we must rewrite the given expression. As in the case of a fraction of polynomials, we can do this by dividing everything through by the largest exponent which appears, which is e^{3x}:

$$\frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} \left(\frac{\frac{1}{e^{3x}}}{\frac{1}{e^{3x}}} \right) = \frac{1 - \frac{e^{-3x}}{e^{3x}}}{1 + \frac{e^{-3x}}{e^{3x}}} = \frac{1 - e^{-6x}}{1 + e^{-6x}},$$

where in the end we use some exponent properties to simplify $\frac{e^{-3x}}{e^{3x}}$:

$$\frac{e^{-3x}}{e^{3x}} = e^{-3x}e^{-3x} = e^{-3x-3x} = e^{-6x}.$$

Thus

$$\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \lim_{x\to\infty} \frac{1 - e^{-6x}}{1 + e^{-6x}} = \frac{1}{1} = 1$$

since e^{-6x} approaches 0 as $x \to \infty$ because the exponent $-6x$ gets more and more negative.

Lecture 14: Inverses and Logarithms

Warm-Up. We find the vertical and horizontal asymptotes of the function

$$f(x) = \frac{4e^{2x} - e^x + 4}{(e^x - 1)^2}.$$

First, the candidate for the vertical asymptote is $x = 0$, since this is the value which makes the denominator 0 but not the numerator. (Recall that $e^0 = 1$, or indeed $a^0 = 1$ for any $a > 0$.) To verify that this is a vertical asymptote we compute the following limit:

$$\lim_{x\to 1} \frac{4e^{2x} - e^x + 4}{(e^x - 1)^2}.$$

(We do not have to consider $x \to 1^+$ and $x \to 1^-$ separately since the denoinator $(e^x - 1)^2$ is always positive due to the squaring effect.) The denominator approaches 0 while the numerator approaches $4e^0 - e^0 + 4 = 7$, so this fraction behaves like

$$\frac{7}{\text{positive numbers approaching 0}}.$$

Hence this limit is ∞:

$$\lim_{x\to 1} \frac{4e^{2x} - e^x + 4}{(e^x - 1)^2} = \infty.$$

Since this is true whether we approach 1 from the left or right, the graph of f approaches this vertical asymptote at $x = 1$ in an upward direction both to the left and right of 1.

Now for horizontal asymptotes we first compute

$$\lim_{x\to\infty} \frac{4e^{2x} - e^x + 4}{(e^x - 1)^2} = \lim_{x\to\infty} \frac{4e^{2x} - e^x + 4}{e^{2x} - 2e^x + 1}.$$

We divide everything through by e^{2x} to get

$$\lim_{x\to\infty} \frac{4e^{2x} - e^x + 4}{e^{2x} - 2e^x + 1} \left(\frac{\frac{1}{e^{2x}}}{\frac{1}{e^{2x}}}\right) = \lim_{x\to\infty} \frac{4 - \frac{1}{e^x} + \frac{4}{e^{2x}}}{1 - \frac{2}{e^x} + \frac{1}{e^{2x}}} = \frac{4}{1} = 4$$

since all the terms with e^x or e^{2x} in the denominator approach 0. Hence f has a horizontal asymptote at $y = 4$ in the positive direction.

But we should also check for horizontal asymptotes in the negative direction by computing

$$\lim_{x\to -\infty} \frac{4e^{2x} - e^x + 4}{(e^x - 1)^2}.$$

In this case however, as x gets more and more negative, e^x gets closer and closer to 0, so the limit in this case is:

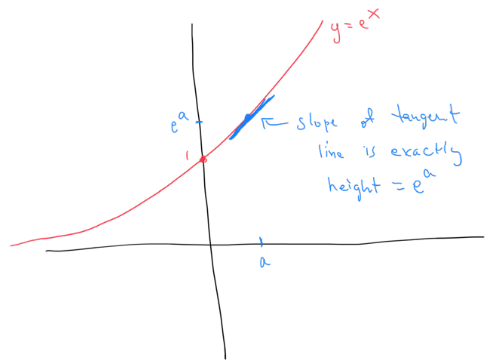
$$\lim_{x\to-\infty} \frac{4e^{2x} - e^x + 4}{(e^x - 1)^2} = \frac{4 \cdot 0 - 0 + 4}{(0-1)^2} = 4.$$

Thus the horizontal asymptote at $y = 4$ also occurs in the negative direction.

Derivative of e^x. As we mentioned last time, e is a number which is about 2.71828, but the real reason why it is so important is the following fact:

$$\frac{d}{dx}(e^x) = e^x.$$

That is, the function $f(x) = e^x$ equals its own derivative. It is not possible to overemphasize that *this* is why mathematicians are in love with the number e, and why it shows up in numerous contexts again, and again, and again. Graphically, this says that the slope at a point on the graph of $y = e^x$ is *exactly* given by the height at that point:



More generally, note that the function $g(x) = 2e^x$ also equals its own derivative, as does any multiple of e^x:

$$\frac{d}{dx}(ce^x) = ce^x.$$

It turns out that these are the *only* (!) functions which equal their own derivative: if $f'(x) = f(x)$ for all x, then $f(x)$ must be a multiple of e^x. This fact will not play a big role in this course, but is absolutely essential in many applications of calculus.

Examples. Combined with the chain rule, we can now compute some nice derivatives. For instance, say we want to compute the derivative of

$$f(x) = xe^{x^2}.$$

First the product rule gives

$$f'(x) = 1 \cdot e^{x^2} + x\frac{d}{dx}(e^{x^2}).$$

Now, the chain rule says that to differentiate e^{x^2}, we first differentiate the $e^?$ part, which gives the exact same thing since e^x equals its own derivative, and then multiply by the derivative of the exponent:

$$\frac{d}{dx}(e^{x^2}) = e^{x^2}(2x).$$

In general,

$$\frac{d}{dx}(e^{\text{something}}) = e^{\text{something}}(\text{derivative of "something"}).$$

Thus we get

$$f'(x) = e^{x^2} + x(e^{x^2}2x) = e^{x^2} + 2x^2e^{x^2}.$$

The derivative of $g(x) = e^{\sin x}$ is

$$g'(x) = e^{\sin x}\cos x.$$

To be clear the first piece $e^{\sin x}$ comes from the derivative of the "exponential", and the $\cos x$ comes from the derivative of the $\sin x$ term in the exponent.

Inverses. There is another key function related to e^x which is important, but before we introduce it we must say something about *inverse functions*. The inverse (if it exists) of a function f is the function f^{-1} which does the "opposite" of what f does. To be clear, if f sends a number x to a number y in the sense that inputting x gives y as the output, the inverse should send y to x in the sense that inputting y into the inverse should give back x. So, the inverse is characterized by the requirement that

$$f(x) = y \quad \text{means the same thing as} \quad x = f^{-1}(y).$$

Not all functions have an inverse, and we'll see in the course of the following examples how to determine those which do.

Example 1. The function $f(x) = 5 + 3x$ has an inverse, as we'll see. To find the inverse we start with

$$y = 5 + 3x,$$

and then figure out what x is *in terms of* y, since this will describe the number x we must input into f in order to obtain y as an output. We have:

$$y = 5 + 3x \iff y - 5 = 3x \iff \frac{y-5}{3} = x.$$

Thus, if f sends x to $y = 5 + 3x$, the inverse f^{-1} should send y to $\frac{y-5}{3} = x$. So the inverse is the function

$$f^{-1}(y) = \frac{y-5}{3}.$$

Since we ordinarily write functions using "x" as the variable, we will instead write that the inverse is the function

$$f^{-1}(x) = \frac{x-5}{3}.$$

The point is that starting with x, applying the function f and then applying f^{-1} to the result will give back the original x: $f^{-1}(f(x)) = x$. Similarly, applying f^{-1} first to x and then f to the result will give back x: $f(f^{-1}(x)) = x$.

Example 2. We find the inverse of $f(c) = \frac{2}{1+x}$, which we'll see does exist. Again we start with

$$y = \frac{2}{1+x}$$

and isolate x:

$$y = \frac{2}{1+x} \iff y(1+x) = 2 \iff y + yx = 2 \iff x = \frac{2-y}{y}.$$

Thus the inverse of f is

$$f^{-1}(y) = \frac{2-y}{y}, \text{ or said another way } f^{-1}(x) = \frac{2-x}{x}.$$

Note that in this case f^{-1} is only defined for $x \neq 0$, which makes sense since there is no value of x we can plug into f to obtain 0 as the result:

$$f(x) = \frac{2}{1+x} \text{ is never equal to 0, so } f^{-1}(0) \text{ is undefined.}$$

Example 3. The function $f(x) = 3x^2 - 12$ does not have an inverse. Indeed, note that

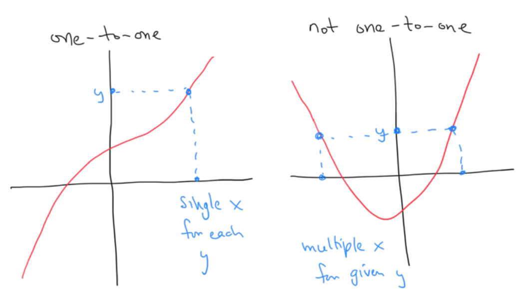
$$f(2) = 0 \quad \text{and} \quad f(-2) = 0.$$

But this means that f^{-1} would have to satisfy

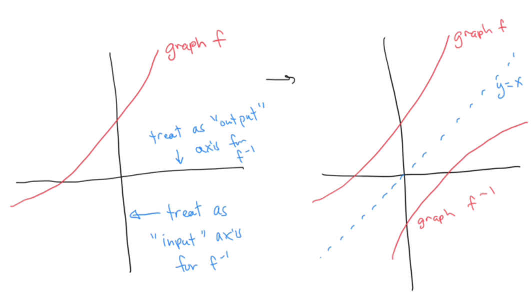
$$f^{-1}(0) = 2 \quad \text{and} \quad f^{-1}(0) = -2,$$

which is not possible if f^{-1} is indeed meant to be a *function*. So, this means that f^{-1} does not exist. We say that the function $f(x) = 3x^2 - 12$ is NOT *one-to-one* since it is possible to have different inputs give the same output. The upshot is that inverses only make senes for function which *are* one-to-one, meaning functions where we cannot have different inputs giving the same output, since it is only for these functions that in $f(x) = y$ it will be possible to solve for x *uniquely* in terms of y.

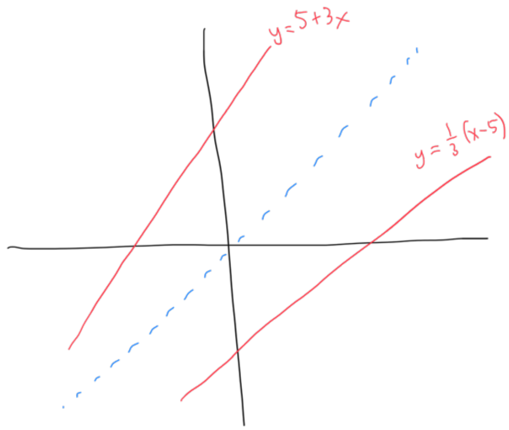
Graphs of inverses. So, inverses only exist for functions which are one-to-one. The graph of such a function can only have one x value corresponding to a given y value:



For a function which is one-to-one, the graph of the inverse is easy to describe geometrically: since $y = f(x)$ corresponds to $f^{-1}(y) = x$, the graph of f^{-1} is obtained by exchanging the roles of the x-axis and y-axis, which amounts to *reflecting* the graph of f across the diagonal line $y = x$:



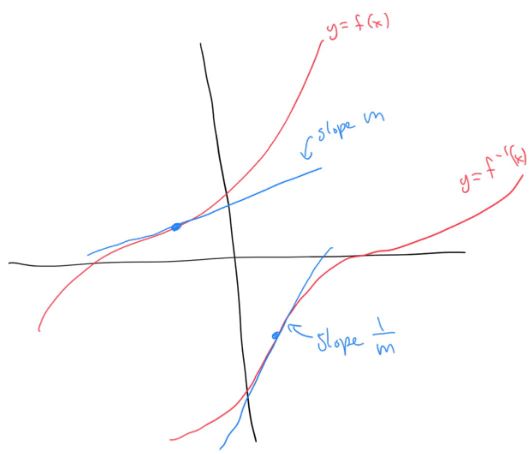
For instance, for the function $f(x) = 5 + 3x$ we looked at in Example 1, the graph of its inverse $f^{-1}(x) = \frac{1}{3}(x - 5)$ looks like:



There is also a nice relation between the derivative of f^{-1} and the derivative of f, in that the derivative of f^{-1} is the reciprocal of the derivative of f evaluated at the appropriate point:

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}.$$

We'll actually see where this comes from in the concrete examples we'll look at later, but for some geometric intuition consider the graph of f and of its inverse, and the graphs of their tangent lines at some points:



The point is that the graph of the tangent line to f^{-1} should also be obtained by reflecting the graph of the tangent line to f across the line $y = x$, which amounts to exchanging y and x in the tangent line equation. So, if a tangent line to the graph of f looks like

$$y = mx + b$$

with slope m, exchanging x and y are manipulating gives:

$$x = my + b \iff y = \frac{1}{m}x + (\text{whatever}),$$

so that the tangent line to the graph of f^{-1} has slope $\frac{1}{m}$, which is indeed the reciprocal of the slope of the tangent line to the graph of f. Again, we'll work out what the derivative of the inverse function is concretely in the main examples we'll care about.

The natural logarithm. Back to $f(x) = e^x$. This function is indeed one-to-one, and so we now know that it should have inverse. This inverse is so important that we give it a special name: the *natural logarithm* function. We denote this natural logarithm using the symbol "ln", so the inverse of e^x is $\ln x$:

$$y = e^x \quad \text{means the same thing as} \quad \ln y = x.$$

In other words, the number $\ln x$ is the exponent we have to raise e to in order to get x:

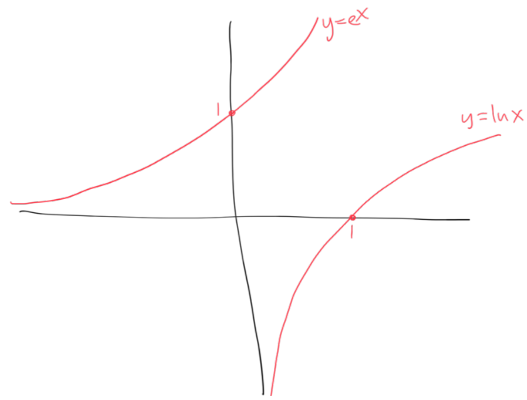
$$e^{\ln x} = x.$$

Similarly we get the identity

$$\ln(e^x) = x$$

since the exponent we have to raise e to in order to get e^x is x itself. (For instance, the exponent we should raise e to in order to get e^3 is 3, so $\ln(e^3) = 3$.) These two identities indeed say that the exponential function e^x and the natural logarithm functon $\ln x$ "undo" one another. Note that, since e^x can never be zero nor negative, its inverse $\ln x$ is only defined for $x > 0$.

Given the graph of $y = e^x$, the graph of $y = \ln x$ comes from reflecting across $y = x$:



Note that $\ln x$ has a vertical asymptote at $x = 0$ since $\lim_{x \to 0^+} \ln x = -\infty$, which mimics the limit value $\lim_{x \to -\infty} e^x = 0$ for the exponential function.

Derivative of $\ln x$. We'll talk about general derivatives involving $\ln x$ more next time, but the key fact is that:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

That is, the derivative of $\ln x$ is the function $x^{-1} = \frac{1}{x}$. Up until now actually, we have never had a function whose derivative was simply the reciprocal of x. At first glance you might think that differentiating x^k might give something like $\frac{1}{x}$ for an appropriate k, since we get others powers of x by differentiating these types of expressions; for instance, differentiating $x^{2/3}$ gives something which involves $\frac{1}{x^{1/3}}$, and differentiating x^{-4} gives something involves $\frac{1}{x^5}$, so why couldn't we find such a x^k to differentiate in order obtain something which has a single power of x in the denominator?

However, since $\frac{d}{dx}(x^k) = kx^{k-1}$, in order for the result to have x^1 in the denominator we need k itself to be 0 since then $x^{0-1} = \frac{1}{x}$ is what we're after. BUT, if we did take $k = 0$, then our original function is $x^0 = 1$, which is a constant with derivative zero! So, there is no way x^k, for any value of k, can ever result in a derivative involving $\frac{1}{x}$, which is why we need a new function to give us this type of derivative. The fact that the derivative of $\ln x$ is $\frac{1}{x}$ is crucial in Math 224 when learning about what are called *integrals*, and we'll look at some derivative computations along these lines next time. We'll also see precisely why the derivative of $\ln x$ is indeed $\frac{1}{x}$.

Lecture 15: Logarithmic Differentiation

Warm-Up 1. We find the value(s) of x satisfying each of the following equations

$$e^{2x^2-1} = 3 \quad \text{and} \quad \ln\sqrt{x-1} = 5.$$

For the first, we need to first "undo" the exponential part. (Analogous to how when solving $(x-1)^2 = 5$, we must first "undo" the squaring by taking a square root.) This is the natural log function allows us to do. Taking natural log of both sides of $e^{x^2-1} = 3$ gives

$$\ln(e^{x^2-1}) = \ln 3.$$

The left side simplifies to $x^2 - 1$ since $\ln(e^a) = a$ no matter what a is. So we get

$$x^2 - 1 = \ln 3, \text{ and thus } x^2 = 1 + \ln 3.$$

Thus $x = \pm\sqrt{1+\ln 3}$ are the values of x satisfying the first equation.

For the second we must "undo" the logarithm, which is what taking an exponential allows us to do. Raising e to the power of both sides of $\ln\sqrt{x-1} = 5$ gives

$$e^{\ln\sqrt{x-1}} = e^5.$$

The left side is just $\sqrt{x-1}$ since $e^{\ln a} = a$ for any a, so we get

$$\sqrt{x-1} = e^5, \text{ and thus } x - 1 = (e^5)^2 = e^{10}.$$

Hence $x = 1 + e^{10}$ is the only value of x satisfying the second equation.

Warm-Up 2. We determine the values of the following limits:

$$\lim_{x\to\infty}[\ln(x^2+1) - \ln(x^2+3)] \quad \text{and} \quad \lim_{x\to 2^+}\ln[(x+2)(x+3)^2].$$

In each we use some key properties of logarithms. For the first this is the fact that

$$\ln a - \ln b = \ln\frac{a}{b}.$$

That is, a difference of two natural logs can always be written as the natural log of a single fraction. You can check the book to see where this comes from, but it is the "inverse" analogue of the property

$$e^{x-y} = e^x e^{-y} = \frac{e^x}{e^y}$$

for exponentials. So, the first limit can be written as

$$\lim_{x\to\infty}[\ln(x^2+1) - \ln(x^2+3)] = \lim_{x\to\infty}\ln\left(\frac{x^2+1}{x^2+3}\right).$$

To determine this we can use the fact that the natural log function is continuous in order to bring the limit "inside". That is, we first consider

$$\lim_{x\to\infty}\frac{x^2+1}{x^2+3}=\lim_{x\to\infty}\frac{1+\frac{1}{x^2}}{1+\frac{3}{x^2}}=1.$$

Thus

$$\lim_{x\to\infty}\ln\left(\frac{x^2+1}{x^2+3}\right)=\ln\left(\lim_{x\to\infty}\frac{x^2+1}{x^2+3}\right)=\ln 1=0,$$

where $\ln 1 = 0$ since $1 = e^0$.

For the second limit we make use of the fact that

$$\ln a+\ln b=\ln(ab),$$

so a *sum* of natural logs is the natural log of a single product. This is the "inverse" analogue of the property

$$e^{x+y}=e^x e^y$$

for exponentials. So, the second limit can be written as

$$\lim_{x\to 2^+}\ln[(x+2)(x+3)^2]=\lim_{x\to 2^+}[\ln(x+2)+\ln(x+3)^2].$$

One more property we make use of now is that:

$$\ln(a^r)=r\ln a,$$

so that when taking the natural log of some "power" expression, the exponent can be brought down in front. This is the "inverse" analogue of

$$(e^x)^k=e^{kx}.$$

So we can rewrite the second limit once more as

$$\lim_{x\to 2^+}[\ln(x+2)+\ln(x+3)^2]=\lim_{x\to 2^+}[\ln(x+2)+2\ln(x+3)].$$

Now we're in business: the $\ln(x+3)$ terms approaches $\ln(2+3)=\ln 5$ as $x\to 2^+$, while the $\ln(x+2)$ term goes to $-\infty$ since $x+2$ itself approaches 0, and ln(something approaching 0) itself gets more and more negative. Thus the overall limit is

$$\lim_{x\to 2^+}[\ln(x+2)+2\ln(x+3)]=-\infty.$$

Note that we could have used the $\ln(a^r)=r\ln a$ property back in Warm-Up 1 when solving $\ln\sqrt{x-1}=5$: write the left side as $\ln\sqrt{x-1}=\ln(x-1)^{1/2}=\frac{1}{2}\ln(x-1)$, so that

$$\frac{1}{2}\ln(x-1)=5 \text{ becomes } \ln(x-1)=10.$$

Then taking exponentials of both sides and solving for x gives the same value we found before.

Warm-Up 3. Finally, we compute the following derivatives:

$$\frac{d}{dx}(\ln(x^3+2x-1)) \quad\text{and}\quad \frac{d}{dx}(\ln(\cos x)).$$

We use the fact we mentioned last time that

$$\frac{d}{dx}(\ln x) = \frac{1}{x},$$

and so according to the chain rule we have more generally:

$$\frac{d}{dx}\ln(\text{inside}) = \frac{1}{\text{inside}}(\text{derivative of inside}).$$

The $\frac{1}{\text{inside}}$ comes from the derivative of the logarithm portion, and the "derivative of inside" comes as a consequence of the chain rule. Thus:

$$\frac{d}{dx}\ln(x^3+2x-1) = \frac{1}{x^3+2x-1}(3x^2+2) = \frac{3x^2+2}{x^3+2x-1},$$

and

$$\frac{d}{dx}\ln(\cos x) = \frac{1}{\cos x}(-\sin x) = -\frac{\sin x}{\cos x} = -\tan x.$$

As a side note, if we use the same technique to compute the derivative of $\ln(3x)$, say, we get:

$$\frac{d}{dx}\ln(3x) = \frac{1}{3x}(3) = \frac{3}{3x} = \frac{1}{x},$$

which is the same as the derivative of $\ln x$ itself! But this makes sense: $\ln(3x)$ is the same as

$$\ln(3x) = \ln 3 + \ln x,$$

so its derivative should be the same as that of $\ln x$ since the derivative of the constant $\ln 3$ is 0.

Other exponentials. From what we've done so far, we can now determine the derivative of an arbitrary exponential function a^x. The point is that it is always possible to write a general exponential function concretely *in terms of* the exponential function e^x if we use some properties of e and ln:

$$a^x = e^{\ln(a^x)} = e^{x\ln a}.$$

Thus, the derivative of a^x should be:

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\ln a}) = e^{x\ln a}(\ln a)$$

where the $\ln a$ at the end comes from the derivative of the exponent $x\ln a$. Since $e^{x\ln a}$ is the same as a^x, we get

$$\frac{d}{dx}(a^x) = a^x\ln a.$$

So, the derivative of any exponential a^x is always just a multiple of that same exponential, and the case of $a=e$ is when we get the "multiple" obtained by multiplying the exponential by $\ln e = 1$.

Just as $\ln x$ was defined to be the inverse function to e^x, we use other types of logarithms to describe the inverse functions of arbitrary exponentials: the inverse function of a^x is the *logarithm base* a and is denoted by $\log_a x$. That is:

$$a^x = y \quad \text{means the same thing as} \quad x = \log_a y.$$

So, $\log_a y$ is the exponent we have to raise a to in order to get y. Again, $a = e$ just gives the natural logarithm $\log_e = \ln$ we've been talking about so far. Indeed, this is called the "natural" logarithm since it is the logarithm which tends to show up most often in practice and is the simplest to work with. As in the case of a^x being expressible in terms of e, an arbitrary logarithm is always expressible in terms of ln via:

$$\log_a x = \frac{\ln x}{\ln a}.$$

You can check the book to see where this comes from, but it is the "inverse" analogue of

$$a^x = e^{x \ln a}.$$

Based on this, we get that the derivative of $\log_a x$ is

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{x \ln a}.$$

In the $a = e$ case we have $\ln e = 1$, so we are just left with $\frac{1}{x}$ as the derivative.

It is absolutely the case that for most purposes the only logarithm we actually care about is the natural logarithm, so $\log_a x$ will not really play a big role going forward, nor in Math 224 or later courses. As we've seen, a key point is that any other logarithm can be directly related to the natural logarithm, just as any exponential can be related to e, which means it is okay to mainly (if not exclusively) focus only on e^x and $\ln x$.

Examples. The derivative of $f(x) = 5^{x^2}$ is

$$f'(x) = 5^{x^2}(\ln 5)(\text{derivative of } x^2) = 2x5^{x^2}(\ln 5).$$

The $5^{x^2}(\ln 5)$ comes from the fact that the derivative of a^x is $a^x \ln a$, and the derivative of x^2 part comes from the chain rule.

Using repeated chain rules, the derivative of $g(x) = 5^{e^{\cos x}}$ is:

$$\begin{aligned} g'(x) &= 5^{e^{\cos x}}(\ln 5)(\text{derivative of } e^{\cos x}) \\ &= 5^{e^{\cos x}}(\ln 5)e^{\cos x}(\text{derivative of } \cos x) \\ &= 5^{e^{\cos x}}(\ln 5)e^{\cos x}(-\sin x). \end{aligned}$$

Logarithmic differentiation. Suppose we want to compute the derivative of the function

$$f(x) = x^x.$$

So far none of the techniques we've looked at will work! As a first guess, you might think to use the $(x^k)' = kx^{k-1}$ property since we are taking a power of x, so you would get:

$$\text{incorrect guess: } f'(x) = x \cdot x^{k-1}$$

by "bringing" the exponent down and then subtracting one from the exponent. However, this doesn't apply here since the exponent itself varies! In other words, in $(x^k) = kx^{k-1}$, we are working with a *constant* exponent, like x^2, or x^5. Our function x^x is not of this type, so this rule does not apply. As a second guess, noticing that the exponent itself is x, we might try to the technique for differentiating exponential functions $(a^x)' = a^x \ln a$ and say that:

$$\text{incorrect guess: } f'(x) = x^x \ln x$$

since we leave the exponential as is and then multiply the natural log of the number of which we are taking a power. However, this only applies when the number of which we are taking powers is also constant, such as in e^x or 5^x, but not when this number itself varies as in our function x^x.

So, we need something new. The new idea is to manipulate our function using logarithms, and then use an implicit differentiation to find the required derivative. We start with $y = x^x$ and first take natural log of both sides:

$$\ln y = \ln(x^x).$$

The point is that the right-side now simplifies after using a key property of natural log:

$$\ln y = \ln(x^x) = x \ln x.$$

Thus the fact that we had x in both the exponent and in the number we are taking powers of which caused trouble in the attempts above is no longer an issue. This new technique we are using is called *logarithmic differentiation*, since we use logarithms to manipulate expressions on our way to finding derivatives.

From $\ln y = x \ln x$ we next use implicit differentiation and take the derivative of both sides with respect to x:

$$\frac{1}{y}(y') = \ln x + x\frac{1}{x} = \ln x + 1.$$

To be clear, the left side came from the derivative of $\ln y$, where we had to use the chain rule (recall that y is a function here), and the right side came from the product rule. So we get

$$\frac{y'}{y} = \ln x + 1.$$

Now we find isolate y':

$$y' = y(\ln x + 1),$$

and finally since $y = x^x$, we can express the right side solely in terms of x:

$$y' = x^x(\ln x + 1).$$

This is then the required derivative. The technique of logarithmic differentiation really turns out to be quite useful in situations where our "standard" techniques would not apply.

Final example. As a final example, we use logarithmic differentiation to find the derivative of

$$y = \frac{(x^2+1)\sqrt{x^3+1}}{(x^2-3x+2)^2}.$$

Of course, in this case we could certainly use the quotient rule to find y', so as opposed to the previous example here it is not this derivative cannot be found any other way, but rather that using logarithmic differentiation gives a way of finding y' which avoids the messy quotient rule. As before, we take natural log of both sides:

$$\ln y = \ln \frac{(x^2+1)\sqrt{x^3+1}}{(x^2-3x+2)^2}.$$

The point is that again we use can properties of logarithms to rewrite the right side in a simpler way:

$$\ln y = \ln[(x^2+1)\sqrt{x^3+1}] - \ln(x^2-3x+2)^2$$

$$= \ln(x^2+1) + \ln(x^3+1)^{1/2} - \ln(x^2-3x+2)^2$$
$$= \ln(x^2+1) + \frac{1}{2}\ln(x^3+1) - 2\ln(x^2-3x+2).$$

So, we are left with an expression which no longer involves square roots or messy-looking exponents. We take derivatives of both sides:

$$\frac{1}{y}y' = \frac{2x}{x^2+1} + \frac{1}{2}\left(\frac{3x^2}{x^3+1}\right) - 2\left(\frac{2x-3}{x^2-3x+2}\right),$$

and finally isolate y':

$$y' = y\left(\frac{2x}{x^2+1} + \frac{3x^2}{2(x^3+1)} - \frac{2(2x-3)}{x^2-3x+2}\right).$$

Plugging back in for y gives the required answer:

$$y' = \frac{(x^2+1)\sqrt{x^3+1}}{(x^2-3x+2)^2}\left(\frac{2x}{x^2+1} + \frac{3x^2}{2(x^3+1)} - \frac{2(2x-3)}{x^2-3x+2}\right).$$

This is what you would find using the quotient rule instead, although written this way it is a bit "cleaner" than what the quotient rule will give.

Lecture 16: Inverse Trig Functions

Warm-Up 1. We find the derivative of $f(x) = (\tan x)^{1/x}$. Since both the exponent and the number of which we are taking powers depends on x, we need to use logarithmic differentiation. Taking natural log of both sides gives:

$$\ln f(x) = \ln(\tan x)^{1/x} = \frac{1}{x}\ln(\tan x).$$

Next, taking derivatives of both sides gives:

$$\frac{1}{f(x)}f'(x) = -\frac{1}{x^2}\ln(\tan x) + \frac{1}{x}\left(\frac{1}{\tan x}\sec^2 x\right).$$

The right side came from the product rule, and in particular $\frac{1}{\tan x}\sec^2 x$ came from the derivative of the $\ln(\tan x)$ portion. Isolating $f'(x)$ gives:

$$f'(x) = f(x)\left[-\frac{1}{x^2}\ln(\tan x) + \frac{1}{x}\left(\frac{1}{\tan x}\sec^2 x\right)\right],$$

and plugging back in for $f(x)$ gives

$$f'(x) = (\tan x)^{1/x}\left[-\frac{1}{x^2}\ln(\tan x) + \frac{1}{x}\left(\frac{1}{\tan x}\sec^2 x\right)\right]$$

as the result. Note that this could be simplified a bit, say using the fact that $\tan x = \frac{\sin x}{\cos x}$ and $\sec^2 x = \frac{1}{\cos^2 x}$, but we'll leave it as it.

Warm-Up 2. Consider the curve with equation $x^y = y^x$. We find the equation for the tangent line to this curve at the point $(2,2)$. Note first that $(2,2)$ does lie on this curve since $x = 2, y = 2$ satisfies the given equation.

To find the slope of this tangent line we need to compute y'. So, we first take natural log of both sides (this is needed since both the exponent and number we are taking powers of vary), which gives:

$$y \ln x = x \ln y$$

after simplifying a bit. Taking derivatives of both sides gives:

$$y' \ln x + y\frac{1}{x} = \ln y + x\frac{1}{y}y'.$$

Grouping the y' terms on the left and other terms on the right gives

$$y' \ln x - y'\frac{x}{y} = \ln y - \frac{y}{x}, \text{ or } y'\left(\ln x - \frac{x}{y}\right) = \ln y - \frac{y}{x}.$$

Thus

$$y' = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}.$$

At the point $(x, y) = (2, 2)$, this then gives a slope of:

$$y' = \frac{\ln 2 - \frac{2}{2}}{\ln 2 - \frac{2}{2}} = 1.$$

Thus the tangent line at $(2, 2)$ is

$$y - 2 = 1(x - 2), \text{ or } y = x.$$

Back to inverses. We will soon discuss *inverse trig functions* and their derivatives, so before that let's come back and see exactly why the derivative of $\ln x$ is $\frac{1}{x}$. The starting point is the identity

$$e^{\ln x} = x$$

which comes from the fact that $\ln x$ is the inverse function of e^x. Taking derivatives of both sides gives:

$$e^{\ln x}\left(\frac{d}{dx}\ln x\right) = 1,$$

where the left side comes from the chain rule. Now, the point is that if we didn't know what $\frac{d}{dx}\ln x$ was yet, this equation gives us a way to determine what it is: dividing both sides by $e^{\ln x}$ gives

$$\frac{d}{dx}(\ln x) = \frac{1}{e^{\ln x}}, \text{ which simplifies to } \frac{d}{dx}(\ln x) = \frac{1}{x}.$$

The point is that the fact that the derivative of $\ln x$ is $\frac{1}{x}$ *is a consequence* of the fact that $\ln x$ is the inverse of e^x, by taking derivatives of both sides of $e^{\ln x} = x$ and isolating the $\frac{d}{dx}(\ln x)$ term.

Actually, we mentioned when first talking about inverses that in general there was a relation between the derivative of the inverse of a function and that of the function itself; namely, if $f(x)$ is some one-to-one function with inverse $f^{-1}(x)$, the claim was that

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

so that the derivative of f^{-1} comes from the reciprocal of the derivative of f, evaluated at the appropriate point. Previously we gave some intuition for this by considering how tangent lines are affected when taking inverses, but now we can give a bit more precise justification, using the same idea as in the computation of the derivative of $\ln x$ above. Start with the following identity:

$$f(f^{-1}(x)) = x.$$

Taking derivatives of both sides gives

$$f'(f^{-1}(x))\left(\frac{d}{dx}f^{-1}(x)\right) = 1,$$

and so after some dividing we get:

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}$$

as claimed. In the case where $f(x) = e^x$, so $f'(x) = e^x$, this formula gives

$$\frac{d}{dx}(\ln x) = \frac{1}{f'(\ln x)} = \frac{1}{e^{\ln x}},$$

which simplifies to $\frac{1}{x}$. This same idea is the one we'll use to compute derivatives of the inverse trig functions, which we'll work out soon enough.

Inverse trigonometric functions. The *inverse trig functions* $\arcsin x, \arccos x$, and $\arctan x$ (pronounced "arc" sine, cosine, and tangent) are the inverse functions to $\sin x, \cos x$, and $\tan x$ respectively. Alternatively, the notations

$$\sin^{-1} x = \arcsin x,\ \cos^{-1} x = \arccos x,\ \tan^{-1} x = \arctan x$$

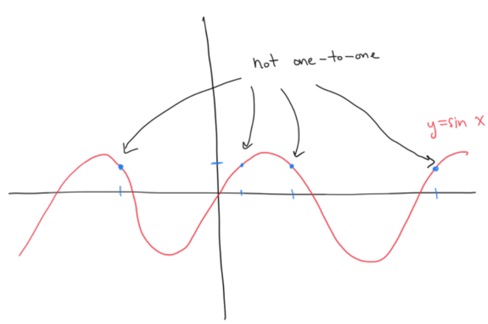
are also commonly used, which reflect the standard f^{-1} notation for inverses. So for instance,

$$\sin^{-1}\left(\tfrac{1}{2}\right) = \tfrac{\pi}{6},\ \text{since } \tfrac{1}{2} = \sin\tfrac{\pi}{6}.$$

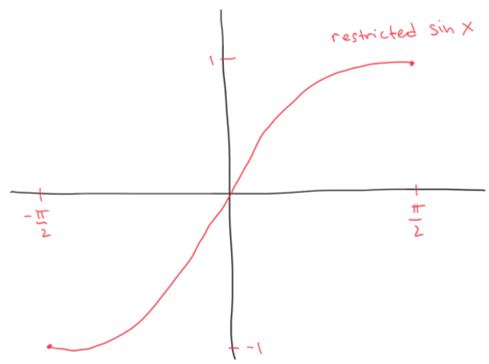
In other words, $\sin^{-1} x$ is meant to be the *angle* whose sine value is x, and similarly for other inverse trig functions.

However, we have to be careful about what we are actually doing here when discussing such inverse functions, since $\sin x, \cos x$, and $\tan x$ are actually NOT one-to-one! A function which is not one-to-one does not have inverse, so what exactly do we mean in this case? The way we get around this is to only consider the standard trig functions on a *restricted* domain, meaning only considering certain ranges of x values over which these functions are one-to-one. We'll go through this for $\sin^{-1} x$ and $\tan^{-1} x$, and you can check book for similar discussions about $\cos^{-1} x$, and other inverse trig functions like $\csc^{-1} x, \sec^{-1} x$, and $\cot^{-1} x$. (Really, arcsin, arccos, and arctan are the only ones we'll care about.)

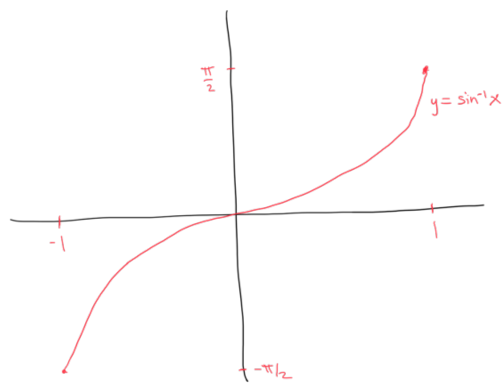
Arcsine. As we said above, $f(x) = \sin x$ is not one-to-one:



So what we do is to only consider this function defined on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$:



This "restricted" function *is* now one-to-one, and so we can talk about its inverse. By definition, $\arcsin x = \sin^{-1} x$ is the inverse of *this* restricted sine function:



So, the domain of $\sin^{-1} x$ is the interval $[-1, 1]$, and the range of $\sin^{-1} x$ (i.e. the range of possible outputs) is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. That is, the result of $\sin^{-1} x$ for some $-1 \leq x \leq 1$ is to actually give the angle *between* $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$ whose sine value is x.

For instance, let us determine the value of

$$\sin^{-1}(\sin(-\tfrac{3\pi}{2})).$$

First, $\sin(-\frac{3\pi}{2}) = 1$ so

$$\sin^{-1}(\sin(-\tfrac{3\pi}{2})) = \sin^{-1}(1).$$

But since $\sin^{-1})(1)$ should be the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ which has sine value 1, we get that

$$\sin^{-1}(\sin(-\tfrac{3\pi}{2})) = \sin^{-1}(1) = \tfrac{\pi}{2}.$$

That is, $\sin^{-1}(\sin(-\frac{3\pi}{2}))$ is not $-\frac{3\pi}{2}$, as a consequence of how we actually define $\sin^{-1} x$. In other words, the equality

$$\sin^{-1}(\sin x) = x$$

is only correct for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. This won't be a big deal, but is something to keep in mind.

Derivative of arcsine. Now we find the derivative of $\arcsin x$, using the same idea we used to find the derivative of $\ln x$. Start with the equality:

$$\sin(\sin^{-1} x) = x.$$

Differentiating both sides, using a chain rule on the left, gives:

$$\cos(\sin^{-1} x)\left(\frac{d}{dx}\sin^{-1} x\right) = 1,$$

so isolating the derivative we're trying to find gives:

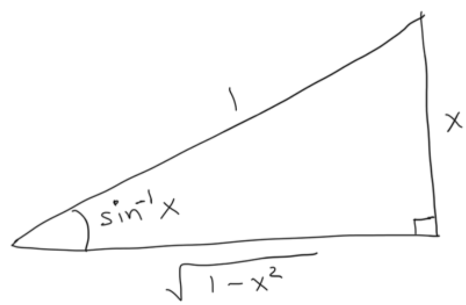
$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\cos(\sin^{-1} x)}.$$

Notice that this is precisely what we would expect from the general expression for the derivative of an inverse:

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}$$

in the case where $f(x) = \sin x$.

Now, the expression we found above is correct, but you will almost never see it written that way in any other source, including our book. The point is that we can find a much better and simpler way to represent the $\cos(\sin^{-1} x)$ term. Consider the following right triangle:



The angle we're considering is $\sin^{-1}(x)$, and we are looking for its cosine value. But this angle has sine value x by definition, and since sine is "opposite over hypotenuse", we can label the opposite side x and hypotenuse 1 since these will give the correct sine value of

$$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{1} = x.$$

Then, the length of the missing side can be found using the Pythagorean Theorem: if we call this missing side length "?" for now, we have

$$?^2 + x^2 = 1^2, \text{ so } ? = \sqrt{1 - x^2}.$$

Thus, the cosine of the angle in question, which is "adjacent over hypotenuse", is $\sqrt{1-x^2}$, so

$$\cos(\sin^{-1} x) = \sqrt{1 - x^2}.$$

Hence the expression we found for the derivative of $\sin^{-1} x$ becomes:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1 - x^2}}.$$

This is the expression you will actually see for this derivative anywhere you look.

Example. We find the derivative of the function

$$f(x) = \sin^{-1}(3x^2 + 2).$$

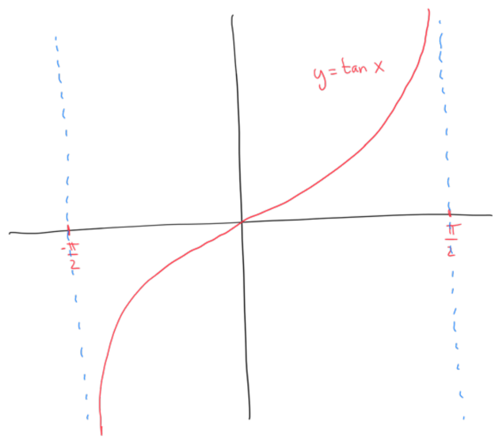
The chain rule gives

$$f'(x) = \frac{1}{\sqrt{1-(3x^2+2)^2}}(\text{derivative of } 3x^2+2) = \frac{6x}{\sqrt{1-(3x^2+2)^2}}.$$

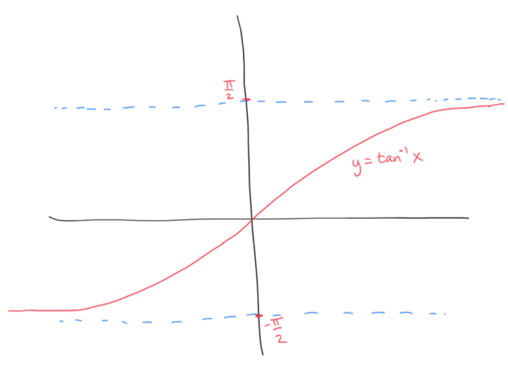
To be clear, the chain rule we are using is:

$$\frac{d}{dx}(\sin^{-1}(\text{inside})) = \frac{1}{\sqrt{1-(\text{inside})^2}}(\text{derivative of inside}).$$

Arctangent. The tangent function is also not one-to-one, so to get an inverse we restrict its domain to be $(-\frac{\pi}{2}, \frac{\pi}{2})$:



Note that $\tan x$ is undefined at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$, and that $\tan x$ has vertical asymptotes at these values. The *arctangent*, or, inverse tangent function is the inverse of this restricted tangent function:



Note two key properties of $\arctan x = \tan^{-1} x$ are:

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2} \quad \text{and} \quad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2},$$

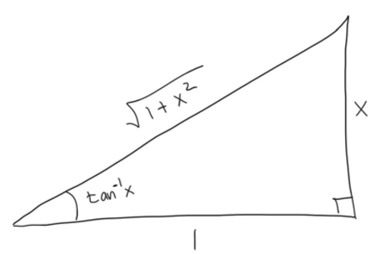
so $\tan^{-1} x$ has horizontal asymptotes at $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$.

Example. The limit
$$\lim_{x\to\infty} \tan^{-1}(e^x)$$
has value $\frac{\pi}{2}$. Indeed, as $x \to \infty$, the e^x term gets larger and larger, and arctangent of something getting larger and larger itself gets closer and closer to $\frac{\pi}{2}$, due to the horizontal asymptote of $\tan^{-1} x$ at $y = \frac{\pi}{2}$ in the positive direction.

On the other hand, the limit
$$\lim_{x\to-\infty} \tan^{-1}(e^x)$$
has value 0, since e^x approaches 0 as $x \to -\infty$, and $\tan^{-1}(0) = 0$.

Derivative of arctangent. Start with
$$\tan(\tan^{-1} x) = x.$$
Taking derivatives of both sides gives:
$$\sec^2(\tan^{-1} x)\left(\frac{d}{dx}(\tan^{-1} x)\right) = 1,$$
so
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{\sec^2(\tan^{-1} x)}.$$
This is also what you get when using the general "derivative of an inverse" formula for $f(x) = \tan x$.

But, as in the case of $\arcsin x$, there is a simpler way of writing this result. Consider the right triangle:



We are looking at the angle $\tan^{-1} x$, whose tangent value is x. Tangent is "opposite over adjacent", so we label the opposite side x and adjacent side 1 so that "opposite over adjacent" is indeed $\frac{x}{1} = x$. Then the hypotenuse is
$$\sqrt{1^2 + x^2} = \sqrt{1 + x^2}.$$
Now, secant of the angle in question is "hypotenuse over adjacent", so we have
$$\sec(\tan^{-1}) = \frac{\sqrt{1+x^2}}{1} = \sqrt{1+x^2}.$$
Thus $\sec^2(\tan^{-1} x) = 1 + x^2$, so the derivative of $\tan^{-1} x$ becomes:
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{\sec^2(\tan^{-1} x)} = \frac{1}{1+x^2}.$$

Example. The derivative of
$$f(x) = x^2 \tan^{-1}(e^x)$$

is

$$f'(x) = 2x\tan^{-1}(e^x) + x^2\left(\frac{1}{1+(e^x)^2}\right)(e^x).$$

To be clear, we used the product rule and then the fact that (via the chain rule):

$$\frac{d}{dx}(\tan^{-1}(\text{inside})) = \frac{1}{1+(\text{inside})^2}(\text{derivative of inside}).$$

Other inverse trig functions. The arccosine function $\arccos x = \cos^{-1} x$ is defined in a similar way to $\sin^{-1} x$, by first restricting the domain and then taking the inverse. The derivative of $\cos^{-1} x$, computed in a similar manner to the derivatives above, ends up being

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}.$$

This is similar to the derivative of $\sin^{-1} x$, only that the extra negative comes from the fact that the derivative of $\cos x$ is $-\sin x$, and not simply $\sin x$.

You can check the book for more details on $\cos^{-1} x$, and also for the inverse functions for secant, cosecant, and cotangent: $\sec^{-1} x, \csc^{-1} x, \cot^{-1} x$. You can also see the book for the derivatives of these final three, but honestly $\sin^{-1} x, \cos^{-1} x$, and $\tan^{-1} x$ are the only ones we'll care about.

Lecture 17: L'Hospital's Rule

Warm-Up 1. We compute the following limit:

$$\lim_{x\to\infty} \tan^{-1}\left(\frac{-3^{2x}+1}{3^x+2}\right).$$

Since arctangent is continuous, we can exchange the limit operation and the arctangent function, so we start by computing the limit of the expression inside:

$$\lim_{x\to\infty} \frac{-3^{2x}+1}{3^x+2}.$$

For this we can divide everything through by the largest exponent which appears, which is 3^{2x}:

$$\frac{-3^{2x}+1}{3^x+2}\left(\frac{\frac{1}{3^{2x}}}{\frac{1}{3^{2x}}}\right) = \frac{-1+\frac{1}{3^{2x}}}{\frac{1}{3^x}+\frac{2}{e^{2x}}}.$$

Thus

$$\lim_{x\to\infty} \frac{-3^{2x}+1}{3^x+2} = \lim_{x\to\infty} \frac{-1+\frac{1}{3^{2x}}}{\frac{1}{3^x}+\frac{2}{e^{2x}}}.$$

The numerator here approaches -1 while the denominator approaches 0, so the entire fraction goes to $-\infty$:

$$\lim_{x\to\infty} \frac{-3^{2x}+2}{3^x+2} = -\infty.$$

Hence in the original limit, $\tan^{-1}$ is being evaluated at values which get more and more negative, and so $\tan^{-1}$ itself should be approaching $-\frac{\pi}{2}$. Thus

$$\lim_{x\to\infty} \tan^{-1}\left(\frac{-3^{2x}+1}{3^x+2}\right) = -\frac{\pi}{2}.$$

Warm-Up 2. Consider the curve with equation

$$\cos^{-1}(xy) = 1 + x^2 y.$$

We find y' using implicit differentiation. Recalling that

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}},$$

differentiating both sides of our curve equation gives:

$$-\frac{1}{\sqrt{1-(xy)^2}}(y + xy') = 0 + 2xy + x^2 y'.$$

To be clear, the left side used the chain rule, and in particular the $y + xy'$ comes from the derivative of xy via the product rule. Grouping the y' terms on one side gives:

$$-\frac{xy'}{\sqrt{1-x^2y^2}} - x^2 y' = 2xy + \frac{y}{\sqrt{1-x^2y^2}},$$

and isolating y' thus gives:

$$y' = \frac{2xy + \frac{y}{\sqrt{1-x^2y^2}}}{-\frac{x}{\sqrt{1-x^2y^2}} - x^2}.$$

Indeterminate forms. We have seen many techniques for computing limits thus far, where the most elaborate ones (i.e. cases where we can't can't simply "plug things in") usually come from situations where we have expressions like:

$$\frac{\text{numerator getting larger and larger}}{\text{denominator getting larger and larger}}, \text{ or } \frac{\text{numerator getting closer to 0}}{\text{denominator getting closer to 0}}.$$

Such expressions (and others we'll see) are called *indeterminate forms*. As a shorthand notation, in the first case we say we have an indeterminate form of the type "$\frac{\infty}{\infty}$", and in the second an indeterminate form of the type "$\frac{0}{0}$". To be clear, these are purely notations we use to indicate a certain type of behavior—we are NOT saying that the values of these limits is literally $\frac{\infty}{\infty}$, or $\frac{0}{0}$.

But, although we have seen ways of handling such indeterminate forms already, not all such forms are susceptible to the techniques we have thus far. For instance, consider

$$\lim_{x\to 0} \frac{\sin 4x}{\tan 5x}.$$

Up until now we would attempt to rewrite the given function in an alternate way which turned it into a form which was not "indeterminate", but we'll have no such luck in this case: there is no simple way of manipulating $\frac{\sin 4x}{\tan 5x}$ algebraically in order to get an alternate expression, since there is no direct easy way of relating $\sin 4x$ to $\tan 5x$. What we need is a new technique, called *L'Hospital's Rule*. (The first 's' in "L'Hospital" is silent!)

L'Hospital's Rule. Whenever we have a limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

which is an indeterminate form, L'Hospital's Rule says that this limit should have the same value as the one where we take the fraction whose numerator is $f'(x)$ and whose denominator is $g'(x)$:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Again, this only applies when the original limit is indeterminate. The point is that often times the resulting new limit will involve simpler expressions than the original, and so the new limit will be simpler to compute.

Going back to the limit we mentioned above:

$$\lim_{x\to 0} \frac{\sin 4x}{\tan 5x},$$

first note that this limit is indeterminate of type $\frac{0}{0}$, since both the numerator and denominator approach 0. Thus, L'Hospital's Rule says that

$$\lim_{x\to 0} \frac{\sin 4x}{\tan 5x} = \lim_{x\to 0} \frac{4\cos 4x}{5\sec^2 5x},$$

where the numerator in the new limit is the derivative of $\sin 4x$, and the denominator is the derivative of $\tan 5x$. Now, recalling that $\sec 5x = \frac{1}{\cos 5x}$, this new limit can be written as

$$\lim_{x\to 0} \frac{4\cos 4x}{5\sec^2 5x} = \lim_{x\to 0} \frac{4}{5}\cos 4x \cos^2 5x = \frac{4}{5}\cdot 1\cdot 1 = \frac{4}{5}.$$

Thus

$$\lim_{x\to 0} \frac{\sin 4x}{\tan 5x} = \frac{4}{5}$$

as well by L'Hospital's Rule.

Example 1. Consider the limit

$$\lim_{x\to\infty} \frac{-3^{2x}+1}{3^x+2}$$

we computed as part of Warm-Up 1. We saw there how to compute this using an alternate expression, but let us now point out that we can also do this using L'Hospital's Rule. First, this limit indeterminate is the numerator goes to $-\infty$ while the denominator goes to ∞, so L'Hospital's Rule applies. Thus after taking derivative of the numerator and of the denominator, we get:

$$\lim_{x\to\infty} \frac{-3^{2x}+1}{3^x+2} = \lim_{x\to\infty} \frac{-3^{2x}(\ln 3)2}{3^x \ln 3} = \lim_{x\to\infty} -2\cdot 3^x = -\infty,$$

just as we computed previously.

Example 2. We compute

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}.$$

The numerator approaches $e^0 - 1 - 0 = 1 - 1 = 0$, as does the denominator. Hence this limit is indeterminate, so L'Hospital's rule gives:

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x}.$$

But now, this new limit is also indeterminate, since the numerator and denominator both go to 0. So, what do we do? Apply L'Hospital's rule again of course! We get:

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}.$$

The upshot is that each time we get a new indeterminate limit, there's no reason why we can't apply L'Hospital's rule again (and again, and again).

Example 3. Next we compute

$$\lim_{x\to\infty} \sqrt{x}e^{-\frac{x}{2}}.$$

Now, we must be careful here: L'Hospital's Rule as stated ONLY applies to indeterminate fractions, and our given expression is not yet written as a fraction. However, in this case we can easily get a fraction by rewriting the function as

$$\sqrt{x}e^{-\frac{x}{2}} = \frac{\sqrt{x}}{e^{\frac{x}{2}}}.$$

Now we have an indeterminate fraction, since the numerator and denominator each go to ∞. Thus L'Hospital's Rule gives:

$$\lim_{x\to\infty} \sqrt{x}e^{-\frac{x}{2}} = \lim_{x\to\infty} \frac{\sqrt{x}}{e^{\frac{x}{2}}} = \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{e^{x/2}\frac{1}{2}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}e^{\frac{x}{2}}} = 0.$$

Example 4. Finally, we look at

$$\lim_{x\to 0^+} x \ln x.$$

This is an indeterminate form, we say, of the type "$0 \cdot \infty$". The point is that we have one portion getting larger and larger (or in this case more and more negative), and another getting closer and closer to zero, so there is no simple way to tell at first glance what happens to their products. (It depends on which part "wins out": whether the term getting larger does so at a faster rate than the term getting smaller, or vice versa.) So, we hope to be able to apply L'Hospital's Rule.

However, L'Hospital's Rule only applies to fractions, and as opposed to the previous example, here there is no obvious "negative exponent" we can treat put in the denominator. But, there IS a way we can write the given expression as a fraction, but noticing that the x in front can be written as 1 over its reciprocal:

$$x = \frac{1}{1/x}.$$

That is,

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}.$$

Now we're in business: the numerator and new denominator here both go to some type of infinity, so this is an indeterminate fraction and L'Hospital's Rule applies:

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0.$$

Tada! Don't be deterred by not having an expression to which L'Hospital's Rule is not immediately obviously applicable—it may take finding a good way of rewriting it to see why L'Hospital's Rule will still work out.

Lecture 18: More on L'Hospital's Rule

Warm-Up 1. We compute the following limit:

$$\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x}.$$

The numerator approaches $e^0 + e^0 - 2 = 1 + 1 - 2 = 0$ and the denominator also approaches $1 - \cos 0 = 1 - 1 = 0$. Thus L'Hospital's Rule is applicable, and we get:

$$\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x} = \lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x}.$$

We still have numerator approaching $e^0 - e^0 = 1 - 1 = 0$ and denominator approaching $\sin 0 = 0$, so we can apply L'Hospital's Rule again to get:

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x} = \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x} = \frac{e^0 + e^0}{\cos 0} = \frac{2}{1} = 2.$$

Thus after two applications of L'Hospital's Rule we see that our original limit equals 2:

$$\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x} = 2.$$

Warm-Up 2. Now we compute

$$\lim_{x\to\infty} x \tan\left(\frac{1}{x}\right).$$

The x factor goes to ∞, while the $\tan(\frac{1}{x})$ approaches 0 (since $\frac{1}{x}$ approaches 0 and $\tan 0 = 0$), so this limit is indeterminate. (Notice this would not be the case if the second portion approached a nonzero number: if we consider the limit of $f(x)g(x)$ where $f(x)$ goes to ∞ and $g(x)$ approaches something like 1, the full limit would still be ∞. We only get an indeterminate form when we have "$\infty \cdot 0''$, meaning something going to ∞ multiplied by something approaching 0.) In order to be able to apply L'Hospital's Rule, we must rewrite our function so that it becomes an indeterminate fraction; for this we rewrite the x in front as 1 divided by $\frac{1}{x}$:

$$\lim_{x\to\infty} x \tan\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\tan(\frac{1}{x})}{\frac{1}{x}}.$$

Now we have numerator and denominator both approaching 0, so we can apply L'Hospital's Rule to get:

$$\lim_{x\to\infty} \frac{\tan(\frac{1}{x})}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1+(\frac{1}{x})^2}(-\frac{1}{x^2})}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{1 + \frac{1}{x^2}} = \frac{1}{1} = 1.$$

Thus $\lim\limits_{x\to\infty} x \tan(\frac{1}{x}) = 1$ as a consequence of L'Hospital's Rule.

Example. Next we consider

$$\lim_{x\to\infty} (x - \ln x).$$

This is a new type of indeterminate form, of the type "$\infty - \infty$". As opposed to the case of an indeterminate product $\infty \cdot 0$, in this case we cannot simply rewrite one portion as 1 over the reciprocal of itself. Instead, observe that we can factor an x term out of each piece:

$$\lim_{x\to\infty} (x - \ln x) = \lim_{x\to\infty} x \left(1 - \frac{\ln x}{x}\right).$$

Before we can think about applying L'Hospital's Rule, we need to verify whether or not this new expression is indeterminate. The x in front definitely goes to ∞, so we must determine what happens to the term in parentheses. But for this, we can actually L'Hospital's Rule on the $\frac{\ln x}{x}$ portion! This fraction is definitely indeterminate since both numerator and denominator go to ∞, and L'Hospital's Rule gives:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Thus the limit of the expression in parentheses above is:

$$\lim_{x\to\infty} \left(1 - \frac{\ln x}{x}\right) = 1 - 0 = 1.$$

Hence the limit

$$\lim_{x\to\infty} x\left(1 - \frac{\ln x}{x}\right)$$

we want is not indeterminate: the first factor goes to ∞ while the second goes to 1, so the product itself goes to ∞:

$$\lim_{x\to\infty} (x - \ln x) = \lim_{x\to\infty} x\left(1 - \frac{\ln x}{x}\right) = \infty.$$

The upshot is that this is an example where we only apply L'Hospital's Rule to a portion of our entire limit, after finding an initial way in which to rewrite it.

Indeterminate powers. Consider now the limit:

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x.$$

This is indeterminate since the number $1 + \frac{1}{x}$ of which we are taking powers is approaching 1 from the right, while the exponent x gets larger and larger; the issue is that in general we expect that taking powers of smaller numbers should give smaller values, unless the power we are taking is getting larger and larger as it is here. It does not make sense to say that the limit is "1^∞" since there are two opposing behaviors at play, due to the fact that both the exponent and number being exponentiated are varying.

As written, this is not in the form to which L'Hospital's Rule is applicable. But, the key idea here is that we can use properties of exponentials and logarithms to rewrite this as follows:

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x\ln(1+\frac{1}{x})}$$

To be clear, we are using the fact that $a = e^{\ln a}$ to write $1 + \frac{1}{x}$ as an exponential:

$$\left(1 + \frac{1}{x}\right)^x = e^{\ln(1+\frac{1}{x})^x} = e^{x\ln(1+\frac{1}{x})}.$$

The benefit is that we now have an expression which no longer as x in an exponent, except for the initial $e^{x(\text{blah})}$. So, we are left computing:

$$\lim_{x\to\infty} e^{x\ln(1+\frac{1}{x})}.$$

Since the exponential function is continuous, we only need to determine the limit of the exponent:

$$\lim_{x\to\infty} e^{x\ln(1+\frac{1}{x})} = e^{\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)}.$$

This expression in the exponent is indeterminate of the type $\infty \cdot 0$, so we can apply L'Hospital's Rule after rewriting the x in front as 1 over $\frac{1}{x}$:

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}(-\frac{1}{x^2})}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1.$$

Thus our original limit is:

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x\ln(1+\frac{1}{x})} = e^{\lim_{x\to\infty} x\ln(1+\frac{1}{x})} = e^1 = e.$$

If you go back to when we first introduced e, the first definition we gave was precisely as the value of this specific limit. Of course, that was not a very enlightening definition, but at least now we can verify that this limit does in fact have the value e.

Final example. For a final indeterminate power example, consider:

$$\lim_{x\to 1^+} x^{\frac{1}{1-x}}.$$

This is indeterminate of the type $1^{-\infty}$, which is just a shorthand way of saying that we need to compute the limit another way. Using the same $a = e^{\ln a}$ idea from before, we get:

$$\lim_{x\to 1^+} x^{\frac{1}{1-x}} = \lim_{x\to 1^+} e^{\frac{1}{1-x}\ln x}.$$

(To be sure, we used $\ln(x^{\frac{1}{1-x}}) = \frac{1}{1-x}\ln x$.) The limit of the exponent is indeterminate of the type $\frac{0}{0}$, so L'Hospital's Rule gies:

$$\lim_{x\to 1^+} \frac{\ln x}{1-x} = \lim_{x\to 1^+} \frac{\frac{1}{x}}{-1} = \frac{x\to 1^+}{-}\frac{1}{x} = -1.$$

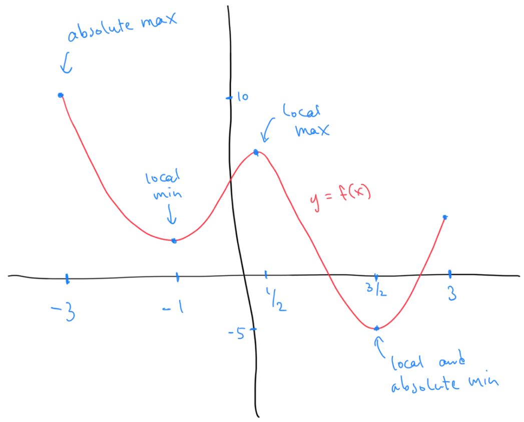
Thus

$$\lim_{x\to 1^+} x^{\frac{1}{1-x}} = \lim_{x\to 1^+} e^{\frac{1}{1-x}\ln x} = e^{\lim_{x\to 1^+} \frac{1}{1-x}\ln x} = e^{-1} = \frac{1}{e}.$$

Extreme Values. At this point in class we began discussing extreme value, but in these notes I'll save this material for the next lecture to keep all the relevant material in one place.

Lecture 19: Extreme Values

Absolute and local extrema. Consider the following graph of a function $f(x)$ for $-3 \le x \le 3$:



There are various points of interest here. For instance, the value at $x = -3$, which is $f(-3) = 10$, is the largest value that f has over the entire interval $[-3, 3]$. We say that over the interval $[-3, 3]$, f has an *absolute maximum* at $x = 3$, and that its absolute maximum *value* is 10. The smallest possible value which f has over this entire interval occurs at $x = \frac{3}{2}$, so we say that f has an *absolute minimum* at $x = \frac{3}{2}$, with absolute minimum *value* being $f(\frac{3}{2}) = -5$. We call the absolute maximum and absolute minimum the *extreme values* of f.

Now, the value of f at $x = -1$ is not the smallest value is has over the entire interval, but this value *is* smaller than the values are points "nearby"; in this case we say that f has a *local minimum* at $x = -1$. Note that f also has a local minimum at $x = \frac{3}{2}$ since its value here is indeed smaller than the values are points nearby—it just so happens that this local minimum at $x = \frac{3}{2}$ is also the absolute minimum as described above. The value of f at $x = \frac{1}{2}$ is larger than its values at points nearby, so f has a *local maximum* at $x = \frac{1}{2}$. In general, local maximums are characterized visually as "humps" (similar to a parabola opening downward), while local minimums are characterized as "valleys" (similar to a parabola opening upward). We refer to local maxima and local minima as being *local extrema* of f.

Without the graph, how could we have found these types of points? The key observation is that at each local maximum or minimum, the tangent line is horizontal, meaning that the derivative of f at these points is zero. Thus, by finding points at which the derivative of f is zero we can hope to find the possible local maxima and minima. Note that this is not to say that a point where the derivative is zero *must* be a local extremum of f, only that local extrema are found *among* the points where the derivative is zero. But, one more thing to be careful of: in this example, the derivative of f at the absolute maximum at $x = -3$ is NOT zero! Indeed, if we imagine the graph of f continuing on upward to the left of $x = -3$, we would see that $f'(-3)$ should actually be negative. The point is that $x = -3$ is not actually a "local maximum" of f over a larger interval since its value at $x = -3$ is NOT larger than at points nearby to the left, so we should not expect that $f'(-3)$ should be zero. Points where the derivative of f is zero only include *local* extrema, but absolute extrema which occur at the endpoints of the interval in question are not necessarily among these local extrema. Also, this idea of using derivatives to find local extrema only works if these derivatives actually exist—we'll see an example soon where derivatives won't help in finding extreme values.

Thus the main takeaways are:

- if f is differentiable, then the points at which local maxima and minima occur are among those satisfying $f'(x) = 0$, but points satisfying $f'(x) = 0$ do not necessarily have to be local extrema; and

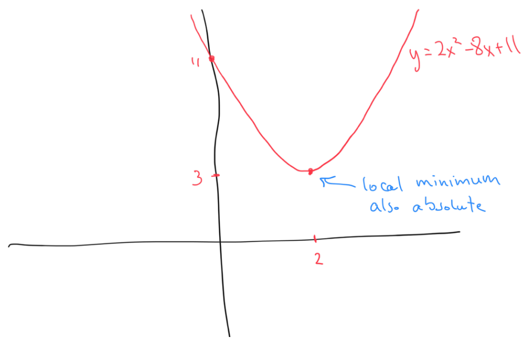
- if an absolute extremum occurs at the endpoints of the interval in question, these might not be included among the points where $f'(x) = 0$ and should be considered separately.

Example 1. Consider the function $f(x) = 2x^2 - 8x + 11$. We find the local extrema of f, which come from solving $f'(x) = 0$. In this case, $f'(x) = 4x - 8$, which is zero only at $x = 2$. Thus $x = 2$ is the only candidate point for a local extremum.

To see what actually happens at $x = 2$, we can proceed one of two ways. First, we might recognize the graph of $f(x)$ as being a parabola, which is always the case for a polynomial of degree 2. In particular, since

$$\lim_{x \to \pm\infty} (2x^2 - 8x + 11) = \infty$$

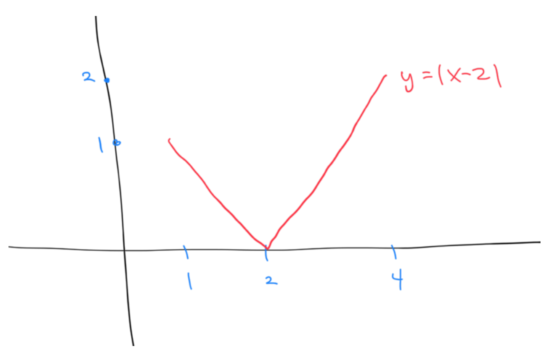
in this case, we see that this should be a parabola which opens upward:



Thus $x = 2$ is a local minimum of f, which is actually also the absolute minimum. The absolute minimum value at this point is $f(2) = 3$. In this case f has no local nor absolute maximum, at least if we consider it over the entire number line.

Alternatively, we might notice that the expression for f can be written as $f(x) = 2(x-2)^2 + 3$. Since $2(x-2)^3$ is never negative, we see that these values of f can get larger by making the $(x-2)^2$ term larger and larger, so f should not have any maximum, but that the smallest value f can have is when the $2(x-2)^2$ term is zero, which indeed happens at $x = 2$. So, $x = 2$ should be a local (and absolute over the entire number line) minimum as we said before.

Example 2. We find the absolute and local extrema of $f(x) = |x - 2|$ on the interval $[1, 4]$. Here we can simply draw the graph:



From this we see that f should both a local and absolute minimum at $x = 2$, and an absolute maximum at $x = 4$, with absolute maximum value being $f(4) = |4 - 2| = 2$.

The important observation is that, in this case, the local minimum value would not be found by finding points where the derivative of f is zero, since there are no points where this derivative is zero! The derivative of f equals -1 for $x < 2$, 1 for $x > 2$, and does NOT exist at $x = 2$ itself! This shows that we have to be careful about assumptions we're making when applying the various criteria we're developing: it is true that if a function f is differentiable over some interval, then points at which local maxima/minima occur must be among those where f' is zero, but this ONLY applies if f' actually exists! In the case of this function, we have to resort to the graph to see what the local minimum actually is.

Extreme Value Theorem. At this point we should say something about existence of absolute extrema, since if we didn't know for sure that such a thing would exist, there would be no point in searching for it. The key fact is what is known as the *Extreme Value Theorem*, which says that for a *continuous* function defined over a *closed* interval, an absolute maximum and absolute minimum always exists. We won't always explicitly point this fact out, but keep in mind that it is often hiding in the background. For instance, this is one way to know that the function $f(x) = |x-2|$ in Example 2 should indeed have an absolute minimum on $[1,4]$.

Example 3. We find the absolute extrema of $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ on $[-2,3]$. This function is differentiable everywhere since it is a polynomial, so we can use the derivative to find local extrema. Also, this function is continuous on the closed interval $[-2,3]$, so absolute extrema should exist.

First we find the candidate points for a local extremum by seeing where $f'(x)$ is zero. We have

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x-2)(x+1),$$

so $f'(x)$ is zero only for $x = -1, 0, 2$. Apart from these, it is possible that absolute extrema might also occur at the endpoints of $[-2,3]$. So second, we test the points found above and these endpoints by plugging them into the function and seeing which gives the largest and smallest values:

$$f(-1) = -4 \qquad f(0) = 1 \qquad f(2) = -31 \qquad f(-2) = 33 \qquad f(3) = 136.$$

Thus we get that the absolute minimum occurs at $x = 2$ (which is also a local minimum) with a value of -31, and the absolute maximum occurs at $x = 3$ (which is not a local maximum) with a value of 136. You might try to imagine what the graph of this function looks like based on this and the possible local maxima/minima we found, which is something we'll come back to later.

Example 4. Finally, we find the absolute maximum and minimum of $f(x) = xe^{-x}$ on $[0,2]$. This function is differentiable everywhere, so we can use the first derivative to find local extrema, and it is continuous, so absolute extrema on the closed interval $[0,2]$ do exist.

We have $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$. Since e^{-x} is never zero, this first derivative is zero only when $1 - x = 0$, so at $x = 1$. The value here is $f(1) = e^{-1}$. The value of f at the endpoints of the interval in question are $f(0) = 0$ and $f(2) = 2e^{-2}$. Thus the absolute minimum of $f(x) = xe^{-x}$ on $[0,2]$ is at $x = 0$ and the absolute maximum is at $x = 1$ since $f(2) = \frac{2}{e^2}$ is less than $f(1) = \frac{1}{e}$.

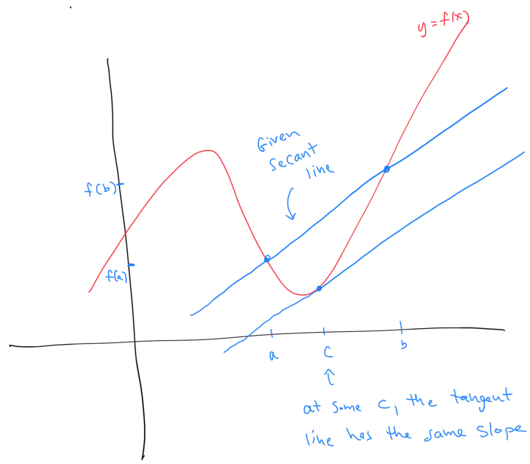
Lecture 20: Mean Value Theorem

Warm-Up. We find the absolute extrema of $f(x) = x - \ln x$ on the interval $[\frac{1}{2}, 2]$. We have $f'(x) = 1 - \frac{1}{x}$, which is zero only at $x = 1$. So $x = 1$ is one candidate point for where an absolute extrema might occur. The value here is $f(1) = 1 - \ln 1 = 1$. Next we compute:

$$f(\tfrac{1}{2}) = \tfrac{1}{2} - \ln \tfrac{1}{2} \quad \text{and} \quad f(2) = 2 - \ln 2.$$

The first value $f(\frac{1}{2})$ is about 1.19, while the second $f(2)$ is about 1.31, so f has an absolute maximum at $x = 2$ and an absolute minimum at $x = 1$.

Mean Value Theorem. We're in the midst of figuring out how to use information about derivatives in order to tell us something meaningful about functions. Using derivatives to find local extrema is the first step in this process. The *Mean Value Theorem* also falls within this topic, in that it too gives a way to derive information about a function from information about its derivative. The idea is actually simple to see visually:



Say we are given a differentiable function like that above. The slope of the *secant line* connecting the point corresponding to $x = a$ on the graph to the point corresponding to $x = b$ is

$$\frac{f(b) - f(a)}{b - a}.$$

The observation is that there seems to a point, labeled c in the picture above, between a and b at which the tangent line is *parallel* to this secant line. The *Mean Value Theorem* states that this is indeed guaranteed to be true. To be precise, the slope of the tangent line at $x = c$ is $f'(c)$, and in order for these two lines to be parallel they should have the same slope, so the Mean Value Theorem states that there is some c between a and b satisfying

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Again, all this says visually is that no matter what secant line we take, there is always a point between where the tangent line has the same slope as this secant line.

The point is that the resulting equality

$$\frac{f(b) - f(a)}{b - a} = f'(c), \text{ which can be rewritten as } f(b) - f(a) = f'(c)(b - a)$$

gives a way to relate values of f (such as the ones on the left side) to values of its derivative. The idea is that the derivative value $f'(c)$ "turns" the change $b - a$ in inputs into the corresponding change $f(b) - f(a)$ in outputs. We don't know what c will necessarily be in this equation, but regardless the power comes in using information about $f'(c)$ to derivative information about $f(b) - f(a)$.

Careful with assumptions. Just one standard thing to point out regarding the Mean Value Theorem, which is the assumption that our function actually be differentiable between the points

$x = a$ and $x = b$ we're considering. For instance, consider $f(x) = |x|$ and the points $x = -1$ and $x = 1$. If the Mean Value Theorem were applicable here, it would say that there should exist some number c between -1 and 1 satisfying

$$\frac{f(1) - f(-1)}{1 - (-1)} = f'(c), \text{ which simplifies to } 0 = f'(c).$$

But this is nonsense: there is no point c satisfying $f'(c) = 0$ for the function $f(x) = |x|$. The issue is that $f(x) = |x|$ is NOT differentiable everywhere between -1 and 1 since $f'(0)$ does not exist. So, the Mean Value Theorem does not apply in this situation.

Example 1. Consider the function $f(x) = 2x^2 - 8x + 11 = 2(x - 2)^2 + 3$ and the points $x = 1$ and $x = 4$. This function is differentiable and continuous everywhere, so the Mean Value Theorem applies and says that in particular there should exist a number c between 1 and 4 satisfying

$$\frac{f(4) - f(1)}{4 - 1} = f'(c).$$

In this case the left side works out to be $\frac{6}{3} = 2$, so the claim is that there is a number at which the derivative of f is 2. But here we can verify this directly: $f'(x) = 4x - 8$, which has the value 2 at $x = \frac{10}{4} = \frac{5}{2}$. This point is indeed between 1 and 4, so the point which the Mean Value Theorem says should exist in this case is $c = \frac{5}{2}$.

The point is that in general finding this point *explicitly* will not usually be possible (as opposed to what happened here), but nonetheless we know it will exist, which for our purposes will be good enough. This is really the key point of a result like this: knowing something exists, without knowing what it actually is.

Example 2. Suppose f is a differentiable function whose derivative satisfies $3 \leq f'(x) \leq 5$ for all x. If $f(2) = 4$, we want to know how large and how small the value of $f(8)$ could be. The underlying idea is that information about the derivative of f-namely in this case that it's values are always between 3 and 5-should give us information about the function itself, and this is indeed what the Mean Value Theorem tells us is true.

The Mean Value Theorem says that there is some number c between 3 and 5 satisfying

$$f(8) - f(2) = f'(c)(8 - 2) = 4f'(c).$$

Whatever c is, we know that $f'(c)$ has to be between 3 and 5, so

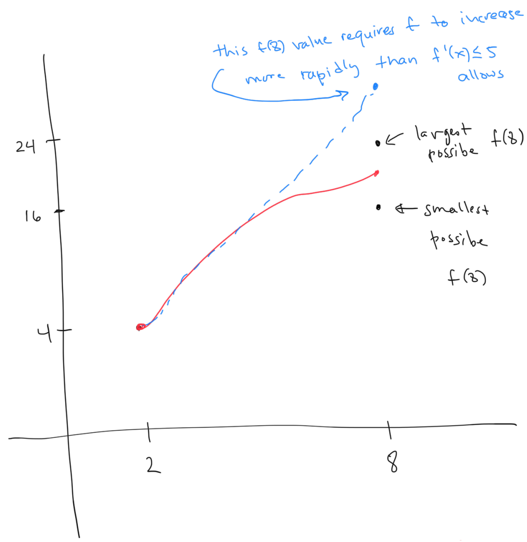
$$4 \cdot 3 \leq 4f'(c) \leq 4 \cdot 5$$

and thus $12 \leq f(8) - f(2) \leq 20$. Since $f(2) = 4$, this gives

$$12 \leq f(8) - 4 \leq 20, \text{ and hence } 16 \leq f(8) \leq 24.$$

Thus the smallest value $f(8)$ can be is 16 and the largest it can be is 24.

The underlying idea in this problem is that given the restriction $3 \leq f'(x) \leq 5$ on the derivative of f, or in other words a restriction on how rapidly f can increase between $x = 2$ and $x = 8$, there is a corresponding restriction on the extent to which the values $f(8)$ and $f(2)$ can differ from one another, in this case they can differ only by an amount between 12 and 20. The point is that if we wanted to $f(8)$ to fall outside the range 16 to 24, say beyond 24, it would have to be true that f should increase by a rate greater than 5 somewhere between $x = 2$ and $x = 8$:



Otherwise the graph of f is not increasing rapidly enough to allow it to move from the point $(2, 4)$ to a point $(8, y)$ with y larger than 24.

Example 3. We now use the Mean Value Theorem to explain why the equation

$$2x + \cos x = 0$$

has exactly one solution. Explaining why this equation has *at least* one solution is something we could have done way back when using the Intermediate Value Theorem. Indeed, the function $f(x) = 2x + \cos x$ is continuous, and since

$$f(-\tfrac{\pi}{2}) = -\pi < 0 < 1 = f(0),$$

the Intermediate Value Theorem guarantees that there is a value of x satisfying $f(x) = 0$, which is thus a value of x satisfying $2x + \cos x = 0$.

The new thing here is explaining why this equation cannot have *more* than one solution, which is where the Mean Value Theorem comes in. This is typical of these types of problems: the Intermediate Value Theorem is used to explain why there is *at least* one solution, and the Mean Value Theorem is used to explain why there is *at most* one solution. Combining these two we get that the given equation indeed has *exactly* one solution.

Suppose the given equation had two different solutions, call them $x = a$ and $x = b$. (We are trying to see why this cannot be possible.) These two values then satisfying $f(a) = 0$ and $f(b) = 0$ where $f(x) = 2x+\cos x$ is the same function we used above. This function is differentiable everywhere, so the Mean Value Theorem applies and says that there would exist a number c between a and b satisfying

$$f(b) - f(a) = f'(c)(b - a).$$

The left side here is $0 - 0 = 0$, so we get $0 = f'(c)(b - a)$, and since b and a are different, $b - a \neq 0$. Thus it must be the case that $f'(c) = 0$, so the Mean Value Theorem implies that there is some c satisfying $f'(c) = 0$. But this is nonsense: $f'(x) = 2 - \sin x$ is never zero in our case since $\sin x$ is never larger than 1. Thus the Mean Value Theorem would lead to an impossibility, so we could not have had two different solutions a, b to start with, and hence $f(x) = 0$ can only have at most one solution. Combined with the fact that this equation has at least one solution, we thus conclude that it has exactly one solution.

Example 4. Finally we explain why the equation $x^4 + 4x + c = 0$, where c is any constant, has at most two solutions. Set $f(x) = x^4 + 4x + c$, which is a differentiable function and hence the Mean Value Theorem applies. If there were three solutions to the given equation, this would give three different numbers a, b, c satisfying

$$f(a) = 0, \ f(b) = 0, \ \text{and } f(c) = 0.$$

Applying the Mean Value Theorem to a, b implies there is some number d between them satisfying

$$f(b) - f(a) = f'(d)(b - a),$$

and applying the Mean Value Theorem to b, c implies there is some number k between them satisfing

$$f(c) - f(b) = f'(k)(c - b).$$

Since $f(b) = f(a) = f(c) = 0$ and $a \neq b \neq c$, this two equations imply that

$$f'(d) = 0 \text{ and } f'(k) = 0.$$

The upshot is that if $f(x) = 0$ had three solutions, there would be two points at which f' had the value zero. But $f'(x) = 4x^3 + 4$ in our case, and this zero only at $x = -1$:

$$4x^3 + 4 = 0 \implies x^3 = -1 \implies x = -1.$$

Thus there cannot be two different points at which $f'(x)$ equals zero, so the Mean Value Theorem should not have been applicable, meaning that we could not have had three different solutions a, b, c of $f(x) = 0$ to start with. So, there are only at most two numbers satisfying $x^4 + 4x + c = 0$.

Lecture 21: Extrema and Concavity

Warm-Up 1. Is there a differentiable function f satisfing $f(0) = -1, f(2) = 4$, and $f'(x) \leq 2$ for all x? In other words, does the condition $f'(x) \leq 2$ allow for a rapid enough increase in order for the graph of f moves from the point $(0, -1)$ to the point $(2, 4)$? The Mean Value Theorem implies that there is some c between 0 and 2 satisfying

$$f(4) - f(2) = f'(c)(4 - 2), \text{ which becomes } 5 = 2f'(c).$$

Hence the derivative of f at c would have to be $f'(c) = \frac{5}{2}$, which is larger than 2 and so violates $f'(x) \leq 2$ for all x. Hence no such function f can exist.

Warm-Up 2. We explain why the equation $x = 2 - \ln x$ has exactly one solution. Set

$$f(x) = x - 2 + \ln x.$$

Then f is differentiable and continuous on the interval $(0, \infty)$. Since

$$f(1) = -1 < 0 < \ln 2 = f(2),$$

the Intermediate Value Theorem implies there is some number x satisfying $f(x) = 0$, which is thus a number x satisfying $x = 2 - \ln x$. Thus this equation has at least one solution.

If this equation had two solutions, say a and b, the Mean Value Theorem would imply that there is some c between a and b satisfying

$$f(b) - f(a) = f'(c)(b - a).$$

Here, $f(b) = f(a) = 0$ since a and b are both meant to be solutions of $f(x) = 0$, and $b - a \neq 0$ since a and b are different, so we would get that $f'(c) = 0$. But $f'(x) = 1 + \frac{1}{x}$ is never zero for $x > 0$ in our case, so $f'(c) = 0$ is not possible. Hence there could not have been two solutions of the given equation, so it has at most one solution. Thus the given equation has exactly one solution.

Derivatives and shapes of graphs. We have already seen back at the start of the course that we can use f' to tell us something about the shape of the graph of f. For instance, having positive derivative means that the graph of f is increasing, while negative derivative means that it is decreasing. We also know that points where the derivative is zero might correspond to local maxima or minima, which graphically are "humps" and "valleys".

Now we push this idea further, using information about *second derivatives* to tell us more about the shape of a graph; in particular, the second derivative can tell us about the *concavity* of the graph. Before going into this further, let us start with an example.

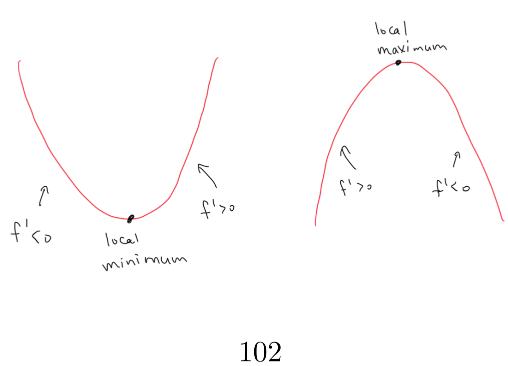
Example 1. Previously we looked at the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ when discussing extreme values. We found that the absolute maximum value of this function on the interval $[-2, 3]$ was 136 and occurred at $x = 3$, while the absolute minimum was -31 and occurred at $x = -2$, and that this absolute minimum was also a local minimum. What remained was to determine the behavior of f at $x = 0$ and $x = -1$, which were the two other points satisfying $f'(x) = 0$.

Consider the following table:

x	-1.5	-1	-0.5	0	1
$f'(x)$	$-$	0	$+$	0	$-$

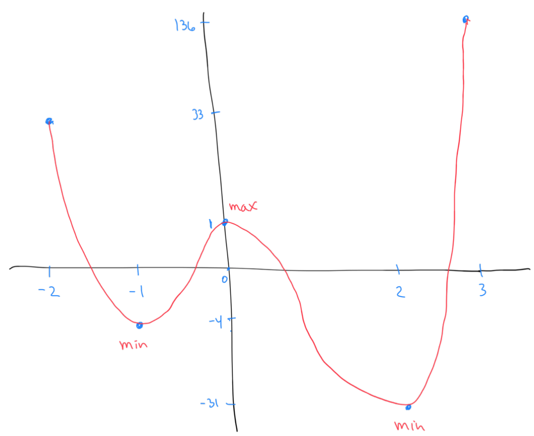
Recall that $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x+1)(x-2)$. This table keeps track of the sign of f' for values to the left and to the right of the critical points $x = -1$ and $x = 0$. For instance, $f'(-1.5)$ is negative, since each of $12x, x+1, x-2$ is negative at this point, so $f'(x) = 12x(x+1)(x-2)$ is negative. We don't care so much about what the actual value of $f'(-1.5)$ is, only that it is negative. Similarly, at $x = -0.5$, the factors $12x$ and $x - 2$ are negative, but $x + 1$ is positive, so overall $f'(-0.5)$ is positive also. And finally, $f'(1)$ is negative since $12x$ and $x+1$ are positive here, but $x - 2$ is negative. Again, all we are determining is whether f' is positive or negative to the left and right of the various critical points; there was nothing special about using -1.5 for instance, we could have used -1.25 or any other value a bit to the left of -1, and similarly for using -0.5 as something between -1 and 0, and using 1 as something a bit to the right of 0.

The point is that these signs tell us whether f is increasing or decreasing before and after our critical points. For instance, before $x = -1$ the graph of f should decrease, since $f' < 0$, but it should increase after $x = -1$, since $f' > 0$. Thus $x = -1$ must actually be a local *minimum* of f:

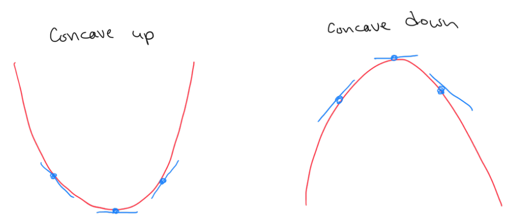


Along the same lines, the graph of f should be increasing up to $x = 0$, after which it should decrease, so $x = 0$ should be a local maximum.

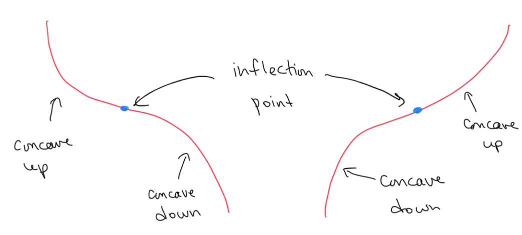
We already know that $x = 2$ is a local minimum because it is actually an absolute minimum, but this we could have also verified using $f'(x) = 12x(x+1)(x-2)$: f' is negative at 1 and positive at $x = 3$, so the graph should decrease before $x = 2$ and increase after, so $x = 2$ is indeed a local minimum. All together this gives enough information to draw a rough sketch of the graph of f, by noting where it should increase and decrease, and where it should have local maxima and minima:



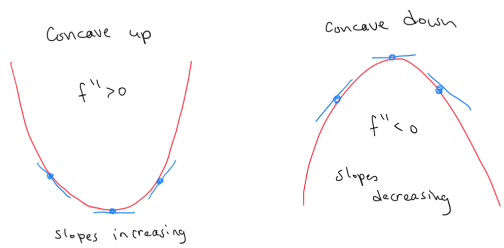
Concavity and second derivatives. In our example above we could have also determined whether $x = -1$ and $x = 0$ gave a local maximum or minimum usinig the *second derivative* of f. The key notion is that of concavity:



We say that the graph of f is *concave up* at a point if it lies above its tangent line there, and *concave down* if it lies below its tangent line there. A point where the concavity changes from concave up to concave down, or from concave down to concave up is called an *inflection point*:



The key observation is that a point where $f''(x) > 0$ should be a point where the graph of f is concave up, while a point where $f''(x) < 0$ is a point where the graph if concave down. Indeed, in the concave up case, the tangent slopes are *increasing* since they are getting larger, so f' should be a increasing function, meaning that its derivative $(f')' = f''$ should be positive, while the opposite happens in the concave down case



Based on how the graph of f looks like at a local maximum or local minimum, we can see that if $f''(x) < 0$ at a critical point x, then x should be a local maximum since its graph is a concave down "hump", while if $f''(x) > 0$ at a critical point, then x should be a local minimum since its graph is a concave up "valley". A point where $f''(x) = 0$ then is a possible inflection point, since at such a point the graph of f has the potential of switching concavity; i.e. f'' could change from being positive to being negative, or vice-versa.

Back to Example 1. Returning to the example of $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$, where $f'(x) = 12x^3 - 12x^2 - 24x$, we now use

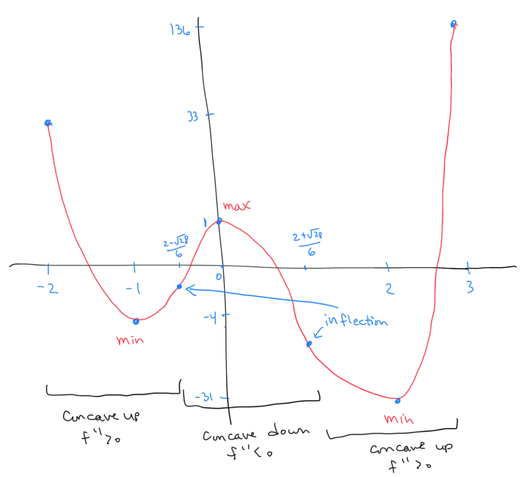
$$f''(x) = 36x^2 - 24x - 24$$

as an alternate way to characterize the local extrema we found. Since $f''(-1) = 36 > 0$, the critical point $x = -1$ should indeed be a local minimum, as we found previously; since $f''(0) = -24 < 0$, $x = 0$ should be a local maximum; and since $f''(2) = 72 > 0$, $x = 2$ should be a local minimum.

In addition, there are two points at which $f''(x)$ is zero:

$$f''(x) = 12(3x^2 - 2x - 2) = 0 \implies x = \frac{2 - \sqrt{28}}{6} \text{ or } x = \frac{2 + \sqrt{28}}{6}$$

where we used the quadratic formula to find the roots of $3x^2 - 2x - 2$. These are possible inflection points, and indeed we can determine that to the left of the first point f'' is positive while to the right it is negative, so the graph changes concavity at the first point and hence this is indeed an inflection point, while at the second point the graph changes from being concave down to being concave up, so this is also an inflection point:



Example 2. Finally, we consider $f(x) = 2 + 2x^2 - x^4$. We have:

$$f'(x) = 4x - 4x^3 = 4x(1 - x^2) = 4x(1 - x)(1 + x),$$

so the critical points of f are $x = -1, 0, 1$. The second derivative of f is $f''(x) = 4 - 12x^2$, and we have:

$$f''(-1) = -8 < 0 \qquad f''(0) = 4 > 0 \qquad f''(1) = -8 < 0.$$

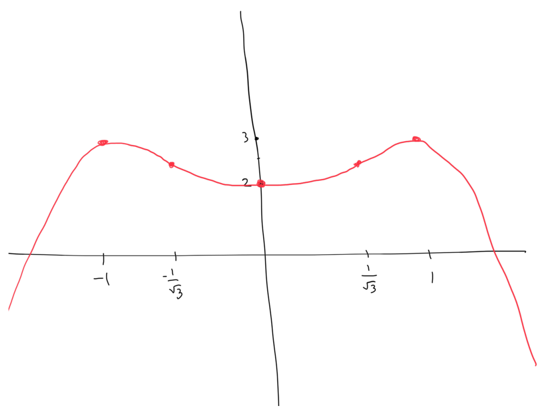
Thus f should be concave down at $x = -1$ and $x = 1$, so these give local maximums, and concave up at $x = 0$, so this is a local minimum. The second derivative is zero when:

$$4 - 12x^2 = 0, \text{ so } x = \pm\frac{1}{\sqrt{3}}.$$

Using the following chart:

x	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1
$f''(x)$	$-$	0	$+$	0	$-$

we see that f switches from being concave down to concave up at $x = -\frac{1}{\sqrt{3}}$, and from being concave up to concave down at $x = \frac{1}{\sqrt{3}}$, so these two are inflection points. The graph of f thus roughly looks like:



Lecture 22: Curve Sketching

Warm-Up 1. We find and classify the critical points of $f(x) = x(x-4)^3$, and also find the inflection points. First we compute the first derivative:

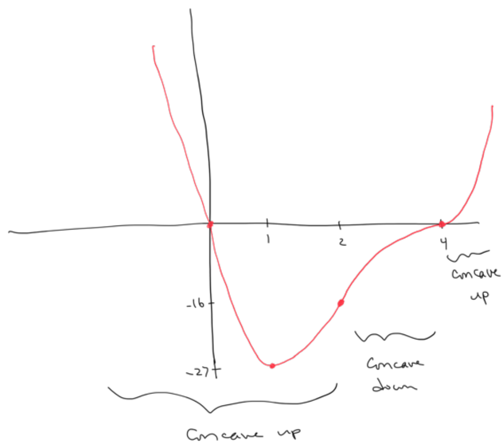
$$f'(x) = (x-4)^3 + 3x(x-4)^2 = (x-4)^2(4x-4) = 4(x-4)^2(x-1).$$

Thus the critical points of f are $x = 4$ and $x = 1$. To determine the behavior of f at these points we use the second derivative:

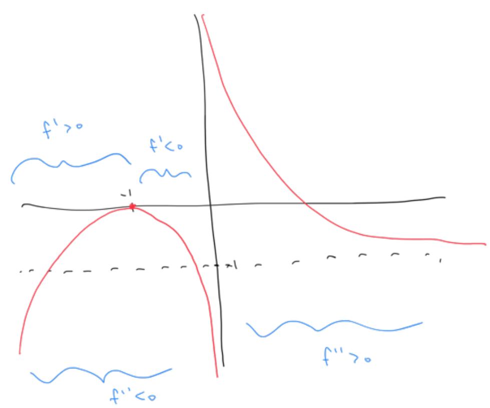
$$f''(x) = 8(x-4)(x-1) + 4(x-4)^2 = (x-4)(12x-24) = 12(x-4)(x-2).$$

Since $f''(1) > 0$, the graph of f is concave up at $x = 1$, so this is a local minimum. Since $f''(4) = 0$, $x = 4$ is actually a candidate inflection point. Indeed, since f'' is negative at $x = 3$ to the left of $x = 4$ and positive at $x = 5$ to the right, the graph switches from being concave down to being concave up $x = 4$, so this is an inflection point. There is another inflection point at $x = 2$, where f'' is also zero, since f'' is positive $x = 1$ to the left of $x = 2$ and negative at $x = 3$ to the right, so the graph switches from being concave up to concave down at $x = 2$.

With this information, together with the values $f(0) = 0, f(1) = -27, f(2) = -16, f(4) = 0$, we can draw a rough sketch of the graph of f as:



Warm-Up 2. We sketch the graph of a function $f(x)$ with vertical asymptote at $x = 0$ and horizontal asymptote at $y = -1$, and for which $f'(x) > 0$ for $x < -2$, $f' < 0$ for $0 > x > -2$, $f''(x) < 0$ for $x < 0$, and $f''(x) > 0$ for $x > 0$. At $x = -2$, f changes from increasing beforehand (since $f' > 0$) to decreasing afterwards (since $f' < 0$), so at $x = -2$ f should have a local maximum. Using the information about asymptotes, the fact that the graph should be concave down to the left of $x = 0$ and concave up to the right, we get a picture such as:



Indeed, this has the correct vertical asymptote and horizontal asymptote, and the correct local maximum at $x = -2$. Note in particular that, since the graph should be concave up to the right of $x = 0$, it must approach the vertical asymptote at $x = 0$ from the right in the upward direction (i.e. $\lim_{x\to 0^+} f(x) = \infty$) instead of the downward direction (what would happen if $\lim_{x\to 0^-} f(x) = -\infty$)) since the latter would require a concave down graph. A similar reason is why the graph approaches $x = 0$ in the downward direction to the left of $x = 0$.

Curve sketching. By combining all techniques we've developed so far to help understand the behavior of functions—looking at asymptotes, maxima and minima, inflection points and concavity, etc—we should be able to give fairly accurate sketches of graphs of various functions. We've already done this a bit so far, and we'll look at a few more examples now.

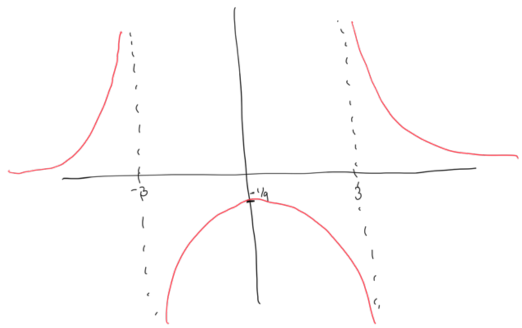
Example 1. We sketch the curve with equation $y = \frac{1}{x^2-9}$. First, this has vertical asymptotes at $x = \pm 3$ since:

$$\lim_{x\to -3^-} \frac{1}{x^2-9} = \infty \qquad \lim_{x\to -3^+} \frac{1}{x^2-9} = -\infty \qquad \lim_{x\to 3^-} \frac{1}{x^2-9} = -\infty \qquad \lim_{x\to 3^+} \frac{1}{x^2-9} = \infty.$$

Since $\lim_{x\to\pm\infty} \frac{1}{x^2-9} = 0$, the curve has a horizontal asymptote at $y = 0$ in both the positive and negative directions. Now:

$$y' = \frac{-2x}{(x^2-9)^2},$$

which is zero at $x = 0$, which is thus a critical point. Since $y' > 0$ to the left of $x = 0$ and $y' < 0$ to the right of $x = 0$, the curve increases to the left of $x = 0$ and increases to the right, so $x = 0$ is a local maximum. Thus the curve looks roughly like:



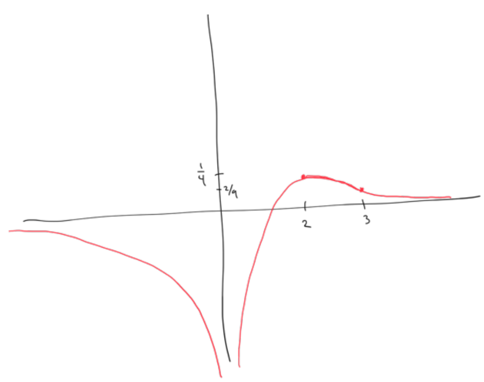
Example 2. Now we sketch the curve with equation $y = \frac{x-1}{x^2}$. Since $\lim_{x\to 0} \frac{x-1}{x^2} = -\infty$, this curve has a vertical asymptote at $x = 0$ which the curve approaches in the downward direction from either side. Since $\lim_{x\to\pm\infty} \frac{x-1}{x^2} = 0$, the curve has a horizontal asymptote at $y = 0$ in both directions. Now, we have:

$$y' = \frac{x^2 - 2x(x-1)}{x^4} = \frac{2}{x^3} - \frac{1}{x^2},$$

the curve has a critical point when $\frac{2}{x} = 1$, so at $x = 2$. Since

$$y'' = -\frac{6}{x^4} + \frac{2}{x^3},$$

is negative at $x = 2$, the curve is concave down at $x = 2$ so this is a local maximum. The second derivative is zero at $x = 3$, and it is negative before this point and positive afterwards, so $x = 3$ is an inflection point since the curve switches from being concave down to concave up at this point. Thus the curve looks like:



Example 3. We sketch the curve with equation $y = \frac{e^x}{x^2}$. We have:

$$\lim_{x\to 0} \frac{e^x}{x^2} = \infty \qquad \lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2} = \infty \qquad \lim_{x\to-\infty} \frac{e^x}{x^2} = 0$$

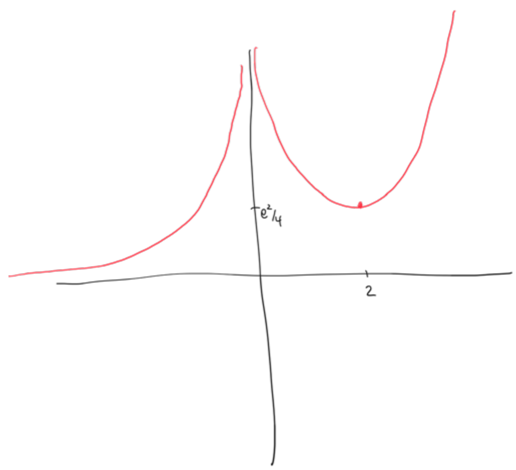
where in the second limit we used L'Hospital's rule twice. Thus the curve has a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$, but only in the negative direction. Next we compute:

$$y' = \frac{x^2 e^x - 2xe^x}{x^4} = \frac{e^x(x-2)}{x^3}.$$

This is zero at $x = 2$, so $x = 2$ is a critical point. Since $y' < 0$ to the left of $x = 2$ and $y' > 0$ to the right, $x = 2$ is a local minimum. The second derivative can be written as:

$$y'' = \frac{x^3[e^x(x-2) + e^x] - e^x(x-2)3x^2}{x^6} = \frac{e^x[(x - \frac{5}{2})^2 + \frac{3}{4}]}{x^4},$$

which is always positive. Thus the curve is always concave up, and $x = 2$ is a local minimum. Hence the curve looks like:



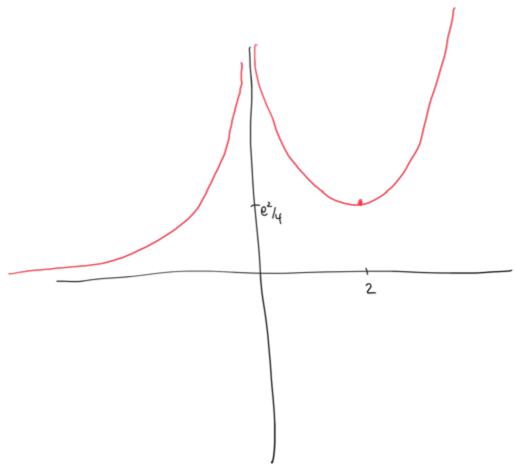
Example 4. Finally we sketch the curve with equation $y = x - 3x^{1/3}$. This has no asymptotes. Since

$$y' = 1 - \frac{1}{x^{2/3}}$$

is zero at $x = \pm 1$, there are two critical points. The second derivative is:

$$y'' = \frac{2}{3x^{5/3}}.$$

This is negative at $x = -1$ and positive at $x = 1$, so $x = -1$ is a local maximum and $x = 1$ is a local minimum. Also, y'' changes sign at $x = 0$ (where it is undefined), being negative to the left and positive to the right. Hence the curve switches from being concave down to the left of $x = 0$ to being concave up to the right, so $x = 0$ is an inflection point. The curve intersects the x-axis at $x = \pm\sqrt{27}$ then $y = x - 3x^{1/3}$ is zero, so all together the curve looks like:



Note that y' is undefined at $x = 0$, which is why at this point the curve has a *vertical* tangent.

Lecture 23: Antiderivatives

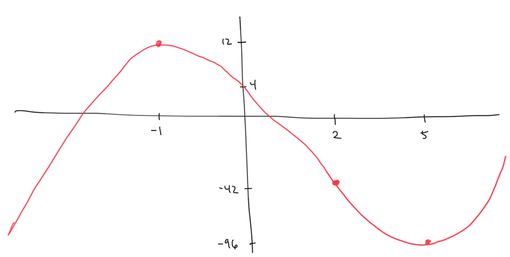
Warm-Up 1. We sketch the curve with equation $y = x^3 - 6x^2 - 15x + 4$. First, this curve has no asymptotes since there is not point we can approach in order to obtain an infinite value, and since as $x \to \pm\infty$, $y \to \pm\infty$. We have:

$$y' = 3x^2 - 12x - 15 = 3(x^2 - 4x - 5) = 3(x-5)(x+1).$$

This is zero when $x = 5$ and $x = -1$, so these are the two critical points. Since

$$y'' = 6x - 12$$

is positive at $x = 5$ and negative at $x = -1$, $x = 5$ is a local minimum and $x = -1$ is a local maximum. Finally, there is an inflection point $x = 2$ since the curve switches from being concave down before this to concave up afterwards. The curve has a y-intercept at $(0, 4)$, so it thus looks like:



Warm-Up 2. We sketch the curve with equation

$$y = \frac{1}{x(x-3)}.$$

First we have:

$$\lim_{x\to 0^-} \frac{1}{x(x-3)} = \infty \qquad \lim_{x\to 0^+} \frac{1}{x(x-3)} = -\infty \qquad \lim_{x\to 3^-} \frac{1}{x(x-3)} = -\infty \qquad \lim_{x\to 3^+} \frac{1}{x(x-3)} = -\infty,$$

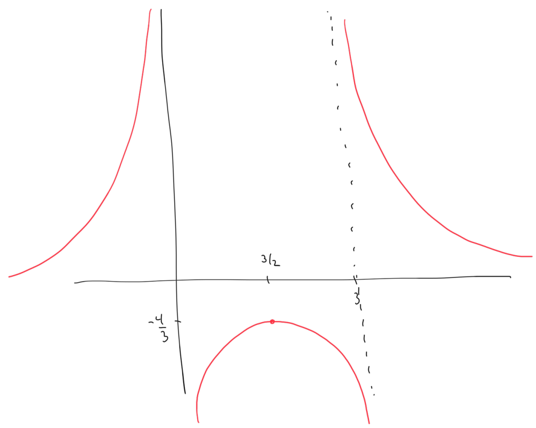
so the curve has vertical asymptotes at $x = 0$ and $x = 3$. Since

$$\lim_{x \to \pm\infty} \frac{1}{x(x-3)} = 0,$$

the curve has a horizontal asymptote at $y = 0$. Now:

$$y' = \frac{-(x-3) - x}{x^2(x-3)^2} = \frac{3-2x}{x^2(x-3)^2}.$$

This is zero when $x = \frac{3}{2}$, so $x = \frac{3}{2}$ is a critical point. For $x = 1$ to the left of $x = \frac{3}{2}$, y' is positive, while for $x = 2$ to the right it is negative, so $x = 1$ i s thus a local maximum. The corresponding local maximum value is $y = -\frac{4}{3}$, so the curve looks like:



Antiderivatives. So far we have looked at problems which involve computing the derivative of some known function: given $f(x)$, find $f'(x)$. Now we ask the opposite question: given $f'(x)$, what is $f(x)$? In other words, we want to "undo" the process of taking a derivative to go back and find the original function.

A function $F(x)$ satisfying $F'(x) = f(x)$ is known as an *antiderivative* of $f(x)$, and the process of finding an antiderivative given $f(x)$ is known as *antidifferentiation*. Antiderivatives play a huge role in the study of what are called *integrals*, and they are what the first half of Math 224 will be devoted to. For now we focus on basic examples which don't involve fancy techniques; more general methods for finding antiderivatives will be left to Math 224.

Example 1. For instance, say $F(x) = xe^x$. Then

$$F'(x) = e^x + xe^x = (1+x)e^x,$$

so we would say that $F(x) = xe^x$ is an antiderivative of $f(x) = (1+x)e^x$. But, and this is a key point, this is not the only possible antiderivative of $f(x) = (1+x)e^x$. Indeed note that the derivative of

$$xe^x + 2, \text{ or of } xe^x - 100$$

is *also* $(1+x)e^x$, since these two functions also count as antiderivatives of $(1+x)e^x$. More generally, any function of the form

$$xe^x + C$$

where C is a constant will be an antiderivative.

So, how can we describe *all* antiderivatives of $f(x) = (1+x)e^x$? Suppose $G(X)$ is another antiderivative of $f(x)$, meaning that $G'(x) = (1+x)e^x$. Then the derivative of $G(x) - F(x)$ (where $F(x) = xe^x$) is:

$$G'(x) - F'(x) = (1+x)e^x + (1+x)e^x = 0.$$

But this means that $G(x) - F(X)$ must be a constant function since it has derivative equal to 0 everywhere, so

$$G(x) - F(x) = C \text{ for some constant } C.$$

Then $G(x) = F(x) + C$, and so the fact is that $G(x)$ is the form $G(x) = xe^x + C$. In other words, adding arbitrary constants to the one antiderivative $F(x) = xe^x$ we already had in fact produces all possible antiderivatives! That is, any function whose derivative is $(1+x)e^x$ must be of the form $xe^x + C$ for some constant C.

Example 2. We find the most general antiderivative of the function

$$f(x) = 3e^x + \frac{1}{\sqrt{1-x^2}} - x^3 - 2$$

by thinking about what kinds of expressions we need to differentiate in order to produce the different portions of $f(x)$. First, in order to obtain $3e^x$ after differentiating we can start with $3e^x$ itself:

$$(3e^x)' = 3e^x.$$

So our antiderivative should have a $3e^x$ term in it:

$$F(x) = 3e^x + \text{other stuff}.$$

Next, in order to obtain the $\frac{1}{\sqrt{1-x^2}}$ term the antiderivative should have $\sin^{-1} x$ in it since

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}.$$

Thus the antiderivative should now look like

$$F(x) = 3e^x + \sin^{-1} x + \text{other stuff}.$$

Now, in order to obtain x^3 we should start with x^4, since after differentiating this will produce a third power of x. But if we simply take x^4, or even better $-x^4$ to deal with the negative in front, we get:

$$(-x^4)' = -4x^3,$$

whereas we want to get $-x^3$. The point is that now we can multiply by some constant in order to balance out the extra factor of 4 the derivative of x^4 would produce. This extra factor can be "cancelled out" using a factor of $\frac{1}{4}$, so we use $-\frac{1}{4}x^4$ as the antiderivative of $-x^3$:

$$\left(-\frac{1}{4}x^4\right)' = -\frac{1}{4}4x^3 = -x^3.$$

Thus our antiderivative so far looks like:

$$F(x) = 3e^x + \sin^{-1} x - \frac{1}{4}x^4 + \text{other stuff}.$$

(More generally, a possible antiderivative of x^k for $k \neq -1$ is given by $\frac{x^{k+1}}{k+1}$; for $k = -1$ an antiderivative of $x^{-1} = \frac{1}{x}$ is given by $\ln|x|$.) Finally, the -2 term at the end of $f(x)$ can be obtained by differentiating $-2x$, so our antiderivative is

$$F(x) = 3e^x + \sin^{-1} x - \frac{1}{4}x^4 - 2x + C$$

where C is an arbitrary constant. Again, each part was obtained by thinking about the types of expressions we would need to start with in order to obtain the different portions of $f(x)$ after differentiating.

Example 3. Suppose $f(x)$ is a function satisfying

$$f'(x) = e^{-x} + 2x^2 + \sin x - \sqrt{x} \quad \text{and} \quad f(0) = 3.$$

We want to find $f(x)$ itself. The point is that $f(x)$ should be an antiderivative of

$$y = e^{-x} + 2x^2 + \sin x - \sqrt{x}$$

and the additional information that $f(0) = 3$ will give us a way to find any arbitrary constants which show up. To obtain e^{-x} after differentiating we might think of starting with e^{-x} itself, since we know that derivatives of exponentials involve the same exact expressions. But

$$(e^{-x})' = -e^{-x}$$

has an extra negative in front, so we balance this out by using $-e^{-x}$ in our antiderivative:

$$(-e^{-x})' = e^{-x}.$$

Next, in order to obtain $2x^2$, we should start with x^3, but this would give the wrong coefficient in front after differentiating:

$$(x^3)' = 3x^2.$$

Thus we use $\frac{2}{3}x^3$ in our antiderivative instead:

$$\left(\frac{2}{3}x^3\right)' = \frac{2}{3}3x^2 = 2x^2.$$

To obtain $\sin x$ we should use $-\cos x$, and to obtain $-\sqrt{x} = -x^{1/2}$, we should use $-\frac{2}{3}x^{3/2}$:

$$\left(-\frac{2}{3}x^{3/2}\right)' = -\frac{2}{3}\frac{3}{2}x^{1/2} = -x^{1/2}.$$

(The fact that this portion of the antiderivative should involve $x^{3/2}$ came from adding one to the exponent of $x^{1/2}$, and the coefficient in front came from wanting the end result to be -1.)

Putting everything together, the most general antiderivative of $y = e^{-x} + 2x^2 + \sin x - \sqrt{x}$ is

$$f(x) = -e^{-x} + \frac{2}{3}x^3 - \cos x - \frac{2}{3}x^{3/2} + C$$

Since we want it to be true that $f(0) = 3$, we must have:

$$3 = f(0) = -e^0 + 0 - \cos 0 - 0 + C, \text{ so } 3 = -2 + C, \text{ so } C = 5.$$

Thus the function $f(x)$ we want is $f(x) = -e^{-x} + \frac{2}{3}x^3 - \cos x - \frac{2}{3}x^{3/2} + 5$.

Example 4. Finally, we find the function $f(x)$ which satisfies

$$f''(x) = e^x - \frac{1}{x^2} + 3, \ f(1) = 3, \text{ and } f'(1) = 0.$$

Since we are given the second derivative of f, this will involve taking *two* antiderivatives. First, in order to obtain $f''(x) = e^x - \frac{1}{x^2} + 3$, $f'(x)$ itself must look like

$$f'(x) = e^x + \frac{1}{x} + 3x + C$$

where C is some constant. Indeed, the e^x term is needed in order to obtain e^x after differentiating; to obtain the $-\frac{1}{x^2} = -x^{-2}$ portion we need $x^{-1} = \frac{1}{x}$ after adding 1 to the exponent; and $3x$ is an antiderivative of 3. Since we want $f'(1) = 0$, we get:

$$0 = e^1 + 1 + 3 + C, \text{ so } C = -e - 4.$$

Thus $f'(x)$ must be

$$f'(x) = e^x + \frac{1}{x} + 3x - (e - 4).$$

We now take one more antiderivative to find $f(x)$:

$$f(x) = e^x + \ln|x| + \frac{3}{2}x^2 - (e-4)x + D$$

where D is some constant. (The $\frac{2}{3}x^2$ term comes from the fact that $\frac{1}{2}x^2$ is an antiderivative of x.) Since we want $f(1) = 3$, we must have:

$$3 = f(1) = e + \ln 1 + \frac{3}{2} - (e - 4) + D = \frac{11}{2} + D.$$

Thus $D = 3 - \frac{11}{2} = -\frac{5}{2}$, so $f(x)$ is

$$f(x) = e^x + \ln|x| + \frac{3}{2}x^2 - (e-4)x - \frac{5}{2}.$$

Lecture 24: Optimization

Warm-Up 1. Suppose $f(x)$ is a function satisfying

$$f'(x) = 3x^{1/3} + 3\cos x - \frac{1}{1+x^2} - 3 \quad \text{and} \quad f(0) = 4.$$

We find $f(x)$ explicitly. The function we want should be an antiderivative of $3x^{1/3} + 3\cos x - \frac{1}{1+x^2} - 3$, so we start by describing such antiderivatives. In order to get an $x^{1/3}$ term, the antiderivative should have an $x^{4/3}$ term. But this would give a coefficient of $\frac{4}{3}$ in its derivative, so to balance this out we use $\frac{9}{4}x^{4/3}$ in the antiderivative:

$$\left(\frac{9}{4}x^{4/3}\right)' = \frac{9}{4}\frac{4}{3}x^{1/3} = 3x^{1/3}.$$

To get $3\cos x$ we need $3\sin x$; to get $-\frac{1}{1+x^2}$ we need $-\tan^{-1} x$, and to get -3 we need $-3x$. Thus the function we want should look like

$$f(x) = \frac{9}{4}x^{4/3} + 3\sin x - \tan^{-1} x - 3x + C$$

for some constant C. Since we also want $f(0) = 4$, it must be true that:

$$4 = f(0) = 0 + 0 + 0 - 0 + C, \text{ so } C = 4.$$

Thus our function is

$$f(x) = \frac{9}{4}x^{4/3} + 3\sin x - \tan^{-1} x - 3x + 4.$$

Warm-Up 2. Suppose a ball is thrown straight up in the air from a height of 100 ft with an initial velocity of 1 ft/sec. We determine the height of the ball 1 sec later. The key point is that if $h(t)$ is a function giving the height of the ball after t seconds, then $h'(t)$ gives the velocity, and $h''(t)$ gives the acceleration. In our case, the acceleration is determined only by the force of gravity pulling down on the ball, and it is a standard physical fact that the acceleration due to gravity is

$$h''(t) = -32 \text{ ft/sec}^2,$$

which we take for granted in this course. Thus we work towards finding $h(t)$ by taking two antiderivatives.

Since $h''9t) = -32$, taking one antiderivative gives

$$h'(t) = -32t + C$$

for some constant C. We are given that the initial velocity of the ball is 1, so this is the value of $h'(0)$. Thus we need:

$$1 = h'(0) = -32(0) + C,$$

which gives $C = 1$. Hence our velocity function is $h'(t) = -32t + 1$. Taking another antiderivative gives

$$h(t) = -16t^2 + t + D$$

for some constant D. Since the initial height of the ball is $h(0) = 100$, we need:

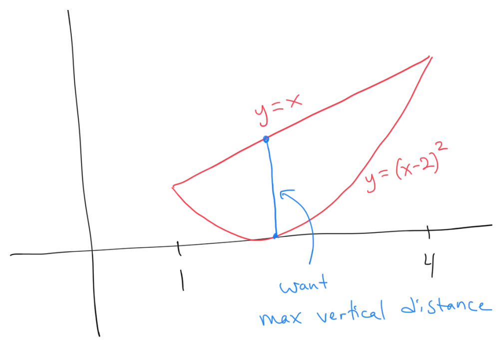
$$10 = h(0) = -16(0^2) + 0 + D,$$

so $D = 100$. Thus the height function is $h(t) = -16t^2 + t + 100$. At $t = 1$ sec, the height of the ball is thus $h(1) = -16 + 1 + 100 = 85$ ft. Note this means that after 1 sec is ball is at a lower height than it was when it was first thrown.

Optimization. We now come to our final topic of the quarter: optimization. Optimization provides one of the historically most important applications of calculus, and one which continues to be essential in other fields. The basic problem is to maximize or minimize some quantity, possibly subject to some constraint on the variables in question.

The setup is almost always the same: first use the given information and constraints to find a function of one-variable which models the quantity we are wanting to optimize; second find the maximal or minimal values of this function using a derivative; and finally check that the value we find does indeed give us the type of extremum (i.e. maximum or minimum) we actually wanted. Much of the work goes into finding the appropriate single-variable function to use.

Example 1. To start with, we find the maximal vertical distance which can occur between points on the curves $y = (x-2)^2$ and $y = x$ for $1 \leq x \leq 4$:



The vertical distance between vertically-aligned points on the two the curves is given by the difference between their y-coordinates, and so is given by the function

$$f(x) = x - (x-2)^2.$$

We want to find the (absolute) maximum of this function over the interval $[1, 4]$. Since:

$$f'(x) = 1 - 2(x-2) = 5 - 2x,$$

we have a critical point at $x = \frac{5}{2}$, which does fall within $[1, 4]$. The points $x = 1$ and $x = 4$ give a vertical distance of 0 since these are the points at which the curves intersect, so these endpoints certainly are not the maximums for which we are looking. Thus $x = \frac{5}{2}$ must indeed give the maximal value we want. Hence, the maximal vertical distance between the two curves over the interval $[1, 4]$ occurs at $x = \frac{5}{2}$, and the value of this maximal vertical distance is

$$f(\tfrac{5}{2}) = \tfrac{5}{2} - \left(\tfrac{5}{2} - 2\right)^2 = \tfrac{5}{2} - \tfrac{1}{4} = \tfrac{9}{4}.$$

Example 2. Consider all positive numbers whose product is 100. We want to find the smallest possible sum among them which can occur. For instance, 100 and 1 are two numbers whose product is 100, and their sum is 101, so we are asking whether a lower sum is possible and what it is. Say that x and y denote our numbers, so we want to minimize

$$x + y.$$

However, the values of x and y we consider are constrained by the fact that xy should be 100. So, we want to:

optimize $x + y$ subject to the constraint $xy = 100$.

From the constraint we get that $y = \frac{100}{x}$, which we ca use to turn the expression we want to optimize into a single-variable one:

$$f(x) = x + \frac{100}{x}.$$

The minimal value we want comes from finding critical points of $f(x)$, so we compute:

$$f'(x) = 1 - \frac{100}{x^2}.$$

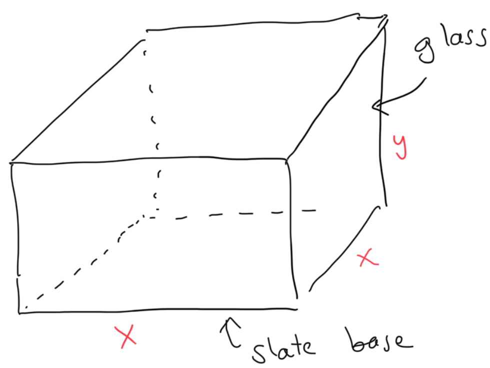
This is zero when $x^2 = 100$, so when $x = \pm 10$. However, we are only considering positive numbers in this problem, we only need to consider $x = 10$. From the constraint, the corresponding value of $y = \frac{100}{x}$ is $y = 10$. Thus it seems that $x = 10$ and $y = 10$ are the numbers we want. But we should make sure these values actually give a minimal sum and not a maximal sum. In this case we already know that 1 and 100 also satisfy the constraint and give a larger sum of 101 than do $x = 10$ and $y = 10$, so $x = 10$ and $y = 10$ definitely do not give a maximum and so do give a minimum instead, at least assuming a minimum actually exists. Alternatively, we can note that

$$f'(x) = 1 - \frac{100}{x^2}$$

is negative for x a bit smaller than 10 and positive for x a bit larger than 10, so $f(x)$ should decrease before $x = 10$ and increase after, so $x = 10$ indeed gives a minimum.

To summarize, the positive numbers whose product is 100 which give the smallest possible sum are 10 and 10, and this smallest sum is $10 + 10 = 20$.

Example 3. Suppose we are aiming to construct an aquarium in the shape of a rectangular box with a square base:



The base is made of slate and costs \$10 per square ft, and the sides are made of class and cost \$5 per square ft. If we have \$100 to spend, we want to find the dimensions of the aquarium which maximize the volume it holds. If x denotes the side length (in feet) of the base and y the height of each of the sides, the volume of the box is x^2y. The base costs $10x^2$ total dollars and each glass side costs $5xy$ in total, so all together we will spend $10x^2 + 4(5xy) = 10x^2 + 20xy$ dollars. We thus want to:

$$\text{maximize } x^2y \text{ subject to the constraint } 10x^2 + 20xy = 100.$$

The constraint gives

$$y = \frac{100 - 10x^2}{20x} = \frac{10 - x^2}{2x},$$

which we can plug into the expression we want to maximize in order to obtain the single-variable function

$$f(x) = x^2\left(\frac{10 - x^2}{2x}\right) = \frac{1}{2}x(10 - x^2) = 5x - \frac{1}{2}x^3.$$

The maximum comes from a critical point, so since:

$$f'(x) = 5 - \frac{3}{2}x^2$$

is zero when $x^2 = \frac{10}{3}$, we get that $x = \sqrt{\frac{10}{3}}$ is our critical point. (We ignore the negative square root since our dimensions should be positive.) This does give a maximum since $f'(x)$ is positive to the left (for instance at $x = 1 < \sqrt{\frac{10}{3}}$ we get $f'(1) = \frac{7}{2}$), while $f'(x)$ is negative to the right (for instance at $x = 2 > \sqrt{\frac{10}{3}}$ we get $f'(2) = -1$). From the constraint the corresponding height is

$$y = \frac{10 - \frac{10}{3}}{2\sqrt{\frac{10}{3}}},$$

so these values of x and y give the dimensions of the aquarium subject to the given constraint which maximize its volume.

Lecture 25: More on Optimization

Warm-Up. We find the point on the line $y = 2x + 3$ which is closest to the origin $(0, 0)$. The distance from a point (x, y) to the point $(0, 0)$ is

$$\sqrt{x^2 + y^2},$$

so we can minimize the expression $x^2 + y^2$ since the square root above is minimized when the term under the square root is minimized. (This avoids having to take a derivative of a square root, which leads to a simpler computation.) But, we want our point (x, y) to satisfy the constraint $y = 2x + 3$, so we obtain the single-variable function

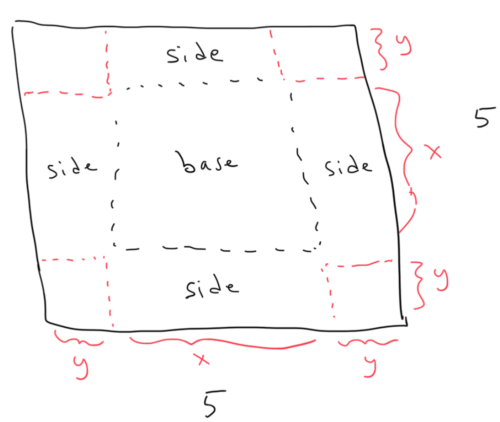
$$f(x) = x^2 + (2x + 3)^2$$

as the one we want to minimize. We have:

$$f'(x) = 2x + 2(2x + 3)2 = 10x + 12,$$

which is zero when $x = -\frac{6}{5}$. The corresponding y-coordinate is $y = 2(-\frac{6}{5}) + 3 = \frac{3}{5}$, so the point we want is $(-\frac{6}{5}, \frac{3}{5})$. To guarantee that this point does give minimal distance, note that $f'(x) = 10x + 12$ is negative at, say, $x = -2$ to the left of $x = \frac{-}{6}5$ and positive at $x = -1$ to the right, so $x = \frac{6}{5}$ should be a point which gives a minimum.

Example 1. Take a 5×5 ft^2 piece of cardboard. We create a cardboard box by cutting out a square out of each corner, and then folding up the resulting "sides":



We want to find the dimensions x, y (respectively the length of the base and height of the sides of the resulting box) which maximize the volume of the box. The volume of the box is given by x^2y, and we are constrained by the requirement that the length of each side in the original piece of cardboard be 5 ft, which translates to $x + 2y = 5$. Thus we want to

$$\text{maximize } x^2y \text{ subject to the constraint } x + 2y = 5.$$

The constraint gives $x = 5 - 2y$, so we will maximize the single-variable function

$$f(y) = (5 - 2y)^2 y.$$

We have

$$f'(y) = 2(5 - 2y)(-2)y + (5 - 2y)^2 = (5 - 2y)(-4y + 5 - 2y) = (5 - 2y)(5 - 6y),$$

which is zero when $y = \frac{5}{2}$ and $y = \frac{5}{6}$. The value $y = \frac{5}{2}$ gives $x = 5 - 2(\frac{5}{2}) = 0$, which certainly does not give maximal volume since it corresponds to a "box" which has zero length. So, we must take $y = \frac{5}{6}$ instead. To guarantee that this gives a maximum, note that $f'(0.5) > 0$ and $f'(1) < 0$, so f increases up to $y = \frac{5}{6}$ and decreases afterwards, so $x = \frac{5}{6}$ is a maximum. The corresponding x value is $x = 5 - 2(\frac{5}{6}) = \frac{10}{3}$, so the dimensions of the box which maximize the volume are $x = \frac{10}{3}$ and $y = \frac{5}{6}$.

Example 2. Consider all rectangles with fixed perimeter P. We claim that the one which maximizes the area is a square. If x and y denote the dimensions of the rectangle, we thus want to maximize the area xy subject to the constraint that $2x + 2y = P$, since $2x + 2y$ is the perimeter of the rectangle. The constraint gives $y = \frac{P}{2} - x$, so we thus optimize the single-variable function

$$f(x) = x(\tfrac{P}{2} - x).$$

We compute $f'(x) = \frac{P}{2} - x - x = \frac{P}{2} - 2x$, which is zero when $x = \frac{P}{4}$. The corresponding value for y is $y = \frac{P}{2} - \frac{P}{4} = \frac{P}{4}$, and so the point is that we do get a square since the dimensions $x = y = \frac{P}{4}$ are equal. This gives maximal area since $f'(x) = \frac{P}{2} - 2x$ is positive for, say $x = \frac{P}{8}$ to the left of $x = \frac{P}{4}$, and negative at $x = P$ to the right.

Example 3. We construct a tin can in the shape of a cylinder, which should have volume 1000π cm^3. The can should have a base but no top. We want to find the dimensions, r radius and h height, which minimize the amount of tin used. The volume of the can is $\pi r^2 h$, and the surface area (which describes the amount of tin which should be used) is $2\pi rh + \pi r^2$, where the $2\pi rh$ gives the area of the cylindrical shell (the cylinder part) and πr^2 the area of the base. Thus we want to

$$\text{minimize } 2\pi rh + \pi r^2 \text{ subject to the constraint } \pi r^2 h = 1000\pi.$$

The constraint gives $h = \frac{1000}{r^2}$, so the single-variable function we want to minimize is

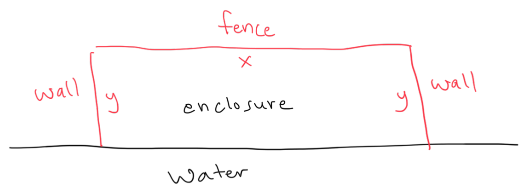
$$f(r) 2\pi r\left(\frac{1000}{r^2}\right) + \pi r^2 = \frac{2000\pi}{r} + \pi r^2.$$

We compute:

$$f'(r) = -\frac{2000\pi}{r^2} + 2\pi r,$$

which is zero when $r^3 = 1000$, and thus when $r = 10$. The corresponding height is $h = \frac{1000}{10^2} = 10$. This gives minimal surface area since $f'(9) < 0$ to the left of $r = 10$ and $f'(11) > 0$ to the right, so f decreases before $r = 10$ and increases after.

Example 4. Finally, suppose an enclosure is to be constructed next to a river:



The wall sides will be y ft long, and cost \$10 per ft to build. (We'll ignore the height of the enclosure.) The fence portion will be x ft long, and costs \$5 per ft to build. If we want to enclose 500 ft^2 in total, we find the dimensions x, y which minimize the cost. The enclosed area is xy, so the constraint is that $xy = 500$. The cost to build the walls is $20y$ ($10y$ for each wall), and the cost to build the fence is $5x$, so the total cost is $20y + 5x$. Thus we want to minimize the function

$$f(x) = 5x + 20\left(\frac{500}{x}\right)$$

which comes from setting $y = \frac{500}{x}$ as a result of the constraint.

We have:

$$f'(x) = 5 - \frac{1}{25x^2},$$

which is zero when $x^2 = 125$, and thus for $x = \sqrt{125}$. (We ignore the negative square root since the dimension x should be positive.) The corresponding length of each wall is $y = \frac{500}{\sqrt{125}}$. These dimensions do minimize cost since $f'(10) < 0$ to the left of $x = \sqrt{125}$ and $f'(12) > 0$ after, so f decreases before and increases after $x = \sqrt{125}$.

Thanks for reading!