Short notes on separation of variables

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1 Separation of variables: brief introduction

Imagine that you are trying to solve a problem whose solution is a function of several variables, e.g.: $u = u(x,\, y\, z)$. Finding this solution directly may be very hard, but you may be able to find other solutions with the simpler structure

$$u = f(x)\, g(y)\, h(z), \tag{1.1}$$

where f, g, and h are some functions. This is called a *separation of variables* solution.

If the problem that you are trying to solve is linear, and you can find enough solutions of the form in (1.1), then you may be able to solve the problem by using a linear combinations of separated variables solutions.

The technique described in the paragraph above is called *separation of variables.* Note that

1. When the problem is a pde and solutions of the form in (1.1) are allowed, the pde reduces to three ode — one for each of f, g, and h. Thus the solution process is enormously simplified.
2. In (1.1) all the three variables are separated. But it is also possible to seek for *partially separated* solutions. For example
$$u = f(x)\, v(y,\, z). \tag{1.2}$$
This is what happens when you look for *normal mode* solutions to time evolution equations of the form
$$u_t = \mathcal{L}\, u, \tag{1.3}$$
where $\mathcal{L}$ is a linear operator acting on the space variables $\vec{r} = (x, y, \dots)$ only — for example: $\mathcal{L} = \partial_x^2 + \partial_y^2 + \dots$. The normal mode solutions have time separated
$$u = e^{\lambda\, t}\phi(\vec{r}), \quad \text{where} \quad \lambda = \text{constant}, \tag{1.4}$$
and reduce the equation to an eigenvalue problem in space only
$$\lambda\phi = \mathcal{L}\, \phi. \tag{1.5}$$
3. *Separation of variables does not always work.* In fact, it rarely works for random problems. But it works for many problems of physical interest. For example: it works for the heat equation, but only for a few very symmetrical domains (rectangles, circles, cylinders, ellipses). But these are enough to build intuition as to how the equation works. Many properties valid for generic domains can be gleaned from the solutions in these domains.

4. Even if you cannot find enough separated solutions to write all the solutions as linear combinations of them, or if the problem is nonlinear and you can get just a few separated solutions, sometimes this is enough to discover interesting physical effects, or gain intuition as to the system behavior.

Remark 1.1 *An* **important point** *is that separation of variables solutions* **do not** *(and cannot) satisfy all the boundary and/or initial conditions that apply in a specific problem. Generic boundary and/or initial data are obtained (when the method works) by doing arbitrary linear combinations of separated solutions. For example:*

Normal Modes.
The normal modes $e^{\lambda t}\,\phi(\vec{r})$ *in (1.4) cannot satisfy general initial data, but only those allowed by the form* $e^{\lambda t}\,\phi(\vec{r})$ *— which is very restrictive, since only functions* ϕ *leading to a solution of (1.3) are allowed. This is illustrated by the problem* $u_t = u_{x\,x}$ *for* $0 < x < 1$*, with* $u(0,\,t) = u(1,\,t) = 0$*. Then the normal modes satisfy initial data of the form* $\phi \propto \sin(n\,\pi\,x)$*,* n *an integer, only. General initial data follow by using sine Fourier series.*
Note also that normal modes make sense only for problems with homogeneous boundary conditions. For example, consider the problem $u_t = u_{x\,x}$ *for* $0 < x < 1$ *and* $t > 0$*, with* $u(0,\,t) = 0$*,* $u(1,\,t) = \sigma(t)$*, and* $u(x,\,0) = u_0(x)$*. If a particular solution satisfying the boundary conditions is known, then the problem can be reduced to one for which normal modes apply. That is, let* $u_p(x,t)$ *be a particular solution.*[1] *Then* $v = u - u_p$ *solves the problem:* $v_t = v_{x\,x}$*, with* $v(0,\,t) = v(1,\,t) = 0$ *and* $v(x,\,0) = u_0(x) - u_p(x,\,0)$ *— which can be solved by normal modes.*

Static, boundary value problems.
As we will see in the examples below, in these problems the separated solutions satisfy only special types of boundary conditions — the reason being, again, that requiring a solution in separated form imposes severe restrictions on what type of functions can occur . More general (in some cases, generic) boundary conditions follow only upon doing linear combinations of separated solutions.

2 Example: heat equation in a square, with zero boundary conditions

Consider the problem

$$T_t = \Delta\,T = T_{xx} + T_{yy}, \tag{2.6}$$

in the square domain $0 < x,\ y < \pi$, with T vanishing along the boundary and initial data $T(x,\,y,\,0) = W(x,\,y)$. To solve this problem by separation of variables, we first look for solutions of the form

$$T = f(t)\,g(x)\,h(y), \tag{2.7}$$

which satisfy the boundary conditions, but **not** *the initial data* — see remark **1.1**. Substituting (2.7) into (2.6) yields

$$f'\,g\,h = f\,g''\,h + f\,g\,h'', \tag{2.8}$$

where the primes indicate derivatives with respect to the respective variables. Dividing this through by u yields

$$\frac{f'}{f} = \frac{g''}{g} + \frac{h''}{h}. \tag{2.9}$$

[1] It is possible to have particular solutions that are themselves separated solutions. For example $u_p = \sin(x)\,e^{-t}$ is a particular solution when $\sigma = \sin(1)\,e^{-t}$.

Since each of the terms in this last equation is a function of a different independent variable, the equation can be satisfied only if each term is a constant. Thus

$$\frac{g''}{g} = c_1, \quad \frac{h''}{h} = c_2, \quad \text{and} \quad \frac{f'}{f} = c_1 + c_2, \tag{2.10}$$

where c_1 and c_2 are constants. Now the problem has been reduced to a set of three simple ode. Furthermore, for (2.7) to satisfy the boundary conditions in (2.6), we need:

$$g(0) = g(\pi) = h(0) = h(\pi) = 0, \tag{2.11}$$

which restricts the possible choices for the constants c_1 and c_2. The equations in (2.10 – 2.11) are easily solved, and yield [2]

$$g = \sin(n\,x), \quad \text{with} \quad c_1 = -n^2, \tag{2.12}$$

$$h = \sin(m\,y), \quad \text{with} \quad c_2 = -m^2, \tag{2.13}$$

$$f = e^{-(n^2+m^2)\,t}, \tag{2.14}$$

where $n = 1, 2, 3, \ldots$ and $m = 1, 2, 3, \ldots$ are natural numbers. The solution to the problem in (2.6) can then be written in the form

$$T = \sum_{n,\,m=1}^{\infty} w_{nm}\,\sin(n\,x)\,\sin(m\,y)\,e^{-(n^2+m^2)\,t}, \tag{2.15}$$

where the coefficients w_{nm} follow from the double sine-Fourier series expansion (of the initial data)

$$W = \sum_{n,\,m=1}^{\infty} w_{nm}\,\sin(n\,x)\,\sin(m\,y). \tag{2.16}$$

That is

$$w_{nm} = \frac{4}{\pi^2}\int_0^{\pi} dx \int_0^{\pi} dy\, W(x,\,y)\,\sin(n\,x)\,\sin(m\,y). \tag{2.17}$$

3 Example: Heat equation in a circle, with zero boundary conditions

Consider now the heat equation with zero boundary conditions, but on a circle instead of a square. That is, using polar coordinates, we want to solve the problem

$$T_t = \Delta\,T = \frac{1}{r^2}\,\left(r(r\,T_r)_r + T_{\theta\theta}\right), \tag{3.18}$$

for $0 \le r < 1$, with $T(1,\,\theta,\,t) = 0$, and some initial data $T(r,\,\theta,\,0) = W(r,\,\theta)$. To solve the problem using separation of variables, we look for solutions of the form

$$T = f(t)\,g(r)\,h(\theta), \tag{3.19}$$

which satisfy the boundary conditions, but **not** *the initial data* — see remark **1.1**. In addition

> Polar coordinates must be used, otherwise solutions of the form (3.19) cannot satisfy any sensible boundary conditions. This exemplifies an *important feature of the separation of variables method:*
> Separation must be done in coordinate systems where the boundaries are coordinate surfaces.

[2] We set the arbitrary multiplicative constants in each of these solutions to one. Given (2.15 – 2.16), there is no loss of generality in this.

Substituting (3.19) into (3.18) yields

$$f'\,g\,h = \frac{1}{r^2}\,\left(f\,r\,(r\,g')'\,h + f\,g\,h''\right), \tag{3.20}$$

where, as before, the primes indicate derivatives. Dividing this through by u yields

$$\frac{f'}{f} = \frac{(r\,g')'}{r\,g} + \frac{h''}{r^2\,h}. \tag{3.21}$$

Since the left side in this equation is a function of time only, while the right side is a function of space only, the two sides must be equal to the same constant. Thus

$$f' = -\lambda\,f \tag{3.22}$$

and

$$\frac{r\,(r\,g')'}{g} + \lambda\,r^2 + \frac{h''}{h} = 0, \tag{3.23}$$

where λ is a constant. Here, again, we have a situation involving two functions of different variables being equal. Hence

$$h'' = \mu\,h, \tag{3.24}$$

and

$$\frac{1}{r}\,(r\,g')' + \left(\lambda + \frac{\mu}{r^2}\right)\,g = 0, \tag{3.25}$$

where μ is another constant. The problem has now been reduced to a set of three ode. Furthermore, from the boundary conditions and the fact that θ is the polar angle, we need:

$$g(1) = 0 \qquad \text{and} \qquad h \;\text{ is periodic of period }\; 2\,\pi. \tag{3.26}$$

In addition, g must be non-singular at $r = 0$ — the singularity that appears for $r = 0$ in equation (3.25) is due to the coordinate system singularity, equation (3.18) is perfectly fine there.

It follows that it should be $\boldsymbol{\mu = -n^2}$ and

$$h = e^{i\,n\,\theta}, \quad \text{where } n \text{ is an integer.} \tag{3.27}$$

Notes:

- Here and below we set the arbitrary multiplicative constants in each of the ode solutions to one. Given (3.30), there is no loss of generality in this.
- Instead of complex exponentials, the solutions to (3.24) could be written in terms of sine and cosines. But complex exponentials provide for a more compact notation.

Then (3.25) takes the form

$$\frac{1}{r}\,(r\,g')' + \left(\lambda - \frac{n^2}{r^2}\right)\,g = 0. \tag{3.28}$$

This is a Bessel equation of integer order. The non-singular (at $r = 0$) solutions of this equation are proportional to the Bessel function of the first kind $J_{|n|}$. Thus we can write

$$g = J_{|n|}(\kappa_{|n|\,m}\,r), \quad \text{and} \quad \lambda = \kappa_{|n|\,m}^2, \tag{3.29}$$

where $m = 1,\,2,\,3,\,\ldots$, and $\boldsymbol{\kappa_{|n|\,m} > 0}$ **is the $\boldsymbol{m}$-th zero of $\boldsymbol{J_{|n|}}$.**

Remark 3.2 *That (3.28) turns out to be a well known equation should not be a surprise. Bessel functions, and many other special functions, were first introduced in the context of problems like the one here!*

Putting it all together, we see that the solution to the problem in (3.18) can be written in the form

$$T = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} w_{n\,m}\, J_{|n|}(\kappa_{|n|\,m}\, r)\, \exp\left(i\, n\, \theta - \kappa_{|n|\,m}^2\, t\right), \tag{3.30}$$

where the coefficients $w_{n\,m}$ follow from the double (Complex Fourier) – (Fourier – Bessel) expansion

$$W = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} w_{n\,m}\, J_{|n|}(\kappa_{|n|\,m}\, r)\, e^{i\,n\,\theta}. \tag{3.31}$$

That is:

$$w_{n\,m} = \frac{1}{\pi\, J_{|n|+1}^2(\kappa_{|n|\,m})} \int_0^{2\pi} d\theta \int_0^1 r\, dr\, W(r,\,\theta)\, J_{|n|}(\kappa_{|n|\,m}\, r)\, e^{-i\,n\,\theta}. \tag{3.32}$$

Remark 3.3 *You may wonder how (3.32) arises. Here is a sketch:*

(i) For θ we use a Complex-Fourier series expansion. For any 2π-periodic function

$$G(\theta) = \sum_{n=-\infty}^{\infty} a_n\, e^{i\,n\,\theta}, \quad \textit{where} \quad a_n = \frac{1}{2\pi} \int_0^{2\pi} G(\theta)\, e^{-i\,n\,\theta}\, d\theta. \tag{3.33}$$

(ii) For r we use the Fourier-Bessel series expansion explained in item (iii).

(iii) Note that (3.28), **for any fixed n,** *is an eigenvalue problem in $0 < r < 1$. Namely*

$$\mathcal{L}\, g = \lambda\, g, \quad \textit{where} \quad \mathcal{L}\, g = -\frac{1}{r}\,(r\, g')' + \frac{n^2}{r^2}\, g, \tag{3.34}$$

g is regular for $r = 0$, and $g(1) = 0$. Without loss of generality, assume that $n \geq 0$, and consider the set of all the (real valued) functions such that $\int_0^1 \tilde{g}^2(r)\, r\, dr < \infty$. Then define the scalar product

$$\langle \tilde{f},\, \tilde{g}\, \rangle = \int_0^1 \tilde{f}(r)\, \tilde{g}(r)\, r\, dr. \tag{3.35}$$

With this scalar product $\mathcal{L}$ is self-adjoint, and it yields a complete set of orthonormal eigenfunctions

$$\phi_m = J_n(\kappa_{n\,m}\, r) \quad \textit{and} \quad \lambda_m = \kappa_{n\,m}^2, \quad \textit{where} \quad m = 1,\, 2,\, 3,\, \ldots \tag{3.36}$$

Thus one can expand

$$F(r) = \sum_{m=1}^{\infty} b_m\, \phi_m(r), \quad \textit{where} \quad b_m = \frac{1}{\int_0^1 r\, \phi_m^2(r)\, dr} \int_0^1 F(r)\, \phi_m(r)\, r\, dr. \tag{3.37}$$

Finally, note that

$$\int_0^1 r\, \phi_m^2(r)\, dr \qquad \textit{follows from} \qquad \int_0^1 r\, J_n^2(\kappa_{n\,m}\, r)\, dr = \frac{1}{2}\, J_{n+1}^2(\kappa_{n\,m}). \tag{3.38}$$

We will not prove this identity here.

4 Example: Laplace equation in a circle sector, with Dirichlet boundary conditions, non-zero on one side

Consider the problem

$$0 = \Delta u = \frac{1}{r^2}\left(r(r\,u_r)_r + u_{\theta\theta}\right), \quad 0 < r < 1 \quad \text{and} \quad 0 < \theta < \alpha, \tag{4.39}$$

where $\boldsymbol{\alpha}$ **is a constant, with** $\boldsymbol{0 < \alpha < 2\pi}$. The boundary conditions are

$$u(1,\,\theta) = 0, \quad u(r,\,0) = 0, \quad \text{and} \quad u(r,\,\alpha) = w(r), \tag{4.40}$$

for some given function w. To solve the problem using separation of variables, we look for solutions of the form

$$u = g(r)\,h(\theta), \tag{4.41}$$

which satisfy the boundary conditions at $r = 1$ *and* $\theta = 0$*, but* **not** *the boundary condition at* $\theta = \alpha$. The reasons are as in remark **1.1** — we aim to obtain the solution for general w using linear combinations of solutions of the form (4.41). Note also that **we use a coordinate system where the boundary is given by coordinate lines.**

Substituting (4.41) into (4.39) yields, after a bit of manipulation

$$\frac{r\,(r\,g')'}{g} + \frac{h''}{h} = 0, \tag{4.42}$$

where each term is a function of a different variable. We conclude that

$$r\,(r\,g')' - \mu\,g = 0 \quad \text{and} \quad h'' + \mu\,h = 0, \tag{4.43}$$

where μ is some constant, $g(1) = 0$, and $h(0) = 0$ — the problem has been reduced to solving ode. In fact, it is easy to see that it should be [3]

$$h = \frac{1}{s}\sinh(s\,\theta) \quad \text{and} \quad g = \frac{r^{i\,s} - r^{-is}}{s}, \quad \text{where} \quad \mu = -s^2, \tag{4.44}$$

and as yet we know nothing about s, other than it is some (possibly complex) constant.

At this point it would seem natural to argue that the solution should not be singular at $r = 0$, since the singularity in equation (4.39) at $r = 0$ — and in the equation for g in (4.43) — is merely a consequence of the coordinate system singularity at $r = 0$. This would lead to the conclusion that s should be selected so that g is not singular at $r = 0$. However, **we cannot make this argument here:** since the origin is on the boundary for the problem in (4.39 – 4.40), *there is no reason why the solutions should be differentiable across the boundary.* We can only state that [4]

$$\boldsymbol{g} \textbf{ should be bounded} \quad \Longleftrightarrow \quad \boldsymbol{s \neq 0} \textbf{ is real.} \tag{4.45}$$

Furthermore, (4.44)is invariant under $s \to -s$. Thus

$$\boldsymbol{0 < s < \infty}. \tag{4.46}$$

In this example the answer involves a continuum of separated solutions, as opposed to examples where discrete sets occur.

Putting it all together, we can now write the solution to the problem in (4.39 – 4.40) as follows

$$u = \int_0^\infty \frac{\sinh(s\,\theta)}{\sinh(s\,\alpha)}\left(r^{i\,s} - r^{-i\,s}\right) W(s)\,ds, \tag{4.47}$$

where W is computed in (4.50) below. *Notice that, for* $\theta < \alpha$*, the factor* $\sinh(s\,\theta)/\sinh(s\,\alpha)$ *in the integral above decays exponentially as* $s \to \infty$*. This results in* u *being smooth inside the wedge, even if the data* w *is not.*

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[3] The multiplicative constant in these solutions is selected so that the limit $s \to 0$ gives the solution for $s = 0$.

[4] As $s \to 0$, (4.44) gives $g = 2\,i\,\ln(r)$. For $s \neq 0$ complex, either $r^{i\,s}$ or $r^{-i\,s}$ becomes unbounded as $r \to 0$.

Remark 4.4 *Start with the complex Fourier Transform*

$$f(\zeta) = \int_{-\infty}^{\infty} \hat{f}(s)\, e^{i\,s\,\zeta}\, ds, \quad \textit{where} \quad \hat{f}(s) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} f(\zeta)\, e^{-i\,s\,\zeta}\, d\zeta. \tag{4.48}$$

Apply it to an odd function. The answer can then be manipulated into the sine Fourier Transform

$$f(\zeta) = \int_{0}^{\infty} F(s)\, \sin(s\,\zeta)\, ds, \quad \textit{where} \quad F(s) = \frac{2}{\pi} \int_{0}^{\infty} f(\zeta)\, \sin(s\,\zeta)\, d\zeta, \tag{4.49}$$

and $0 < \zeta,\ s < \infty$. *Let* $0 < r = e^{-\zeta} < 1$, $w(r) = f(\zeta)$, *and* $W(s) = -\dfrac{1}{2\,i}\, F(s)$. *Then*

$$\begin{aligned} \boldsymbol{w(r)} &= \boldsymbol{\int_{0}^{\infty} W(s)\, \left(r^{i\,s} - r^{-i\,s}\right)\, ds,} \\ \textit{where} \quad \boldsymbol{W(s)} &= \boldsymbol{\frac{1}{2\,\pi} \int_{0}^{1} \left(\frac{r^{-i\,s} - r^{i\,s}}{r}\right) w(r)\, dr,} \end{aligned} \tag{4.50}$$

which is another example of a transform pair associated with the spectrum of an operator *(see below).*

Finally: What is behind (4.50)? Why should we expect something like this? Note that the problem for g can be written in the form

$$\mathcal{L}\, g = \mu\, g, \quad \text{where} \quad \mathcal{L}\, g = r\,(r\, g')', \tag{4.51}$$

$g(1) = 0$ and g is bounded (more accurately: the inequality in (4.52) applies). This is an eigenvalue problem in $0 < r < 1$. Further, consider consider the set of all the functions such that

$$\int_{0}^{1} |\tilde{g}|^2(r)\, \frac{dr}{r} < \infty, \tag{4.52}$$

and define the scalar product

$$\langle \tilde{f},\, \tilde{g}\, \rangle = \int_{0}^{1} \tilde{f}^*(r)\, \tilde{g}(r)\, \frac{dr}{r}. \tag{4.53}$$

With this scalar product $\mathcal{L}$ is self-adjoint. However, it does not have any discrete spectrum,[5] only continuum spectrum — with the pseudo-eigenfunctions given in (4.44), for $0 < s < \infty$. This continuum spectrum is what is associated with the formulas in (4.50). In particular, note the presence of the scalar product (4.53) in the formula for W.

THE END.

[5] No solutions that satisfy $g(1) = 0$ and (4.52).