3.2   THE DEFINITION OF DERIVATIVE

The Formal Definition

The graphical idea of a **slope of a tangent line** is very useful, but for some uses we need a more algebraic definition of the **derivative of a function.** We will use this definition to calculate the derivatives of several functions and see that the results from the definition agree with our graphical understanding. We will also look at several different interpretations for the derivative, and derive a theorem which will allow us to easily and quickly determine the derivative of any fixed power of x.

In the last section we found the slope of the tangent line to the graph of the function $f(x) = x^2$ at an arbitrary point (x, f(x)) by calculating the slope of the secant line through the points (x, f(x)) and (x+h, f(x+h)),

$$m_{sec} \ = \ \frac{f(x+h) - f(x)}{(x+h) - (x)} \ ,$$

and then by taking the limit of m_{sec} as h approached 0 (Fig. 1).   That approach to calculating slopes of tangent lines is the definition of the derivative of a function.
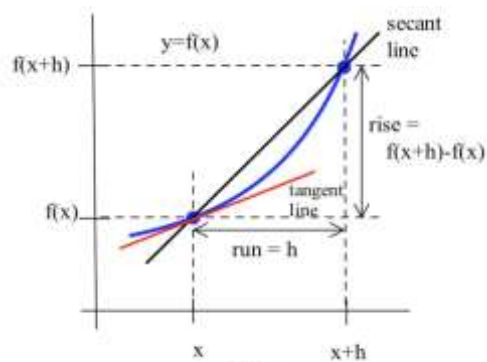

Fig. 1

> **Definition of the Derivative:**
>
> The **derivative** of a function f is a new function, **f '** (pronounced "eff prime"),
>
> whose value at x is **f '(x) =** $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ if the limit exists and is finite.

This is <u>the</u> definition of differential calculus, and you must know it and understand what it says. The rest of this chapter and all of Chapter 3 are built on this definition as is much of what appears in later chapters. It is remarkable that such a simple idea (the slope of a tangent line) and such a simple definition (for the derivative f ') will lead to so many important ideas and applications.

Notation: There are three commonly used notations for the **derivative of y = f(x)**:

f '(x)   emphasizes that the derivative is a **function** related to f

D(f)   emphasizes that we perform an **operation** on f to get the derivative of f

$\dfrac{\mathbf{df}}{\mathbf{dx}}$ emphasizes that the derivative is the **limit of** $\dfrac{\Delta\mathbf{f}}{\Delta\mathbf{x}} = \dfrac{f(x+h) - f(x)}{h}$.

We will use all three notations so you can get used to working with each of them.

f '(x) represents the slope of the tangent line to the graph of
y = f(x) at the point (x, f(x)) or the instantaneous rate of
change of the function f at the point (x, f(x)).

If, in Fig. 2, we let x be the point a+h, then
h = x – a. As h→0, we see that x→a and

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ so}$$

$$\mathbf{f\,'(a) =} \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \ .$$

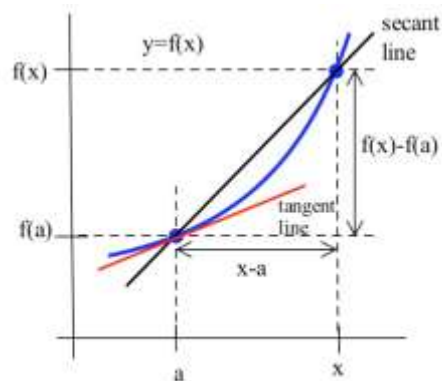We will use whichever of these two forms is more convenient algebraically.

Calculating Some Derivatives Using The Definition

Fortunately, we will soon have some quick and easy ways to calculate most derivatives, but first we will have to use the definition to determine the derivatives of a few basic functions. In Section 2.2 we will use those results and some properties of derivatives to calculate derivatives of combinations of the basic functions. Let's begin by using the graphs and then the definition to find a few derivatives.

Example 1: Graph $y = f(x) = 5$ and estimate the **slope** of the tangent line at each point on the graph. Then use the definition of the derivative to calculate the exact slope of the tangent line at each point. Your graphic estimate and the exact result from the definition should agree.
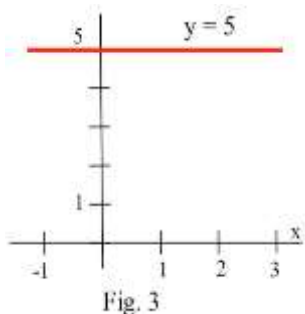
Solution: The graph of $y = f(x) = 5$ is a horizontal line (Fig. 3) which has slope 0 so we should expect that its tangent line will also have slope 0.

Using the definition: Since $f(x) = 5$, then $f(x+h) = 5$, so

$$\mathbf{D(\ f(x)\ )} \quad \equiv \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{5-5}{h} = \lim_{h \to 0} \frac{0}{h}$$

$$= 0 .$$

Fig. 3

Using similar steps, it is easy to show that the derivative of any constant function is 0.

> **Theorem:** If $f(x) = k$, then $f'(x) = 0$.

Practice 1: Graph $y = f(x) = 7x$ and estimate the slope of the tangent line at each point on the graph. Then use the definition of the derivative to calculate the exact slope of the tangent line at each point.

Example 2: Determine the derivative of $y = f(x) = 5x^3$ graphically and using the definition. Find the equation of the line tangent to $y = 5x^3$ at the point $(1,5)$.

Solution: It appears that the graph of $y = f(x) = 5x^3$ (Fig. 4) is increasing so the slopes of the tangent lines are positive except perhaps at $x = 0$ where the graph seems to flatten out.
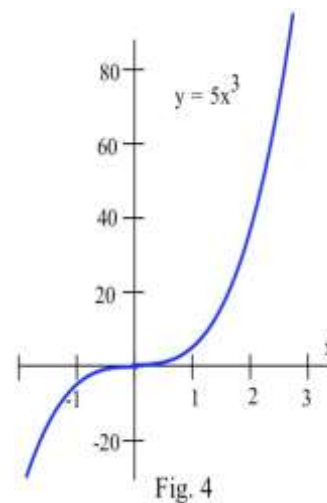
Using the definition: Since $f(x) = 5x^3$, then

$f(x+h) = 5(x+h)^3 = 5(x^3 + 3x^2h + 3xh^2 + h^3)$ so

$$\mathbf{f'(x)} \quad \equiv \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{the definition}$$

$$= \lim_{h \to 0} \frac{5(x^3 + 3x^2h + 3xh^2 + h^3) - 5(x^3)}{h} \qquad \text{eliminate } 5x^3 - 5x^3$$

$$= \lim_{h \to 0} \frac{15x^2h + 15xh^2 + 5h^3}{h}$$

$$= \lim_{h \to 0} \frac{h(15x^2 + 15xh + 5h^2)}{h} \qquad \text{divide by } h$$

Fig. 4

$$= \lim_{h \to 0} (15x^2 + 15xh + 5h^2) = 15x^2 + 0 + 0 = \mathbf{15x^2}$$

so $\mathbf{D(\ 5x^3\ )} = 15x^2$ which is positive except when x=0, and then $15x^2 = 0$.

$f'(x) = 15x^2$ is the slope of the line tangent to the graph of f at the point (x , f(x)). At the point (1,5), the slope of the tangent line is $f'(1) = 15(1)^2 = 15$. From the point–slope formula, the equation of the tangent line to f is $y - 5 = 15(\ x - 1)$ or $y = 15x - 10$.

Practice 2:   Use the definition to show that the derivative of $y = x^3$ is $\dfrac{dy}{dx} = 3x^2$. Find the equation of the line tangent to the graph of $y = x^3$ at the point (2, 8).

If f has a derivative at x, we say that f is **differentiable** at x. If we have a point on the graph of a differentiable function and a slope (the derivative evaluated at the point), it is easy to write the equation of the tangent line.

> **Tangent Line Formula**
>
>   If   f is differentiable at **a**
>
>   then   the equation of the tangent line to f at the point (a ,f(a)) is **y = f(a) + f '(a)(x − a)** .

Proof: The tangent line goes through the point (a , f(a)) with slope f '(a) so, using the point–slope formula, $y - f(a) = f'(a)\ (x - a)$ or $y = f(a) + f'(a)\ (x - a)$.

Practice 3: The derivatives $\mathbf{D(\ x\ )} = 1$, $\mathbf{D(\ x^2\ )} = 2x$, $\mathbf{D(\ x^3\ )} = 3x^2$ exhibit the start of a pattern. <u>Without</u> using the definition of the derivative, what do you <u>think</u> the following derivatives will be? $\mathbf{D(\ x^4\ )}$, $\mathbf{D(\ x^5\ )}$, $\mathbf{D(\ x^{43}\ )}$, $\mathbf{D(\ \sqrt{x}\ )} = \mathbf{D(\ x^{1/2}\ )}$ and $\mathbf{D(\ x^\pi\ )}$.

(Just make an intelligent "guess" based on the pattern of the previous examples.)

Before going on to the "pattern" for the derivatives of powers of x and the general properties of derivatives, let's try the derivatives of two functions which are not powers of x: sin(x) and | x | .

> **Theorem:** $\mathbf{D(\ sin(x)\ )} = \mathbf{cos(x)}$.

The graph of $y = f(x) = sin(x)$ is well–known (Fig. 5). The graph has horizontal tangent lines (slope = 0) when $x = \pm\dfrac{\pi}{2}$ and $x = \pm\dfrac{3\pi}{2}$ and
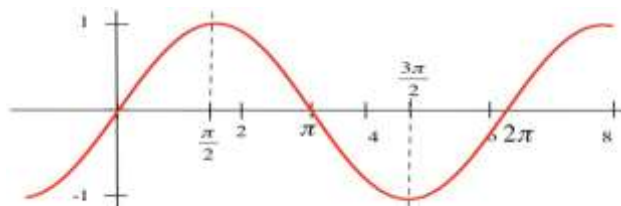

Fig. 5

so on. If $0 < x < \frac{\pi}{2}$, then the slopes of the tangent lines to the graph of y = sin(x) are positive. Similarly, if $\frac{\pi}{2} < x$

$< \frac{3\pi}{2}$, then the slopes of the tangent lines are negative. Finally, since the graph of y = sin(x) is periodic, we

expect that the derivative of y = sin(x) will also be periodic.

Proof of the theorem: Since f(x) = sin(x), f(x+h) = sin(x+h) = sin(x)cos(h) + cos(x)sin(h)   so

$$\mathbf{f\,'}(x) \quad \equiv \quad \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\{\sin(x)\cos(h)+\cos(x)\sin(h)\}-\{\sin(x)\}}{h}$$

this limit looks formidable, but if we

just collect the terms containing sin(x)

and then those containing cos(x) we get

$$= \lim_{h \to 0} \left\{ \sin(x) \cdot \frac{\cos(h)-1}{h} + \cos(x) \cdot \frac{\sin(h)}{h} \right\}$$

now calculate the limits separately

$$= \left\{ \lim_{h \to 0} \sin(x) \right\} \cdot \left\{ \lim_{h \to 0} \frac{\cos(h)-1}{h} \right\} + \left\{ \lim_{h \to 0} \cos(x) \right\} \cdot \left\{ \lim_{h \to 0} \frac{\sin(h)}{h} \right\}$$

the first and third limits do not

depend on h, and we calculated the

second and fourth limits in Section 1.2

$= $ sin(x)·(0) + cos(x)·(1) $=$ **cos (x)**.

So **D(sin(x))** = cos(x) , and the various properties we expected of the derivative of y = sin(x) by examining its
graph are true of cos(x).

Practice 4:   Use the definition to show that **D(cos(x))** $= -$ sin(x) . (This is similar to the situation for f(x) = sin(x).
You will need the formula cos(x+h) = cos(x)·cos(h) – sin(x)·sin(h). Then collect all the terms containing cos(x)
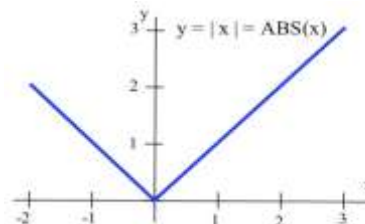and all the terms with sin(x). At that point you should recognize and be able to evaluate the limits.)

Fig. 6

Example 3: For y = |x| find dy/dx .

Solution: The graph of y = f(x) = | x | (Fig. 6) is a "V" with its vertex
at the origin. When x > 0, the graph is just y = |x| = x which is a line
with slope +1 so we should expect the derivative of |x| to be +1. When
x < 0, the graph is y = |x| = –x which is a line with slope –1, so we
expect the derivative of |x| to be –1. When x = 0, the graph has a corner,
and we should expect the derivative of |x| to be undefined at x = 0.

Using the definition: It is easiest to consider 3 cases in the definition of | x | : x > 0, x < 0 and x = 0.

If x > 0, then, for small values of h, x + h > 0 so **D**f(x) $\equiv \lim\limits_{h\to 0} \dfrac{|x+h|-|x|}{h} = \lim\limits_{h\to 0} \dfrac{h}{h} = 1.$

If x < 0, then, for small values of h, we also know that x + h < 0 so **D**f(x) $= \lim\limits_{h\to 0} \dfrac{-h}{h} = -1.$

When x = 0, the situation is a bit more complicated and

Df(x) $\equiv \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{|0+h|-|0|}{h} = \lim\limits_{h\to 0} \dfrac{|h|}{h}$ which is undefined

since $\lim\limits_{h\to 0^+} \dfrac{|h|}{h} = +1$ and $\lim\limits_{h\to 0^-} \dfrac{|h|}{h} = -1$. **D**(|x|) $= \begin{cases} +1 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$.

Practice 5: Graph y = | x – 2 | and y = | 2x | and use the graphs to determine **D**(|x – 2|) and **D**(| 2x |).

INTERPRETATIONS OF THE DERIVATIVE

So far we have emphasized the derivative as the slope of the line tangent to a graph . That interpretation is very visual
and useful when examining the graph of a function, and we will continue to use it. Derivatives, however, are used in
a wide variety of fields and applications, and some of these fields use other interpretations. The following are a few
interpretations of the derivative which are commonly used.

General

Rate of Change   f '(x) is the **rate of change** of the function at x. If the units for x are years and the units for f(x) are people, then the units for $\frac{df}{dx}$ are $\frac{people}{year}$, a rate of change in population.

Graphical

Slope   f '(x) is the **slope of the line tangent to the graph of f at the point (x, f(x)).**

Physical

Velocity   If f(x) is the position of an object at time x, then f '(x) is the **velocity** of the object at time x. If the units for x are hours and f(x) is distance measured in miles, then the units for

$\mathbf{f\,'(x)} = \frac{df}{dx}$ are $\frac{miles}{hour}$, miles per hour, which is a measure of velocity.

Acceleration   If f(x) is the velocity of an object at time x, then f '(x) is the **acceleration** of the object at time x. If the units are for x are hours and f(x) has the units $\frac{miles}{hour}$, then the units for the acceleration **f '(x)**

$= \frac{df}{dx}$ are $\frac{miles/hour}{hour}$ $= \frac{miles}{hour^2}$, miles per hour per hour.

Magnification   **f '(x)** is the **magnification factor** of the function f for points which are close to x.

If a and b are two points very close to x, then the distance between f(a) and f(b) will be close to f '(x) times the original distance between a and b: f(b) – f(a) $\approx$ f '(x) (b – a) .

Business

Marginal Cost   If f(x) is the total cost of x objects, then **f '(x)** is the **marginal cost**, at a production level of x. This marginal cost is approximately the <u>additional</u> cost of making one more object once we have already made x objects. If the units for x are bicycles and the units for f(x) are dollars, then the units for f '(x) = $\frac{df}{dx}$ are $\frac{dollars}{bicycle}$, the cost per bicycle.

Marginal Profit   If f(x) is the total profit from producing and selling x objects, then **f '(x)** is the **marginal profit**, the profit to be made from producing and selling one more object.

If the units for x are bicycles and the units for f(x) are dollars, then the units for f '(x) = $\frac{df}{dx}$

are $\frac{dollars}{bicycle}$, dollars per bicycle, which is the profit per bicycle.

    In business contexts, the word "marginal" usually means the derivative or rate of change of some quantity.

One of the strengths of calculus is that it provides a unity and economy of ideas among diverse applications. The vocabulary and problems may be different, but the ideas and even the notations of calculus are still useful.
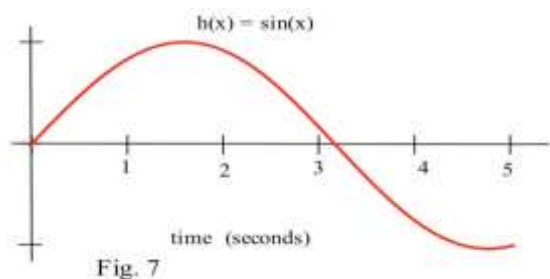
Fig. 7

Example 4:   A small cork is bobbing up and down, and at time t seconds it is h(t) = sin(t) feet above the mean water level (Fig. 7). Find the height, velocity and acceleration of the cork when t = 2 seconds. (Include the proper units for each answer.)

Solution: h(t) = sin(t) represents the height of the cork at any time t , so the height of the cork when t = 2 is h(2) = sin(2) ≈ 0.91 feet.

The velocity is the derivative of the position, so $v(t) = \dfrac{d\ h(t)}{dt} = \dfrac{d\ \sin(t)}{dt} = \cos(t)$. The derivative of position is the limit of (Δh)/(Δt) , so the units are (feet)/(seconds). After 2 seconds the velocity is v(2) = cos(2) ≈ –0.42 feet per second = –0.42 ft/s .

The acceleration is the derivative of the velocity, so $a(t) = \dfrac{d\ v(t)}{dt} = \dfrac{d\ \cos(t)}{dt} = -\sin(t)$. The derivative of velocity is the limit of (Δv)/(Δt) , so the units are (feet/second) / (seconds) or feet/second^2. After 2 seconds the acceleration is a(2) = – sin(2) ≈ –0.91 ft/s^2 .

Practice 6:   Find the height, velocity and acceleration of the cork in the previous example after 1 second?

A USEFUL FORMULA: $D(x^n)$

Functions which include powers of x are very common (every polynomial is a sum of terms which include powers of x), and, fortunately, it is easy to calculate the derivatives of such powers. The "pattern" emerging from the first few examples in this section is, in fact, true for all powers of x. We will only state and prove the "pattern" here for positive integer powers of x, but it is also true for other powers as we will prove later.

> **Theorem: If n is a positive integer, then $D(x^n) = n \cdot x^{n-1}$.**

This theorem is an example of the power of <u>generality</u> and <u>proof</u> in mathematics. Rather than resorting to the definition when we encounter a new power of x (imagine using the definition to calculate the derivative of x^{307}), we can justify the pattern for all positive integer exponents n , and then simply apply the result for whatever exponent we have. We know, from the first examples in this section, that the theorem is true for n= 1, 2 and 3 , but no number of <u>examples</u> would guarantee that the pattern is true for all exponents. We need a proof that what we think is true really is true.

Proof of the theorem: Since $f(x) = x^n$, then $f(x+h) = (x+h)^n$, and in order to simplify

$f(x+h) - f(x) = (x+h)^n - x^n$, we will need to expand $(x+h)^n$. However, we really only need to know the first

two terms of the expansion and to know that all of the other terms of the expansion contain a power of h of at

least 2. The Binomial Theorem from algebra says (for $n > 3$) that

$(x+h)^n = x^n + n{\cdot}x^{n-1}h + a{\cdot}x^{n-2}h^2 + b{\cdot}x^{n-3}h^3 + ... + h^n$ where a and b   represent numerical coefficients.

(Expand $(x+h)^n$ for at least a few different values of n to convince yourself of this result.)

Then $\mathbf{D}(\,f(x)\,) \;\equiv\; \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} \;=\; \lim\limits_{h\to 0} \dfrac{(x+h)^n - x^n}{h}$ then expand to get

$= \lim\limits_{h\to 0} \dfrac{\left\{x^n + n\cdot x^{n-1}h + a\cdot x^{n-2}h^2 + b\cdot x^{n-3}h^3 + ... + h^n\right\} - x^n}{h}$ eliminate $x^n - x^n$

$= \lim\limits_{h\to 0} \dfrac{\left\{n\cdot x^{n-1}h + a\cdot x^{n-2}h^2 + b\cdot x^{n-3}h^3 + ... + h^n\right\}}{h}$ factor h out of the numerator

$= \lim\limits_{h\to 0} \dfrac{h\cdot\left\{n\cdot x^{n-1} + a\cdot x^{n-2}h + b\cdot x^{n-3}h^2 + ... + h^{n-1}\right\}}{h}$ divide by the factor h

$= \lim\limits_{h\to 0} \; \{\, n{\cdot}x^{n-1} + a{\cdot}x^{n-2}h + b{\cdot}x^{n-3}h^2 + ... + h^{n-1} \,\}$ separate the limits

$= n{\cdot}x^{n-1} + \lim\limits_{h\to 0} \; \{\, a{\cdot}x^{n-2}\mathbf{h} + b{\cdot}x^{n-3}\mathbf{h^2} + ... + \mathbf{h^{n-1}} \,\}$ each term has a factor of h, and $h\to 0$

$= n{\cdot}x^{n-1} + \mathbf{0} \;=\; \mathbf{n{\cdot}x^{n-1}}$ so $\mathbf{D}(\,x^n\,) = \mathbf{n{\cdot}x^{n-1}}$.

Practice 7:   Use the theorem to calculate $\mathbf{D}(\,x^5\,)$, $\dfrac{\mathbf{d}}{\mathbf{dx}}(\,x^2\,)$, $\mathbf{D}(\,x^{100}\,)$, $\dfrac{\mathbf{d}}{\mathbf{dt}}(\,t^{31}\,)$, and $\mathbf{D}(\,x^0\,)\mathbf{.}$

We will occasionally use the result of the theorem for the derivatives of **all** constant powers of x even though it has

only been proven for positive integer powers, so far. The result for all constant powers of x is proved in Section

2.9

Example 5: Find $\mathbf{D}(\,1/x\,)$ and $\dfrac{\mathbf{d}}{\mathbf{dx}}(\,\sqrt{x}\,)$.

Solution: $\mathbf{D}(\dfrac{1}{x}) = \mathbf{D}(x^{-1}) = -1x^{(-1)-1} = -1x^{-2} = \dfrac{-1}{x^2}$. $\dfrac{\mathbf{d}}{\mathbf{dx}}(\,\sqrt{x}\,) = \mathbf{D}(\,x^{1/2}\,) = (1/2)x^{-1/2} = \dfrac{1}{2\sqrt{x}}$.

These results can be obtained by using the definition of the derivative, but the algebra is slightly awkward.

Practice 8:   Use the pattern of the theorem to find $D(x^{3/2})$, $\dfrac{d}{dx}(x^{1/3})$, $D(\dfrac{1}{\sqrt{x}})$ and $\dfrac{d}{dt}(t^{\pi})$.

Example 6: It costs $\sqrt{x}$ hundred dollars to run a training program for x employees.

   (a) How much does it cost to train 100 employees? 101 employees? If you already need to train 100

   employees, how much additional will it cost to add 1 more employee to those being trained?

   (b) For $f(x) = \sqrt{x}$, calculate f '(x) and evaluate f ' at x = 100. How does f '(100) compare with the last

   answer in part (a)?

Solution: (a) Put $f(x) = \sqrt{x} = x^{1/2}$ hundred dollars, the cost to train x employees. Then f(100) = $1000

   and f(101) = $1004.99 , so it costs $4.99 additional to train the 101st employee.

   (b) $f'(x) = \dfrac{1}{2} x^{-1/2} = \dfrac{1}{2\sqrt{x}}$ so $f'(100) = \dfrac{1}{2\sqrt{100}} = \dfrac{1}{20}$ hundred dollars = $5.00 .

   Clearly f '(100) is very close to the actual additional cost of training the 101st employee.

IMPORTANT DEFINITIONS AND RESULTS

Definition of Derivative: $f'(x) \equiv \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ if the limit exists and is finite.

Notations For The Derivative: $f'(x)$, $Df(x)$, $\dfrac{d\,f(x)}{dx}$

Tangent Line Equation: The line $y = f(a) + f'(a)\cdot(x - a)$ is tangent to the graph of f at (a , f(a)) .

Formulas: $D(\text{constant}) = 0$

      $D(x^n) = n{\cdot}x^{n-1}$ (proven for n = positive integer: true for all constants n)

      $D(\sin(x)) = \cos(x)$ and $D(\cos(x)) = -\sin(x)$

      $D(|x|) = \begin{cases} +1 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$.
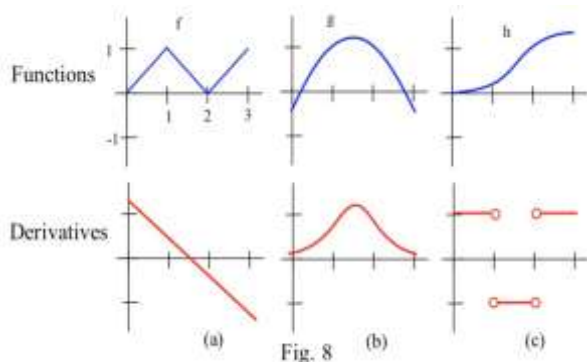
Interpretations of f '(x):   Slope of a line tangent to a graph

          Instantaneous rate of change of a function at a point

          Velocity or acceleration

          Magnification factor
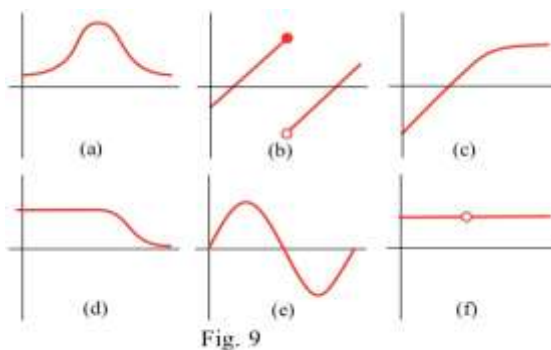
          Marginal change

Problems for Solution

1. Match the graphs of the three functions in Fig. 8 with the graphs of their derivatives.

2. Fig. 9 shows six graphs, three of which are derivatives of the other three. Match the functions with their derivatives.


Fig. 8

Fig. 9

In problems 3 – 6, find the slope m_{sec} of the secant line through the two given points and then calculate $m_{tan} = \lim\limits_{h \to 0} m_{sec}$.

3. $f(x) = x^2$ (a) $(-2,4), (-2+h, (-2+h)^2)$ (b) $(0.5, 0.25), (0.5+h, (0.5+h)^2)$

4. $f(x) = 3 + x^2$ (a) $(1,4), (1+h, 3+(1+h)^2)$ (b) $(x, 3 + x^2), (x+h, 3 + (x+h)^2)$

5. $f(x) = 7x - x^2$ (a) $(1, 6), (1+h, 7(1+h) - (1+h)^2)$ (b) $(x, 7x - x^2), (x+h, 7(x+h) - (x+h)^2)$

6. $f(x) = x^3 + 4x$ (a) $(1, 5), (1+h, (1+h)^3 + 4(1+h))$ (b) $(x, x^3 + 4x), (x+h, (x+h)^3 + 4(x+h))$
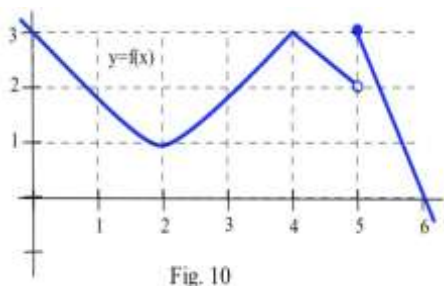
Fig. 10

7. Use the graph in Fig. 10 to estimate the values of these limits. (It helps to recognize what the limit represents.)

(a) $\lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$ (b) $\lim\limits_{h \to 0} \dfrac{f(1+h) - f(1)}{h}$ (c) $\lim\limits_{h \to 0} \dfrac{f(2+h) - 1}{h}$

(d) $\lim\limits_{w \to 0} \dfrac{f(3+w) - f(3)}{w}$ (e) $\lim\limits_{h \to 0} \dfrac{f(4+h) - f(4)}{h}$ (f) $\lim\limits_{s \to 0} \dfrac{f(5+s) - f(5)}{s}$
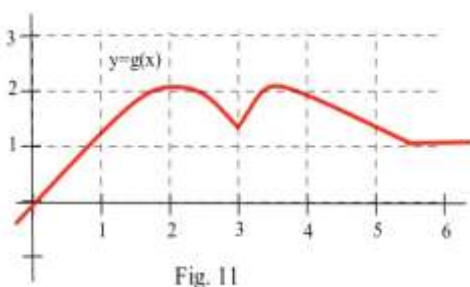
Fig. 11

8. Use the graph in Fig. 11 to estimate the values of these limits.

(a) $\lim\limits_{h \to 0} \dfrac{g(h) - g(0)}{h}$ (b) $\lim\limits_{h \to 0} \dfrac{g(1+h) - g(1)}{h}$ (c) $\lim\limits_{h \to 0} \dfrac{g(2+h) - 2}{h}$

(d) $\lim\limits_{w \to 0} \dfrac{g(3+w) - g(3)}{w}$ (e) $\lim\limits_{h \to 0} \dfrac{g(4+h) - g(4)}{h}$ (f) $\lim\limits_{s \to 0} \dfrac{g(5+s) - g(5)}{s}$

In problems 9 – 12 , use the Definition of the derivative to calculate $f\,'(x)$ and then evaluate $f\,'(3)$.

9. $f(x) = x^2 + 8$ 10. $f(x) = 5x^2 - 2x$

11. $f(x) = 2x^3 - 5x$ 12. $f(x) = 7x^3 + x$

13. Graph $f(x) = x^2$, $g(x) = x^2 + 3$ and $h(x) = x^2 - 5$. Calculate the derivatives of f, g, and h.

14. Graph $f(x) = 5x$, $g(x) = 5x + 2$ and $h(x) = 5x - 7$. Calculate the derivatives of f, g, and h.

In problems 15 – 18, find the slopes and equations of the lines tangent to $y = f(x)$ at the given points.

15. $f(x) = x^2 + 8$ at (1,9) and (–2,12)   16. $f(x) = 5x^2 - 2x$ at (2, 16) and (0,0)

17. $f(x) = \sin(x)$ at $(\pi, 0)$ and $(\pi/2,1)$ 18. $f(x) = |\,x + 3\,|$ at (0,3) and (–3,0)

19.   (a)   Find the equation of the line tangent to the graph of $y = x^2 + 1$ at the point (2,5).
     (b)   Find the equation of the line perpendicular to the graph of $y = x^2 + 1$ at (2,5).
     (c)   Where is the tangent to the graph of $y = x^2 + 1$ horizontal?
     (d)   Find the equation of the line tangent to the graph of $y = x^2 + 1$ at the point (p,q).
     (e)   Find the point(s) (p,q) on the graph of $y = x^2 + 1$ so the tangent line to the curve at (p,q) goes through the point (1, –7).

20.   (a)   Find the equation of the line tangent to the graph of $y = x^3$ at the point (2,8).
     (b)   Where, if ever, is the tangent to the graph of $y = x^3$ horizontal?
     (c)   Find the equation of the line tangent to the graph of $y = x^3$ at the point (p,q).
     (d)   Find the point(s) (p,q) on the graph of $y = x^3$ so the tangent line to the curve at (p,q) goes through the point (16,0).

21.   (a)   Find the angle that the tangent line to $y = x^2$ at (1,1) makes with the x–axis.
     (b)   Find the angle that the tangent line to $y = x^3$ at (1,1) makes with the x–axis.
     (c)   The curves $y = x^2$ and $y = x^3$ intersect at the point (1,1). Find the angle of intersection of the two curves (actually the angle between their tangent lines) at the point (1,1) .

22.   Fig. 12 shows the graph of $y = f(x)$. Sketch the graph of $y = f\,'(x)$.

23.   Fig. 13 shows the graph of the height of an object at time t. Sketch the graph of the

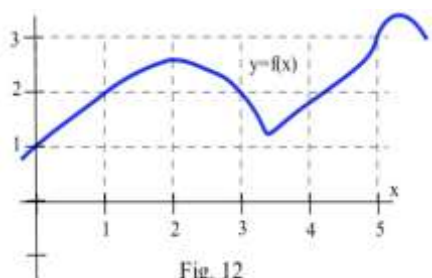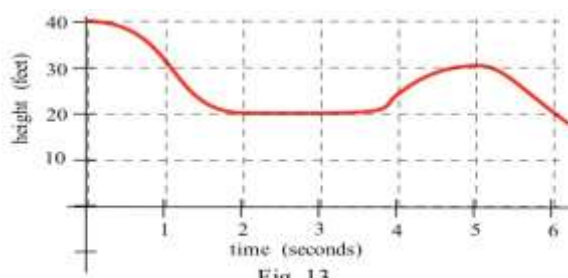object's upward velocity. What are the units for each axis on the velocity graph?

Fig. 12

Fig. 13

24.   Fill in the table with the appropriate units for f '(x).

units for x	units for f(x)	units for f '(x)
hours	miles	
people	automobiles	
dollars	pancakes	
days	trout	
seconds	miles per second	
seconds	gallons	
study hours	test points	

25.   A rock dropped into a deep hole will drop $d(x) = 16x^2$ feet in x seconds.

(a)   How far into the hole will the rock be after 4 seconds? 5 seconds?

(b)   How fast will it be falling at exactly 4 seconds? 5 seconds? x seconds?

26.   It takes $T(x) = x^2$ hours to weave x small rugs. What is the marginal production time to weave

a rug? (Be sure to include the units with your answer.)

27.   It costs $C(x) = \sqrt{x}$ dollars to produce x golf balls. What is the marginal production cost to make

a golf ball? What is the marginal production cost when x = 25? when x = 100? (Include units.)

28.   Define A(x) to be the **area** bounded by the x and y axes, the

line y = 5, and a vertical line at x (Fig. 14).

(a)   Evaluate A(0), A(1), A(2) and A(3).

(b)   Find a formula for A(x) for x ≥ 0: A(x) = ?

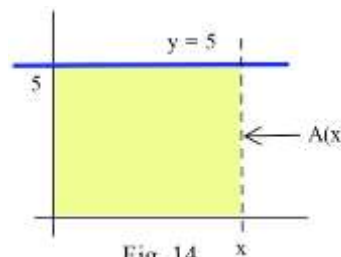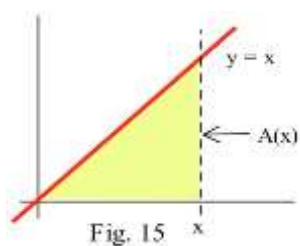(c)   Determine $\dfrac{d\ A(x)}{dx}$.   (d)   What does $\dfrac{d\ A(x)}{dx}$ represent?

Fig. 14

29. Define A(x) to be the **area** bounded by the x–axis,

the line y = x, and a vertical line at x (Fig. 15).

(a) Evaluate A(0), A(1), A(2) and A(3).

(b) Find a formula which represents A(x) for all x ≥ 0: A(x) = ?

(c) Determine $\dfrac{d\,A(x)}{dx}$.   (d) What does $\dfrac{d\,A(x)}{dx}$ represent?

Fig. 15

30. Find (a) $\mathbf{D}(x^{12})$ (b) $\dfrac{\mathbf{d}}{\mathbf{dx}}(\sqrt[7]{x})$ (c) $\mathbf{D}(\dfrac{1}{x^3})$ (d) $\dfrac{\mathbf{d}\,x^e}{\mathbf{dx}}$ (e) $\mathbf{D}(\,|\,x{-}2\,|\,)$

31. Find (a) $\mathbf{D}(x^9)$ (b) $\dfrac{\mathbf{d}\,x^{2/3}}{\mathbf{dx}}$ (c) $\mathbf{D}(\dfrac{1}{x^4})$ (d) $\mathbf{D}(x^\pi)$ (e) $\dfrac{\mathbf{d}\,|\,x{+}5\,|}{\mathbf{dx}}$

In problems 32 – 37, find a function f which has the given derivative. (Each problem has several correct answers, just find one of them.)

32. $f\,'(x) = 4x + 3$ 33. $f\,'(x) = 3x^2 + 8x$ 34. $\mathbf{D}(\,f(x)\,) = 12x^2 - 7$

35. $\dfrac{\mathbf{d}\,f(t)}{\mathbf{dt}} = 5\cos(t)$ 36. $\dfrac{\mathbf{d}\,f(x)}{\mathbf{dx}} = 2x - \sin(x)$ 37. $\mathbf{D}(\,f(x)\,) = x + x^2$

Section 3.2   **PRACTICE Answers**

Practice 1:   The graph of f(x) = 7x is a line through the origin. The slope of the line is 7.

For all x, $m_{\tan} = \lim\limits_{h\to0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to0} \dfrac{7(x+h)-7x}{h} = \lim\limits_{h\to0} \dfrac{7h}{h} = \lim\limits_{h\to0} 7 = \mathbf{7}$.

Practice 2: $f(x) = x^3$ so $f(x + h) = (x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$.

$\dfrac{dy}{dx} = \lim\limits_{h\to0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to0} \dfrac{\left\{x^3 + 3x^2h + 3xh^2 + h^3\right\} - x^n}{h}$

$= \lim\limits_{h\to0} \dfrac{3x^2h + 3xh^2 + h^3}{h} = \lim\limits_{h\to0} (3x^2 + 3xh + h^2) = \mathbf{3x^2}$.

At the point (2,8), the slope of the tangent line is $3(2)^2 = 12$ so the equation of the tangent
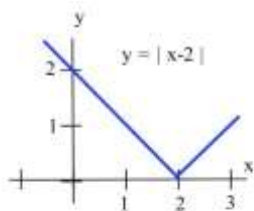
line is y – 8 = 12(x – 2) or y = 12x –16.

Practice 3: $D(x^4) = 4x^3$, $D(x^5) = 5x^4$, $D(x^{43}) = 43x^{42}$, $D(x^{1/2}) = \frac{1}{2} x^{-1/2}$, $D(x^\pi) = \pi x^{\pi-1}$

Practice 4: $D(\cos(x)) = \lim\limits_{h\to 0} \dfrac{\cos(x+h) - \cos(x)}{h} = \lim\limits_{h\to 0} \dfrac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$

$$= \lim\limits_{h\to 0} \cos(x)\frac{\cos(h)-1}{h} - \sin(x)\frac{\sin(h)}{h} \longrightarrow \cos(x){\cdot}(0) - \sin(x){\cdot}(1) = -\sin(x) .$$

Fig. 16

Practice 5: See Fig. 16 for the graphs of $y = |x - 2|$ and $y = |2x|$.

$$D(|x-2|) = \begin{cases} +1 & \text{if } x > 2 \\ \text{undefined} & \text{if } x = 2 \\ -1 & \text{if } x < 2 \end{cases}$$

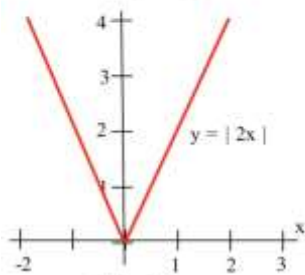$$D(|2x|) = \begin{cases} +2 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ -2 & \text{if } x < 0 \end{cases}$$

Practice 6: $h(t) = \sin(t)$ so $h(1) = \sin(1) \approx 0.84$ feet,

$v(t) = \cos(t)$ so $v(1) = \cos(1) \approx 0.54$ feet/second.

$a(t) = -\sin(t)$ so $a(1) = -\sin(1) \approx -0.84$ feet/second^2 .

Practice 7: $D(x^5) = 5x^4$, $\dfrac{d\,x^2}{dx} = 2x^1 = 2x$, $\dfrac{d\,x^{100}}{dx} = 100x^{99}$, $\dfrac{d\,t^{31}}{dt} = 31t^{30}$,

$D(x^0) = 0x^{-1} = 0$ or $D(x^0) = D(1) = 0$.

Practice 8: $D(x^{3/2}) = \dfrac{3}{2} x^{1/2}$, $\dfrac{d\,x^{1/3}}{dx} = \dfrac{1}{3} x^{-2/3}$, $D(x^{-1/2}) = \dfrac{-1}{2} x^{-3/2}$, $\dfrac{d\,t^\pi}{dt} = \pi\, t^{\pi-1}$.