4.2 Directional Derivative

For a function of 2 variables $f(x, y)$, we have seen that the function can be used to represent the surface

$$z = f(x, y)$$

and recall the geometric interpretation of the partials:

(i) $f_x(a, b)$-represents the rate of change of the function $f(x, y)$ as we vary x and hold $y = b$ fixed.

(ii) $f_y(a, b)$-represents the rate of change of the function $f(x, y)$ as we vary y and hold $x = a$ fixed.

We now ask, at a point P can we calculate the slope of f in an arbitrary direction?

Recall the definition of the vector function ∇f,

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

We observe that,

$$\begin{aligned} \nabla f \cdot \hat{i} &= f_x \\ \nabla f \cdot \hat{j} &= f_y \end{aligned}$$

This enables us to calculate the directional derivative in an arbitrary direction, by taking the dot product of ∇f with a unit vector, $\vec{u}$, in the desired direction.

Definition. *The directional derivative of the function f in the direction $\vec{u}$ denoted by $D_{\vec{u}} f$, is defined to be,*

$$D_{\vec{u}} f = \frac{\nabla f \cdot \vec{u}}{|\vec{u}|}$$

Example. What is the directional derivative of $f(x, y) = x^2 + xy$, in the direction $\vec{i} + 2\vec{j}$ at the point $(1, 1)$?

Solution: We first find ∇f.

$$\begin{aligned}\nabla f &= (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \\ &= (2x+y, x) \\ \nabla f(1,1) &= (3,1)\end{aligned}$$

Let $u = \vec{i} + 2\vec{j}$.

$$|\vec{u}| = \sqrt{1^2+2^2} = \sqrt{1+4} = \sqrt{5}.$$

$$\begin{aligned}D_{\vec{u}} f(1,1) &= \frac{\nabla f \cdot \vec{u}}{|\vec{u}|} \\ &= \frac{(3,1).(1,2)}{\sqrt{5}} \\ &= \frac{(3)(1)+(1)(2)}{\sqrt{5}} \\ &= \frac{5}{\sqrt{5}} \\ &= \sqrt{5}\end{aligned}$$

Properties of the Gradient deduced from the formula of Directional Derivatives

$$\begin{aligned}D_{\vec{u}} f &= \frac{\nabla f \cdot \vec{u}}{|\vec{u}|} \\ &= \frac{|\nabla f||\vec{u}| \cos\theta}{|\vec{u}|} \\ &= |\nabla f| \cos\theta\end{aligned}$$

1. If $\theta = 0$, i.e. i.e. $\vec{u}$ points in the same direction as ∇f, then $D_{\vec{u}} f$ is maximum. Therefore we may conclude that

   (i) ∇f **points in the steepest direction**.

   (ii) The **magnitude of ∇f gives the slope in the steepest direction**.

2. At any point P, $\nabla f(P)$ **is perpendicular to the level set** through that point.

EXAMPLE. 1. Let $f(x,y) = x^2 + y^2$ and let $P = (1,2,5)$. Then P lies on the graph of f since $f(1,2) = 5$. Find the slope and the direction of the steepest ascent at P on the graph of f

SOLUTION:

- We use the first property of the Gradient vector. The direction of the steepest ascent at P on the graph of f is the direction of the gradient vector at the point $(1,2)$.

$$\begin{aligned} \nabla f &= (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \\ &= (2x, 2y) \\ \nabla f(1,2) &= (2,4). \end{aligned}$$

- The slope of the steepest ascent at P on the graph of f is the magnitude of the gradient vector at the point $(1,2)$.

$$|\nabla f(1,2)| = \sqrt{2^2 + 4^2} = \sqrt{20}.$$

2. Find a normal vector to the graph of the equation $f(x,y) = x^2 + y^2$ at the point $(1,2,5)$. Hence write an equation for the tangent plane at the point $(1,2,5)$.

SOLUTION: We use the second property of the gradient vector. For a function g, $\nabla g(P)$ **is perpendicular to the level set**. So we want our surface $z = x^2 + y^2$ to be the level set of a function.
Therefore we define a new function,

$$g(x,y,z) = x^2 + y^2 - z.$$

Then our surface is the level set

$$\begin{aligned} g(x,y,z) &= 0 \\ x^2 + y^2 - z &= 0 \\ z &= x^2 + y^2 \end{aligned}$$

$$
\begin{aligned}
\nabla g &= (\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}) \\
&= (2x, 2y, -1) \\
\nabla g(1,2,5) &= (2, 4, -1)
\end{aligned}
$$

By the above property, $\nabla g(P)$ is perpendicular to the level set $g(x, y, z) = 0$. Therefore $\nabla g(P)$ is the required normal vector.
Finally an equation for the tangent plane at the point $(1, 2, 5)$ on the surface is given by

$$2(x - 1) + 4(y - 2) - 1(z - 5) = 0.$$

4.3 Curl and Divergence

We denoted the gradient of a scalar function $f(x, y, z)$ as

$$\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$$

Let us separate or isolate the operator $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$. We can then define various physical quantities such as div, curl by specifying the action of the operator ∇.

Divergence

DEFINITION. *Given a vector field $\vec{v}(x, y, z) = (v_1(x, y, z), v_2(x, y, z), v_3(x, y, z))$, the divergence of $\vec{v}$ is a scalar function defined as the dot product of the vector operator ∇ and $\vec{v}$,*

$$
\begin{aligned}
\text{Div}\, \vec{v} &= \nabla \cdot \vec{v} \\
&= (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot (v_1, v_2, v_3) \\
&= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}
\end{aligned}
$$

EXAMPLE. Compute the divergence of $(x - y)\vec{i} + (x + y)\vec{j} + z\vec{k}$.

Solution:

$$\begin{aligned}
\vec{v} &= ((x-y),(x+y),z) \\
\nabla &= (\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}) \\
\operatorname{Div}\vec{v} &= \nabla\cdot\vec{v} \\
&= (\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})\cdot((x-y),(x+y),z) \\
&= \frac{\partial(x-y)}{\partial x}+\frac{\partial(x+y)}{\partial y}+\frac{\partial z}{\partial z} \\
&= 1+1+1 \\
&= 3
\end{aligned}$$

Curl

Definition. *The curl of a vector field is a vector function defined as the cross product of the vector operator* ∇ *and* $\vec{v}$*,*

$$\begin{aligned}
\operatorname{Curl}\vec{v}=\nabla\times\vec{v} &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \\
&= (\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z})i-(\frac{\partial v_3}{\partial x}-\frac{\partial v_1}{\partial z})j+(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y})k
\end{aligned}$$

Example. Compute the curl of the vector function $(x-y)\vec{i}+(x+y)\vec{j}+z\vec{k}$.

Solution:

$$\begin{aligned}
\operatorname{Curl}\vec{v}=\nabla\times\vec{v} &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x-y) & (x+y) & z \end{vmatrix} \\
&= (\frac{\partial z}{\partial y}-\frac{\partial(x+y)}{\partial z})i-(\frac{\partial z}{\partial x}-\frac{\partial(x-y)}{\partial z})j+(\frac{\partial(x+y)}{\partial x}-\frac{\partial(x-y)}{\partial y})k \\
&= (0-0)\vec{i}-(0-0)\vec{j}+(1-(-1))\vec{k} \\
&= 2\vec{k}
\end{aligned}$$

4.4 Laplacian

We have seen above that given a vector function, we can calculate the divergence and curl of that function. A scalar function f has a vector function ∇f associated to it. We now look at $\text{Curl}(\nabla f)$ and $\text{Div}(\nabla f)$.

$$\begin{aligned}
\text{Curl}(\nabla f) &= \nabla \times \nabla f \\
&= (\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z})i + (\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x})j + (\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y})k \\
&= (f_{yz} - f_{zy})i + (f_{zx} - f_{xz})j + (f_{xy} - f_{yx})k \\
&= 0 \\
\text{Div}(\nabla f) &= \nabla \cdot \nabla f \\
&= (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

DEFINITION. *The Laplacian of a scalar function $f(x, y)$ of two variables is defined to be* $\text{Div}(\nabla f)$ *and is denoted by* $\nabla^2 f$,

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

The Laplacian of a scalar function $f(x, y, z)$ of three variables is defined to be $\text{Div}(\nabla f)$ *and is denoted by* $\nabla^2 f$,

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

EXAMPLE. Compute the Laplacian of $f(x, y, z) = x^2 + y^2 + z^2$.

SOLUTION:

$$\begin{aligned}
\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\
&= \frac{\partial 2x}{\partial x} + \frac{\partial 2y}{\partial y} + \frac{\partial 2z}{\partial z} \\
&= 2 + 2 + 2 \\
&= 6.
\end{aligned}$$

We have the following identities for the Laplacian in different coordinate systems:

$$\begin{aligned}
\textit{Rectangular} &: \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\
\textit{Polar} &: \nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\big(r\frac{\partial f}{\partial r}\big) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \\
\textit{Cylindrical} &: \nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\big(r\frac{\partial f}{\partial r}\big) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \\
\textit{Spherical} &: \nabla^2 f = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\big(\rho^2\frac{\partial f}{\partial \rho}\big) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\big(\sin\theta\frac{\partial f}{\partial \theta}\big) + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}
\end{aligned}$$

EXAMPLE. Consider the same function $f(x, y, z) = x^2 + y^2 + z^2$. We have seen that in rectangular coordinates we get

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 6.$$

We now calculate this in cylindrical and spherical coordinate systems, using the formulas given above.

1. Cylindrical Coordinates.
   We have $x = r\cos\theta$ and $y = r\sin\theta$ so

   $$f(r, \theta, z) = r^2\cos^2\theta + r^2\sin^2\theta + z^2 = r^2 + z^2.$$

   Using the above formula:

   $$\begin{aligned}
   \nabla^2 f &= \frac{1}{r}\frac{\partial}{\partial r}\big(r\frac{\partial f}{\partial r}\big) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \\
   &= \frac{1}{r}\frac{\partial}{\partial r}(r2r) + 0 + \frac{\partial(2z)}{\partial z} \\
   &= \frac{1}{r}(4r) + 2 \\
   &= 4 + 2 \\
   &= 6
   \end{aligned}$$

2. Spherical Coordinates.
   We have $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$ and $\rho = \sqrt{x^2 + y^2 + z^2}$, so

   $$f(r, \theta, z) = \rho^2.$$

Using the above formula:

$$\begin{aligned}
\nabla^2 f &= \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\big(\rho^2\frac{\partial f}{\partial\rho}\big) + \frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\big(\sin\theta\frac{\partial f}{\partial\theta}\big) + \frac{1}{\rho^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} \\
&= \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2 2\rho) + 0 + 0 \\
&= \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(2\rho^3) \\
&= \frac{1}{\rho^2}(6\rho^2) \\
&= 6.
\end{aligned}$$

These three different calculations all produce the same result because ∇^2 is a derivative with a real physical meaning, and does not depend on the coordinate system being used.

References

1. A briliant animated example, showing that the maximum slope at a point occurs in the direction of the gradient vector. The animation shows:

   - a surface
   - a unit vector rotating about the point (1, 1, 0), (shown as a rotating black arrow at the base of the figure)
   - a rotating plane parallel to the unit vector, (shown as a grey grid)
   - the traces of the planes in the surface, (shown as a black curve on the surface)
   - the tangent lines to the traces at (1, 1, f (1, 1)), (shown as a blue line)
   - the gradient vector (shown in green at the base of the figure)

   http://archives.math.utk.edu/ICTCM/VOL10/C009/dd.gif

2. A complete set of notes on Pre-Calculus, Single Variable Calculus, Multivariable Calculus and Linear Algebra.
   Here is a link to the chapter on Directional Derivatives.
   http://tutorial.math.lamar.edu/Classes/CalcIII/DirectionalDeriv.aspx.

   Here is a link to the chapter on Curl and Divergence.
   http://tutorial.math.lamar.edu/Classes/CalcIII/CurlDivergence.aspx