2.20 – Marine Hydrodynamics
Lecture 2

Chapter 1 - Basic Equations

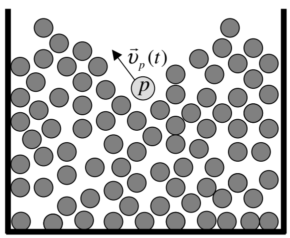
1.1 Description of a Flow

To define a flow we use either the 'Lagrangian' description or the 'Eulerian' description.

- Lagrangian description: Picture a fluid flow where each fluid particle caries its own properties such as density, momentum, etc. As the particle advances its properties may change in time. The procedure of describing the entire flow by recording the detailed histories of each fluid particle is the Lagrangian description. A neutrally buoyant probe is an example of a Lagrangian measuring device.

  The particle properties density, velocity, pressure, ...can be mathematically represented as follows: $\rho_p(t), \vec{v}_p(t), p_p(t), \ldots$

  The Lagrangian description is simple to understand: conservation of mass and Newton's laws apply directly to each fluid particle . However, it is computationally expensive to keep track of the trajectories of all the fluid particles in a flow and therefore the Lagrangian description is used only in some numerical simulations.



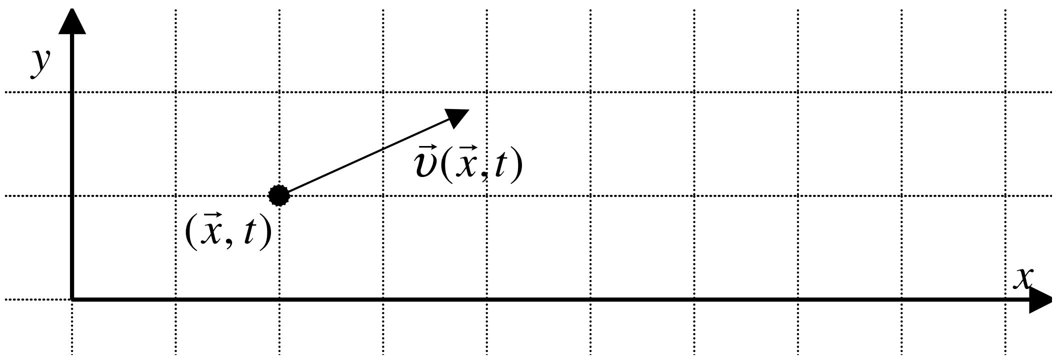
Lagrangian description; snapshot

- Eulerian description: Rather than following each fluid particle we can record the evolution of the flow properties at every point in space as time varies. This is the Eulerian description. It is a *field* description. A probe fixed in space is an example of an Eulerian measuring device.

  This means that the flow properties at a specified location depend on the location and on time. For example, the density, velocity, pressure, ... can be mathematically represented as follows: $\vec{v}(\vec{x}, t), p(\vec{x}, t), \rho(\vec{x}, t), \ldots$

  The aforementioned locations are described in coordinate systems. In 2.20 we use the cartesian, cylindrical and spherical coordinate systems.

  The Eulerian description is harder to understand: how do we apply the conservation laws? However, it turns out that it is mathematically simpler to apply. For this reason, in Fluid Mechanics we use mainly the Eulerian description.



Eulerian description; Cartesian grid

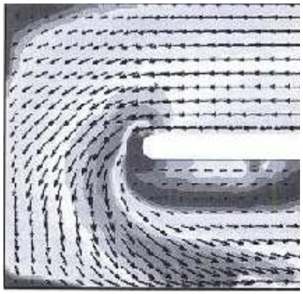
1.2 Flow visualization - Flow lines

- Streamline: A line everywhere tangent to the fluid velocity $\vec{v}$ at a given instant (flow snapshot). It is a strictly Eulerian concept.

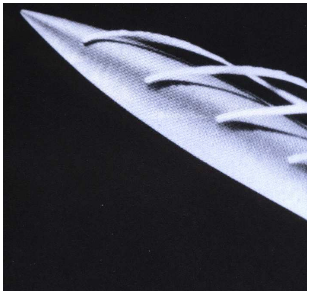
- Streakline: Instantaneous locus of all fluid particles that have passed a given point (snapshot of *certain* fluid particles).

- Pathline: The trajectory of a given particle P in time. The photograph analogy would be a long time exposure of a marked particle. It is a strictly Lagrangian concept.

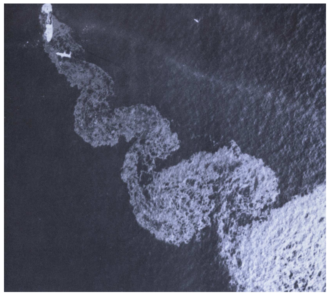
Can you tell whether any of the following figures ([1] Van Dyke, *An Album of Fluid Motion* 1982 (p.52, 100)) show streamlines/streaklines/pathlines?



Velocity vectors



Emitted colored fluid [1]



Grounded *Argo Merchant* [1]

1.3 Some Quantities of Interest

- Einstein Notation

  - Range convention: Whenever a subscript appears only *once* in a term, the subscript takes *all* possible values. E.g. in 3D space:

  $$x_i \quad (i = 1, 2, 3) \quad \longrightarrow \quad x_1, \quad x_2, \quad x_3$$

  - Summation convention: Whenever a subscript appears *twice* in the *same term* the repeated index is summed over the index parameter space. E.g. in 3D space:

  $$a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (i = 1, 2, 3)$$

  Non repeated subscripts remain fixed during the summation. E.g. in 3D space $a_i = x_{ij} \hat{n}_j$ denotes three equations, one for each $i = 1, 2, 3$ and j is the **dummy index**.

  Note 1: To avoid confusion between fixed and repeated indices or different repeated indices, etc, no index can be repeated more than twice.

  Note 2: Number of free indices shows how many quantities are represented by a single term.

  Note 3: If the equation looks like this: $(u_i)(\hat{x}_i)$, the indices are **not** summed.

  - Comma convention: A subscript comma followed by an index indicates partial differentiation with respect to each coordinate. Summation and range conventions apply to indices following a comma as well. E.g. in 3D space:

  $$u_{i,i} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

- Scalars, Vectors and Tensors

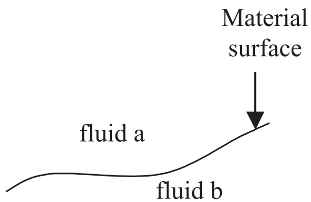
Scalars	**Vectors** (a_i)	**Tensors** (a_{ij})
magnitude	*magnitude* *direction*	*magnitude* *direction* *orientation*
density $\rho(\vec{x}, t)$ pressure $p(\vec{x}, t)$	velocity $\vec{v}(\vec{x}, t)$ /momentum mass flux	momentum flux stress $\tau_{ij}(\vec{x}, t)$

1.4 Concept and Consequences of Continuous Flow

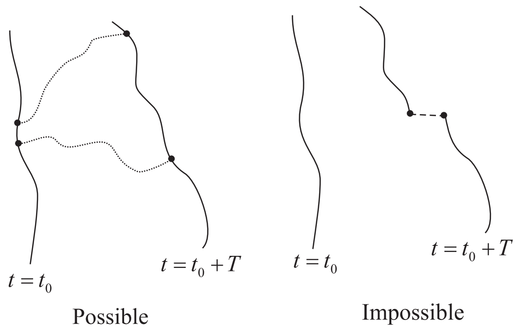
For a fluid flow to be continuous, we require that the velocity $\vec{v}(\vec{x}, t)$ be a finite and continuous function of $\vec{x}$ and t.
i.e. $\nabla \cdot \vec{v}$ and $\frac{\partial \vec{v}}{\partial t}$ are finite but **not** necessarily continuous.
Since $\nabla \cdot \vec{v}$ and $\frac{\partial \vec{v}}{\partial t} < \infty$, there is no infinite acceleration i.e. no infinite forces , which is physically consistent.

1.4.1 Consequences of Continuous Flow

- Material volume remains material. No segment of fluid can be joined or broken apart.
- Material surface remains material. The interface between two material volumes always exists.
- Material line remains material. The interface of two material surfaces always exists.



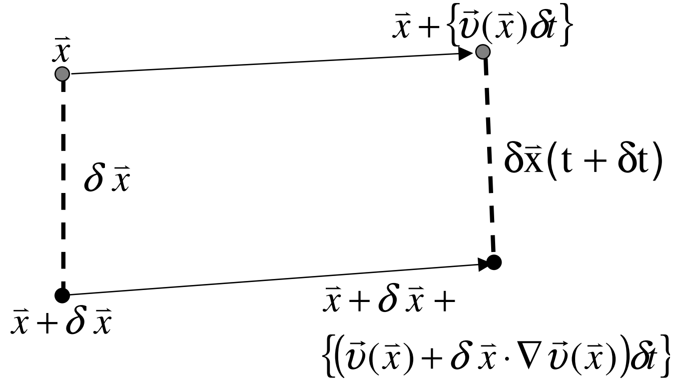
- Material neighbors remain neighbors. To prove this mathematically, we must prove that, given two particles, the distance between them at time t is small, and the distance between them at time $t + \delta t$ is still small.



Assumptions At time t, assume a continuous flow ($\nabla \cdot \vec{v}, \frac{\partial \vec{v}}{\partial t} << \infty$) with fluid velocity $\vec{v}(\vec{x}, t)$. Two arbitrary particles are located at $\vec{x}$ and $\vec{x} + \delta\vec{x}(t)$, respectively.

Result If $\delta\vec{x}(t) \equiv \delta\vec{x} \to 0$ then $\delta\vec{x}(t+T) \to 0$, for all subsequent times $t + T$.

Proof



After a small time δt:

- The particle initially located at $\vec{x}$ will have travelled a distance $\vec{v}(\vec{x})\delta t$ and at time $t + \delta t$ will be located at $\vec{x} + \{\vec{v}(\vec{x})\delta t\}$.

- The particle initially located at $\vec{x}+\delta\vec{x}$ will have travelled a distance $(\vec{v}(\vec{x}) + \delta\vec{x} \cdot \nabla\vec{v}(\vec{x}))\, \delta t$. (Show this using Taylor Series Expansion about$(\vec{x}, t)$). Therefore after a small time δt this particle will be located at $\vec{x} + \delta\vec{x} \ + \{(\vec{v}(\vec{x}) + \delta\vec{x} \cdot \nabla\vec{v}(\vec{x}))\delta t\}$.

- The difference in position $\delta\vec{x}(t + \delta t)$ between the two particles after a small time δt will be:

$$\begin{aligned} \delta\vec{x}(t + \delta t) &= \vec{x} + \delta\vec{x} + \{(\vec{v}(\vec{x}) + \delta\vec{x} \cdot \nabla\vec{v}(\vec{x}))\delta t\} - (\vec{x} + \{\vec{v}(\vec{x})\delta t\}) \\ \Rightarrow \quad \delta\vec{x}(t + \delta t) &= \delta\vec{x} + (\delta\vec{x} \cdot \nabla\vec{v}(\vec{x}))\delta t \ \propto \ \delta\vec{x} \end{aligned}$$

Therefore $\delta\vec{x}(t + \delta t) \propto \delta\vec{x}$ because $\nabla\vec{v}$ is finite (from continuous flow assumption). Thus, if $\delta\vec{x} \to 0$, then $\delta\vec{x}(t + \delta t) \to 0$. In fact, for any subsequent time $t + T$:

$$\delta\vec{x}(t + T) \propto \delta\vec{x} + \int_t^{t+T} \delta\vec{x} \cdot \nabla\vec{v}dt \ \propto \ \delta\vec{x},$$

and $\delta\vec{x}(t+T) \to 0$ as $\delta\vec{x} \to 0$. In other words the particles will never be an infinite distance apart. *Thus, if the flow is continuous two particles that are neighbors will always remain neighbors.*

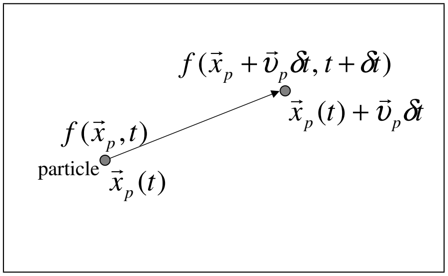
1.5 Material/Substantial/Total Time Derivative: D/Dt

A material derivative is the time derivative – rate of change – of a property following a fluid particle ‘p’. The material derivative is a Lagrangian concept.

By expressing the material derivative in terms of Eulerian quantities we will be able to apply the conservation laws in the Eulerian reference frame.

Consider an Eulerian quantity $f(\vec{x}, t)$. The time rate of change of f as experienced by a particle ‘p’ travelling with velocity $\overrightarrow{v_p}$ is the substantial derivative of f and is given by:

$$\frac{Df(\overrightarrow{x_p}(t), t)}{Dt} = \lim_{\delta t \to 0} \frac{f(\overrightarrow{x_p} + \overrightarrow{v_p}\delta t, t + \delta t) - f(\overrightarrow{x_p}, t)}{\delta t} \quad (1)$$



Performing a Taylor Series Expansion about $(\overrightarrow{x_p}, t)$ and taking into account that $\overrightarrow{v_p}\delta t = \delta\vec{x}$, we obtain:

$$f(\overrightarrow{x_p} + \overrightarrow{v_p}\delta t, t + \delta t) = f(\vec{x}, t) + \delta t \frac{\partial f(\vec{x}, t)}{\partial t} + \delta\vec{x} \cdot \nabla f(\vec{x}, t) + O(\delta^2)(\text{higher order terms})(2)$$

From Eq.(1, 2) we see that the substantial derivative of f as experienced by a particle travelling with $\overrightarrow{v_p}$ is given by:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \overrightarrow{v_p} \cdot \nabla f$$

The generalized notation:

$$\underbrace{\frac{D}{Dt}}_{\text{Lagrangian}} \equiv \underbrace{\frac{\partial}{\partial t} + \overrightarrow{v_p} \cdot \nabla}_{\text{Eulerian}}$$

Example 1: Material derivative of a fluid property $\vec{G}(\vec{x},t)$ as experienced by a fluid particle.

Let 'p' denote a fluid particle. A fluid particle is always travelling with the local fluid velocity $\vec{v}_p(t) = \vec{v}(\vec{x}_p,t)$. The material derivative of a fluid property $\vec{G}(\vec{x},t)$ as experienced by this fluid particle is given by:

$$\underbrace{\frac{D\vec{G}}{Dt}}_{\substack{\text{Lagragian} \\ \text{rate of change}}} = \underbrace{\frac{\partial\vec{G}}{\partial t}}_{\substack{\text{Eulerian} \\ \text{rate of change}}} + \underbrace{\vec{v}\cdot\nabla\vec{G}}_{\substack{\text{Convective} \\ \text{rate of change}}}$$

Example 2: Material derivative of the fluid velocity $\vec{v}(\vec{x},t)$ as experienced by a fluid particle. This is the Lagrangian acceleration of a particle and is the acceleration that appears in Newton's laws. It is therefore evident that its Eulerian representation will be used in the Eulerian reference frame.

Let 'p' denote a fluid particle. A fluid particle is always travelling with the local fluid velocity $\vec{v}_p(t) = \vec{v}(\vec{x}_p,t)$. The Lagrangian acceleration $\frac{D\vec{v}(\vec{x},t)}{Dt}$ as experienced by this fluid particle is given by:

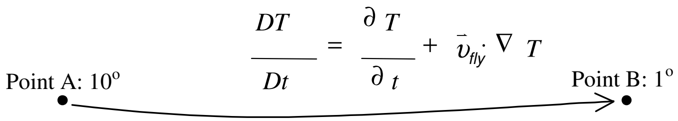
$$\underbrace{\frac{D\vec{v}}{Dt}}_{\substack{\text{Lagragian} \\ \text{acceleration}}} = \underbrace{\frac{\partial\vec{v}}{\partial t}}_{\substack{\text{Eulerian} \\ \text{acceleration}}} + \underbrace{\vec{v}\cdot\nabla\vec{v}}_{\substack{\text{Convective} \\ \text{acceleration}}}$$

1.6 Difference Between Lagrangian Time Derivative and Eulerian Time Derivative

Example 1: Consider an Eulerian quantity, temperature, in a room at points A and B where the temperature is different at each point.

At a fixed in space point C, the temperature rate of change is $\frac{\partial T}{\partial t}$ which is an Eulerian time derivative.

Example 2: Consider the same example as above: an Eulerian quantity, temperature, in a room at points A and B where the temperature varies with time.

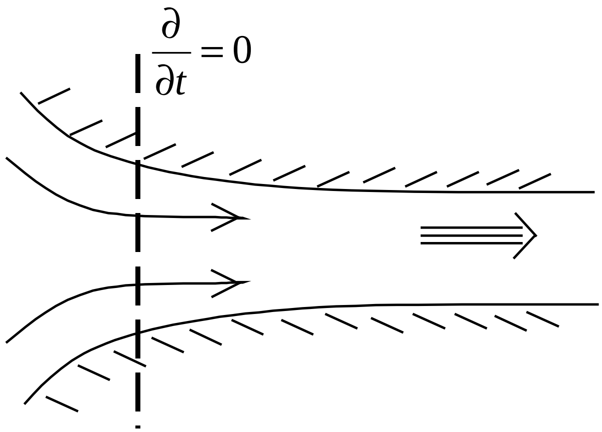


Following a fly from point A to B, the Lagrangian time derivative would need to include the temperature gradient as both time and position changes: $\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{v}_{fly} \cdot \nabla T$

1.6.1 **Concept of a Steady Flow** ($\frac{\partial}{\partial t} \equiv 0$)

A steady flow is a strictly **Eulerian** concept.

Assume a steady flow where the flow is observed from a fixed position. This is like watching from a river bank, i.e. $\frac{\partial}{\partial t} = 0$. Be careful not to confuse this with $\frac{D}{Dt}$ which is more like following a twig in the water. Note that $\frac{D}{Dt} = 0$ does not mean steady since the flow could speed up at some points and slow down at others.



1.6.2 **Concept of an Incompressible Flow** ($\frac{D\rho}{Dt} \equiv 0$)

An incompressible flow is a strictly **Lagrangian** concept.

Assume a flow where the density of *each* fluid particle is constant in time. Be careful not to confuse this with $\frac{\partial \rho}{\partial t} = 0$, which means that the density at a particular point in the flow is constant and would allow particles to change density as they flow from point to point. Also, do not confuse this with $\rho = const$, which for example does not allow a flow of two incompressible fluids.