Module 2

Kinematics of deformation and Strain

Learning Objectives

- develop a mathematical description of the local state of deformation at a material point
- understand the tensorial character of the resulting strain tensor
- distinguish between a compatible and an incompatible strain field and understand the mathematical requirements for strain compatibility
- describe the local state of strain from experimental strain-gage measurements
- understand the limitations of the linearized theory and discern situations where non-linear effects need to be considered.

2.1 Local state of deformation at a material point

Readings: BC 1.4.1

Deformation described by *deformation mapping*:

$$\mathbf{x}' = \varphi(\mathbf{x}) \tag{2.1}$$

We seek to characterize the local state of deformation of the material in a neighborhood of a point P. Consider two points P and Q in the undeformed:

$$P : \mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = x_i\mathbf{e}_i \tag{2.2}$$

$$Q : \mathbf{x} + \mathbf{dx} = (x_i + dx_i)\mathbf{e}_i \tag{2.3}$$

and deformed

$$P' : \mathbf{x}' = \varphi_1(\mathbf{x})\mathbf{e}_1 + \varphi_2(\mathbf{x})\mathbf{e}_2 + \varphi_3(\mathbf{x})\mathbf{e}_3 = \varphi_i(\mathbf{x})\mathbf{e}_i \tag{2.4}$$

$$Q' : \mathbf{x}' + \mathbf{dx}' = \big(\varphi_i(\mathbf{x}) + d\varphi_i\big)\mathbf{e}_i \tag{2.5}$$

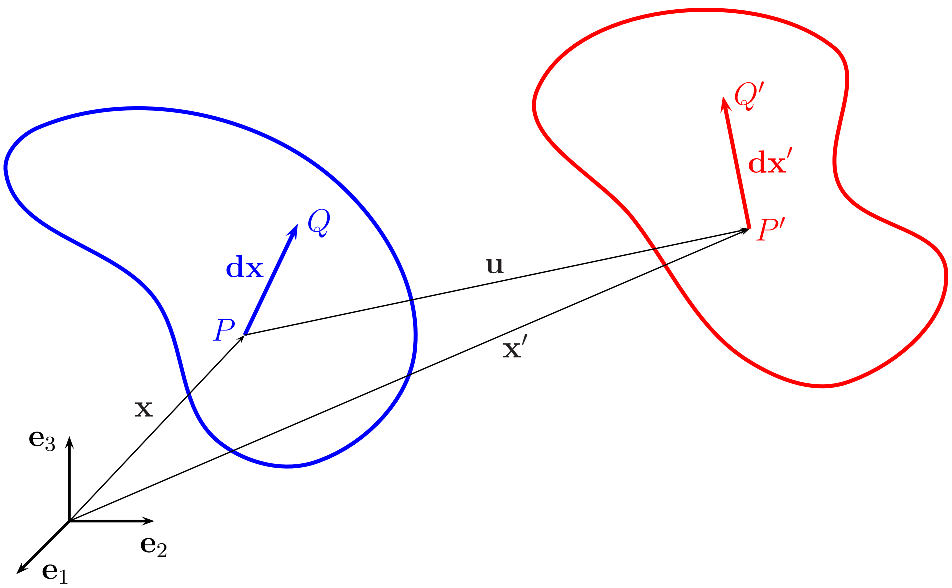


Figure 2.1: Kinematics of deformable bodies

configurations. In this expression,

$$\mathbf{dx}' = d\varphi_i \mathbf{e}_i \tag{2.6}$$

Expressing the differentials $d\varphi_i$ in terms of the partial derivatives of the functions $\varphi_i(x_j \mathbf{e}_j)$:

$$d\varphi_1 = \frac{\partial \varphi_1}{\partial x_1} dx_1 + \frac{\partial \varphi_1}{\partial x_2} dx_2 + \frac{\partial \varphi_1}{\partial x_3} dx_3, \tag{2.7}$$

and similarly for $d\varphi_2, d\varphi_3$, in index notation:

$$d\varphi_i = \frac{\partial \varphi_i}{\partial x_j} dx_j \tag{2.8}$$

Replacing in equation (2.5):

$$Q' : \mathbf{x}' + \mathbf{dx}' = \Big(\varphi_i + \frac{\partial \varphi_i}{\partial x_j} dx_j \Big) \mathbf{e}_i \tag{2.9}$$

$$\mathbf{dx}'_i = \frac{\partial \varphi_i}{\partial x_j} dx_j \mathbf{e}_i \tag{2.10}$$

We now try to compute the change in length of the segment $\overrightarrow{PQ}$ which deformed into segment $\overrightarrow{P'Q'}$. Undeformed length (to the square):

$$ds^2 = \|\mathbf{dx}\|^2 = \mathbf{dx} \cdot \mathbf{dx} = dx_i dx_i \tag{2.11}$$

Deformed length (to the square):

$$(ds')^2 = \|\mathbf{dx}'\|^2 = \mathbf{dx}' \cdot \mathbf{dx}' = \frac{\partial \varphi_i}{\partial x_j} dx_j \frac{\partial \varphi_i}{\partial x_k} dx_k \tag{2.12}$$

The change in length of segment $\overrightarrow{PQ}$ is then given by the difference between equations (2.12) and (2.11):

$$(ds')^2 - ds^2 = \frac{\partial \varphi_i}{\partial x_j} dx_j \frac{\partial \varphi_i}{\partial x_k} dx_k - dx_i dx_i \tag{2.13}$$

We want to extract as common factor the differentials. To this end we observe that:

$$dx_i dx_i = dx_j dx_k \delta_{jk} \tag{2.14}$$

Then:

$$\begin{aligned}(ds')^2 - ds^2 &= \frac{\partial \varphi_i}{\partial x_j} dx_j \frac{\partial \varphi_i}{\partial x_k} dx_k - dx_j dx_k \delta_{jk} \\ &= \underbrace{\left(\frac{\partial \varphi_i}{\partial x_j}\frac{\partial \varphi_i}{\partial x_k} - \delta_{jk}\right)}_{2\epsilon_{jk}:\ \textit{Green-Lagrange strain tensor}} dx_j dx_k\end{aligned} \tag{2.15}$$

Assume that the deformation mapping $\varphi(\mathbf{x})$ has the form:

$$\varphi(\mathbf{x}) = \mathbf{x} + \mathbf{u} \tag{2.16}$$

where $\mathbf{u}$ is the *displacement field*. Then,

$$\frac{\partial \varphi_i}{\partial x_j} = \frac{\partial x_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j} = \delta_{ij} + \frac{\partial u_i}{\partial x_j} \tag{2.17}$$

and the Green-Lagrange strain tensor becomes:

$$\begin{aligned}2\epsilon_{ij} &= \left(\delta_{mi} + \frac{\partial u_m}{\partial x_i}\right)\left(\delta_{mj} + \frac{\partial u_m}{\partial x_j}\right) - \delta_{ij} \\ &= \cancel{\delta_{ij}} + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j} - \cancel{\delta_{ij}}\end{aligned} \tag{2.18}$$

$$\text{Green-Lagrange strain tensor :} \quad \boxed{\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right)} \tag{2.19}$$

When the absolute values of the derivatives of the displacement field are much smaller than 1, their products (nonlinear part of the strain) are even smaller and we'll neglect them. We will make this assumption throughout this course (See accompanying Mathematica notebook evaluating the limits of this assumption). Mathematically:

$$\left\|\frac{\partial u_i}{\partial x_j}\right\| \ll 1 \ \Rightarrow \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j} \sim 0 \tag{2.20}$$

We will define the *linear part* of the Green-Lagrange strain tensor as the *small strain tensor*:

$$\boxed{\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} \tag{2.21}$$

Concept Question 2.1.1. *Strain fields from displacements.*

The purpose of this exercice is to determine strain fields from given displacements.

1. Find the linear and nonlinear strain fields associated with the following displacements

$$\begin{aligned} u_1^a &= x_1x_2(2-x_1) - c_1x_2 + c_2x_2^3, \\ u_2^a &= -c_3x_2^2(1-x_1) - (3-x_1)\frac{x_1^2}{3} - c_1x_1. \end{aligned}$$

2. Find the linear strain fields associated with the following displacements

$$\begin{aligned} u_1^b &= x_1^3x_2 + 2c_1c_2^3x_1 + 3c_1c_2^2x_1x_2 - c_1x_1x_2^3, \\ u_2^b &= -2c_2^3x_2 - \frac{3}{2}c_2^2x_2^2 + \frac{1}{4}x_2^4 - \frac{3}{2}c_1x_1^2x_2^2. \end{aligned}$$

■ **Solution:** The expression to calculate the nonlinear (nl) strains in function of the displacements is

$$\varepsilon_{ij}^{nl} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right). \tag{2.22}$$

When the derivatives of the displacement components are small in comparison to one, i.e. $\frac{\partial u_m}{\partial x_i}, \frac{\partial u_m}{\partial x_j} \ll 1$, the product $\left(\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right)$ can be neglected, and the previous equation simplifies to the following linear (l) expression

$$\varepsilon_{ij}^{l} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2.23}$$

When we apply the Equation 2.23 to the field (u_1^a, u_2^a), we obtain the following linear (l) strain tensor

$$\varepsilon_a^l = \begin{bmatrix} 2x_2(1-x_1) & -c_1 + \frac{(3c_2+c_3)}{2}x_2^2 \\ -c_1 + \frac{(3c_2+c_3)}{2}x_2^2 & -2c_3x_2(1-x_1) \end{bmatrix}.$$

On the other hand, the Equation 2.22 allows us to calculate the nonlinear (nl) strain tensor for the field (u_1^a, u_2^a)

$$\varepsilon_a^{nl} = \begin{bmatrix} \varepsilon_{11}^{nl} & \varepsilon_{12}^{nl} \\ \varepsilon_{12}^{nl} & \varepsilon_{22}^{nl} \end{bmatrix},$$

where

$$\begin{aligned} \varepsilon_{11}^{nl} &= 2x_2(1-x_1)\left[1 + x_2(1-x_1)\right] + \frac{1}{2}\left[-c_1 + c_3x_2^2 - x_1(2-x_1)\right]^2, \\ \varepsilon_{22}^{nl} &= 2c_3x_2(1-x_1)\left[-1 + c_3x_2(1-x_1)\right] + \frac{1}{2}\left[-c_1 + 3c_2x_2^2 + x_1(2-x_1)\right]^2, \\ \varepsilon_{12}^{nl} &= -c_1 + \frac{(3c_2+c_3)}{2}x_2^2 \\ &\quad + x_2(1-x_1)\left[x_1(2-x_1)(1+c_3) + c_1(-1+c_3) + (3c_2 - c_3^2)x_2^2\right]. \end{aligned}$$

The linear (l) strain tensor for the displacement field (u_1^b, u_2^b) is

$$\varepsilon_b^l = \begin{bmatrix} 3x_1^2x_2 + 2c_1c_2^3 + 3c_1c_2^2x_2 - c_1x_2^3 & \frac{1}{2}x_1^3 + \frac{3}{2}c_1c_2^2x_1 - 3c_1x_1x_2^2 \\ \frac{1}{2}x_1^3 + \frac{3}{2}c_1c_2^2x_1 - 3c_1x_1x_2^2 & -3c_1x_1^2x_2 + x_2^3 - 3c_2^2x_2 - 2c_2^3 \end{bmatrix}.$$

■

2.2 Transformation of strain components

Readings: BC 1.5.1, 1.6.2, 1.5.2, 1.6.3, 1.6.4

Given: $\epsilon_{ij}, \mathbf{e}_i$ and a new basis $\tilde{\mathbf{e}}_k$, determine the components of strain in the new basis $\tilde{\epsilon}_{kl}$

$$\tilde{\epsilon}_{ij} = \frac{1}{2}\Big(\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} + \frac{\partial \tilde{u}_j}{\partial \tilde{x}_i}\Big) \tag{2.24}$$

We want to express the quantities with tilde on the right-hand side in terms of their non-tilde counterparts. Start by applying the chain rule of differentiation:

$$\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = \frac{\partial \tilde{u}_i}{\partial x_k}\frac{\partial x_k}{\partial \tilde{x}_j} \tag{2.25}$$

Transform the displacement components:

$$\mathbf{u} = \tilde{u}_m\tilde{\mathbf{e}}_m = u_l\mathbf{e}_l \tag{2.26}$$

$$\tilde{u}_m(\tilde{\mathbf{e}}_m \cdot \tilde{\mathbf{e}}_i) = u_l(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) \tag{2.27}$$

$$\tilde{u}_m\delta_{mi} = u_l(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) \tag{2.28}$$

$$\tilde{u}_i = u_l(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) \tag{2.29}$$

take the derivative of $\tilde{u}_i$ with respect to x_k, as required by equation (2.25):

$$\frac{\partial \tilde{u}_i}{\partial x_k} = \frac{\partial u_l}{\partial x_k}(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i) \tag{2.30}$$

and take the derivative of the reverse transformation of the components of the position vector $\mathbf{x}$:

$$\mathbf{x} = x_j\mathbf{e}_j = \tilde{x}_k\tilde{\mathbf{e}}_k \tag{2.31}$$

$$x_j(\mathbf{e}_j \cdot \mathbf{e}_i) = \tilde{x}_k(\tilde{\mathbf{e}}_k \cdot \mathbf{e}_i) \tag{2.32}$$

$$x_j\delta_{ji} = \tilde{x}_k(\tilde{\mathbf{e}}_k \cdot \mathbf{e}_i) \tag{2.33}$$

$$x_i = \tilde{x}_k(\tilde{\mathbf{e}}_k \cdot \mathbf{e}_i) \tag{2.34}$$

$$\frac{\partial x_i}{\partial \tilde{x}_j} = \frac{\partial \tilde{x}_k}{\partial \tilde{x}_j}(\tilde{\mathbf{e}}_k \cdot \mathbf{e}_i) = \delta_{kj}(\tilde{\mathbf{e}}_k \cdot \mathbf{e}_i) = (\tilde{\mathbf{e}}_j \cdot \mathbf{e}_i) \tag{2.35}$$

Replacing equations (2.30) and (2.35) in (2.25):

$$\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = \frac{\partial \tilde{u}_i}{\partial x_k}\frac{\partial x_k}{\partial \tilde{x}_j} = \frac{\partial u_l}{\partial x_k}(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i)(\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k) \tag{2.36}$$

Replacing in equation (2.24):

$$\tilde{\epsilon}_{ij} = \frac{1}{2}\Big[\frac{\partial u_l}{\partial x_k}(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i)(\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k) + \frac{\partial u_l}{\partial x_k}(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_j)(\tilde{\mathbf{e}}_i \cdot \mathbf{e}_k)\Big] \tag{2.37}$$

Exchange indices l and k in second term:

$$\begin{aligned}\tilde{\epsilon}_{ij} &= \frac{1}{2}\Big[\frac{\partial u_l}{\partial x_k}(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i)(\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k) + \frac{\partial u_k}{\partial x_l}(\mathbf{e}_k \cdot \tilde{\mathbf{e}}_j)(\tilde{\mathbf{e}}_i \cdot \mathbf{e}_l)\Big] \\ &= \frac{1}{2}\Big(\frac{\partial u_l}{\partial x_k} + \frac{\partial u_k}{\partial x_l}\Big)(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i)(\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k)\end{aligned} \tag{2.38}$$

Or, finally:

$$\boxed{\tilde{\epsilon}_{ij} = \epsilon_{lk}(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i)(\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k)} \tag{2.39}$$

Concept Question 2.2.1. *2d relations for strain tensor rotation.*

In two dimensions, let us consider two basis $\mathbf{e_i}$ and $\tilde{\mathbf{e}}_\mathbf{k}$ such that $\tilde{\mathbf{e}}_\mathbf{1}$ is oriented at an angle θ with respect to the axis $\mathbf{e_1}$. ϵ_{ij} and $\tilde{\epsilon}_{ij}$ are, respectively, the components of a strain tensor ϵ expressed in the $\mathbf{e_i}$ and $\tilde{\mathbf{e}}_\mathbf{k}$ bases (i.e. they correspond to the same state of deformation. Using the following expression introduced in the class notes,

$$\tilde{\epsilon}_{ij} = \epsilon_{lk}(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_i)(\tilde{\mathbf{e}}_j \cdot \mathbf{e}_k)$$

derive the following relations:

$$\begin{aligned}\tilde{\epsilon}_{11} &= \epsilon_{11}\cos^2\theta + \epsilon_{22}\sin^2\theta + \epsilon_{12}\sin 2\theta \\ \tilde{\epsilon}_{22} &= \epsilon_{11}\sin^2\theta + \epsilon_{22}\cos^2\theta - \epsilon_{12}\sin 2\theta \\ \tilde{\epsilon}_{12} &= -\frac{\epsilon_{11} - \epsilon_{22}}{2}\sin 2\theta + \epsilon_{12}\cos 2\theta\end{aligned}$$

Note: It is also usual to find the following expressions for $\tilde{\epsilon}_{11}$ and $\tilde{\epsilon}_{22}$ in textbooks:

$$\begin{aligned}\tilde{\epsilon}_{11} &= \frac{\epsilon_{11} + \epsilon_{22}}{2} + \frac{\epsilon_{11} - \epsilon_{22}}{2}\cos 2\theta + \epsilon_{12}\sin 2\theta \\ \tilde{\epsilon}_{22} &= \frac{\epsilon_{11} + \epsilon_{22}}{2} + \frac{\epsilon_{22} - \epsilon_{11}}{2}\cos 2\theta - \epsilon_{12}\sin 2\theta\end{aligned}$$

■ **Solution:** First, let us recall the following trigonometric relations between the vectors of $\mathbf{e_i}$ and $\tilde{\mathbf{e}}_\mathbf{k}$:

$$\begin{aligned}\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_1 &= \cos\theta \qquad & \mathbf{e}_1 \cdot \tilde{\mathbf{e}}_2 &= -\sin\theta \\ \mathbf{e}_2 \cdot \tilde{\mathbf{e}}_2 &= \cos\theta \qquad & \mathbf{e}_2 \cdot \tilde{\mathbf{e}}_1 &= \sin\theta\end{aligned}$$

Using (2.39), it is possible to write the following:

$$\begin{aligned}
\tilde{\epsilon}_{11} &= \epsilon_{11}(\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_1)^2 + \epsilon_{22}(\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_2)^2 + 2\epsilon_{12}(\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_1)(\tilde{\mathbf{e}}_1 \cdot \mathbf{e}_2) \\
&= \epsilon_{11}\cos^2\theta + \epsilon_{22}\sin^2\theta + \epsilon_{12}\sin 2\theta \\
\tilde{\epsilon}_{22} &= \epsilon_{11}(\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_2)^2 + \epsilon_{22}(\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_2)^2 + 2\epsilon_{12}(\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_2)(\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2) \\
&= \epsilon_{11}\sin^2\theta + \epsilon_{22}\cos^2\theta - \epsilon_{12}\sin 2\theta \\
\tilde{\epsilon}_{22} &= \epsilon_{11}(\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_1)(\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_1) + \epsilon_{22}(\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_1)(\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2) \\
&+ \epsilon_{12}(\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_1)(\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_2) + \epsilon_{21}(\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_1)(\tilde{\mathbf{e}}_2 \cdot \mathbf{e}_1) \\
&= -\frac{\epsilon_{11}}{2}\sin 2\theta + \frac{\epsilon_{22}}{2}\sin 2\theta + \epsilon_{12}(\cos^2\theta - \sin^2\theta) \\
&= -\frac{\epsilon_{11} - \epsilon_{22}}{2}\sin 2\theta + \epsilon_{12}\cos 2\theta
\end{aligned}$$

The expresssions given in the remark can be derived from these using the following trigonometric relations:

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

■

Concept Question 2.2.2. *Principal strains and maximum shear strain in 2d.*

Using the relations introduced in Problem 2.2.1, show that given the components ϵ_{ij} of a 2d strain tensor in a basis $\mathbf{e_i}$:

1. The principal strains can be computed as follows:

$$\epsilon_{1,2} = \frac{\epsilon_{11}+\epsilon_{22}}{2} \pm \sqrt{\left(\frac{\epsilon_{11}-\epsilon_{22}}{2}\right)^2 + \epsilon_{12}^2}$$

   and the principal directions of strain for angles with respect to $\mathbf{e_1}$ satisfy:

$$\tan 2\theta^p = \frac{2\epsilon_{12}}{\epsilon_{11}-\epsilon_{22}}$$

2. The maximum shear strain can be computed as follows:

$$\epsilon_{12}^{max} = \sqrt{\left(\frac{\epsilon_{11}-\epsilon_{22}}{2}\right)^2 + \epsilon_{12}^2}$$

   and the normal of the planes of maximum shear form angles with respect to $\mathbf{e_1}$

$$\tan 2\theta^s = -\frac{\epsilon_{11}-\epsilon_{22}}{2\epsilon_{12}}.$$

   Conclude that the direction of maximum shear is always oriented at an angle equal to 45^o with respect to the principal directions of strain.

■ **Solution: Principal strains:** The characteristic polynomial $\chi(\epsilon)$ corresponding to the strain tensor components ϵ_{ij} is:

$$\begin{aligned}\chi(\epsilon) &= \det(\epsilon_{ij} - \epsilon\delta_{ij}) = (\epsilon_{11} - \epsilon)(\epsilon_{22} - \epsilon) - \epsilon_{12}^2 \\ &= \epsilon^2 - (\epsilon_{11} + \epsilon_{22})\epsilon + (\epsilon_{11}\epsilon_{22} - \epsilon_{12}^2)\end{aligned}$$

The roots of the characteristic polynomial are:

$$\epsilon_{1,2} = \frac{\epsilon_{11} + \epsilon_{22}}{2} \pm \sqrt{\left(\frac{\epsilon_{11} - \epsilon_{22}}{2}\right)^2 + \epsilon_{12}^2}$$

To find the angle θ^p formed by the principal directions and the basis vecto $\mathbf{e}_1$, use the fact that the shear strains vanish in principal directions:

$$0 = -\frac{\epsilon_{11} - \epsilon_{22}}{2}\sin 2\theta^p + \epsilon_{12}\cos 2\theta^p \quad \Rightarrow \quad \tan 2\theta^p = \frac{2\epsilon_{12}}{\epsilon_{11} - \epsilon_{22}}$$

Maximum shear strain: The maximum shear strain can be found by simply finding the value of the argument θ in the expression for transforming the shear strain component which makes the derivative of ϵ_{12} with respect to θ vanish:

$$\begin{aligned}\epsilon_{12}^{\max} &= -\frac{\epsilon_{11} - \epsilon_{22}}{2}\sin 2\theta^s + \epsilon_{12}\cos 2\theta^s \\ \frac{\partial\epsilon_{12}}{\partial\theta} &= -2\left(\frac{\epsilon_{11} - \epsilon_{22}}{2}\cos 2\theta^s + \epsilon_{12}\sin 2\theta^s\right) = 0\end{aligned}$$

By taking the square of the two previous equations and summing them, it is easy to show that:

$$\epsilon_{12}^{\max\,2} = \left(\frac{\epsilon_{11} - \epsilon_{22}}{2}\right)^2 + \epsilon_{12}^2$$

The second equation leads directly to the angular relation:

$$\tan 2\theta^s = \frac{\epsilon_{22} - \epsilon_{11}}{2\epsilon_{12}}$$

From the trigonometric relation: $\tan\left(\alpha + \frac{\pi}{2}\right) = -\frac{1}{\tan\alpha}$ it is also easy to see that:

$$\tan\left(2\left(\theta^p + \frac{\pi}{4}\right)\right) = -\frac{1}{\tan 2\theta^p} = -\frac{\epsilon_{11} - \epsilon_{22}}{2\epsilon_{12}} = \tan 2\theta^s$$

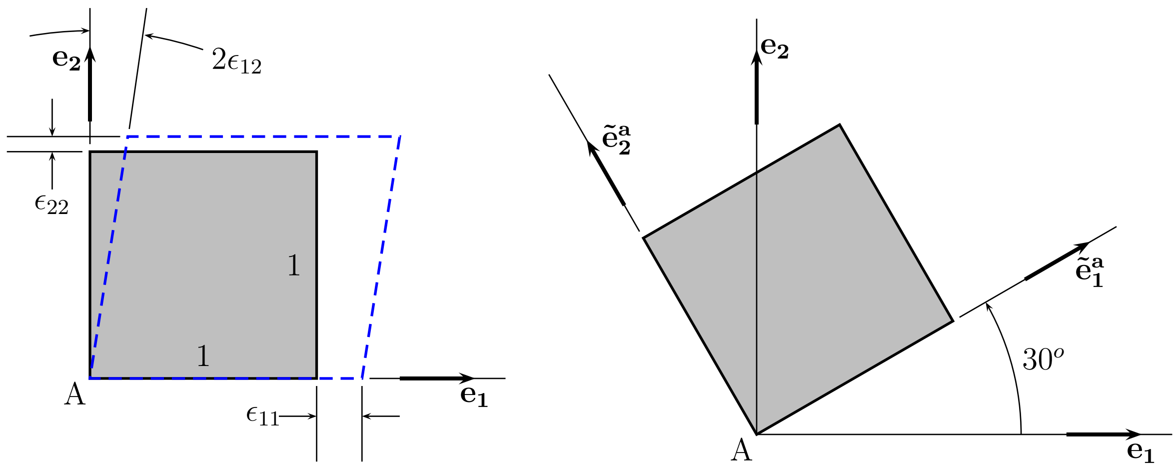
Thus, proving that $\theta^s = \theta^p + \frac{\pi}{4}$. ■

Concept Question 2.2.3. *Strain tensor rotation.*

Consider the following problem of a square of unit area subject to the following strain components in the basis given, Figure 2.3(a). :

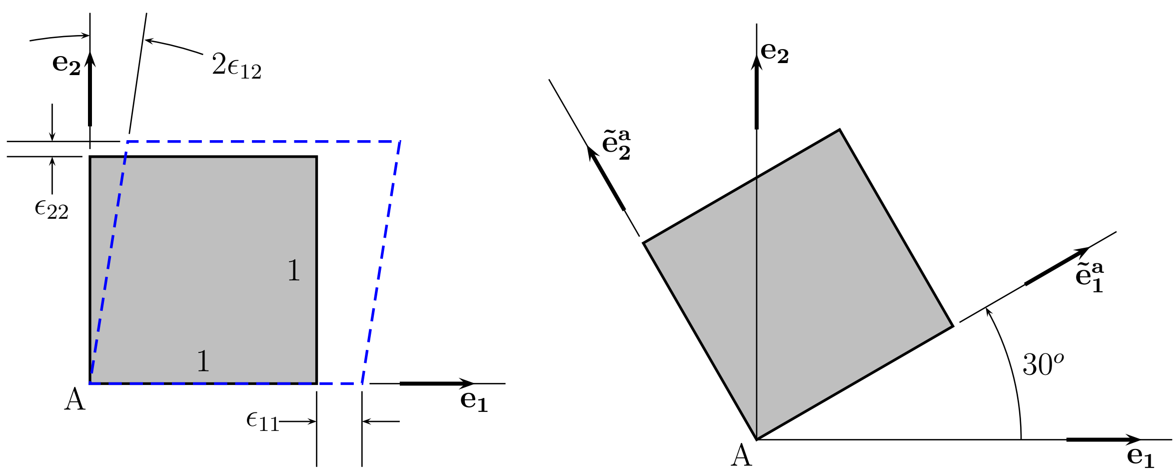
$$\epsilon_{11} = 3.4 \times 10^{-4} \qquad \epsilon_{22} = 1.1 \times 10^{-4} \qquad \epsilon_{12} = 9.0 \times 10^{-5}$$

Since the square has its edge of unit length, the changes in length in the directions $\mathbf{e_1}$ and $\mathbf{e_2}$ are directly equal to ϵ_{11} and ϵ_{22}, respectively. The shear strain ϵ_{12} is equal to half of the decrease in angle in A (for infinitesimal angles).



(a) Deformed unit square (b) different initial orientation of the unit square

Figure 2.2: Deformed unit square and oriented new initial configuration.



(a) Deformed unit square (b) different initial orientation of the unit square

Figure 2.3: Deformed unit square and oriented new initial configuration.

1. Determine the strain components on a square initialy oriented at an angle equal to 30^o to the axis $\mathbf{e_1}$ as shown on Figure 2.3(b). Sketch in this case, the deformed configuration.

2. Determine the principal strains and sketch the deformed configuration.

3. Determine the maximum shear strain and sketch the deformed configuration.

■ **Solution:** For the solution of this problem, we are going to use extensively the relations introduced in Problem 2.2.1. Let us first compute the two following ratios:

$$\frac{\epsilon_{11}+\epsilon_{22}}{2} = 2.25 \times 10^{-4} \qquad \frac{\epsilon_{11}-\epsilon_{22}}{2} = 1.15 \times 10^{-4}$$

Orientation at an angle $\theta = 30^o$: The value of the strain tensor in the basis $\mathbf{e_i^a}$ are as follows:

$$\begin{aligned}
\tilde{\epsilon}^a_{11} &= \frac{\epsilon_{11}+\epsilon_{22}}{2} + \frac{\epsilon_{11}-\epsilon_{22}}{2}\cos 2\theta + \epsilon_{12}\sin 2\theta \\
&= 2.25 \times 10^{-4} + 1.15 \times 10^{-4} \times \frac{1}{2} + 9.0 \times 10^{-5} \times \frac{\sqrt{3}}{2} = 3.6 \times 10^{-4} \\
\tilde{\epsilon}^a_{22} &= \frac{\epsilon_{11}+\epsilon_{22}}{2} + \frac{\epsilon_{22}-\epsilon_{11}}{2}\cos 2\theta - \epsilon_{12}\sin 2\theta \\
&= 2.25 \times 10^{-4} - 1.15 \times 10^{-4} \times \frac{1}{2} + 9.0 \times 10^{-5} \times \frac{\sqrt{3}}{2} = 9.0 \times 10^{-5} \\
\tilde{\epsilon}^a_{12} &= -\frac{\epsilon_{11}-\epsilon_{22}}{2}\sin 2\theta + \epsilon_{12}\cos 2\theta \\
&= -1.15 \times 10^{-4} \times \frac{\sqrt{3}}{2} + 9.0 \times 10^{-5} \times \frac{1}{2} = -5.5 \times 10^{-5}
\end{aligned}$$

Figure 2.4(a) shows the deformed configuration corresponding to this case.
Principal strains: Using the relation introduced in Problem 2.2.2, the principal strains are:

$$\epsilon_{1,2} = \frac{\epsilon_{11}+\epsilon_{22}}{2} \pm \sqrt{\left(\frac{\epsilon_{11}-\epsilon_{22}}{2}\right)^2 + \epsilon_{12}^2} = \begin{cases} = 3.7 \times 10^{-4} : \epsilon_1 \\ = 8.0 \times 10^{-5} : \epsilon_2 \end{cases}$$

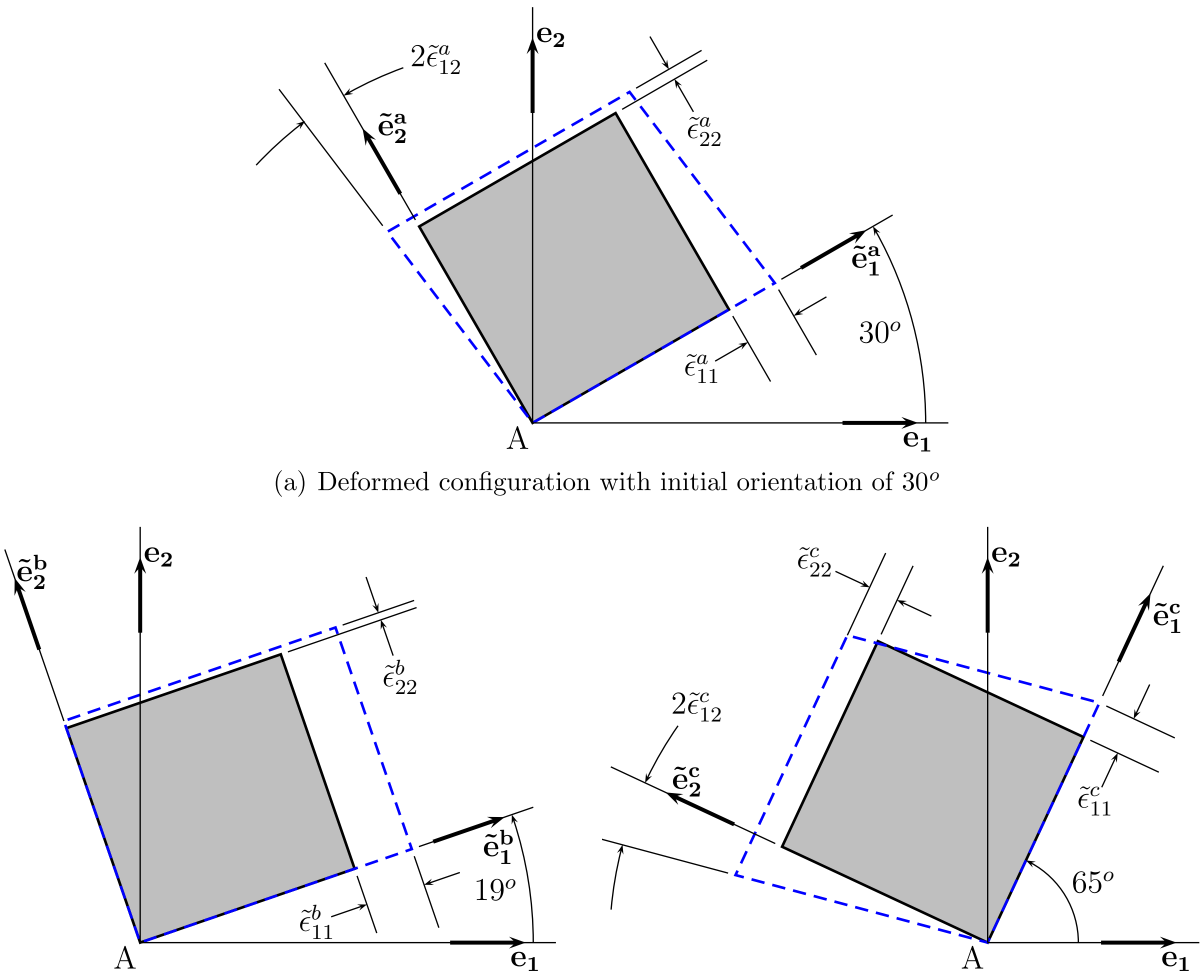
and their respective direction can be computed as:

$$\tan 2\theta^b = \frac{2\epsilon_{12}}{\epsilon_{11}-\epsilon_{22}} \quad \Rightarrow \quad \begin{cases} \theta^b_1 \approx 19^o \\ \theta^b_2 \approx 109^o \end{cases}$$

In order to find which of the two angles solution of the equation above is associated with which value of principal strain, one can test these values of θ^b in the expression of $\tilde{\epsilon}_{11}$ given in Problem 2.2.1. Figure 2.4(b) shows the deformed configuration corresponding to this case.
Maximum shear strain: Following the relations introduced in Problem 2.2.2, we can compute the absolute value of the maximal shear strain as:

$$\epsilon^{max}_{12} = \sqrt{\left(\frac{\epsilon_{11}-\epsilon_{22}}{2}\right)^2 + \epsilon_{12}^2} = \sqrt{(1.15 \times 10^{-4})^2 + (9.0 \times 10^{-5})^2} = 1.46 \times 10^{-4}.$$

Using the fact that the maximum shear direction is oriented at an angle of 45^o to one of the principal strain direction, let us consider the case of maximum shear obtained for an angle $\theta^c = 19^o + 45^o = 64^o$ starting from $\mathbf{e_1}$. We obtain $\epsilon_{12}(\theta^c) = -\epsilon_{12}^{\max}$ and contend that for this angle the maximum negative shear strain is obtained. Figure 2.4(c) shows the deformed configuration corresponding to this case. ■



(a) Deformed configuration with initial orientation of 30^o

(b) Deformed configuration of principal strain

(c) Deformed configuration of maximum shear

Figure 2.4: Several deformed configuration of a unit square.

2.3 Compatibility of strains

Readings: BC 1.8

Given displacement field $\mathbf{u}$, expression (2.21) allows to compute the strains components ϵ_{ij}. How does one answer the reverse question? Note analogy with potential-gradient field. In this section, we will restrain ourselves to small perturbation theory where the displacements and the rotations of a deformable solid are infinitesimal. Let us first restrict the analysis to two dimensions. The small strain tensor is defined as the symmetric part of the displacement gradient $\frac{\partial u_i}{\partial x_j}$:

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.40}$$

We define the skew-symmetric part of $\frac{\partial u_i}{\partial x_j}$ as:

$$\omega_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{2.41}$$

Concept Question 2.3.1. *Properties of ω_{ij}*

1. Verify that $\omega_{ji} = -\omega_{ij}$, i.e. ω_{ij} is skew-symmetric ■ **Solution:**

$$\omega_{ji} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right) = -\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \omega_{ij}$$

■

2. Verify that $\epsilon_{ij} + \omega_{ij} = \frac{\partial u_i}{\partial x_j}$ ■ **Solution:**

$$\epsilon_{ij} + \omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \frac{\partial u_i}{\partial x_j}$$

■

For the two-dimensional setting, the components are as follows:

$$\omega_{11} = \omega_{22} = 0, \quad \omega_{12} = -\omega_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) \tag{2.42}$$

We have seen in a previous section of this module, that ϵ_{ij} describes the change of length of a vector $\mathbf{dx}$ due to deformation. We will now see that ω_{ij} represents the infinitesimal rotation of the vector $\mathbf{dx}$ from the initial to the deformed configuration. ω_{ij} is thus named the *infinitesimal rotation tensor*.

Consider an infinitesimal rotation of a vector $\overrightarrow{PQ}$ in the neighborhood of a point P. For this transformation, the strain tensor ϵ vanishes. Such a transformation can only be a rotation of $\overrightarrow{PQ}$ into $\overrightarrow{PQ'}$ by an angle θ ($\theta \ll 1$) as depicted in the following figure:

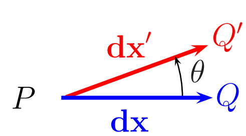


Figure 2.5: infinitesimal rotation of a vector **dx**

From Figure 2.5, it is possible to express $\mathbf{dx}'$ in terms of θ and $\mathbf{dx}$:

$$\mathbf{dx}' = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \mathbf{dx} \approx \begin{bmatrix} 1 & \theta \\ -\theta & 1 \end{bmatrix} \mathbf{dx} \tag{2.43}$$

Altenatively, from (2.17), it is possible to express $\mathbf{dx}'$ in terms of ω_{12} and $\mathbf{dx}$:

$$\mathbf{dx}' = (\delta_{ij} + \omega_{ij})\, dx_j = \begin{bmatrix} 1 & \omega_{12} \\ -\omega_{12} & 1 \end{bmatrix} \mathbf{dx} \tag{2.44}$$

By identification of the transformation matrix components, we conclude that $\omega_{12} = -\omega_{21} \approx \theta$ corresponds indeed to an infinitesimal rotation in the plane of normal $\mathbf{e}_3$. Similar conclusions can be drawn on the remaining components: $\omega_{31} = -\omega_{13}$ corresponds to an infinitesimal rotation in the plane of normal $\mathbf{e}_2$ and $\omega_{23} = -\omega_{32}$ corresponds to an infinitesimal rotation in the plane of normal $\mathbf{e}_1$.

The compatibility of strain is intricately related to the continuity of infinitesimal rotations. In two dimensions, this can be readily expressed by requiring the equality of the mixed partials of ω_{12}: $\frac{\partial^2 \omega_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 \omega_{12}}{\partial x_2 \partial x_1}$. To this end, differentiate ω_{12} with respect to x_1:

$$\begin{aligned} \frac{\partial \omega_{12}}{\partial x_1} &= \frac{1}{2}\left(\frac{\partial^2 u_1}{\partial x_2 \partial x_1} - \frac{\partial^2 u_2}{\partial x_1^2} \right) && (2.45) \\ &= \frac{1}{2}\left(\frac{\partial^2 u_1}{\partial x_2 \partial x_1} + \frac{\partial^2 u_1}{\partial x_2 \partial x_1} - \left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2 \partial x_1} \right) \right) && (2.46) \\ &= \frac{\partial \epsilon_{11}}{\partial x_2} - \frac{\partial \epsilon_{12}}{\partial x_1} && (2.47) \end{aligned}$$

and now with respect to x_2:

$$\frac{\partial^2 \omega_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 \epsilon_{11}}{\partial {x_2}^2} - \frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} \tag{2.48}$$

Similarly, we can find that:

$$\frac{\partial \omega_{12}}{\partial x_2} = \frac{\partial \epsilon_{12}}{\partial x_2} - \frac{\partial \epsilon_{22}}{\partial x_1} \tag{2.49}$$

which differentiated with respect to x_1 gives:

$$\frac{\partial^2 \omega_{12}}{\partial x_2 \partial x_1} = \frac{\partial^2 \epsilon_{12}}{\partial x_2 \partial x_1} - \frac{\partial^2 \epsilon_{22}}{\partial {x_1}^2} \tag{2.50}$$

Equating the mixed partials in equations (2.48) and (2.50) we obtain:

$$\boxed{2\frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2}} \tag{2.51}$$

The following concept question generalizes this result to obtain all of the equations of strain compatibility in three dimensions.

Concept Question 2.3.2. *Strain compatibility equation in 3d.*

The purpose of this exercise is to derive the strain compatibility equations in 3d using the approach followed in class for the 2d case.

1. Apply the equality of mixed partials to the small rotation tensor:
$$\frac{\partial^2 \omega_{ij}}{\partial x^k \partial x^l} = \frac{\partial^2 \omega_{ij}}{\partial x^l \partial x^k}$$
and show that the following relations hold:
$$\frac{\partial^2 \epsilon_{ik}}{\partial x_j \partial x_l} - \frac{\partial^2 \epsilon_{jk}}{\partial x_i \partial x_l} = \frac{\partial^2 \epsilon_{il}}{\partial x_j \partial x_k} - \frac{\partial^2 \epsilon_{jl}}{\partial x_i \partial x_k} \tag{2.52}$$

2. How many relations are defined by (2.52) and how many strain compatibility equations are required in order to ensure that a unique displacement may be computed from a given small strain tensor?

3. Notice that for $i = j$ **or** $l = k$, (2.52) is automatically verified. How many non-trivial relations can be derived from (2.52)? Are all these relation independant?

■ **Solution:** Let us remind first that the small rotation tensor is defined as:
$$\omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right)$$
Thus, the gradient of small rotation reads:
$$\omega_{ij,k} = \frac{1}{2}\left(u_{i,jk} - u_{j,ik}\right)$$
By adding and substracting $u_{k,ij}$ form the right-hand side of the previous relation, it is to express the gradient of small rotation only in terms of the derivatives of the componenent of the small strain tensor:
$$\omega_{ij,k} = \frac{1}{2}\Big(\underbrace{u_{i,jk} + u_{k,ij}}_{2\epsilon_{ik,j}} - \underbrace{(u_{j,ik} - u_{k,ij})}_{2\epsilon_{jk,i}}\Big) = \frac{1}{2}\big(\epsilon_{ik,j} - \epsilon_{jk,i}\big)$$
Thus, the mixed derivatives: $\omega_{ij,kl}$ and $\omega_{ij,lk}$ of the small rotation tensor have the following expressions:
$$\begin{aligned} \omega_{ij,kl} &= \frac{1}{2}\big(\epsilon_{ik,jl} - \epsilon_{jk,il}\big) \\ \omega_{ij,lk} &= \frac{1}{2}\big(\epsilon_{il,jk} - \epsilon_{jl,ik}\big) \end{aligned}$$
The equality of mixed partials implies:
$$\epsilon_{ik,jl} - \epsilon_{jk,il} = \epsilon_{il,jk} - \epsilon_{jl,ik}$$
□

Since i, j, k, l can take any value in $\{1, 2, 3\}$ respectively, (2.52) comprises $3^4 = 81$ relations. It is easy to verify that the only non-trivial relations from (2.52) can be obtained for $i \neq j$ **and** $k \neq l$.

i	$\neq$	j	and	k	$\neq$	l
1		2	and	1		2
2		3	and	2		3
3		1	and	2		1
1		2	and	1		3
2		3	and	2		1
3		1	and	3		2

Thus, obtaining the 6 following relations:

$$\begin{aligned}
\epsilon_{11,22} + \epsilon_{22,11} &= 2\epsilon_{12,12} \\
\epsilon_{22,33} + \epsilon_{33,22} &= 2\epsilon_{23,23} \\
\epsilon_{33,11} + \epsilon_{11,33} &= 2\epsilon_{31,31} \\
\epsilon_{12,23} + \epsilon_{23,12} &= \epsilon_{22,31} + \epsilon_{31,22} \\
\epsilon_{23,31} + \epsilon_{31,23} &= \epsilon_{33,12} + \epsilon_{12,33} \\
\epsilon_{31,12} + \epsilon_{12,31} &= \epsilon_{11,23} + \epsilon_{23,11}
\end{aligned}$$

These six relations are linearly dependent and it is possible to show that if only three are them are verifed then the remaining three are. ■
