Chapter 3

Static Longitudinal Stability and Control

The most critical aspects of static longitudinal stability relate to control forces required for changing trim or performing maneuvers. Our textbook [1] treats primarily the situation when the controls are fixed. This is, of course, and idealization, even for the case of powered, irreversible controls, as the position of the control surfaces can he held fixed only to the extent of the maximum available control forces. The opposite limit – that of free control surfaces – also is an idealization, limited by the assumptions of zero friction in the control positioning mechanisms. But, just as the control fixed limit is useful in determining control *position* gradients, the control free limit is useful in determining control *force* gradients. And these latter are among the most important vehicle properties in determining handling qualities.

3.1 Control Fixed Stability

Even for the controls-fixed case, our text is a bit careless with nomenclature and equations, so we review the most important results for this case here. We have seen that for the analysis of longitudinal stability, terms involving products of the drag coefficient and either vertical displacements of the vehicle center-of-gravity or sines of the angle of attack can be neglected. Then, with the axial locations as specified in Fig. 3.1 the pitching moment about the vehicle c.g. can be written

$$\mathbf{C}_{m\,\mathrm{cg}} = \mathbf{C}_{m0_w} + \mathbf{C}_{Lw}\left(\frac{x_{\mathrm{cg}}}{\bar{c}} - \frac{x_{\mathrm{ac}}}{\bar{c}}\right) - \eta\frac{S_t}{S}\mathbf{C}_{Lt}\left[\frac{\ell_t}{\bar{c}} - \left(\frac{x_{\mathrm{cg}}}{\bar{c}} - \frac{x_{\mathrm{ac}}}{\bar{c}}\right)\right] + \mathbf{C}_{mf} \tag{3.1}$$

where we assume that $\mathbf{C}_{m0_t} = 0$, since the tail is usually symmetrical. Note that, as is the usual convention when analyzing *static* longitudinal stability and control, the positive direction of the x-axis is taken to be *aft*;[1]thus, e.g., the second term on the right-hand side of Eq. (3.1) contributes to a positive (nose-up) pitching moment for positive lift when the c.g. is aft of the wing aerodynamic center.

[1]Also, the origin of the x-axis is taken, by convention, to be at the leading edge of the mean aerodynamic chord of the wing, and distances are normalized by the length of the wing mean aerodynamic chord. Thus, for example, we might specify the location of the vehicle center-of-gravity as being at *30 per cent m.a.c.*

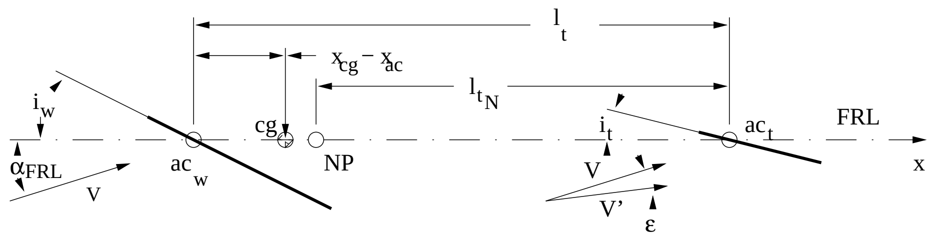

Figure 3.1: Geometry of wing and tail with respect to vehicle c.g., basic neutral point, and wing aerodynamic center. Note that positive direction of the x-axis is aft.

Grouping the terms involving the c.g. location, this equation can be written

$$\mathbf{C}_{m\mathrm{cg}} = \mathbf{C}_{m0_w} + \left(\frac{x_{\mathrm{cg}}}{\bar{c}} - \frac{x_{\mathrm{ac}}}{\bar{c}}\right)\left[\mathbf{C}_{Lw} + \eta\frac{S_t}{S}\mathbf{C}_{Lt}\right] - \eta V_H \mathbf{C}_{Lt} + \mathbf{C}_{mf} \tag{3.2}$$

where $V_H = \frac{\ell_t S_t}{\bar{c}S}$ is the *tail volume parameter*. Note that this definition is based on the distance between the aerodynamic centers of the wing and tail, and is therefore independent of the vehicle c.g. location. Note that the total *vehicle* lift coefficient is

$$\mathbf{C}_L = \frac{L_w + L_t}{QS} = \mathbf{C}_{Lw} + \eta\frac{S_t}{S}\mathbf{C}_{Lt} \tag{3.3}$$

where $\eta = Q_t/Q$ is the tail efficiency factor, and this total vehicle lift coefficient is exactly the quantity appearing in the square brackets in Eq. (3.2). Now, we can introduce the dependence of the lift coefficients on angle of attack as

$$\begin{aligned}\mathbf{C}_{Lw} &= \mathbf{C}_{L\alpha_w}\left(\alpha_{FRL} + i_w - \alpha_{0_w}\right)\\ \mathbf{C}_{Lt} &= \mathbf{C}_{L\alpha_t}\left(\alpha_{FRL} + i_t - \left[\varepsilon_0 + \frac{\mathrm{d}\varepsilon}{\mathrm{d}\alpha}\alpha_{FRL}\right]\right)\end{aligned} \tag{3.4}$$

Note that, consistent with the usual use of symmetric sections for the horizontal tail, we have assumed $\alpha_{0_t} = 0$. Introducing these expressions into Eq. (3.3), the latter can be expressed as

$$\mathbf{C}_L = \mathbf{C}_{L\alpha_w}\left(i_w - \alpha_{0_w}\right) + \eta\frac{S_t}{S}\mathbf{C}_{L\alpha_t}\left(i_t - \varepsilon_0\right) + \left(\mathbf{C}_{L\alpha_w} + \eta\frac{S_t}{S}\left[1 - \frac{\mathrm{d}\varepsilon}{\mathrm{d}\alpha}\right]\mathbf{C}_{L\alpha_t}\right)\alpha_{FRL} \tag{3.5}$$

This equation has the form

$$\mathbf{C}_L = \mathbf{C}_{L0} + \mathbf{C}_{L\alpha}\alpha_{FRL} \tag{3.6}$$

where the *vehicle* lift curve slope is

$$\mathbf{C}_{L\alpha} = \mathbf{C}_{L\alpha_w} + \eta\frac{S_t}{S}\left(1 - \frac{\mathrm{d}\varepsilon}{\mathrm{d}\alpha}\right)\mathbf{C}_{L\alpha_t} \tag{3.7}$$

and

$$\mathbf{C}_{L0} = \mathbf{C}_{L\alpha_w}\left(i_w - \alpha_{0_w}\right) + \eta\frac{S_t}{S}\mathbf{C}_{L\alpha_t}\left(i_t - \varepsilon_0\right) \tag{3.8}$$

is the vehicle lift coefficient at zero (fuselage reference line) angle of attack. Finally, if we define the vehicle angle of attack relative to the angle of attack for zero *vehicle* lift, i.e.,

$$\alpha \equiv \alpha_{FRL} - \alpha_0 \tag{3.9}$$

where

$$\alpha_0 = -\frac{\mathbf{C}_{L0}}{\mathbf{C}_{L\alpha}} \tag{3.10}$$

then

$$\mathbf{C}_L = \mathbf{C}_{L\alpha}\alpha \tag{3.11}$$

where $\mathbf{C}_{L\alpha}$ is the vehicle lift curve slope, given by Eq. (3.7).

Introducing the angle of attack into Eq. (3.2), the expression for the vehicle pitching moment coefficient becomes

$$\begin{aligned}\mathbf{C}_{m\text{cg}} =& \mathbf{C}_{m0_w} + \left(\frac{x_{\text{cg}}}{\bar{c}} - \frac{x_{\text{ac}}}{\bar{c}}\right)\left[\mathbf{C}_{L\alpha_w}(i_w - \alpha_{0_w}) + \eta\frac{S_t}{S}\mathbf{C}_{L\alpha_t}(i_t - \varepsilon_0)\right] - \eta V_H \mathbf{C}_{L\alpha_t}(i_t - \varepsilon_0) + \\ & \left\{\left(\frac{x_{\text{cg}}}{\bar{c}} - \frac{x_{\text{ac}}}{\bar{c}}\right)\left[\mathbf{C}_{L\alpha_w} + \eta\frac{S_t}{S}\left(1 - \frac{\mathrm{d}\varepsilon}{\mathrm{d}\alpha}\right)\mathbf{C}_{L\alpha_t}\right] - \eta V_H\left(1 - \frac{\mathrm{d}\varepsilon}{\mathrm{d}\alpha}\right)\mathbf{C}_{L\alpha_t} + \mathbf{C}_{m\alpha_f}\right\}\alpha_{FRL}\end{aligned} \tag{3.12}$$

This can be expressed in terms of the angle of attack from zero vehicle lift as

$$\begin{aligned}\mathbf{C}_{m\text{cg}} = & \mathbf{C}_{m0_w} + \left(\frac{x_{\text{cg}}}{\bar{c}} - \frac{x_{\text{ac}}}{\bar{c}}\right)\left[\mathbf{C}_{L\alpha_w}(i_w - \alpha_{0_w}) + \eta\frac{S_t}{S}\mathbf{C}_{L\alpha_t}(i_t - \varepsilon_0)\right] - \eta V_H \mathbf{C}_{L\alpha_t}(i_t - \varepsilon_0) \\ & + \mathbf{C}_{m\alpha}\alpha_0 + \left\{\left(\frac{x_{\text{cg}}}{\bar{c}} - \frac{x_{\text{ac}}}{\bar{c}}\right)\mathbf{C}_{L\alpha} - \eta V_H \mathbf{C}_{L\alpha_t}\left(1 - \frac{\mathrm{d}\varepsilon}{\mathrm{d}\alpha}\right) + \mathbf{C}_{m\alpha_f}\right\}\alpha\end{aligned} \tag{3.13}$$

This equation has the form

$$\mathbf{C}_m = \mathbf{C}_{m0} + \mathbf{C}_{m\alpha}\alpha \tag{3.14}$$

with the *vehicle* pitching moment coefficient at zero lift

$$\mathbf{C}_{m0} = \mathbf{C}_{m0_w} + \left(\frac{x_{\text{cg}}}{\bar{c}} - \frac{x_{\text{ac}}}{\bar{c}}\right)\left[\mathbf{C}_{L\alpha_w}(i_w - \alpha_{0_w}) + \eta\frac{S_t}{S}\mathbf{C}_{L\alpha_t}(i_t - \varepsilon_0)\right] - \eta V_H \mathbf{C}_{L\alpha_t}(i_t - \varepsilon_0) + \mathbf{C}_{m\alpha}\alpha_0 \tag{3.15}$$

and the vehicle pitch stiffness

$$\mathbf{C}_{m\alpha} = \left(\frac{x_{\text{cg}}}{\bar{c}} - \frac{x_{\text{ac}}}{\bar{c}}\right)\mathbf{C}_{L\alpha} - \eta V_H \mathbf{C}_{L\alpha_t}\left(1 - \frac{\mathrm{d}\varepsilon}{\mathrm{d}\alpha}\right) + \mathbf{C}_{m\alpha_f} \tag{3.16}$$

Note that Eq. (3.15) can be simplified (using Eq. (3.16)) to

$$\mathbf{C}_{m0} = \mathbf{C}_{m0_w} - \eta V_H \mathbf{C}_{L\alpha_t}\left[i_t - \varepsilon_0 + \left(1 - \frac{\mathrm{d}\varepsilon}{\mathrm{d}\alpha}\right)\alpha_0\right] + \mathbf{C}_{m\alpha_f}\alpha_0 \tag{3.17}$$

Note that Eq. (3.17) correctly shows that the pitching moment at zero net vehicle lift is independent of the c.g. location, as it must be (since at zero lift the resultant aerodynamic force must sum to a pure couple).

The basic (or control-fixed) *neutral point* is defined as the c.g. location for which the vehicle is neutrally stable in pitch – i.e., the c.g. location for which the pitch stiffness goes to zero. From Eq. (3.16) the neutral point is seen to be located at

$$\frac{x_{NP}}{\bar{c}} = \frac{x_{\text{ac}}}{\bar{c}} + \eta V_H \frac{\mathbf{C}_{L\alpha_t}}{\mathbf{C}_{L\alpha}}\left(1 - \frac{\mathrm{d}\varepsilon}{\mathrm{d}\alpha}\right) - \frac{\mathbf{C}_{m\alpha_f}}{\mathbf{C}_{L\alpha}} \tag{3.18}$$

Note that Eq. (3.16) for the pitch stiffness can be expressed as

$$\mathbf{C}_{m\alpha} = \left\{ \frac{x_{\text{cg}}}{\bar{c}} - \left[\frac{x_{\text{ac}}}{\bar{c}} + \eta V_H \frac{\mathbf{C}_{L\alpha_t}}{\mathbf{C}_{L\alpha}} \left(1 - \frac{\mathrm{d}\varepsilon}{\mathrm{d}\alpha} \right) - \frac{\mathbf{C}_{m\alpha_f}}{\mathbf{C}_{L\alpha}} \right] \right\} \mathbf{C}_{L\alpha} \tag{3.19}$$

where the quantity in square brackets is exactly the location of the basic neutral point, as shown in Eq. (3.18). Thus, we can write

$$\mathbf{C}_{m\alpha} = \left\{ \frac{x_{\text{cg}}}{\bar{c}} - \frac{x_{NP}}{\bar{c}} \right\} \mathbf{C}_{L\alpha} \tag{3.20}$$

or, alternatively,

$$\frac{\partial \mathbf{C}_m}{\partial \mathbf{C}_L} = - \left(\frac{x_{NP}}{\bar{c}} - \frac{x_{\text{cg}}}{\bar{c}} \right) \tag{3.21}$$

Thus, the pitch stiffness, measured with respect to changes in vehicle lift coefficient, is proportional to the distance between the c.g. and the basic neutral point. The quantity in parentheses on the right-hand side of Eq. (3.21), i.e., the distance between the vehicle c.g. and the basic neutral point, expressed as a percentage of the wing mean aerodynamic chord, is called the vehicle *static margin*.[2]

3.2 Static Longitudinal Control

The elevator is the aerodynamic control for pitch angle of the vehicle, and its effect is described in terms of the *elevator effectiveness*

$$a_e = \frac{\partial \mathbf{C}_{Lt}}{\partial \delta_e} \tag{3.22}$$

where $\mathbf{C}_{Lt}$ is the lift coefficient of the horizontal tail and δ_e is the elevator deflection, considered positive trailing edge down. The horizontal tail lift coefficient is then given by

$$\mathbf{C}_{Lt} = \frac{\partial \mathbf{C}_{Lt}}{\partial \alpha_t} (\alpha + i_t - \varepsilon) + a_e \delta_e \tag{3.23}$$

and the change in *vehicle* lift coefficient due to elevator deflection is

$$\mathbf{C}_{L\delta_e} = \eta \frac{S_t}{S} a_e \tag{3.24}$$

while the change in vehicle pitching moment due to elevator deflection is

$$\begin{aligned} \mathbf{C}_{m\delta_e} &= -\eta \frac{S_t}{S} a_e \left[\frac{\ell_t}{\bar{c}} + \frac{x_{\text{ac}} - x_{\text{cg}}}{\bar{c}} \right] \\ &= -\mathbf{C}_{L\delta_e} \left[\frac{\ell_t}{\bar{c}} + \frac{x_{\text{ac}} - x_{\text{cg}}}{\bar{c}} \right] \end{aligned} \tag{3.25}$$

The geometry of the moment arm of the tail lift relative to the vehicle c.g. (which justifies the second term in Eq. (3.25)) is shown in Fig. 3.1.

The vehicle is in equilibrium (i.e., is trimmed) at a given lift coefficient $\mathbf{C}_{L\text{trim}}$ when

$$\begin{aligned} \mathbf{C}_{L\alpha} \alpha + \mathbf{C}_{L\delta_e} \delta_e &= \mathbf{C}_{L\text{trim}} \\ \mathbf{C}_{m\alpha} \alpha + \mathbf{C}_{m\delta_e} \delta_e &= -\mathbf{C}_{m0} \end{aligned} \tag{3.26}$$

[2] Again, it is worth emphasizing that the location of the basic neutral point, and other special c.g. locations to be introduced later, are usually described as fractional distances along the wing mean aerodynamic chord; e.g. we might say that the basic neutral point is located at 40 per cent m.a.c.

These two equations can be solved for the unknown angle of attack and elevator deflection to give

$$\begin{aligned}\alpha_{\text{trim}} &= \frac{-\mathbf{C}_{L\delta_e}\mathbf{C}_{m0} - \mathbf{C}_{m\delta_e}\mathbf{C}_{L\text{trim}}}{\Delta}\\ \delta_{\text{trim}} &= \frac{\mathbf{C}_{L\alpha}\mathbf{C}_{m0} + \mathbf{C}_{m\alpha}\mathbf{C}_{L\text{trim}}}{\Delta}\end{aligned} \tag{3.27}$$

where

$$\Delta = -\mathbf{C}_{L\alpha}\mathbf{C}_{m\delta_e} + \mathbf{C}_{m\alpha}\mathbf{C}_{L\delta_e} \tag{3.28}$$

Note that the parameter

$$\begin{aligned}\Delta &= -\mathbf{C}_{L\alpha}\mathbf{C}_{m\delta_e} + \mathbf{C}_{m\alpha}\mathbf{C}_{L\delta_e}\\ &= -\mathbf{C}_{L\alpha}\left[-\mathbf{C}_{L\delta_e}\left(\frac{\ell_t}{\bar{c}} + \frac{x_{\text{ac}} - x_{\text{cg}}}{\bar{c}}\right)\right] + \mathbf{C}_{L\alpha}\left(\frac{x_{\text{cg}} - x_{NP}}{\bar{c}}\right)\mathbf{C}_{L\delta_e}\\ &= \mathbf{C}_{L\alpha}\mathbf{C}_{L\delta_e}\left(\frac{\ell_t}{\bar{c}} + \frac{x_{\text{ac}} - x_{NP}}{\bar{c}}\right) = \mathbf{C}_{L\alpha}\mathbf{C}_{L\delta_e}\frac{\ell_{t_N}}{\bar{c}}\end{aligned} \tag{3.29}$$

where

$$\ell_{t_N} = \ell_t + x_{\text{ac}} - x_{NP} \tag{3.30}$$

is the distance from the basic neutral point to the tail aerodynamic center. Thus, the parameter Δ is independent of the vehicle c.g. location, and is seen to be positive for conventional (aft tail) configurations, and negative for canard (forward tail) configurations.

An important derivative related to handling qualities is the control position gradient for trim, which can be seen from the second of Eqs. (3.27) to be given by

$$\left.\frac{\mathrm{d}\delta_e}{\mathrm{d}\mathbf{C}_L}\right)_{\text{trim}} = \frac{\mathbf{C}_{m\alpha}}{\Delta} \tag{3.31}$$

It is seen from Eq. (3.31) that the control position gradient, which measures the sensitivity of trimmed lift coefficient to control position, is negative for stable, aft tail configurations, and is proportional to the static margin (since Δ is independent of c.g. location and $\mathbf{C}_{m\alpha}$ is directly proportional to the static margin). In fact, using Eq. 3.29, we can see that

$$\left.\frac{\mathrm{d}\delta_e}{\mathrm{d}\mathbf{C}_L}\right)_{\text{trim}} = \frac{-1}{\mathbf{C}_{L\delta_e}}\frac{x_{NP} - x_{\text{c.g.}}}{\ell_{t_N}} \tag{3.32}$$

Thus, the control position gradient is seen to be determined by the static margin, normalized by ℓ_{t_N}, scaled by the effectiveness of the control deflection at generating lift $\mathbf{C}_{L\delta_e}$.

These results can be used in flight tests to determine the location of the basic neutral point. For each of several different c.g. positions the value of lift coefficient $\mathbf{C}_L$ is determined as a function of control position (as indicated by the data points in Fig. 3.2 (a).) For each c.g. location the value of the control position gradient is estimated by the best straight-line fit through these data, and is then plotted as a function of c.g. location. A best-fit straight line to these data, illustrated in Fig. 3.2 (b), is then extrapolated to zero control position gradient, which corresponds to the basic neutral point.

3.2.1 Longitudinal Maneuvers – the Pull-up

Another important criterion for vehicle handling qualities is the sensitivity of vehicle normal acceleration to control input. This can be analyzed by considering the vehicle in a steady pull-up. This

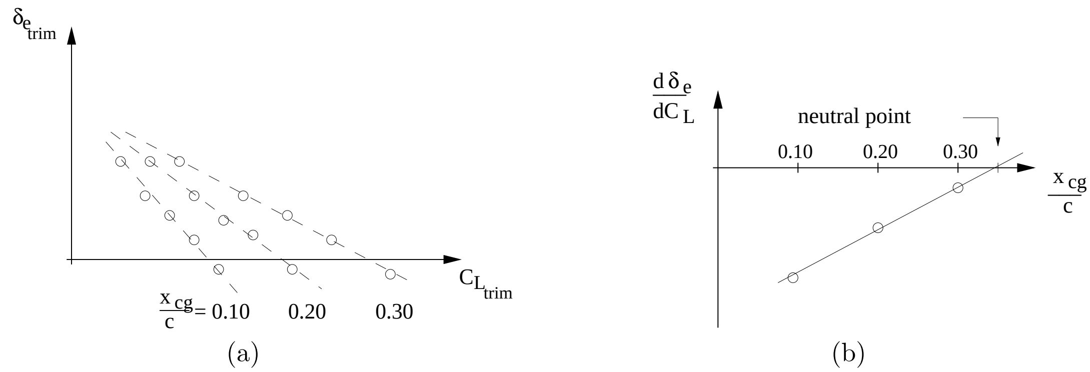

Figure 3.2: Schematic of procedure to estimate the location of the basic neutral point using control position gradient, measured in flight-test.

is a longitudinal maneuver in which the vehicle follows a curved flight path of constant radius R at constant angle of attack, as sketched in Fig. 3.3. For this maneuver, the pitch rate q is constant, and is given by

$$q = \frac{V}{R} \tag{3.33}$$

We define the dimensionless pitch rate

$$\hat{q} = \frac{q}{\frac{2V}{\bar{c}}} = \frac{\bar{c}q}{2V} \tag{3.34}$$

and will need to estimate the additional stability derivatives

$$\mathbf{C}_{L_q} \equiv \frac{\partial \mathbf{C}_L}{\partial \hat{q}} \tag{3.35}$$

and

$$\mathbf{C}_{m_q} \equiv \frac{\partial \mathbf{C}_m}{\partial \hat{q}} \tag{3.36}$$

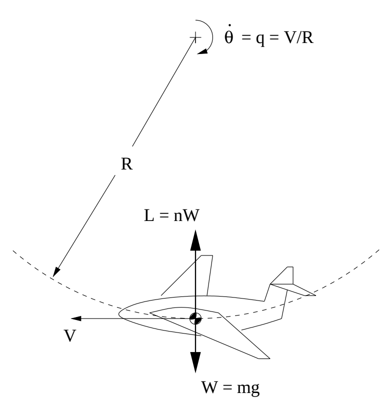

Figure 3.3: Schematic of flight path and forces acting on vehicle in a steady pull-up.

These derivatives characterize the sensitivity of vehicle lift and pitching moment to pitch rate. For vehicles with tails (either aft or canard), the largest contribution to these derivatives comes from the increment in tail lift due to the change in angle of attack of the tail arising from the rotation rate. This change in angle of attack is approximately[3]

$$\Delta\alpha_t = \frac{\ell_t}{V} q = \frac{2\ell_t}{\bar{c}} \hat{q} \tag{3.37}$$

and the resulting change in vehicle lift coefficient is

$$\Delta\mathbf{C}_L = \eta \frac{S_t}{S} \frac{\partial \mathbf{C}_{L\,t}}{\partial \alpha_t} \Delta\alpha_t = 2\eta V_H \frac{\partial \mathbf{C}_{L\,t}}{\partial \alpha_t} \hat{q} \tag{3.38}$$

so

$$\mathbf{C}_{L\,q} = 2\eta V_H \frac{\partial \mathbf{C}_{L\,t}}{\partial \alpha_t} \tag{3.39}$$

This increment in tail lift acts through the moment arm ℓ_t, so the corresponding estimate for the tail contribution to pitch damping is

$$\mathbf{C}_{m\,q} = -\frac{\ell_t}{\bar{c}} \mathbf{C}_{L\,q} = -2\eta \frac{\ell_t}{\bar{c}} V_H \frac{\partial \mathbf{C}_{L\,t}}{\partial \alpha_t} \tag{3.40}$$

The fuselage and wing (especially if the wing is swept) also contribute to the vehicle pitch damping, but it is difficult to develop simple formulas of general applicability, so these contributions will be neglected here. Note that the derivative $\mathbf{C}_{L\,q}$ will be positive for aft tail configurations (and negative for canard configurations), but the pitch damping $\mathbf{C}_{m\,q}$ will be always be negative, regardless of whether the tail is ahead or behind the vehicle center of gravity.

We analyze the motion at the point on the trajectory when the velocity vector is horizontal, so the balance of forces acting at the vehicle c.g. is

$$L - W = m\frac{V^2}{R} = mVq = \frac{2mV^2}{\bar{c}} \hat{q} \tag{3.41}$$

This equation can be written as

$$QS\left\{\mathbf{C}_{L\,\alpha}(\alpha + \Delta\alpha) + \mathbf{C}_{L\,\delta_e}(\delta_e + \Delta\delta_e) + \mathbf{C}_{L\,q}\hat{q}\right\} - W = \frac{2mV^2}{\bar{c}} \hat{q} \tag{3.42}$$

where α and δ_e are the angle of attack and elevator deflection for trim in the unaccelerated case, and $\Delta\alpha$ and $\Delta\delta_e$ correspond to the increments in these angles due to the maneuver. If we introduce the *weight coefficient*

$$\mathbf{C}_W \equiv \frac{W/S}{Q} \tag{3.43}$$

the dimensionless form of this equation can be written

$$\left\{\mathbf{C}_{L\,\alpha}(\alpha + \Delta\alpha) + \mathbf{C}_{L\,\delta_e}(\delta_e + \Delta\delta_e) + \mathbf{C}_{L\,q}\hat{q}\right\} - \mathbf{C}_W = 2\mu\hat{q} \tag{3.44}$$

where

$$\mu \equiv \frac{2m}{\rho S \bar{c}} \tag{3.45}$$

[3] Here, and in the equations through Eq. (3.40), the distance ℓ_t should represent the distance from the vehicle center-of-gravity to the aerodynamic center of the tail. The distance ℓ_t is a good approximation so long as the c.g. is near the wing aerodynamic center, which is usually the case.

is the vehicle *relative mass parameter*, which depends on ρ, the local fluid (air) density. As a result of this dependence on air density, the relative mass parameter is a function of flight altitude.

Subtracting the equilibrium values for the unaccelerated case

$$\mathbf{C}_{L\alpha}\alpha + \mathbf{C}_{L\delta_e}\delta_e - \mathbf{C}_W = 0 \tag{3.46}$$

from Eq. (3.44) gives

$$\mathbf{C}_{L\alpha}\Delta\alpha + \mathbf{C}_{L\delta_e}\Delta\delta_e = \left(2\mu - \mathbf{C}_{Lq}\right)\hat{q} \tag{3.47}$$

Finally, if we introduce the *normal acceleration parameter* n such that $L = nW$, then the force balance of Eq. (3.41) can be written in the dimensionless form

$$(n-1)\mathbf{C}_W = 2\mu\hat{q} \tag{3.48}$$

which provides a direct relation between the normal acceleration and the pitch rate, so that the lift equilibrium equation can be written

$$\mathbf{C}_{L\alpha}\Delta\alpha + \mathbf{C}_{L\delta_e}\Delta\delta_e = (n-1)\mathbf{C}_W\left(1 - \frac{\mathbf{C}_{Lq}}{2\mu}\right) \tag{3.49}$$

The pitching moment must also remain zero for equilibrium (since $\dot{q} = 0$), so

$$\mathbf{C}_{m\alpha}\Delta\alpha + \mathbf{C}_{m\delta_e}\Delta\delta_e + \mathbf{C}_{mq}\hat{q} = 0 \tag{3.50}$$

or

$$\mathbf{C}_{m\alpha}\Delta\alpha + \mathbf{C}_{m\delta_e}\Delta\delta_e = -\mathbf{C}_{mq}\frac{(n-1)\mathbf{C}_W}{2\mu} \tag{3.51}$$

Equations (3.49) and (3.51) provide two equations that can be solved for the unknowns $\Delta\alpha$ and $\Delta\delta_e$ to give

$$\begin{aligned}\Delta\alpha &= \frac{-(n-1)\mathbf{C}_W}{\Delta}\left[\left(1 - \frac{\mathbf{C}_{Lq}}{2\mu}\right)\mathbf{C}_{m\delta_e} + \frac{\mathbf{C}_{mq}}{2\mu}\mathbf{C}_{L\delta_e}\right]\\ \Delta\delta_e &= \frac{(n-1)\mathbf{C}_W}{\Delta}\left[\left(1 - \frac{\mathbf{C}_{Lq}}{2\mu}\right)\mathbf{C}_{m\alpha} + \frac{\mathbf{C}_{mq}}{2\mu}\mathbf{C}_{L\alpha}\right]\end{aligned} \tag{3.52}$$

where

$$\Delta = -\mathbf{C}_{L\alpha}\mathbf{C}_{m\delta_e} + \mathbf{C}_{m\alpha}\mathbf{C}_{L\delta_e} \tag{3.53}$$

is the same parameter as earlier (in Eq. (3.28)).

The control position derivative for normal acceleration is therefore given by

$$\frac{\mathrm{d}\delta_e}{\mathrm{d}n} = \frac{\mathbf{C}_W}{\Delta}\left[\left(1 - \frac{\mathbf{C}_{Lq}}{2\mu}\right)\mathbf{C}_{m\alpha} + \frac{\mathbf{C}_{mq}}{2\mu}\mathbf{C}_{L\alpha}\right] \tag{3.54}$$

Using Eq. (3.20) to express the pitch stiffness in terms of the c.g. location, we have

$$\frac{\mathrm{d}\delta_e}{\mathrm{d}n} = \frac{\mathbf{C}_W}{\Delta}\left[\left(1 - \frac{\mathbf{C}_{Lq}}{2\mu}\right)\left(\frac{x_{\mathrm{cg}}}{\bar{c}} - \frac{x_{NP}}{\bar{c}}\right) + \frac{\mathbf{C}_{mq}}{2\mu}\right]\mathbf{C}_{L\alpha} \tag{3.55}$$

The c.g. location for which this derivative vanishes is called the *basic maneuver point*, and its location, relative to the basic neutral point, is seen to be given by

$$\frac{x_{NP}}{\bar{c}} - \frac{x_{MP}}{\bar{c}} = \frac{\frac{\mathbf{C}_{mq}}{2\mu}}{1 - \frac{\mathbf{C}_{Lq}}{2\mu}} \approx \frac{\mathbf{C}_{mq}}{2\mu} \tag{3.56}$$

Since for all configurations the pitch damping $\mathbf{C}_{mq} < 0$, the maneuver point is aft of the neutral point. Also, since the vehicle relative mass parameter μ increases with altitude, the maneuver point approaches the neutral point with increasing altitude. If Eq. (3.56) is used to eliminate the variable x_{NP} from Eq. (3.55), we have

$$\frac{\mathrm{d}\delta_e}{\mathrm{d}n} = -\frac{\mathbf{C}_W \mathbf{C}_{L\alpha}}{\Delta}\left(1 - \frac{\mathbf{C}_{Lq}}{2\mu}\right)\left(\frac{x_{MP}}{\bar{c}} - \frac{x_{\mathrm{cg}}}{\bar{c}}\right) \tag{3.57}$$

where

$$\left(\frac{x_{MP}}{\bar{c}} - \frac{x_{\mathrm{cg}}}{\bar{c}}\right) \tag{3.58}$$

is called the *maneuver margin*.

3.3 Control Surface Hinge Moments

Just as the control *position* gradient is related to the pitch stiffness of the vehicle when the controls are fixed, the control *force* gradients are related to the pitch stiffness of the vehicle when the controls are allowed to float free.

3.3.1 Control Surface Hinge Moments

Since elevator deflection corresponds to rotation about a hinge line, the forces required to cause a specific control deflection are related to the aerodynamic moments about the hinge line. A free control will float, in the static case, to the position at which the elevator hinge moment is zero:

$$H_e = 0.$$

The elevator hinge moment is usually expressed in terms of the *hinge moment coefficient*

$$\mathbf{C}_{he} = \frac{H_e}{QS_e\bar{c}_e} \tag{3.59}$$

where the reference area S_e and moment arm $\bar{c}_e$ correspond to the planform area and mean chord of the control surface aft of the hinge line. Assuming that the hinge moment is a linear function of angle of attack, control deflection, etc., we write

$$\mathbf{C}_{he} = \mathbf{C}_{he_0} + \mathbf{C}_{h\alpha}\alpha + \mathbf{C}_{h\delta_e}\delta_e + \mathbf{C}_{h\delta_t}\delta_t \tag{3.60}$$

In this equation, α is the angle of attack (from angle for zero vehicle lift), δ_e is the elevator deflection, and δ_t is the deflection of the *control tab* (to be described in greater detail later).

The derivative $\mathbf{C}_{h\alpha}$ characterizes the hinge moment created by changes in angle of attack; it is called the *floating tendency*, as the hinge moment generated by an increase in angle of attack

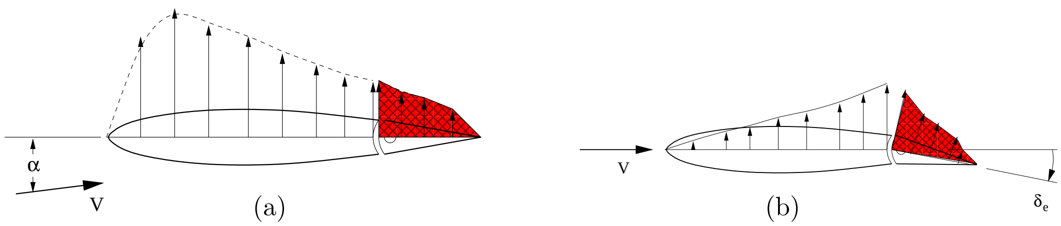

Figure 3.4: Schematic illustration of aerodynamic forces responsible for (a) floating and (b) restoring tendencies of trailing edge control surfaces. Floating (or restoring) tendency represents moment about hinge line of (shaded) lift distribution acting on control surface per unit angle of attack (or control deflection).

generally causes the control surface to float upward. The derivative $\mathbf{C}_{h\delta_e}$ characterizes the hinge moment created by a deflection of the control (considered positive trailing edge down); it is called the *restoring tendency*, as the nose-down hinge moment generated by a positive control deflection tends to restore the control to its original position. The floating tendency in Eq. (3.60) is referred to the vehicle angle of attack, and so it is related to the derivative based on tail angle of attack α_t by

$$\mathbf{C}_{h\alpha} = \left(1 - \frac{\mathrm{d}\epsilon}{\mathrm{d}\alpha}\right)\mathbf{C}_{h\alpha_t} \tag{3.61}$$

which accounts for the effects of wing induced downwash at the tail. The aerodynamic forces responsible for generating the hinge moments reflected in the floating and restoring tendencies are sketched in Fig. 3.4. Only the shaded portion of the lift distribution in these figures acts on the control surface and contributes to the hinge moment.

The angle at which the free elevator floats is determined by the fact that the hinge moment (and, therefore, the hinge moment coefficient) must be zero

$$\mathbf{C}_{he} = 0 = \mathbf{C}_{he_0} + \mathbf{C}_{h\alpha}\alpha + \mathbf{C}_{h\delta_e}\delta_{e\text{free}} + \mathbf{C}_{h\delta_t}\delta_t$$

or

$$\delta_{e\text{free}} = -\frac{1}{\mathbf{C}_{h\delta_e}}\left(\mathbf{C}_{he_0} + \mathbf{C}_{h\alpha}\alpha + \mathbf{C}_{h\delta_t}\delta_t\right) \tag{3.62}$$

The corresponding lift and moment coefficients are

$$\begin{aligned}\mathbf{C}_{L\text{free}} &= \mathbf{C}_{L\alpha}\alpha + \mathbf{C}_{L\delta_e}\delta_{e\text{free}}\\ \mathbf{C}_{m\text{free}} &= \mathbf{C}_{m0} + \mathbf{C}_{m\alpha}\alpha + \mathbf{C}_{m\delta_e}\delta_{e\text{free}}\end{aligned} \tag{3.63}$$

which, upon substituting from Eq. (3.62), can be written

$$\begin{aligned}\mathbf{C}_{L\text{free}} &= \mathbf{C}_{L\alpha}\left(1 - \frac{\mathbf{C}_{L\delta_e}\mathbf{C}_{h\alpha}}{\mathbf{C}_{L\alpha}\mathbf{C}_{h\delta_e}}\right)\alpha - \frac{\mathbf{C}_{L\delta_e}}{\mathbf{C}_{h\delta_e}}\left(\mathbf{C}_{he_0} + \mathbf{C}_{h\delta_t}\delta_t\right)\\ \mathbf{C}_{m\text{free}} &= \mathbf{C}_{m\alpha}\left(1 - \frac{\mathbf{C}_{m\delta_e}\mathbf{C}_{h\alpha}}{\mathbf{C}_{m\alpha}\mathbf{C}_{h\delta_e}}\right)\alpha + \mathbf{C}_{m0} - \frac{\mathbf{C}_{m\delta_e}}{\mathbf{C}_{h\delta_e}}\left(\mathbf{C}_{he_0} + \mathbf{C}_{h\delta_t}\delta_t\right)\end{aligned} \tag{3.64}$$

Thus, if we denote the control free lift curve slope and pitch stiffness using primes, we see from the above equations that

$$\begin{aligned}\mathbf{C}_{L\alpha}' &= \mathbf{C}_{L\alpha}\left(1 - \frac{\mathbf{C}_{L\delta_e}\mathbf{C}_{h\alpha}}{\mathbf{C}_{L\alpha}\mathbf{C}_{h\delta_e}}\right)\\ \mathbf{C}_{m\alpha}' &= \mathbf{C}_{m\alpha}\left(1 - \frac{\mathbf{C}_{m\delta_e}\mathbf{C}_{h\alpha}}{\mathbf{C}_{m\alpha}\mathbf{C}_{h\delta_e}}\right)\end{aligned} \tag{3.65}$$

Inspection of these equations shows that the lift curve slope is always reduced by freeing the controls, and the pitch stiffness of a stable configuration is reduced in magnitude by freeing the controls for an aft tail configuration, and increased in magnitude for a forward tail (canard) configuration (in all cases assuming that the floating and restoring tendencies both are negative).

3.3.2 Control free Neutral Point

The c.g. location at which the control free pitch stiffness vanishes is called the *control free neutral point*. The location of the control free neutral point x'_{NP} can be determined by expressing the pitch stiffness in the second of Eqs. (3.65)

$$\mathbf{C}_{m_\alpha}' = \mathbf{C}_{m\alpha} - \frac{\mathbf{C}_{m\delta_e}\mathbf{C}_{h\alpha}}{\mathbf{C}_{h\delta_e}}$$

as

$$\begin{aligned} \mathbf{C}_{m_\alpha}' &= \left(\frac{x_{\text{cg}}}{\bar{c}} - \frac{x_{NP}}{\bar{c}}\right)\mathbf{C}_{L\alpha} + \frac{\mathbf{C}_{h\alpha}\mathbf{C}_{L\delta_e}}{\mathbf{C}_{h\delta_e}}\left(\frac{\ell_t}{\bar{c}} + \frac{x_{\text{ac}}}{\bar{c}} - \frac{x_{\text{cg}}}{\bar{c}}\right) \\ &= \left(\frac{x_{\text{cg}}}{\bar{c}} - \frac{x_{NP}}{\bar{c}}\right)\mathbf{C}_{L\alpha} + \frac{\mathbf{C}_{h\alpha}}{\mathbf{C}_{h\delta_e}}\eta\frac{S_t}{S}a_e\left(\frac{\ell_t + x_{\text{ac}} - x_{NP}}{\bar{c}} + \frac{x_{NP} - x_{\text{cg}}}{\bar{c}}\right) \\ &= \left(\frac{x_{\text{cg}}}{\bar{c}} - \frac{x_{NP}}{\bar{c}}\right)\left[\mathbf{C}_{L\alpha} - \frac{\mathbf{C}_{L\delta_e}\mathbf{C}_{h\alpha}}{\mathbf{C}_{h\delta_e}}\right] + \eta V_{H_N}\frac{\mathbf{C}_{h\alpha}a_e}{\mathbf{C}_{h\delta_e}} \end{aligned} \tag{3.66}$$

where $a_e = \partial\mathbf{C}_{Lt}/\partial\delta_e$ is the elevator effectiveness and

$$V_{H_N} = \left(\frac{\ell_t}{\bar{c}} + \frac{x_{\text{ac}}}{\bar{c}} - \frac{x_{NP}}{\bar{c}}\right)\frac{S_t}{S} \tag{3.67}$$

is the tail volume ratio based on ℓ_{t_N}, the distance between the tail aerodynamic center and the basic neutral point, as defined in Eq. (3.30). The quantity in square brackets in the final version of Eq. (3.66) is seen to be simply the control free vehicle lift curve slope $\mathbf{C}_{L_\alpha}'$, so we have

$$\mathbf{C}_{m_\alpha}' = \left(\frac{x_{\text{cg}}}{\bar{c}} - \frac{x_{NP}}{\bar{c}}\right)\mathbf{C}_{L_\alpha}' + \eta V_{H_N}\frac{\mathbf{C}_{h\alpha}a_e}{\mathbf{C}_{h\delta_e}} \tag{3.68}$$

Setting the control free pitch stiffness $\mathbf{C}_{m_\alpha}'$ to zero gives the distance between the control free and basic neutral points as

$$\frac{x_{NP}}{\bar{c}} - \frac{x'_{NP}}{\bar{c}} = \eta V_{H_N}\frac{a_e}{\mathbf{C}_{L_\alpha}'}\frac{\mathbf{C}_{h\alpha}}{\mathbf{C}_{h\delta_e}} \tag{3.69}$$

Finally, if Eq. (3.69) is substituted back into Eq. (3.68) to eliminate the variable x_{NP}, we have

$$\mathbf{C}_{m_\alpha}' = -\left(\frac{x'_{NP}}{\bar{c}} - \frac{x_{\text{cg}}}{\bar{c}}\right)\mathbf{C}_{L_\alpha}' \tag{3.70}$$

showing that the control free pitch stiffness is directly proportional to the *control free static margin*

$$\left(\frac{x'_{NP}}{\bar{c}} - \frac{x_{\text{cg}}}{\bar{c}}\right)$$

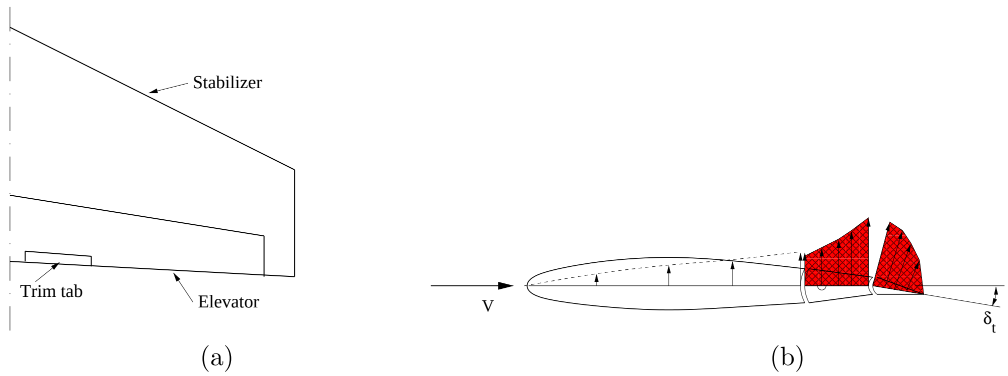

Figure 3.5: (a) Typical location of trim tab on horizontal control (elevator), and (b) schematic illustration of aerodynamic forces responsible for hinge moment due to trim tab deflection.

3.3.3 Trim Tabs

Trim tabs can be used by the pilot to trim the vehicle at zero control force for any desired speed. Trim tabs are small control surfaces mounted at the trailing edges of primary control surfaces. A linkage is provided that allows the pilot to set the angle of the trim tab, relative to the primary control surface, in a way that is independent of the deflection of the primary control surface. Deflection of the trim tab creates a hinge moment that causes the elevator to float at the angle desired for trim. The geometry of a typical trim tab arrangement is shown in Fig. 3.5.

Zero control force corresponds to zero hinge moment, or

$$\mathbf{C}_{h_e} = 0 = \mathbf{C}_{h_{e_0}} + \mathbf{C}_{h\alpha}\alpha + \mathbf{C}_{h\delta_e}\delta_e + \mathbf{C}_{h\delta_t}\delta_t$$

and the trim tab deflection that achieves this for arbitrary angle of attack and control deflection is

$$\delta_t = -\frac{1}{\mathbf{C}_{h\delta_t}}\left(\mathbf{C}_{h_{e_0}} + \mathbf{C}_{h\alpha}\alpha + \mathbf{C}_{h\delta_e}\delta_e\right) \tag{3.71}$$

so the tab setting required for zero control force at trim is

$$\delta_{t_{\text{trim}}} = -\frac{1}{\mathbf{C}_{h\delta_t}}\left(\mathbf{C}_{h_{e_0}} + \mathbf{C}_{h\alpha}\alpha_{\text{trim}} + \mathbf{C}_{h\delta_e}\delta_{e\text{trim}}\right) \tag{3.72}$$

The values of α_{trim} and $\delta_{e\text{trim}}$ are given by Eqs. (3.27)

$$\begin{aligned}\alpha_{\text{trim}} &= \frac{-\mathbf{C}_{L\delta_e}\mathbf{C}_{m0} - \mathbf{C}_{m\delta_e}\mathbf{C}_{L\text{trim}}}{\Delta}\\ \delta_{e\text{trim}} &= \frac{\mathbf{C}_{L\alpha}\mathbf{C}_{m0} + \mathbf{C}_{m\alpha}\mathbf{C}_{L\text{trim}}}{\Delta}\end{aligned} \tag{3.73}$$

Substituting these values into Eq. (3.72) gives the required trim tab setting as

$$\delta_{t_{\text{trim}}} = -\frac{1}{\mathbf{C}_{h\delta_t}}\left(\mathbf{C}_{h_{e_0}} + \frac{\mathbf{C}_{m0}}{\Delta}\left(-\mathbf{C}_{h\alpha}\mathbf{C}_{L\delta_e} + \mathbf{C}_{h\delta_e}\mathbf{C}_{L\alpha}\right) + \frac{1}{\Delta}\left(-\mathbf{C}_{h\alpha}\mathbf{C}_{m\delta_e} + \mathbf{C}_{h\delta_e}\mathbf{C}_{m\alpha}\right)\mathbf{C}_{L\text{trim}}\right) \tag{3.74}$$

Note that the coefficient of $\mathbf{C}_{L\text{trim}}$ in this equation – which gives the sensitivity of the trim tab setting to the trim lift coefficient – can be written as

$$\frac{d\delta_t}{d\mathbf{C}_L} = -\frac{\mathbf{C}_{h\delta_e}}{\mathbf{C}_{h\delta_t}\Delta}\left(\mathbf{C}_{m\alpha} - \frac{\mathbf{C}_{h\alpha}\mathbf{C}_{m\delta_e}}{\mathbf{C}_{h\delta_e}}\right) = -\frac{\mathbf{C}_{h\delta_e}}{\mathbf{C}_{h\delta_t}\Delta}\mathbf{C}_{m\alpha}' = -\frac{\mathbf{C}_{h\delta_e}}{\mathbf{C}_{h\delta_t}\Delta}\left(\frac{x_{NP}'}{\bar{c}} - \frac{x_{\text{cg}}}{\bar{c}}\right)\mathbf{C}_{L\alpha}' \tag{3.75}$$

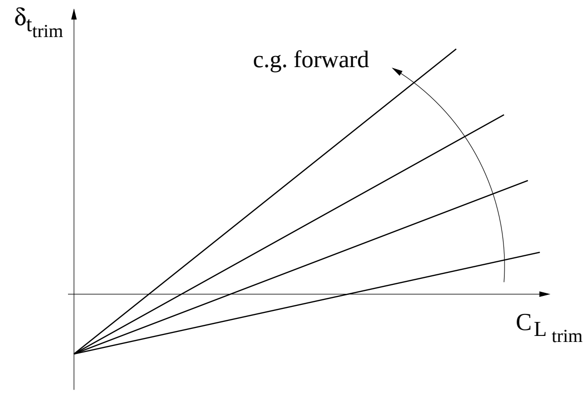

Figure 3.6: Variation in trim tab setting as function of velocity for stable, aft tail vehicle.

and Eq. (3.74) can be written

$$\delta_{t_{\rm trim}} = -\frac{1}{\mathbf{C}_{h\delta_t}}\left[\mathbf{C}_{he_0} + \frac{\mathbf{C}_{m0}}{\Delta}\left(-\mathbf{C}_{h\alpha}\mathbf{C}_{L\delta_e} + \mathbf{C}_{h\delta_e}\mathbf{C}_{L\alpha}\right) + \frac{\mathbf{C}_{h\delta_e}}{\Delta}\mathbf{C}_{L_\alpha}{}'\left(\frac{x'_{NP}}{\bar{c}} - \frac{x_{\rm cg}}{\bar{c}}\right)\mathbf{C}_{L{\rm trim}}\right] \tag{3.76}$$

Thus, the tab setting for trim is a linear function of trimmed lift coefficient whose slope is proportional to the control free static margin. This variation is shown schematically for a conventional (aft tail) configuration in Fig. 3.6.

3.3.4 Control Force for Trim

As mentioned earlier, the most important aspects of stability relating to handling qualities of the vehicle are related to control *forces*. For longitudinal control, the control force F is related to the elevator hinge moment H_e through a gearing constant G, so that

$$F = GH_e \tag{3.77}$$

This equation defines a positive control force as a *pull*, corresponding to the force required to balance a positive (nose up) elevator hinge moment.[4] The units of the gearing constant G are inverse length, which can be interpreted as a mechanical advantage corresponding to radians of control deflection per unit distance (foot) of control yoke displacement.

Expressing the hinge moment in terms of the corresponding dimensionless coefficient, we have

$$F = GS_e\bar{c}_eQ\mathbf{C}_{he} = GS_e\bar{c}_eQ\left(\mathbf{C}_{he_0} + \mathbf{C}_{h\alpha}\alpha + \mathbf{C}_{h\delta_e}\delta_e + \mathbf{C}_{h\delta_t}\delta_t\right) \tag{3.78}$$

Since this equation is linear in tab deflection, the control force required for a tab setting other than the trim value is

$$F = GS_e\bar{c}_eQ\mathbf{C}_{h\delta_t}\left(\delta_t - \delta_{t_{\rm trim}}\right) \tag{3.79}$$

[4]It is important to be careful when reading other books; positive control force is sometimes defined as a *push*, in which case there is a minus sign inserted on the right hand side of Eq. (3.77) and subsequently throughout the analysis.

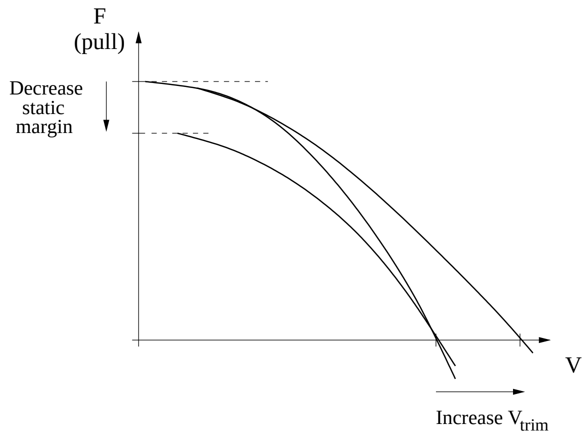

Figure 3.7: Typical variation in control force as function of vehicle velocity for stable configuration.

and, substituting the tab setting required for trim from Eq. (3.76), we have

$$F = GS_e\bar{c}_eQ\left[\mathbf{C}_{h\delta_t}\delta_t + \mathbf{C}_{he_0} + \frac{\mathbf{C}_{m0}}{\Delta}\left(-\mathbf{C}_{h\alpha}\mathbf{C}_{L\delta_e} + \mathbf{C}_{h\delta_e}\mathbf{C}_{L\alpha}\right) + \frac{\mathbf{C}_{h\delta_e}}{\Delta}\mathbf{C}_{L}{'}_{\alpha}\left(\frac{x_{\text{cg}} - x'_{NP}}{\bar{c}}\right)\mathbf{C}_{L\text{trim}}\right] \tag{3.80}$$

Finally, substituting

$$\mathbf{C}_{L\text{trim}} = \frac{W/S}{Q} \tag{3.81}$$

for level flight with $L = W$, we have

$$\begin{aligned} F =& GS_e\bar{c}_e(W/S)\frac{\mathbf{C}_{h\delta_e}\mathbf{C}_{L}{'}_{\alpha}}{\Delta}\left(\frac{x_{\text{cg}} - x'_{NP}}{\bar{c}}\right) + \\ & GS_e\bar{c}_e\left[\mathbf{C}_{h\delta_t}\delta_t + \mathbf{C}_{he_0} + \frac{\mathbf{C}_{m0}}{\Delta}\left(-\mathbf{C}_{h\alpha}\mathbf{C}_{L\delta_e} + \mathbf{C}_{h\delta_e}\mathbf{C}_{L\alpha}\right)\right]\frac{1}{2}\rho V^2 \end{aligned} \tag{3.82}$$

The dependence of control force on velocity described by this equation is sketched in Fig. 3.7. Note from the equation that:

1. The control force $F \propto S_e\bar{c}_e$, i.e, is proportional to the *cube* of the size of the vehicle; control forces grow rapidly with aircraft size, and large aircraft require powered (or power-assisted) control systems.
2. The location of the c.g. (i.e., the control free static margin) affects only the constant term in the equation.
3. The vehicle weight enters only in the ratio W/S.
4. The effect of trim tab deflection δ_t is to change the coefficient of the V^2 term, and hence controls the intercept of the curve with the velocity axis.

If we denote the velocity at which the control force is zero as V_{trim}, then Eq. (3.82) gives

$$\begin{aligned} & GS_e\bar{c}_e\left(\mathbf{C}_{h\delta_t}\delta_t + \mathbf{C}_{he_0} + \frac{\mathbf{C}_{m0}}{\Delta}\left(-\mathbf{C}_{h\alpha}\mathbf{C}_{L\delta_e} + \mathbf{C}_{h\delta_e}\mathbf{C}_{L\alpha}\right)\right)\frac{1}{2}\rho V_{\text{trim}}^2 = \\ & - GS_e\bar{c}_e(W/S)\frac{\mathbf{C}_{h\delta_e}\mathbf{C}_{L}{'}_{\alpha}}{\Delta}\left(\frac{x_{\text{cg}} - x'_{NP}}{\bar{c}}\right) \end{aligned} \tag{3.83}$$

so

$$F = GS_e\bar{c}_e(W/S)\frac{\mathbf{C}_{h\delta_e}\mathbf{C}_{L\alpha}'}{\Delta}\left(\frac{x_{\text{cg}} - x_{NP}'}{\bar{c}}\right)\left[1 - (V/V_{\text{trim}})^2\right] \tag{3.84}$$

and

$$\left.\frac{\mathrm{d}F}{\mathrm{d}V}\right)_{V_{\text{trim}}} = -\frac{2}{V_{\text{trim}}}GS_e\bar{c}_e(W/S)\frac{\mathbf{C}_{h\delta_e}\mathbf{C}_{L\alpha}'}{\Delta}\left(\frac{x_{\text{cg}} - x_{NP}'}{\bar{c}}\right) \tag{3.85}$$

These last two equations, which also can be interpreted in terms of Fig. 3.7, show that:

1. For a given control free static margin (or c.g. position) the control force gradient decreases with increasing flight velocity; and
2. At a given trim velocity, the control force gradient decreases as the c.g. is moved aft toward the control free neutral point (i.e., as the static margin is reduced).

3.3.5 Control-force for Maneuver

Perhaps the single most important stability property of an aircraft, in terms of handling properties, describes the control force required to perform a maneuver. This force must not be too small to avoid over-stressing the airframe, nor too large to avoid making the pilot work too hard.

We will again consider the steady pull-up. The change in control force required to effect the maneuver is

$$\Delta F = GS_e\bar{c}_eQ\Delta\mathbf{C}_{he} \tag{3.86}$$

where

$$\Delta\mathbf{C}_{he} = \mathbf{C}_{h\alpha}\Delta\alpha + \mathbf{C}_{h\delta_e}\Delta\delta_e + \mathbf{C}_{hq}\hat{q} \tag{3.87}$$

where $\hat{q}$ is the dimensionless pitch rate, as defined in Section 3.2.1. It was also seen in that section that the dimensionless pitch rate for a pull-up could be related directly to the excess load factor $(n-1)$, so, using Eq. (3.48), we have

$$\Delta\mathbf{C}_{he} = \mathbf{C}_{h\alpha}\Delta\alpha + \mathbf{C}_{h\delta_e}\Delta\delta_e + \frac{(n-1)\mathbf{C}_W}{2\mu}\mathbf{C}_{hq} \tag{3.88}$$

The derivative $\mathbf{C}_{hq}$ arises from the change in hinge moment due to the change in tail angle of attack arising from the pitch rate. Thus

$$\Delta\mathbf{C}_{he} = \mathbf{C}_{h\alpha_t}\Delta\alpha_t = \mathbf{C}_{h\alpha_t}\frac{2\ell_t}{\bar{c}}\hat{q} \tag{3.89}$$

and

$$\mathbf{C}_{hq} \equiv \frac{\partial\mathbf{C}_{he}}{\partial\hat{q}} = 2\frac{\ell_t}{\bar{c}}\mathbf{C}_{h\alpha_t} \tag{3.90}$$

Now, we can use the solution for $\Delta\delta_e$ from Eq. (3.52)

$$\Delta\delta_e = \frac{(n-1)\mathbf{C}_W}{\Delta}\left[\left(1 - \frac{\mathbf{C}_{Lq}}{2\mu}\right)\mathbf{C}_{m\alpha} + \frac{\mathbf{C}_{mq}}{2\mu}\mathbf{C}_{L\alpha}\right] \tag{3.91}$$

along with the lift coefficient equation, Eq. (3.49), which can be written

$$\Delta\alpha = \frac{1}{\mathbf{C}_{L\alpha}}\left[(n-1)\mathbf{C}_W\left(1-\frac{\mathbf{C}_{Lq}}{2\mu}\right)-\mathbf{C}_{L\delta_e}\Delta\delta_e\right] \tag{3.92}$$

in the hinge moment equation to give

$$\Delta\mathbf{C}_{he} = \mathbf{C}_{h\alpha}\frac{n-1}{\mathbf{C}_{L\alpha}}\left[\left(1-\frac{\mathbf{C}_{Lq}}{2\mu}\right)\mathbf{C}_W - \mathbf{C}_{L\delta_e}\frac{\Delta\delta_e}{n-1}\right]+\mathbf{C}_{h\delta_e}\Delta\delta_e+\frac{(n-1)\mathbf{C}_W}{2\mu}\mathbf{C}_{hq} \tag{3.93}$$

which can be rearranged into the form

$$\frac{\Delta\mathbf{C}_{he}}{n-1} = \frac{\mathbf{C}_W}{\mathbf{C}_{L\alpha}}\left[\left(1-\frac{\mathbf{C}_{Lq}}{2\mu}\right)\mathbf{C}_{h\alpha}+\frac{\mathbf{C}_{hq}}{2\mu}\mathbf{C}_{L\alpha}\right]+\frac{\Delta\delta_e}{n-1}\mathbf{C}_{h\delta_e}\frac{\mathbf{C}_{L\alpha}'}{\mathbf{C}_{L\alpha}} \tag{3.94}$$

Finally, using Eq. (3.57) for $\Delta\delta_e/(n-1)$, the equation for the hinge moment increment can be written

$$\frac{\Delta\mathbf{C}_{he}}{n-1} = \frac{\mathbf{C}_W\mathbf{C}_{L\alpha}'\mathbf{C}_{h\delta_e}}{\Delta}\left(1-\frac{\mathbf{C}_{Lq}}{2\mu}\right)\left[\frac{x_{\text{cg}}-x_{MP}}{\bar{c}}+\frac{\Delta}{\mathbf{C}_{L\alpha}'\mathbf{C}_{h\delta_e}}\left(\frac{\mathbf{C}_{h\alpha}}{\mathbf{C}_{L\alpha}}+\frac{\mathbf{C}_{hq}}{2\mu-\mathbf{C}_{Lq}}\right)\right] \tag{3.95}$$

The *control free maneuver point* is defined as the c.g. location for which the control force gradient (per g) (or, equivalently, the hinge moment coefficient gradient) vanishes. This is seen from Eq. (3.95) to give

$$\frac{x_{MP}-x_{MP}'}{\bar{c}} = \frac{\Delta}{\mathbf{C}_{L\alpha}'\mathbf{C}_{h\delta_e}}\left(\frac{\mathbf{C}_{h\alpha}}{\mathbf{C}_{L\alpha}}+\frac{\mathbf{C}_{hq}}{2\mu-\mathbf{C}_{Lq}}\right) \tag{3.96}$$

Note that this quantity is positive for aft tail configurations, and negative for forward tail (canard) configurations. Substitution of this expression back into Eq. (3.95) then gives

$$\frac{\Delta\mathbf{C}_{he}}{n-1} = \frac{\mathbf{C}_W\mathbf{C}_{L\alpha}'\mathbf{C}_{h\delta_e}}{\Delta}\left(1-\frac{\mathbf{C}_{Lq}}{2\mu}\right)\left(\frac{x_{\text{cg}}-x_{MP}'}{\bar{c}}\right) \tag{3.97}$$

Finally, the control force gradient (per g) is

$$\begin{aligned}\frac{\partial F}{\partial n} &= \frac{\Delta F}{n-1} = GS_e\bar{c}_eQ\frac{\Delta\mathbf{C}_{he}}{n-1}\\ &= GS_e\bar{c}_eQ\frac{\mathbf{C}_W\mathbf{C}_{L\alpha}'\mathbf{C}_{h\delta_e}}{\Delta}\left(1-\frac{\mathbf{C}_{Lq}}{2\mu}\right)\left(\frac{x_{\text{cg}}-x_{MP}'}{\bar{c}}\right)\end{aligned} \tag{3.98}$$

or, since $Q\mathbf{C}_W = W/S$,

$$\frac{\partial F}{\partial n} = GS_e\bar{c}_e(W/S)\frac{\mathbf{C}_{L\alpha}'\mathbf{C}_{h\delta_e}}{\Delta}\left(1-\frac{\mathbf{C}_{Lq}}{2\mu}\right)\left(\frac{x_{\text{cg}}-x_{MP}'}{\bar{c}}\right) \tag{3.99}$$

The distance $\frac{x_{MP}'-x_{\text{cg}}}{\bar{c}}$, seen from the above equation to be directly related to the sensitivity of normal acceleration of the vehicle to control force, is called the *control free maneuver margin.*

Note that the control force gradient (per g) is

1. Directly proportional to the vehicle wing loading W/S;

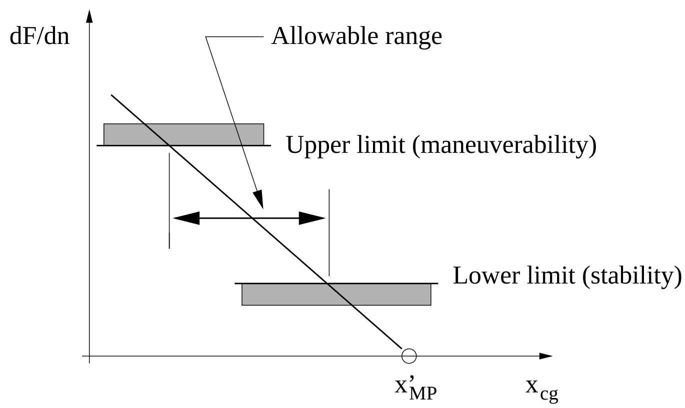

Figure 3.8: Allowable c.g. travel as imposed by limits on control force gradient (per g).

2. Directly proportional to the *cube* of the linear size of the vehicle;
3. Directly proportional to the (control free) maneuver margin $(x'_{MP} - x_{\text{cg}})/\bar{c}$; and
4. Independent of airspeed.

The control force gradient should be neither too small nor too large. If the gradient is too small, the vehicle will be overly sensitive to small control inputs and it will be too easy for the pilot to over stress the airframe. At the same time, the control forces required for normal maneuvers must not be larger than the pilot can supply (or so large that the pilot becomes unduly tired performing normal maneuvers). The lower and upper limits on control force gradient (per g) determine allowable rearward and forward limits on c.g. travel, as sketched in Fig. 3.8. The values of these limits will depend on the vehicle mission; in general the limits will be higher for transport aircraft, and lower for vehicles which require greater maneuverability (such as military fighters or aerobatic aircraft).

3.4 Forward and Aft Limits of C.G. Position

The various control position and force gradients impose limits on the acceptable range of travel of the vehicle center of gravity. These include (for most vehicles):

- **Rearward limits:**
  1. The vehicle must be statically stable; i.e., the c.g. must be ahead of the basic and control free neutral points.
  2. The sensitivity of vehicle velocity to control position must not be too small; i.e., the c.g. must be sufficiently far ahead of the basic neutral point.
  3. The sensitivity of vehicle normal acceleration to control force must not be too small; i.e., the c.g. must be sufficiently far ahead of the control free neutral point.
- **Forward limits:**

1. The vehicle must be trimmable at $\mathbf{C}_{L\max}$; i.e., the c.g. must not be so far forward that there is insufficient elevator power to trim the vehicle at maximum lift coefficient.
2. The sensitivity of vehicle normal acceleration to control force must not be too high; i.e., the c.g. must not be so far forward that excessive control force is required to perform maneuvers for which the vehicle is intended.

Bibliography

[1] Robert C. Nelson, **Aircraft Stability and Automatic Control**, McGraw-Hill, Second edition, 1998.