MATHEMATICS

TEACHER GUIDE
GRADE 10

FEDERAL DEMOCRATIC REPUBLIC OF ETHIOPIA
MINISTRY OF EDUCATION

Mathematics Teacher Guide Grade 10

MATHEMATICS

TEACHER GUIDE
GRADE 10

FEDERAL DEMOCRATIC REPUBLIC OF ETHIOPIA
MINISTRY OF EDUCATION

MATHEMATICS

TEACHER GUIDE
GRADE 10

Authors:

Tadele Mekonnen Yihun (MSc)

Mamo Teketel Mekasha (MSc)

Editors:

Mohammed Yiha Dawud (PhD) **(Content Editor)**

Akalu Chaka Mekuria (MA) **(Curriculum Editor)**

Endalfer Melese Moges (MA) **(Language Editor)**

Illustrator:

Bahiru Chane Tamiru (MSc)

Designer:

Aknaw H/mariam Habte (MSc)

Evaluators:

Matebie Alemayehu Wasihun (MED)

Mustefa Kedir Edao (BED)

Dawit Ayalneh Tebkew (MSc)

FEDERAL DEMOCRATIC REPUBLIC OF ETHIOPIA
MINISTRY OF EDUCATION

HAWASSA UNIVERSITY

First Published xxxxx 2022 by the Federal Democratic Republic of Ethiopia, Ministry of Education, under the General Education Quality Improvement Program for Equity (GEQIP-E) supported by the World Bank, UK's Department for International Development/DFID-now merged with the Foreign, Common wealth and Development Office/FCDO, Finland Ministry for Foreign Affairs, the Royal Norwegian Embassy, United Nations Children's Fund/UNICEF), the Global Partnership for Education (GPE), and Danish Ministry of Foreign Affairs, through a Multi Donor Trust Fund.

The Ministry of Education wishes to thank the many individuals, groups and other bodies involved – directly or indirectly – in publishing this Textbook. Special thanks are due to Hawassa University for their huge contribution in the development of this textbook in collaboration with Addis Ababa University, Bahir Dar University, Jimma University and JICA must project.

Printed by:

xxxxxxxx PRINTING

P.O.Box xxxxxx

xxxxxxx, ETHIOPIA

Under Ministry of Education Contract no. xxxxxxxxxxx

ISBN: 978-999944-2-046-9

Foreword

Education and development are closely related endeavors. This is the main reason why it is said that education is the key instrument in Ethiopia's development and social transformation. The fast and globalized world we now live in requires new knowledge, skill and attitude on the part of each individual. It is with this objective in view that the curriculum, which is not only the Blueprint but also a reflection of a country's education system, must be responsive to changing conditions.

It has been almost three decades since Ethiopia launched and implemented new *Education and Training Policy*. Since the 1994 *Education and Training Policy* our country has recorded remarkable progress in terms of access, equity and relevance. Vigorous efforts also have been made, and continue to be made, to improve the quality of education.

To continue this progress, the Ministry of Education has developed a new General Education Curriculum Framework in 2021. The Framework covers all pre-primary, primary, Middle level and secondary level grades and subjects. It aims to reinforce the basic tenets and principles outlined in the *Education and Training Policy*, and provides guidance on the preparation of all subsequent curriculum materials – including this Teacher Guide and the Student Textbook that come with it – to be based on active-learning methods and a competency-based approach.

In the development of this new curriculum, recommendations of the education Road Map studies conducted in 2018 are used as milestones. The new curriculum materials balance the content with students' age, incorporate indigenous knowledge where necessary, use technology for learning and teaching, integrate vocational contents, incorporate the moral education as a subject and incorporate career and technical education as a subject in order to accommodate the diverse needs of learners.

Publication of a new framework, textbooks and teacher guides are by no means the sole solution to improving the quality of education in any country. Continued improvement calls for the efforts of all stakeholders. The teacher's role must become more flexible ranging from lecturer to motivator, guider and facilitator. To assist this, teachers have been given, and will continue to receive, training on the strategies suggested in the Framework and in this teacher guide.

Teachers are urged to read this Guide carefully and to support their students by putting into action the strategies and activities suggested in it.

For systemic reform and continuous improvement in the quality of curriculum materials, the Ministry of Education welcomes comments and suggestions which will enable us to undertake further review and refinement.

ADDIS ABABA, ETHIOPIA *FEDERAL DEMOCRATIC REPUBLIC OF ETHIOPIA*

xxxxx 2022 *MINISTRY OF EDUCATION*

Welcoming message to the teacher

Dear teacher, this teacher guide is a curriculum material prepared for you to use with your students. It is a material separately prepared for grade 10 Mathematics teacher. Grade 10 textbook has 7 units namely: Relations and Functions, Polynomial Functions, Exponential and Logarithmic Functions, Trigonometric Functions, Circles, Solid figures and Coordinate Geometry respectively. Since the students' textbook is basically unitized, you are advised to follow the four components of each lesson and provide the required assistance to the students regularly. Generally, the four components of each lesson are activity, definition/theorem/note, example and exercises. Therefore, you are expected to play your role accordingly.

The components of grade 10 teacher's guide includes:

1. List of general contents and sub-contents of grade 10 students' text book
2. List of general objectives of each of unit
3. Suggested teaching aids of each unit
4. Expected students' competencies at the end of each sub-unit
5. Elaborated presentation of each main topic focusing on:
    - Some selected topics are elaborated for you to use as reference material;
    - Key ideas to be stressed in each topic are identified;
    - Suggested strategies and sequences of presenting key ideas;
    - Suggestion of alternative methods and techniques for teaching particular topics;
    - Suggested plan of each unit and
    - List of answer keys for each of the activities and exercises of every unit.

Contents

Unit 1 RELATIONS AND FUNCTIONS. ……………… 2

1.1 Relations …………………………………………………… 3

1.2 Functions …………………………………………………… 12

Unit 2 Polynomial functions …………………………… 43

2.1 Introduction of polynomial functions ……………………… 43

2.2 Operations on polynomial functions ……………………… 47

2.3 Theorem on polynomial functions ………………………… 56

2.4 Zeros of polynomial functions …………………………… 63

2.5 Graphs of polynomial functions.………………………… 70

2.6 Applications ……………………………………………… 76

Unit 3 Exponential and Logarithmic Functions ……… 81

3.1 Exponents and Logarithms ……………………………… 81

3.2 The Exponential Functions and Their Graphs …………… 94

3.3 The Logarithmic Functions and Their Graphs…………… 102

3.4 Solving Exponential and Logarithmic Equations………… 107

3.5 Relation between Exponential and Logarithmic functions ……………………………………………… 116

3.6 Applications ……………………………………………… 118

Unit 4 Trigonometric functions………………………… 128

4.1 Radian Measure of angle………………………………… 128

4.2 Basic Trigonometric Function …………………………… 133

4.3 Trigonometric Identities & Equation…………………… 149

4.4 Application ……………………………………………… 152

Unit 5 Circles 162

5.1 Symmetrical properties of circles.................................... 163

5.2 Angle properties of circles 166

5.3 Arc length, perimeters and areas of segments and sectors Arc length... 169

5.4 Theorems on angles and arcs determined by lines intersecting inside, on and outside a circle.................... 172

Unit 6 Solid Figures 181

6.1 Revision Regular Polygons 181

6.2 Pyramids, cones and Spheres.. 184

6.3 Frustum of pyramids and cones.................................... 196

6.4 Surface areas and volumes of composed solids............. 202

6.5 Applications 208

Unit 7 Coordinate Geometry 210

7.1 Distance between two points 210

7.2 Division of a line segment... 213

7.3 Equation of a line... 216

7.4 Slopes of parallel and perpendicular lines.................... 221

Introduction

The study of mathematics at this cycle, grades 9 -12, prepares our students for the future, both practically and philosophically. Studying mathematics provides them not only with specific skills in mathematics, but also with tools and attitudes for constructing the future of our society. As well as learning to think efficiently and effectively, our students come to understand how mathematics underlies daily life and, on a higher level, the dynamics of national and international activity. The students automatically begin to apply high-level reasoning and values to daily life and also to their understanding of the social, economic, political and cultural realities of the country. In turn, this will help them to actively and effectively participate in the ongoing process of developing the nation.

At this cycle, our students gain a solid knowledge of the fundamental mathematical theories, theorems, rules and procedures. They also develop reliable skills for using this knowledge to solve problems independently. To this end, the objectives of mathematics learning at this cycle are to enable students to

- gain a solid knowledge of mathematics.
- appreciate the power, elegance and structure of mathematics.
- use mathematics in daily life.
- understand the essential contributions of mathematics to the fields of engineering, science, economics and so on.

Recent research gives strong arguments for changing the way in which mathematics has been taught. The rote-learning paradigm has been replaced by the student-centered model. A student-centered classroom stimulates student inquiry, and the teacher serves as a mentor who guides students as they construct their own knowledge base and skills. A primary goal when you teach a concept is for the students to discover the concept for themselves, particularly as they recognize threads and patterns in the data and theories that they encounter under your guidance.

One of our teaching goals is particularly fostered by the student-oriented approach. We want our students to develop personal qualities that will help them in real life. For example, student-oriented teachers encourage students' self-confidence and their confidence in their knowledge, skills and general abilities. We motivate our students to express their ideas and observations with courage and confidence. Because we want them to feel comfortable addressing individuals and groups and to present themselves and their ideas well, we give them safe opportunities to stand before the class and present their work. Similarly, we help them learn to learn to answer questions posed directly to them by other members of the class.

Unit 1

RELATIONS AND FUNCTIONS

The main objective of this unit is to express different types of relations in every aspect of our daily life. For instance, motherhood, fatherhood, brotherhood, neighborhood…etc. we have also discussed a relation between any two numbers that possess some related phrase. We may consider; 7 is less than 9. Here, 7and 9 are those that relate one another, and "is less than" is a relating phrase.

A relation in Mathematics has many fundamental details for investigating other consequences like functions. In this unit, learners are expected to see relations and relations in their mathematical sense. Behind discussing relations, they need to proceed to functions. In addition to the mathematical sense of a relation and a function, the notions of Cartesian coordinate system, notions of relations and their graphs, functions and their combinations, and their graphs will be treated in this unit. Participation of students is very important that help for a better realization of the concepts as well. Considering this fact, try to explore local issues that can best describe a relation and a function along with the discussions of the ideas and examples delivered in the student textbook.

Unit outcomes: At the end of this unit, the students will be able to:

- define relation
- define function
- identify types of functions
- sketch graphs of linear and quadratic functions.

Suggested Teaching Aids in Unit 1

It is expected that all students are aware of a relation in its meaning from daily life. The discussion hold in class along this class may not be exhaustive in themselves. So, it may be essential to seek for various inputs via teaching aids and active participation of students. Therefore, arranging different groups, students can develop local examples which will help us an additional teaching aid for a better and easy understanding of the notions of a relation and a function. You can also use charts, that describe relations of different type, and also graphs of relations. You can also use software(s) such as Geometers', Mathematica, MATLAB, etc.

Unit-1 RELATIONS AND FUNCTIONS (24 periods)

Introduction

You can consider a relation as in its dictionary meaning "the existence of connection, correspondence, feeling prevailing between persons or things", you can take some examples such as "mother-father relation, friendship, sister-brother", etc. as examples in their daily life. Mathematically, you can explain a relation as a connection between two things. Here, there are two concepts, namely "those two things that relate one another" and, "a relating phrase". You can ask students to mention more examples apart from the examples given in the textbook. You can ask students to more examples from their own understanding as an activity. You can also let them do the given activity in the student's textbook. Encourage them to give as many examples of relations from their daily life and guide their view, how ordered pair representation helps in describing a relation.

1.1 Relations (8 periods)

Competencies

At the end of this subunit, students will be able to:

- define patterns.
- produce sequences of patterns.
- define a relation.
- determine the domain and range of a relation.

Introduction

In this sub-unit, students are learned about the mathematics of patterns. Patterns are repetitive sequences and can be found in nature, shapes, events, sets of numbers and almost everywhere you care to look. For example, the number sequence 0, 4, 8, 12, 16, …. We come across with the word 'relation'. Generally speaking, by relation we usually understand some connection between the two living or non-living things, like the relations of mother -daughter, brother-sister, teacher -student etc. We are quite familiar with these relations. Today, students will learn about a new concept of "relations" in mathematics.

Teaching Notes

You may start the lesson by giving chance to the students to explain their understanding about pattern and relation from their daily life.

1.1.1 Revision of patterns

In this subsection, you will help students to recall important facts about patterns of numbers. They have learnt about these facts in the previous grades. Group the students and ask them to do the activity. Let some of the groups present their work to the class. Then start discussing the answer to each question with the students. Make sure that students understand the definitions and concepts given in the lesson; in particular, make sure that they can distinguish between the set of natural numbers, and integers, prime numbers and composite numbers. Some students confuse prime numbers with odd numbers. Here, you are expected to make sure that students are able to distinguish between a prime number and an odd number.

Answers for activity 1.1

1. 11, 13, 15 …etc.
2. D
3.

n	1	2	3	4	5	6
$2n+3$	5	7	9	11	13	15

Output $=2n+3$, where $n = 1, 2, 3, 4, \ldots, etc.$

4. If we have n inputs, then we get $3n-1$ outputs.

Answers for exercise1.1

1. Here, the number is decreasing by 5

   The previous number of a is 45. So, a will be 45 – 5, a = 40

   The previous number of b is 35. So, b will be 35 – 5, b = 30

   Therefore, the value of a is 40 and b is 30.

2. Pattern 4, 8, 12, 16, 20 is an arithmetic pattern or arithmetic sequence, as each term in the pattern is obtained by adding 4 to the previous term.

3. Here, the number is increasing by +7

   The previous number of A is 43. So, A will be 43 + 7, A = 50

   The previous number of B is 85. So, B will be 85 + 7, B = 92

   Therefore, the value of A is 50 and B is 92.

4. In this geometric pattern, the rule used is "Divide the previous term by 2 to get the next term".

   (i.e.,) 24/2 = 12. Then, the first missing term = 24/2 = 12. Second Missing term = 6/2 = 3.

   Hence, the geometric sequence is 96, 48, 24, **12**, 6**, 3**.

5. In the first diagram four sticks, second diagram seven sticks, third diagram ten sticks, the fourth diagram thirteen sticks and in the tenth diagram you have thirty-one sticks. So, the n^{th} diagram contains $3n + 1$ sticks.

1.1.2 Cartesian coordinate system in two dimensions

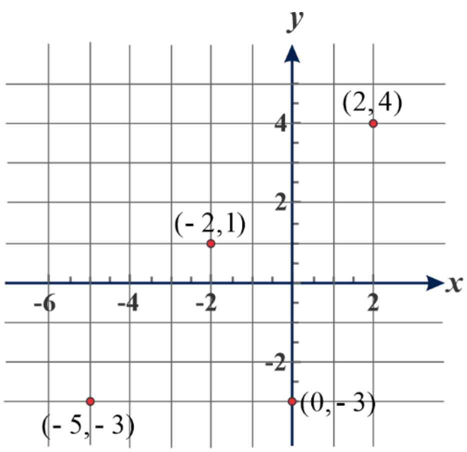
In this subsection, you will help students to recall important facts about coordinate system. Group the students and ask them to do the activity. Let some of the groups present their work to the class. Then start discussing the answer to each question with the students. Make sure that students understand the concepts given in the lesson. Create opportunities for students to discuss and analyze the activity by asking students to describe the quadrants and plot different points in the xy-plane.

Answers for activity 1.2

1.

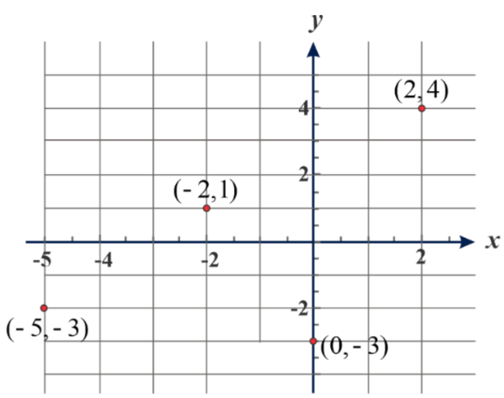
Quadrant	axis	
	x	y
Quadrant-I	positive	positive
Quadrant-II	negative	positive
Quadrant-III	negative	negative
Quadrant-IV	positive	negative
Positive x-axis	positive	0
Positive y-axis	0	positive
Negative x-axis	negative	0
Negative y-axis	0	negative

2. P (-2,2) Q$(2,\ 4)$, R$(0,\ -3)$, S$(-2{,}1)$ and T$(-5,-3)$

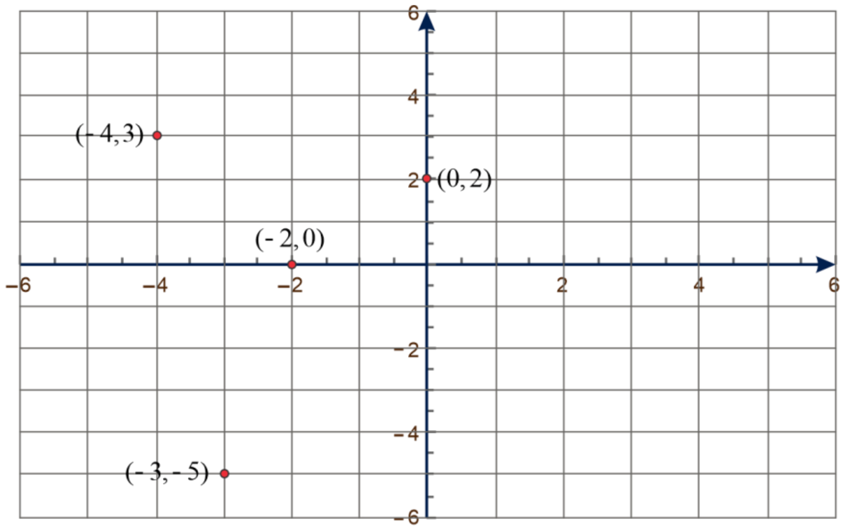


Answers for exercise1.2

1. a.



b.



2. **a. SNNR** **b. Oromia** **c. Addis Ababa** **d. Amhara**

ASSESSMENT

You can use any one of assessment techniques such as: class activities, group discussions, homework/assignments, exercise problems, and/or tests/quizzes.

describe the quadrants and plot different points in the xy-plane.

1.1.3 Basic concept of a relations

In this subsection, you will help students to recall important facts about patterns of numbers. Make sure that students understand the definitions and concepts given in the lesson. An ordered pair is represented as (input, output): The relation shows the relationship between input and output.

Answers for activity 1.3

a. $(4,3),(6,3),\ (6,5)(7,3)\ and\ (7,5\ ).$

b. $(1,12),(1,8),(4,7),(4,9),(2,5),(2,3),(2,7),(2,9),(4,3),(4,5),(6,3),(6,5),(6,7),$
$(6,9),(7,12),(7,8).$

c. $(4,8\ ),(6,12).$

d. $(6,12),\ (2,8)\ and\ (4,8\ ).$

Answers for exercise1.3

a. $(2,1),(-2,-3),(-0.2,-0.21),(0,-5)$

b. (i) the set of positive real numbers n. **(ii)** The set of negative real numbers n.

Answers for exercise1.4

1. $R = \{(-3,-11),\ (-2,-8),\ (-1,-5),\ (0,-2),\ (1,1),\ (2,4),\ (3,7)\}$.

2. **a.** $R = \left\{(0,0), (1,1), (2,8), (-2,-8), (8,512), \left(\frac{1}{5}, \frac{1}{125}\right), (3,27), (-3,-27)\right\}$.

b. $R = \left\{(2,8), (-1,-1), (-2,-8),\ (-3,-27), \left(\frac{1}{3}, \frac{1}{27}\right)\right\}$

Answers for activity 1.4

The domain of R is the set containing the integers -2 and 5.

The range of R is the set containing the integers 2, 3 and 4.

Answers for exercise 1.5

1. **a.** $R = \{(-1,-3), (-2,-6), (0,0), (1,3), (2,6), (3\ ,9)\}$. Domain of the relation $R\ = \{-1, 0, 2, 3\}$ and Range of the relation $R = \{-3, -6, 0, 3, 6, 9\}$.

b. $R = \{(-1,2), (-2,4), (0,0), (1,-2), (2,-4), (3\ ,-6)\}$. Domain of the relation $R\ = \{-1, 0, 2, 3\}$ and Range of the relation $R = \{2, 4, 0, -2, -4, -6\}$.

2. Domain of the relation $= \{-1, 0, 2, 3, 4\}$ and Range of the relation $= \{-2, 3, 4, 7\}$.

3. a) $R = \left\{(0,1), (0,-1), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \left(\frac{1}{2}\ , \frac{\sqrt{3}}{2}\right)\right\}$.

b) Domain $= \left\{0,\ \frac{\sqrt{2}}{2},\ \frac{-\sqrt{2}}{2}, -\frac{\sqrt{3}}{2}\ , \frac{1}{2}\right\}$ and Range $= \left\{1, -1,\ \frac{-1}{2},\ \frac{-\sqrt{2}}{2},\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{3}}{2}\right\}$.

While students doing the exercises, you may need to help students realize that, in a relation, there are two fundamental conceptions: the related objects and the relating phrase. In a relation, the issues of order and the establishment of pairing between objects are fundamentals that every student needs to understand.

1.1.4 Graphs of relations

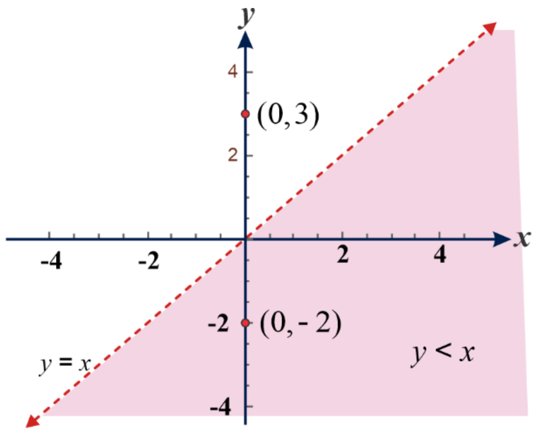
Up to now, students have discussed meaning of a relation and its representation in a form of ordered pairs. They also have discussed domain and range. Consistent with the meaning of a relation, and domain and range, it is now possible to discuss how to sketch a graph of a relation. Graphs are alternative representations of relations. In order to proceed, you can let students do activity 1.3 so that they can discuss coordinate plane, a point on a coordinate plane and a region on a coordinate plane.

For this purpose, you can prepare a flipchart that consists of a coordinate plane and points and a region on it. Encourage students to discuss plots of points on a coordinate plane and their representation as an ordered pair. You can also ask them to list the pairs of numbers represented as a point on a coordinate plane. After you do these, group students (different ability groups) and assign them to do Activity1.3 so that they can help each other. The purpose of this group work is to enable them determine a region representing a relation. Assist the students to discuss and plot the points they evaluated

Answers for activity 1.5

x	-4	-3	-2	-1	0	1	2	3	4
y	-4	-3	-2	-1	0	1	2	3	4

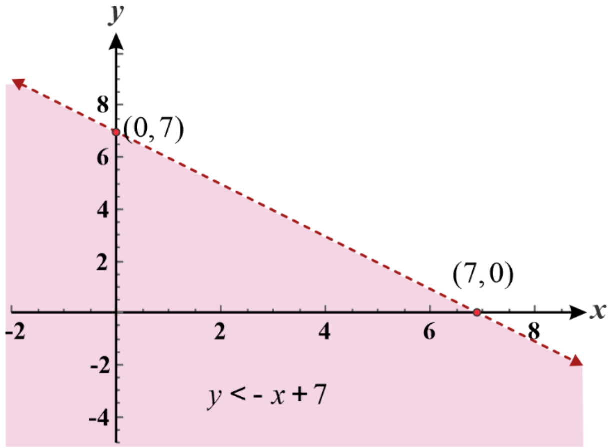
1. Read from above table.
2. Since the relation involves $y < x$, use the broken line. Draw the line $y = x$.
3. Take points representing ordered pairs, say $(0,3)$ and $(0,-2)$ from above and below the line $y = x$.
4. The ordered pair $(0,3)$ satisfies the relation. Hence, the region above, where the point representing $(0,3)$ is contained, is the graph of the relationR.
5. The domain and the range of $y < x$ is the set of all real numbers.



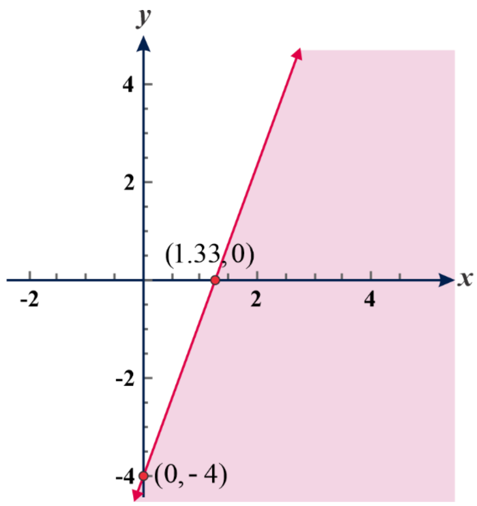
ASSESSMENT

You can use any one of assessment techniques such as: class activities, group discussions, homework/assignments, exercise problems, and/or tests/quizzes.

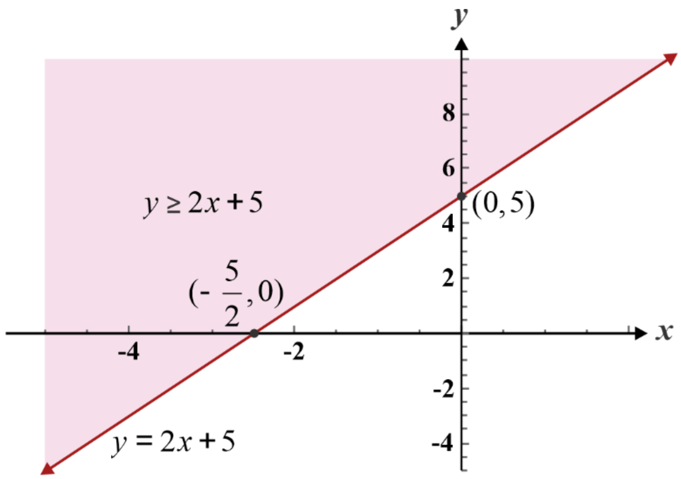
Answers for exercise 1.6



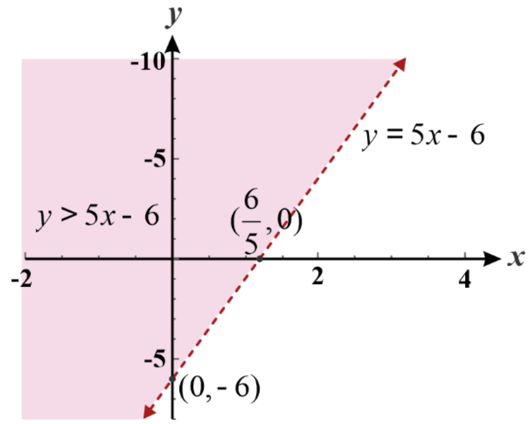
a. The domain and range are the set of real numbers.



b. The domain and range are the set of all real numbers.



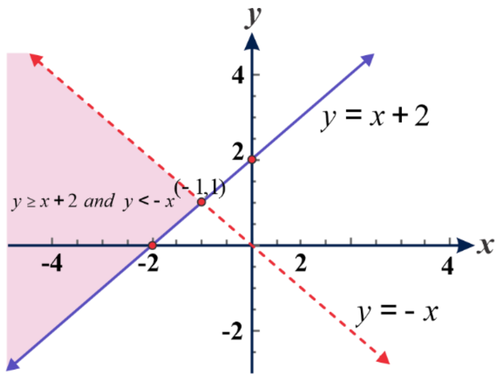
c. The domain and range are the set of all real numbers.



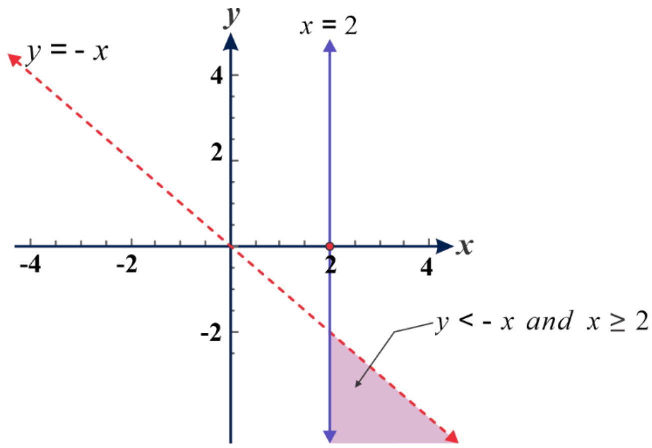
d. The domain and range are the set of all real numbers.

Answers for exercise1.7

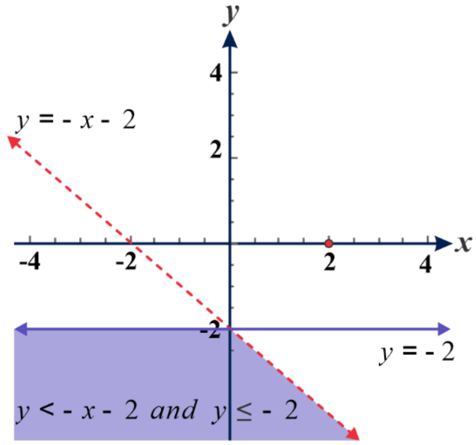
1. a.



b. R is set of ordered pairs $(x, y): x \geq 2$ and $y < -x$



c. R is set of ordered pairs $(x, y): y < -x - 2$ and $y \leq -2$



2. $R = \{(x, y): y \geq x + 1 \text{ and } x \geq 2\}$; Domain $= \{x: x \geq 2\}$ and Range $= \{y: y \geq 3\}$.

ASSESSMENT

You can give them different relations and ask them to sketch their graphs and determine domain and range for each graph. Let them also submit their work and you check and keep record. You can post the best works in the class.

1.2 Functions (15 periods)

Competencies

At the end of this subunit, students will be able to:

- define function
- sketch the graph of the function
- find the domain and range of a function.

Introduction

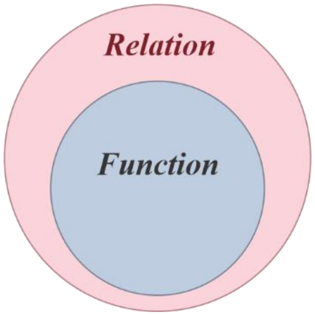
The concept of a function is one of the most important in all mathematics, and in this sub-topic we shall take a much closer look at it. As we proceed you will find yourself learning a great deal more about what a function really is, and how functions arise in mathematics and in everyday life.

Have you ever thought that a particular person has particular jobs or functions to do? Consider the functions or roles of postmen. They deliver letters, postcards, telegrams and invites etc. What do firemen do? They are responsible for responding to fire accidents. In mathematics also, we can define functions. They are responsible for assigning every single object of one set to that of another.

If a stone is dropped out of a window, then in any given period of time it will fall a certain distance. The physicists tell us that in t second it will fall about 4.9meters, in two seconds about 19.6 meters, in t seconds about $4.9t^2$meters. If the length of a square is x meters, the area of the square is x^2 square meters. All of these examples illustrate one of the most important ideas in mathematics; the idea of a function. In this topic we take up the idea in general.

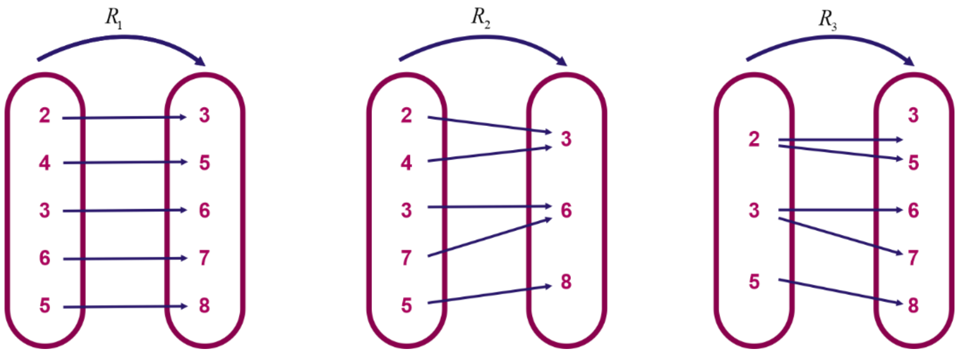
1.2.1 Notions of functions

In this section, you will find the basic definition of functions and relations, special functions, different types of relations and some of the solved examples. Group the students and ask them to do the activity. Let some of the groups present their work to the class. Then start discussing the answer to each question with the students. Make sure that students understand the definitions and concepts given in the lesson; in particular, make sure that they can distinguish between relation and function. Let them critically understand the difference between relations and functions.



Answers for activity 1.6

a. The arrow diagram:



b. In R_1 every first element of the coordinate mapped to distinct second element of the coordinate. Whereas in R_2 distinct first element of the coordinate mapped to the same second coordinate and in $R_3$the same first element of the coordinate mapped to distinct second element of the coordinate.

c. Yes, 2 is mapped to 3 and 5 ; 3 is mapped to 6 and 7.

Answers for exercise1.8

1.

a. R is a function.

b. R is not a function. The same first element of R is mapped to distinct two second elements of R, that is -5 is mapped to 9 and 3.

c. R is not a function. For $= 3$, multiples of x are $3, 6, 9, 12, \ldots$ that is, 3 is mapped to $3,6,9, \ldots$.

d. R is not a function. The same first element of R is mapped to distinct second element of R that is, $x = 4$ is mapped to -2 and 2.

2. Yes, because function is a special type of relation.

Answers for exercise1.9

1. **a.** R is a function.

b. The relation R with an ordered pair (x, y): y is the father of x is a function because no child has more than one father.

c. This relation is not a function since everybody(x) has two grandmothers.

2. Yes. Since for every number there is unique absolute value, each number x mapped to one and only one number y, so the relation R is a function.

Answers for activity 1.7

1. f contains $(0,0), (1,3), (2,6), (4,12)$ and the rule is an ordered pair (x, y) such that $y = 3x$.

ASSESSMENT

You can assess you students by giving them various exercises on relation and function. You can let students do these as homework and present their work.

Answers for exercise1.10

1. Domain of F is a collection of $2, 0, -4$ and -5 and the range of F is a collection of $-1, 0, 2$ and 3.

2. a. The function is defined for every real number x and hence the domain of the function is the set of all real numbers. The range of f is also the set of real numbers since every real number y has a pre-image in the real number x.
   b. The function is defined for every real number x and hence the domain of the function is the set of all real numbers. The range of f is also the set of real numbers since every real number y has a pre-image in the real number x.
   c. The function is defined for every real number x and hence the domain of the function is the set of all real numbers. The range of f is also the set of real numbers since every real number y has a pre-image in the real number x.
   d. The function is defined for every real number x and hence the domain of the function is the set of all real numbers. The range is the set of positive real numbers.
   e. The function is defined for every real number x and hence the domain of the function is the set of all real numbers. The range is the set of real numbers $y: y \geq 1$.
   f. The domain of f is the set of real numbers $x: x \neq 0$ and the range of f is the set of real numbers $y: y \neq 0$.

Answers for exercise1.11

1. a. The domain is the set of real numbers. Range $= \{y: y \leq 1\}$.
   b. The domain is the set of real numbers. Range $= \{y: y \geq 1\}$.
   c. $Dom(f) = \{x: x \leq 2\}$. Range$= \{y: y \geq 0\}$.
   d. Dom$(f) = \{x: -1 \leq x \leq 1\}$. Range $= \{y: y \geq 0\}$.
   e. Dom$(f) = \{x:$ x is real numbers and $x \neq 2\}$. Range is the set of real numbers y such that $y \neq 1$.
2. $f(x) = 2x + \sqrt{4 - x}$
   a. $f(-5) = 2(-5) + \sqrt{4 - (-5)} = -10 + 3 = -7.$
   b. $f(2) = 2 \times 2 + \sqrt{4 - 2} = 4 + \sqrt{2}.$

ASSESSMENT

You can use any one of assessment techniques such as: class activities, group discussions, homework/assignments, exercise problems on functions, and/or tests/quizzes.

1.2.2 Combination of functions

Competency

At the end of this subunit, students will be able to:

- Add or subtract two or functions
- Find the product or quotient of two or more functions

To that extent, students are aware of combination of relations. In order to begin, let each student perform **Activity 1.7**

Answers for activity 1.8

$f(x) = 2x - 3$ and $g(x) = -1$.

a. $(f+g)(x) = f(x) + g(x) = 2x - 3 + (-1) = 2x - 4$,

$(f-g)(x) = 2x - 3 - (-1) = 2x - 2$,

$(f.g) = (2x-3)(-1) = -2x + 3$, and

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x-3}{-1} = -2x + 3$.

b. The domain and range of f and g is the set of real numbers.

c. Domain of f and g, and $f + g$ are the same.

Following the above activity, it will be advisable to practice more on combinations through the examples from the student textbook and more others.

Assessment

For the purpose of checking students understanding and assessing their level, you can give them Exercise 1.5 as homework and you then check and give them your feedback. You need also keep record.

Answers for exercise 1.12

1. Given: $f(x) = 3x + 3$ and $g(x) = x - 1$.

a. $(f+g)(x) = f(x) + g(x) = (3x+3) + (x-1)$

$= 3x + 3 + x - 1 = 4x + 2$.

$(f+g)(2) = 4 \times 2 + 2 = 10$.

b. $(f-g)(x) = f(x) - g(x) = (3x+3) - (x-1)$

$= 3x + 3 - x + 1 = 2x + 4$.

$(f+g)(2) = 2 \times 2 + 4 = 8.$

2. Given: $f(x) = 2x - 5$ and $g(x) = 4x + 1$.

a. $(f+g)(x) = f(x) + g(x) = (2x-5) + (4x+1)$

$= 2x - 5 + 4x + 1$ Removing bracket

$= 6x - 4$ Simplified

b. $(f-g)(x) = f(x) - g(x) = (2x-5) - (4x+1)$

$= 2x - 5 - 4x - 1$ Removing bracket

$= -2x - 6$ Simplified.

c. The domain of $f+g$, and $f-g$ is the set of all real numbers.

3. Given: $f(x) = x^2 + 5$ and $g(x) = \sqrt{1-x}$.

a. $(f+g)(x) = f(x) + g(x)$

$= (x^2+5) + \left(\sqrt{1-x}\right)$

$= x^2 + 5 + \sqrt{1-x}.$

b. $(f-g)(x) = f(x) - g(x)$

$= (x^2+5) - \left(\sqrt{1-x}\right)$

$= x^2 + 5 - \sqrt{1-x}.$

c. The domain of $f+g$ and $f-g$ is the set of all real numbers less than or equal to one.

Answers for exercise 1.13

a. $(f \cdot g)(x) = (x-1) \cdot x^2 = x^3 - x^2$ and $(f \cdot g)(3) = 27 - 9 = 18$.

b. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x-1}{x^2}$ and $\left(\frac{f}{g}\right)(3) = \frac{2}{9}$.

c. $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{x^2}{x-1}$ and $\left(\frac{g}{f}\right)(3) = \frac{9}{2}$.

d. the domain of $f \cdot g$ is the set all of real numbers, whereas the domain of $\frac{f}{g}$ is the set of all real numbers except 0 and the domain of $\frac{g}{f}$ is the set of all real numbers except -1.

Answers for exercise 1.14

1. Let $f(x) = 2x - 5$ and $g(x) = x + 1$.

a. $(f+g)(x) = f(x) + g(x) = (2x-5) + (x+1) = 3x - 4.$

b. $(f-g)(x) = f(x) - g(x) = (2x-5) - (x+1) = x - 6.$

c. $(f \cdot g)(x) = f(x) \cdot g(x) = (2x-5) \cdot (x+1)$

$= 2x^2 + 2x - 5x - 5$ Product

$= 2x^2 - 3x - 5$ Simplified

d. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x-5}{x+1.}$

e. The domain of $f+g, f-g, f.g$ is the set of all real numbers. whereas the domain of $\frac{f}{g}$ is the set of all real numbers except -1 .

2. $f(x) = x^2 + 5$ and $g(x) = \sqrt{1-x}.$

a. $(f+g)(x) = x^2 + 5 + \sqrt{1-x}.$

b. $(f-g)(x) = x^2 + 5 - \sqrt{1-x}.$

c. $(f.g)(x) = (x^2+5)(\sqrt{1-x}) = x^2\sqrt{1-x} + 5\sqrt{1-x}.$

d. $\left(\frac{f(x)}{g(X)}\right) = \frac{x^2+5}{\sqrt{1-x}}.$

e. Domain of $(f+g)$ is the common members of domain of f and domain$g$, i.e. the set of real number x less than or equal to one;

Domain of $(f-g)$ is the common members of domain of f and domain $g =$ the set of real number x less than or equal to one;

Domain of (fg) is the common members of domain of f and domain$g$, i.e. The set of real number x less than or equal to one;

Domain of (f/g) is the common members of domain of f and domain$g$, i.e. the set of real number x less than one.

Answers for exercise 1.15

1. Given: $f(x) = 2 - x$ and $f(x) = -2x + 3.$

a. $(f+2g)(x) = f(x) + g(x) = 2 - x - 2x + 3 = -3x + 5$ and
$(f+2g)(2) = -3 \times 2 + 5 = -1.$

b. $(2f-g)(x) = 2(2-x) - (-2x+3) = 4 - 2x + 2x - 3 = 1$ and
$(2f-g)(2) = 1.$

c. $(2f \cdot g)(x) = 2(2-x)(-2x+3) = (4-2x)(-2x+3)$
$= -8x + 12 + 4x^2 - 6x = 4x^2 - 14x + 12$ and
$(2f \cdot g)(2) = 4 \times 2^2 - 14 \times 2 + 12 = 0$

d. $\frac{3f}{2g}(x) = \frac{3(2-x)}{2(-2x+3)} = \frac{6-3x}{-4x+6}$ and $\frac{3f}{2g}(2) = \frac{3(2-2)}{2(-2\times2+3)} = \frac{0}{-2} = 0$

2. $f(x) = 3x - 3,\ g(x) = \frac{2}{x-1}$.

- **a.** $(2f \cdot g)(x) = 2f(x) \cdot g(x) = 2(3x-3)\left(\frac{2}{x-1}\right) = \frac{12x-12}{x-1} = 12$. So, $(2fg)(2) = \frac{12(2)-12}{2-1} = 12.$
- **b.** $\left(\frac{f}{g} - 2f\right)(3) = \left(\frac{3x-3}{2/x-1} - 2f(3x-3)\right)(x=3) = \frac{6}{1} - 2(6) = -6.$
- **c.** $(f-g)(4) = \left(3x - 3 - \frac{2}{x-1}\right)(x=4) = 9 - \frac{2}{3} = \frac{25}{3}.$
- **d.** $\left(\frac{f}{g}\right)(4) = \left(\frac{3x-3}{\frac{2}{x-1}}\right)(x=4) = \frac{9}{\frac{2}{3}} = \frac{27}{2}.$
- **e.** The domain of $f+g, f-g, f \cdot g$, and $\frac{f}{g}$ is the set of all real numbers except 1.

3. Yes, consider $f(x) = 3x - 1$ and $g(x) = x - 2$. The domain of $f\ and\ g$ is the set of real numbers. But the domain of $\frac{f}{g}$ is the set of real numbers $x \neq 2$.

4. b and g define the same formula while a, c, d, e, f and h do not.

ASSESSMENT

You can use any one of assessment techniques such as: class activities, group discussions, homework/assignments, and/or tests/quizzes.

Answers for activity1.9

Based on the graph and using vertical line test, you are required to demonstrate the activity to the students to answer the questions.

a, c and e are functions. But b, d and f are **not** functions. Because the vertical line crosses the graph of the function more than one point.

1.2.3 Types of functions

Competency

At the end of this subunit, students will be able to:

- Identify different types of functions

Introduction

Answers for exercise 1.16

1. a. A is not one-to-one.

b. B is not one-to-one.

c. C is one-to-one.

d. D is one-to-one.

2. a. onto

b. not onto

3. f neither g injective, surjective and bijective n injective m surjective

4. a. Injective, surjective and bijective **b.** injective, surjective and bijective

1.2.4 Graphs functions

Competency

At the end of this subunit, students will be able to:

- find x and y-intercept of the graph of a function
- determine the domain and range of a function
- sketch the graph of a function

Introduction

In previous sub-unit functions, domain and range of functions, and combinations were discussed. In this sub-unit students will discuss graphs of linear and quadratic functions and will proceed to characterizing graphs and develop properties.

Teaching Notes

Start the lesson by asking students 'Have you learnt graph of relations and if so, give time to think and get two or three students reply. Then list some relations which are functions. Since students have discussed linear and quadratic equations in grade 9, you may ask students to state the definition of linear function and quadratic function. Following this discussion, you may proceed to writing down the following definition of a linear function:

Definition. If a and b are fixed real numbers, then $f(x) = ax + b$; a $\neq 0$ for $x \in \mathbb{R}$ is called a linear function. If $a = 0$ then $f(x) = b$is called a constant function. Before proceeding further, it may be important to let students revise evaluating values of functions and plotting some of the evaluated coordinate points on a coordinate plane to draw graphs of linear functions.

Tell the students, a good approach to **graphing a function** is to make a table of a handful of possible inputs and outputs. We'll graph the points we get from our table and connect them according to the pattern we see.

When we make our table, the inputs (x) will be the x-values in our coordinate pairs, and the outputs $f(x)$ will be the y-values.

Let's try an example $f(x) = x + 1$. First, we're going to set up our table. It's a good idea to pick symmetrical values to plug in for x. This means, for example, if I decide to try out $x = 2$, I should also include $x = -2$ in my table.

It's also a wise choice to include $x = 0$ in every table. This gives you the y-intercept for your graph, where the line crosses the y-axis.

If you need help figuring out how to get output values from the function, look the following table.

x	$f(x)$	(x, y)
-2	-1	$(-2, -1)$
-1	0	$(-1, 0)$
0	1	$(0,1)$
1	2	$(1, 2)$
2	3	$(2, 3)$

Assist the students to discuss the following activity. Give this activity as in group and arrange a discussion among groups. Finally let each group reflect for the solution of each activity. You also let them do more examples and exercise of these types. Whenever it is necessary, you can narrate all the properties at the end. At this moment, you can ask the following to the fast learners:

In $f(x) = -2x + 1$, if $f(x) = 3$ it true for all x, then solve for x.

Answers for exercises 1.17

1. $f(x) = 4x + 1$; $x = -3, -2, -1, 0, 1, 2, 3$

a.

x	-3	-2	-1	0	1	2	3
$f(x) = 4x + 1$	-11	-7	-3	1	5	9	13

b.

x	-1	-0.5	0	2	3	4
$f(x) = -2x + 5$	7	4	5	1	-1	-3

c.

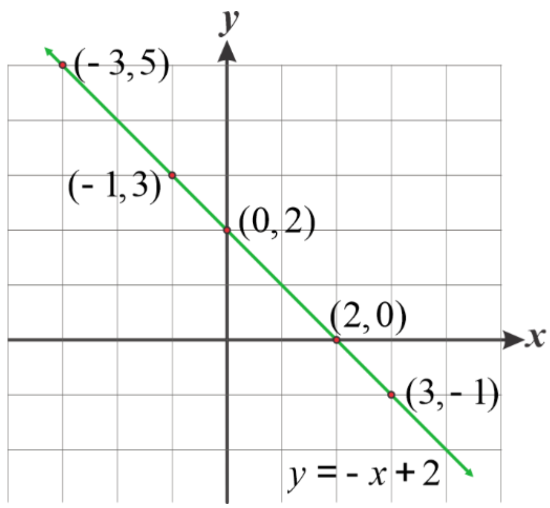
x	-1	0	1	2	3	4
$f(x) = 7 - 3x$	10	7	4	1	-2	-5

d.

x	-8	-4	-2	0	2	4
$f(x) = \frac{1}{4}x + 1$	-1	0	0.5	1	1.5	2

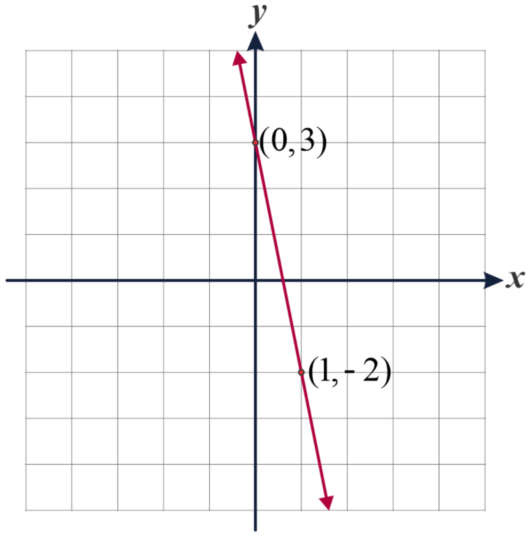
2. a.

x	-4	-3	-2	-1	0	1	2	3	4
$2y + 4x + 7 = 5$	7	5	3	1	-1	-3	-5	-7	-9



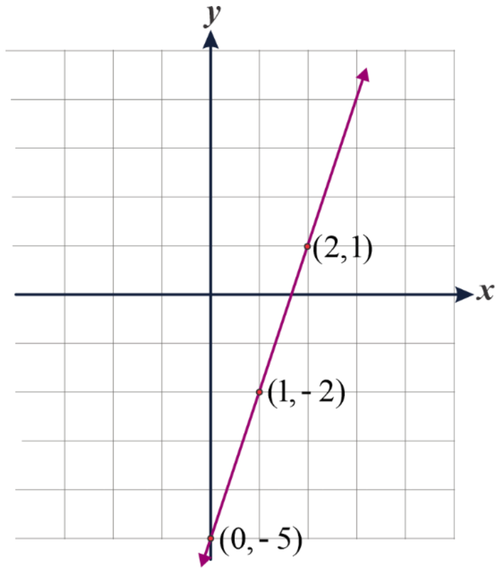
b.

x	-4	-3	-2	-1	0	1	2	3	4
$y = 3 - 5x$	23	18	13	8	3	-2	-7	-12	-17



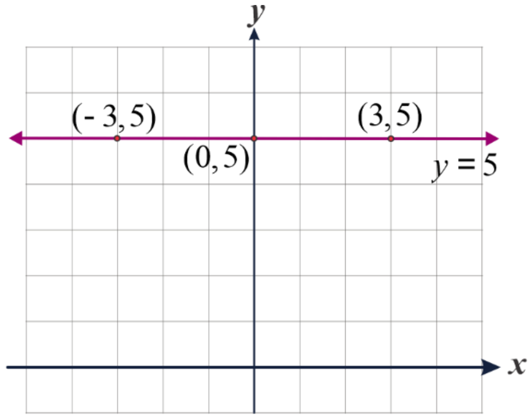
c.

x	-4	-3	-2	-1	0	1	2	3	4
$y-3x+7=2$	-17	-14	-11	-8	-5	-2	1	4	7



d.

x	-2	-1	0	1	2
$f(x)=5$	5	5	5	5	5



Answers for exercise 1.18

1. a. Slope is -1, x-intercept is $(2,\ 0)$ and y-intercept is $(0,\ 2)$.

   b. Slope is -1, x-intercept is $\left(\frac{3}{2},\ 0\right)$ and y-intercept is $\left(0,\ \frac{3}{2}\right)$.

   c. Slope is 2, x-intercept is $(\frac{7}{2},\ 0)$ and y-intercept is (0,-7).

   d. Slope is -3, x-intercept is $(\frac{-5}{3},\ 0)$ and y-intercept (o,-5)

2. a. g, h and k

   b. slope of f is 3, x-intercept is $\left(\frac{1}{3}, 0\right)$ and y-intercept is $(0, -1)$

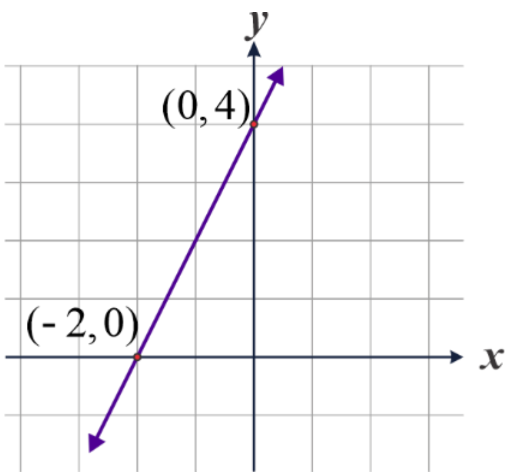
   slope of g is -1, x-intercept is $(2, 0)$ and y-intercept is $(0, 2)$

   slope of h is -2, x-intercept is $(0, 0)$ and y-intercept is $(0, 0)$

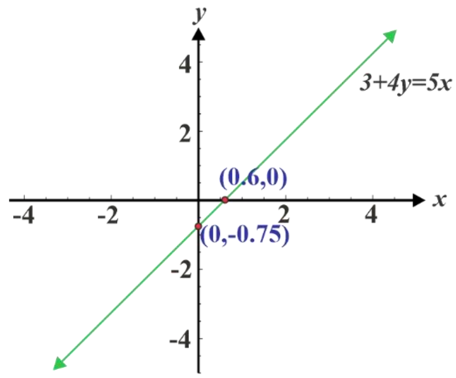
   slope of k is 0, no x-intercept and y-intercept is $(0, 1)$

3. x-intercept is (-2, 0) and y-intercept is (0,4).

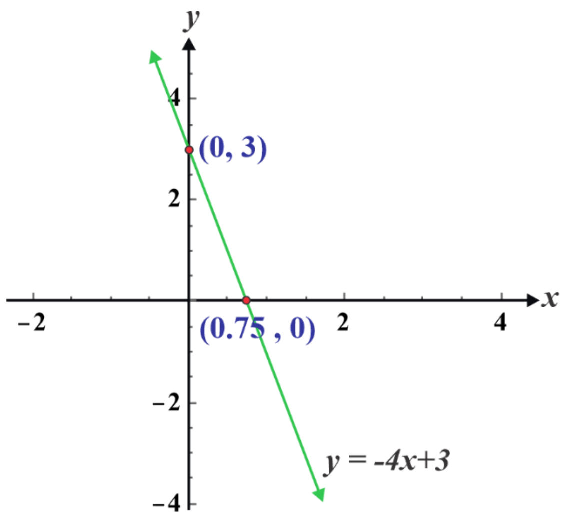
   a.



b.



c.



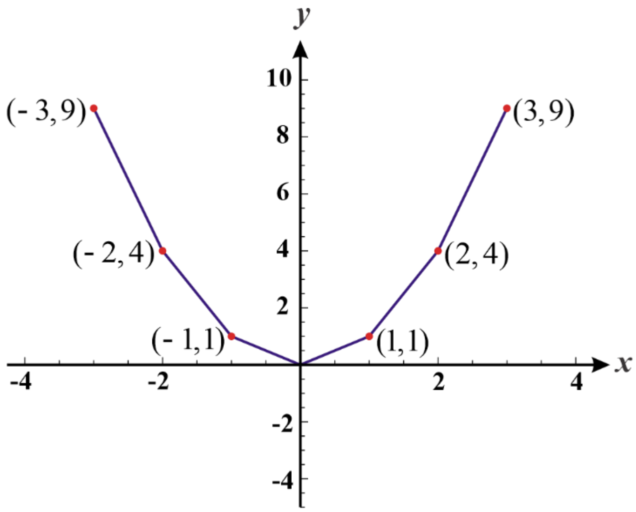
Graphs of Quadratic function

Teaching Notes

The general technique for graphing quadratics is the same as for graphing linear functions. However, since quadratics graph as curvy lines (called "parabolas"), rather than the straight lines generated by linear equations, there are some additional considerations. The most basic quadratic is $y = x^2$. When you graphed straight lines, you only needed two points to graph your line, though you generally plotted three or more points just to be on the safe side. However, three points will almost certainly *not* be enough points for graphing a quadratic, at least not until you are *very* experienced. For example, suppose a student compute these three points:

x	$y = x^2$
0	0
1	1
2	4

He got the graph wrong. You, on the other hand, are more careful. You find many points. Even if you'd forgotten that quadratics graph as curvy parabolas, these points will remind you of this fact. Urge your students to draw a nicely smooth curving line passing neatly through the plotted points. Some students will plot the points correctly, but will then connect the points with straight line segments.



This graph is not correct.

You do still need a ruler for doing your graphing, but only for drawing the axes, not for drawing the parabolas. Parabola's graph are smooth curved lines, not as jointed segments. The correct graph shown in the following activity.

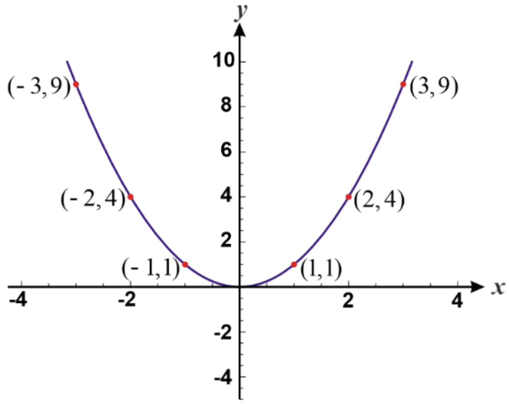
Answers for activity 1.11

a.

x	-3	-2	-1	0	1	2	3
$f(x) = x^2$	9	4	1	0	1	4	9

b. The point (0,0) is x-intercept and y-intercept of the graph

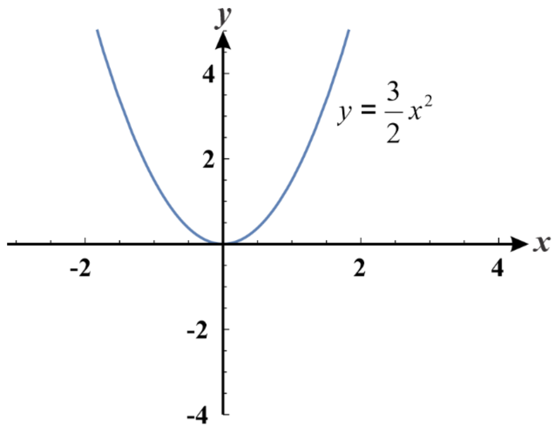
c. $y = x^2$



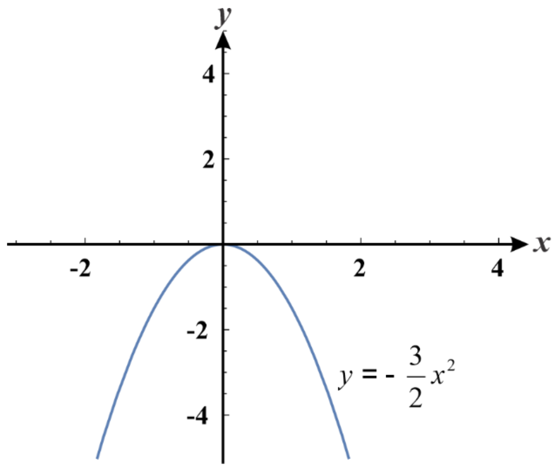
The graph of $y = x^2$

a. The domain is the set of real numbers and the range is the set of real number y greater than or equal to zero.

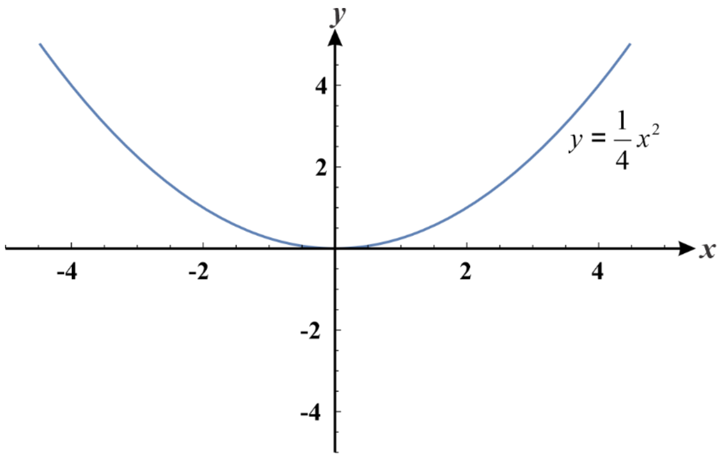
Answers for exercise 1.19



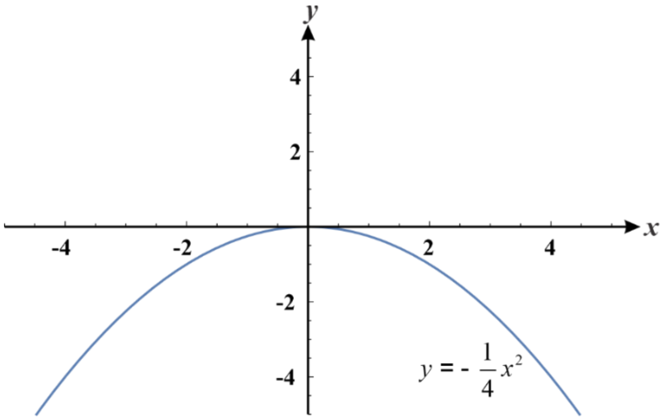
The graph of $f(x) = \frac{3}{2}x^2$



The graph of $f(x) = -\frac{3}{2}x^2$



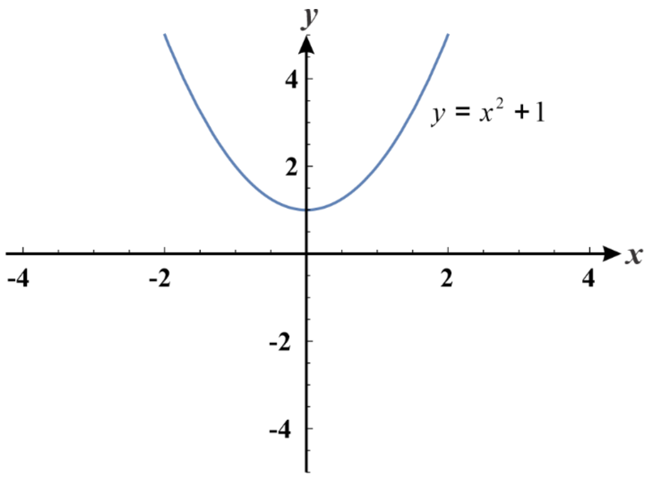
The graph of $f(x) = \frac{1}{4}x^2$



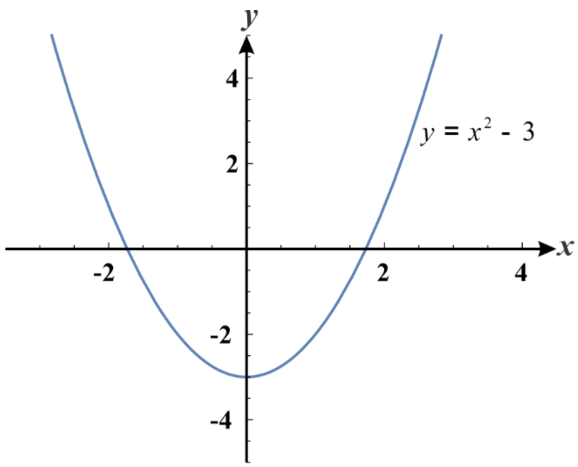
The graph of $f(x) = -\frac{1}{4}x^2$

Answers for exercise 1.20

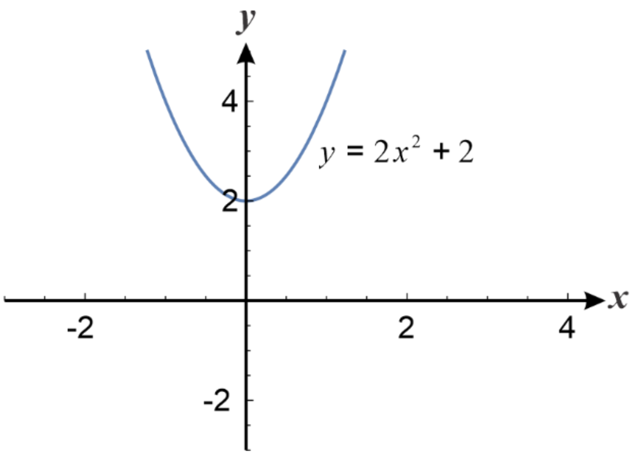
a. The graph of $f(x) = x^2 + 1$



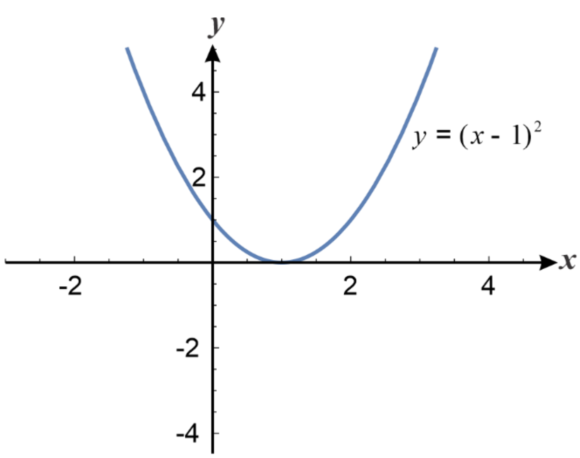
b. The graph of $g(x) = x^2 - 3$



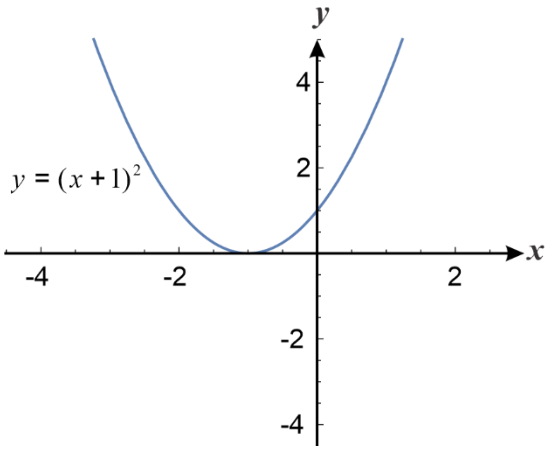
c. The graph of $h(x) = 2x^2 + 2$



d. The graph of $k(x) = (x - 1)^2$



e. The graph of $m(x) = (x + 1)^2$

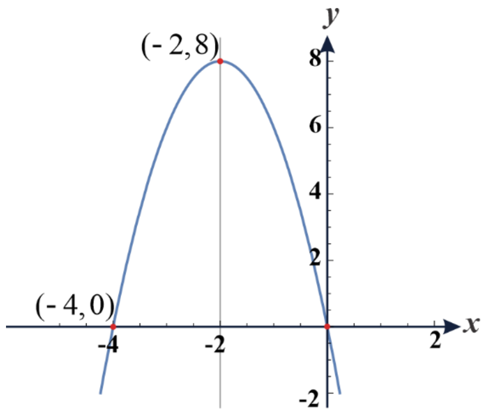


Answers for exercise 1.21

1. **a.** The new curve is symmetric about the line $x = -2$,

and when it is shifted up by 8 units we obtain the equation is

$y = -2(x + 2)^2 + 8$.



b. The curve crosses the y −axis, when $x = 0$. Putting this into the equation $y = -2(x + 2)^2 + 8$, the corresponding value of y is $y = -2(0 + 2)^2 + 8 = 0$, so the curve crosses the $y -$ axis at $y = 0$.

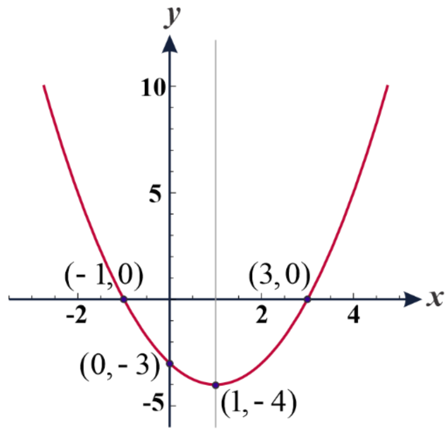
c. The curve sketched at the right.

The graph of $y = -2(x + 2)^2 + 8$.

2. **I. a.** The new curve is symmetric about $x = 1$ and is shifted down by -4 units so its equation is $y = (x - 1)^2 - 4$.

b. The curve crosses the y −axis, when $x = 0$. Putting this into the equation $y = (x - 1)^2 - 4$, the corresponding value of y is $y = (0 - 1)^2 - 4 = -3$, so, the curve crosses the $y -$ axis at $y = -3$.

c. The curve is sketched below.



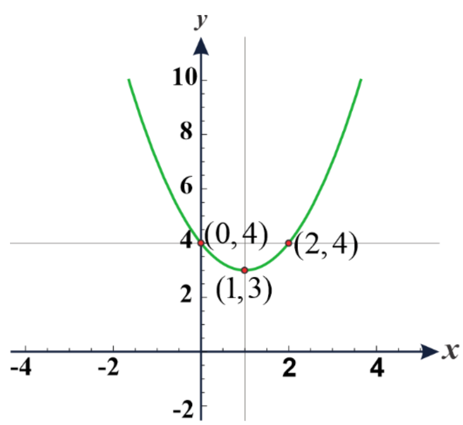
The graph of $y = (x-1)^2 - 4$

2) II) a. The new curve is symmetric about $x = 1$ and is shifted up by 3 units so its equation is

$y = (x-1)^2 + 3.$

b. The curve crosses the y −axis, when $x = 0$. Putting this into the equation $y = (x-1)^2 + 3$, the corresponding value of y is $y = (0-1)^2 + 3 = 4$, so the curve crosses the $y-$ axis at $y = 4$.

c. The curve is sketched below.

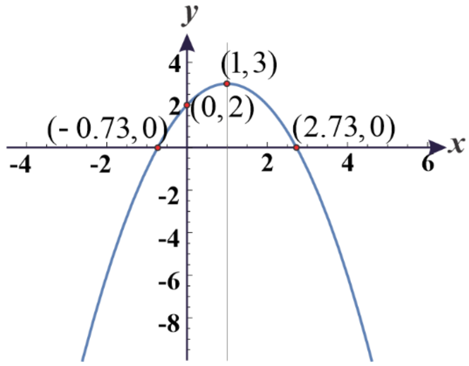


The graph of $y = (x-1)^2 + 3$

2)III) a. The new curve is symmetric about $x = 1$ and is shifted up by 5 units so its equation is $y = -(x-1)^2 + 3$.

b. The curve crosses the y-axis, when $x = 0$. Putting this into the equation $y = -(x-1)^2 + 3$, the corresponding value of y is $y = -(1-1)^2 + 3 = 2$, so, the curve crosses the y-axisat $y = 2$.

c. The curve is sketched below



The graph of $y = -(x-1)^2 + 3$

ASSESSMENT

You can use any one of assessment techniques such as: class activities, group discussions, homework/assignments and/or tests/quizzes.

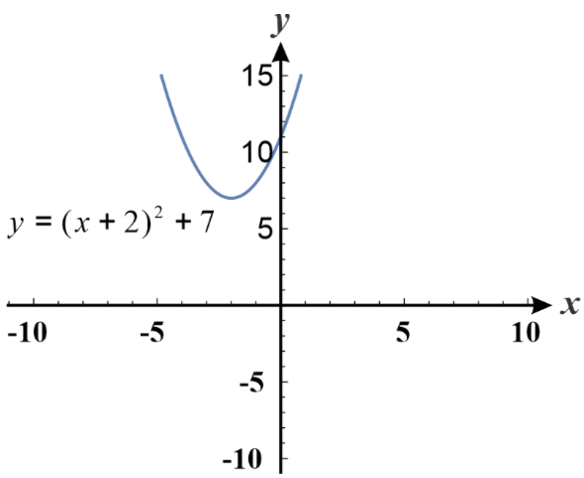
Answers for exercise 1.22

1. a. $y = x^2 + 4x + 3 = (x+2)^2 + 7$.

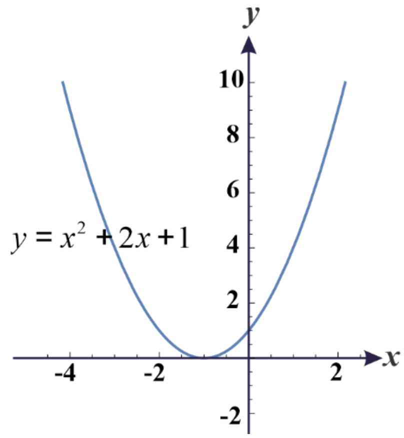
   b. This is the function $y = x^2$ moved to left so that its axis of symmetry is $x = -2$ and shifted up by 7, i.e., its orthogonal axis is $y = 7$.

   c. The function is $y = (x+2)^2 + 7$. It will not cross the x-axis, i.e., the graph has not x-intercept. Putting $x = 0$ into the original form of the function $y = x^2 + 4x + 7$, gives $y = 11$, i.e., it crosses the y-axis at $y = 7$.

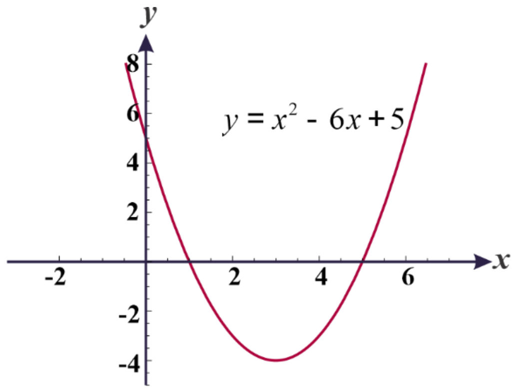
   d. $y = (x+2)^2 + 7$



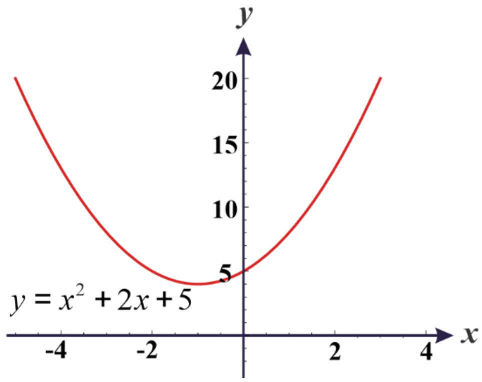
2. a.



b.



c.



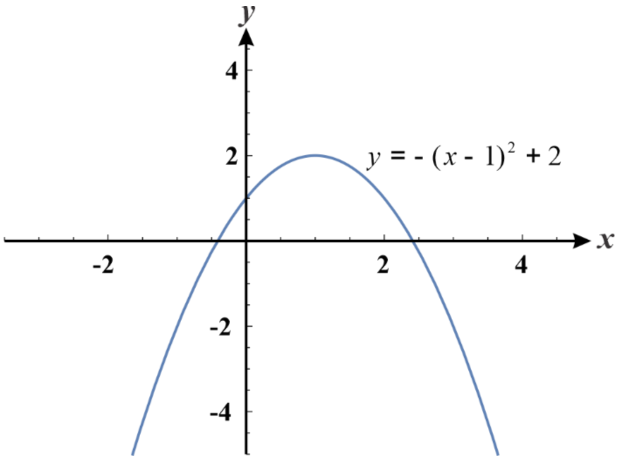
Answers for exercise 1.23

1. **a.** $y = -x^2 - 2x + 1 = -(x-1)^2 + 2.$

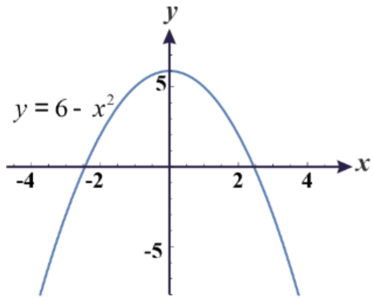
   b. This is the function $y = x^2$ moved to right so that its axis of symmetry is $x = 1$ and shifted up by 2, i.e., its orthogonal axis is $y = 2$.

c. The function is $y = -(x-1)^2 + 2$. It will not cross the x-axis, i.e., the graph has not x-intercept. Putting $x = 0$ into the original form of the function $y = -x^2 - 2x + 1$, gives $y = 2$, i.e., it crosses the y-axis at $y = 2$.

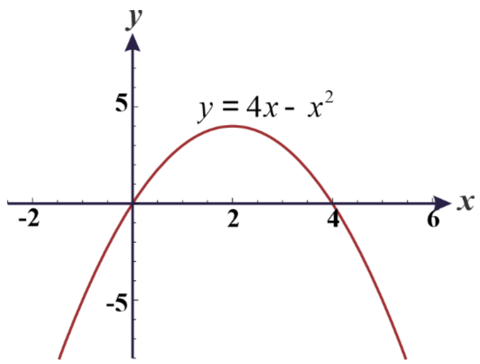
d. $y = -(x-1)^2 + 2$



2. a.



b.



3. a. $y = -2x^2 - 8x$

$= -2(x^2 + 4x)$

$= -2(x^2 + 4x + 4 - 4)$

$= -2(x+2)^2 + 8$

$y = -2(x+2)^2 + 8.$

x- intercept is $(0,0)$ and $(-4,0)$.

y- intercept is $(0,0)$.

$Axis$ of symmetry is $x = -2$ and its orthogonal axis is $y = 8$.

b. $y = -x^2 + 6x + 7$

$= -(x^2 - 6x + 9 - 9) + 7 = -((x-3)^2 - 9) + 3$

$= -(x-3)^2 + 9 + 3$

$y = -(x-3)^2 + 12.$

x- intercept is $(3 - 2\sqrt{3}, 0)$ and $(3 + 2\sqrt{3}, 0)$.

y- intercept is $(0,7)$.

$Axis$ of symmetry is $x = 3$ and its orthogonal axis is $y = 12$.

Answers for activity 1.12

1. For a quadratic function $f(x) = ax^2 + bx + c$

a. If $a > 0$, $f(x)$ has a minimum value of the vertex. Or

b. $If\ a < 0, f(x)$ has a maximum value at the vertex.

Therefore, the maximum value or minimum value is $f\left(-\frac{b}{a}\right)$.

Answers for exercise 1.24

1. **a.** Vertex= $\left(-\frac{b}{a}, f\left(-\frac{b}{a}\right)\right) = (4,-3)$ and Axis of symmetry $x = 4$.

b. Vertex= $\left(-\frac{b}{a}, f\left(-\frac{b}{a}\right)\right) = \left(\frac{5}{2}, \frac{7}{4}\right)$ and Axis of symmetry $x = \frac{5}{2}$.

2. **a.** f has minimum value at $f\left(-\frac{b}{a}\right) = f\left(-\frac{4}{2}\right) = f(-2) = -3$.

b. f has minimum value at $f\left(-\frac{b}{a}\right) = f\left(-\frac{1}{4}\right) = \frac{15}{4}$

c. f has maximum value at $f\left(-\frac{b}{a}\right) = f(-2) = 4$

d. f has maximum value at $f\left(-\frac{b}{a}\right) = f(-2) = -2$

3. Let the x and y be the larger and smaller lengths respectively.

$x + y = 40$ ....(1)

After bending to form a square, area of each square are: $\left(\frac{x}{4}\right)^2$ and $\left(\frac{y}{4}\right)^2$ respectively.

$\frac{1}{16}(x^2+y^2) = 58$. Which implies $x^2+y^2 = 928$ ....(2)

Consider, $(x+y)^2 + (x-y)^2 = x^2+y^2+2xy+x^2+y^2-2xy$.

Therefore, $(x+y)^2 + (x-y)^2 = 2(x^2+y^2)$. Using the values from equations (1) and (2)

$(40)^2 + (x-y)^2 = 2(928)$

$(x-y)^2 = 256 \Rightarrow x-y = 16$. Since$x > y$, the difference $x-y$ cannot be negative.

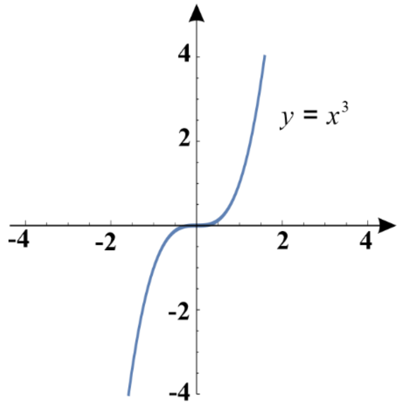
$x - y = 16$...(3)

Solving simultaneous equations (1) and (3), $x = 28$ and $y = 12$

Lengths of pieces are 28 cm and 12 cm.

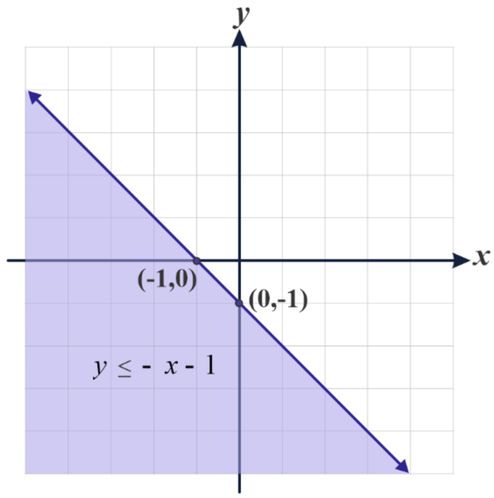
Answers for review exercises

1. a is a function, while b, c and d are **not** functions.
2. **a.** No **b.** No **c.** Yes
3. **a.** The domain and the range are the set of all real numbers.
   b. domain of R is the set of real numbers $x = -2, -1, 0, 1, 3, 5$ and the range of R is the set of real numbers $y = 4, 3, 2, 1, -1, -3$.
   c. domain of R is the set of real numbers $x: -1 \leq x \leq 1$. and the range of R is the set of real numbers $y: y \geq 0$.
   d. The domain is the set of brothers and sisters in a family and range is the set of sisters in a family.
   e. The domain is the set of pupils in a y class and the range is the set of classes in a y class.
4. **a.**



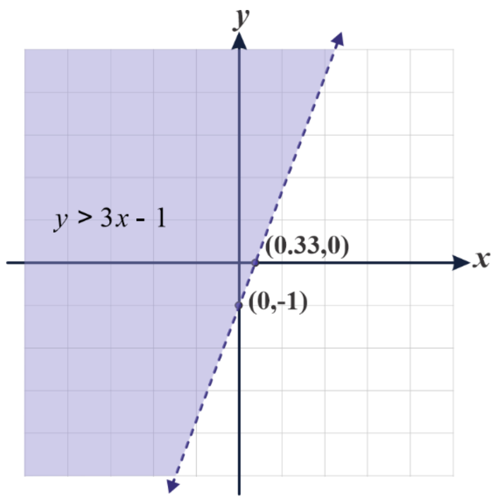
The domain and range is the set of all real numbers.

b.



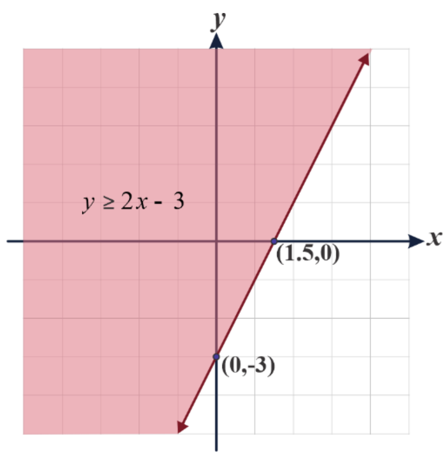
The domain and range is the set of all real numbers.

c.

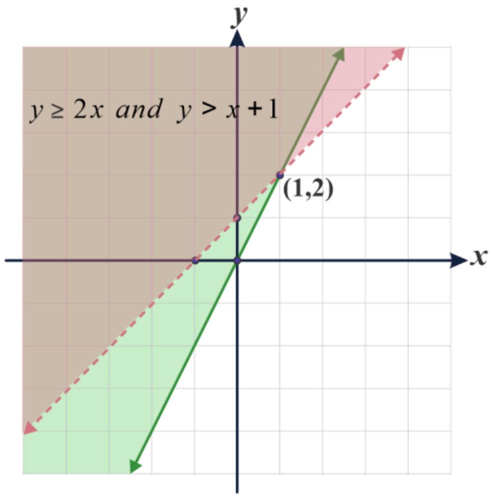


The domain and range is the set of real numbers.

d.

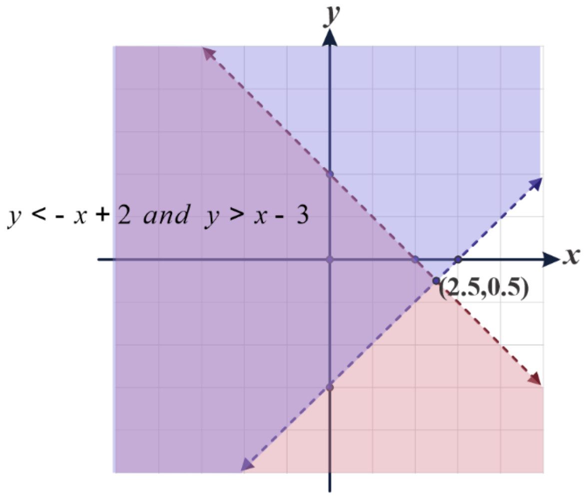


e.



The domain is the set of real numbers x : $x < 1$. The range is the set of all real numbers.

f.



The domain is the set of real numbers x: $x < 2.5$. The range is the set of all real numbers.

5. $R = \{(x, y): y > -x - 3 \text{ and } y \leq x - 1\}$.

The domain is the set of real numbers x: $x > -1$. The range is the set of all real numbers.

6. **i.** $(f + g)(x) = f(x) + g(x) = (2x^2 - 5x - 3) + (-2x^2 + x + 7)$

$= -5x + x - 3 + 7$ Removing bracket

$= -4x + 4.$

ii. $(f - g)(x) = f(x) - g(x) = (2x^2 - 5x - 3) - (-2x^2 + x + 7)$

$= 2x^2 + 2x^2 - 5x - x - 3 - 7$

$= 4x^2 - 6x - 10.$

iii. $(f + g)(-1) = -4(-1) + 4 = 8.$

iv. $(2f-g)(3) = 2(2(3)^2 - 5(3) - 3) - (-2(3)^2 + 3 + 7)$

$= 8$

b. The domain of $f - g$ is the set of all real numbers.

7. Given: $f(x) = 2x^2 - 1$ and $g(x) = x - 3$

a. **i.** $(fg)(x) = f(x)g(x) = (2x^2-1)(x-3) = 2x^3 - 6x^2 - x + 3.$

ii. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x^2-1}{x-3}$

iii. $(fg)(2) = 2(2)^3 - 6(2)^2 - 2 + 3 = -7$

iv. $\left(\frac{f}{g}\right)(5) = \frac{2(5)^2-1}{5-3} = \frac{49}{2}$

b. The domain of $\frac{f}{g}$ is the set of all real numbers $x: x \neq 3$.

8. Given: $f(x) = \frac{2x-3}{x-1}$ and $g(x) = \frac{x+8}{x}$

a. **i.** $(fg)(x) = f(x)g(x) = \left(\frac{2x-3}{x-1}\right)\left(\frac{x+8}{x}\right) = \frac{2x^2+13x-24}{x^2-x}$

ii. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x-3}{x-1} / \frac{x+8}{x} = \frac{2x^2-3x}{x^2+7x-8}$

iii. The domain of fg is the set of real numbers $x: x \neq o, 1$.

b. **i.** $(fg)(-3) = \frac{2(-3)^2+13(-3)-24}{(-3)^2-(-3)} = 45/15 = 3.$

ii. $\left(\frac{f}{g}\right)(3) = \frac{2(3)^2-3\times3}{3^2+7\times3-8} = 9/22.$

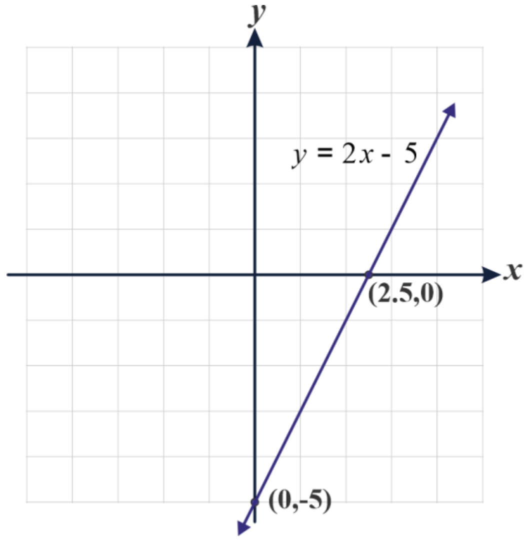
iii. $\left(3f - \frac{f}{g}\right)(-1) = 3\left(\frac{2\times(-1)-3}{-1-1}\right) - \frac{2(-1)^2-3\times-1}{(-1)^2+7(-1)-8}$

$$= 3 \times \frac{5}{2} + \frac{5}{14} = \frac{15}{2} + \frac{5}{14}$$

$$= \frac{110}{14} = \frac{55}{7}$$

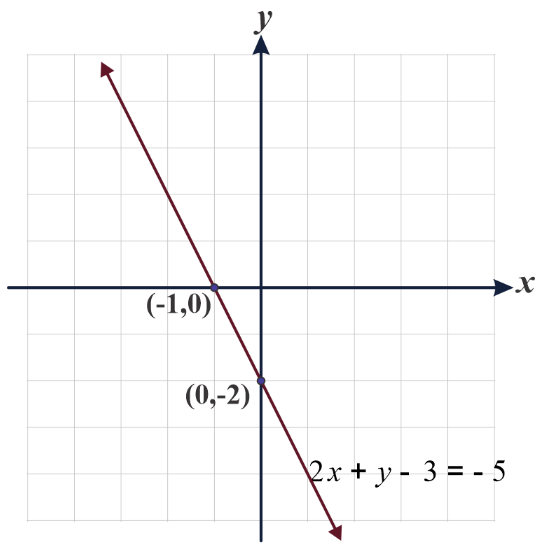
9. **a.**

x	-2	-1	-0	1	2	3	4
$f(x) = 2x - 5$	-9	-7	-5	-3	-1	1	3



b.

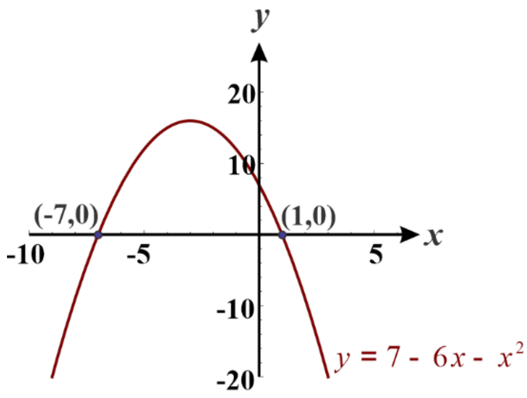
x	-3	-2	-1	0	1	2	3
$2x + y - 3 = -5$	4	2	0	-2	-4	-6	-8



The graph of $2x + y - 3 = -5$

c.

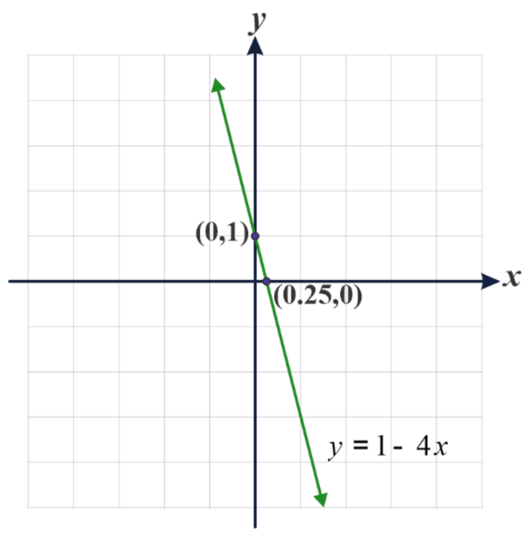
x	-8	-7	-6	-5	-4	-3	-1	0	1	2	3
$f(x) = 7 - 6x - x^2$	-11	0	7	12	15	16	12	7	0	-9	-20



The graph of $y = 7 - 6x - x^2$

d. $f(x) = 1 - 4x$

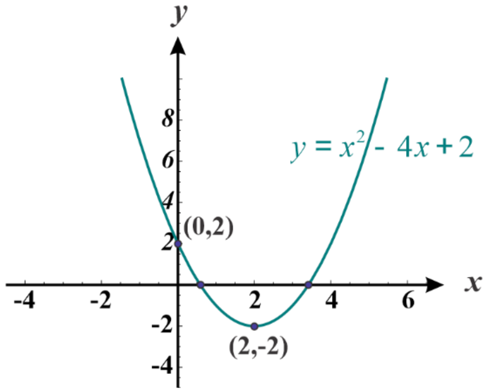
x	-1	$-1/2$	$-1/4$	0	1/4	1/2	1
$f(x) = 1 - 4x$	5	3	2	1	0	-1	-3



The graph of $f(x) = 1 - 4x$

10. a. $y = x^2 - 4x + 2 = x^2 - 4x + 4 - 4 + 2 = (x - 2)^2 - 2$

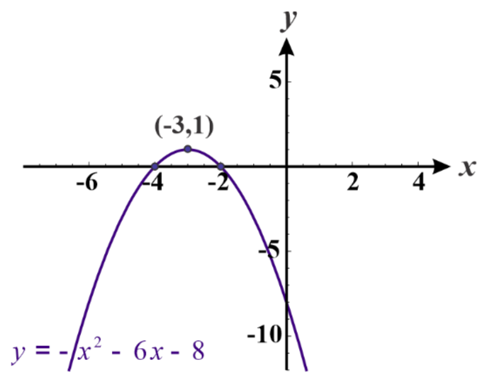
Now, by shifting $y = x^2$ to right and down by 2 we obtain the graph of $y = x^2 - 4x + 2$.



The graph of . $y = x^2 - 4x + 2$

b. $y = -x^2 - 6x - 8$

Now, by shifting $y = x^2$ to left 3 and up by 1 we obtain the graph of $y = -x^2 - 6x - 8$.



The graph of $y = -x^2 - 6x - 8$

11. $C(t) = 2t + 10$, if $t = 3$, then $C(3) = 2 \times 3 + 10 = 16Birr$.

Unit 2

Polynomial functions

Periods allotted: 22 periods

Introduction

This unit requires familiarity to function. Under this unit definition of polynomial function, operations on polynomial functions, theorems on polynomial functions, zeros of polynomial function, graph of polynomial function and application of polynomial function are covered.

Unit outcomes: At the end of this unit, students will be able to: -

- define polynomial functions.
- perform the four fundamental operations on polynomials.
- apply theorems on polynomial functions to solve related problems.
- determine the number of rational and irrational zeros of polynomials.
- sketch the graphs of polynomial functions.

Suggested teaching aids in unit 2

- charts showing model graphs of a polynomial function
- graph calculating software like GeoGebra

2.1 Introduction to polynomial functions

Periods allotted: 2 periods

Competencies

At the end of this sub-unit students will be able to:

- define the polynomial functions of one variable.
- identify the degree, leading coefficient and constant terms of a given polynomial functions.
- give different forms of polynomial functions.

Introduction

In this sub-unit, functions like constant functions, linear functions, and quadratic functions will be discussed as a revision to relate them with a special case of a class of function called polynomial function. The definition of polynomial functions and terms like degree, leading coefficient will be defined.

Teaching Guide

Students are expected to have backgrounds on the definition and properties of constant functions, linear functions and quadratic functions.

For the purpose of revision, you could ask students to do activity 2.

After that you may start the lesson by writing the general definition of these function from the textbook. You can also ask students to give their own examples of linear functions and quadratic functions.

Answer for Activity 2.1

a. Linear **b.** Linear

c. None **d.** Quadratic

e. None **f.** Constant

g. Quadratic **h.** Quadratic

i. Linear **j.** None

After giving the definition of polynomial function ask students to give answer to Activity 2.2 so that students can reach the generalization that constant, linear and quadratic functions are special cases of polynomial functions.

Answer for Activity 2.2

1. One
2. Two
3. Zero

After knowing students' understanding about constant, linear and quadratic function in relation with polynomial function, you can do the examples. Allow students to participate while doing the examples. After this ask students to do exercise 2.1. You are supposed to go around the class to facilitate their work and help students who could not solve it properly. At this end students are expected find out the degree, leading coefficient and constant term of a polynomial function.

Answer for exercise 2.1

1. $f(x) = 12x^3 - 9x^2 + 3x + 4$. The degree is 3, the leading coefficient is 12 and the constant term is 4.
2. $g(x) = 9x^4 + 15x^3 + 7x^2 + 2$. The degree is 4, the leading coefficient is 9 and the constant term is 2.
3. $h(x) = \frac{5}{3}x^3 + 2x^2 + \frac{1}{3}$. The degree is 3, the leading coefficient is $\frac{5}{3}$ and the constant term is $\frac{1}{3}$.

Discuss with students about the domain of a polynomial function. Given a polynomial function $f(x)$, all possible values of x that can give corresponding values of f are included in the domain of f. You could ask students to determine the domain and range of constant, linear and quadratic functions and finally generalize the domain of polynomial function. You could also ask students to do exercise 2.2 to increase their understanding of differentiating polynomial functions from other functions and to have more understanding on the degree, leading coefficient and constant term of a polynomial function. You could form groups so that you can share exercise 2.2 to the groups and finally the groups can present their answer through their representative.

Answer for exercise 2.2

a. Polynomial function with degree 5, leading coefficient 3 and constant term -5.

b. Polynomial function with degree 4, leading coefficient $\frac{3}{8}$ and constant term 4.

c. Not polynomial function because -3, -2 and -1 are not positive integer.

d. Polynomial function with degree 1, leading coefficient $\sqrt{2}$ and constant term $\sqrt{3}$.

e. Polynomial function with degree 2, leading coefficient $\sqrt{2}$ and constant term 5.

f. Not polynomial function because f can be rewritten as $f(x) = x^{-2} + x^{-1} + 1$ and -2 and -1 are not non negative integers.

g. Polynomial function with degree 5, leading coefficient $\frac{5}{3}$ and constant term $\frac{4}{3}$.

h. Polynomial function with degree 2, leading coefficient 1 and constant term -1.

i. Polynomial function with degree 4, leading coefficient 1 and constant term 25.

j. Polynomial function with degree 2, leading coefficient 2 and constant term 7.

Based on their knowledge on integer, rational and real numbers students, students are expected to when to say a polynomial function is a polynomial function over integer, over rational number and over real number and have their own understanding the example in the text book and do exercise 2.3. You could move round in the class to examine their progress and help students who could not solve it properly.

Answer for exercise 2.3

1. **a.** Yes, because we can rewrite $\frac{6x^3-2x^2+9}{3} + 3x^3 - 2x$ as

$$\frac{6x^3}{3} - \frac{2x^2}{3} + \frac{9}{3} + 3x^3 - 2x = 2x^3 - \frac{2}{3}x^2 + \frac{9}{3} + 3x^3 - 2x$$

$$= (2x^3 + 3x^3) - \frac{2}{3}x^2 + \frac{9}{3}$$

$$= 5x^3 - \frac{2}{3}x^2 + \frac{9}{3}$$

b. The degree 3, the leading coefficient is 5 and the constant term is $\frac{9}{3}$.

c. $-\frac{2}{3}$

2. **a.** Polynomial function over real numbers.

b. Polynomial function over integers, over rational numbers and over real numbers.

c. Polynomial function over rational numbers and over real numbers.

At this point you can ask students to create their own examples of a polynomial function over real numbers, over integers and over rational numbers with different degree.

Assessment

At the end of this lesson, you can give class activities, home-work, test or quiz to assess their level of understanding. The assessment should include the types of problems that enable you to know slow, medium and fast learners so that you may arrange appropriate support for those in need.

You can give the following exercise to students and check their level of understanding.

Let them identify whether the following functions are polynomial or not, for those which are polynomial let them indicate the degree, leading coefficient and constant term.

1. $f(x) = 5x^2 + \frac{2}{3}x^3 - \frac{1}{3}x - \frac{4x^3+9x^2-2x+9}{3}$
2. $f(x) = 2\left(\frac{1}{x}\right)^2 + 3\left(\frac{1}{x}\right) - 6$
3. $f(x) = 3(x^2)^2 - 4(-x)^5 + 9x^2 - 5$
4. $f(x) = -2\left(\sqrt{x}\right)^3 + 5\sqrt{x} - 10$
5. $f(x) = 3\pi^2 + 4$

2.2 Operations on Polynomial functions

Periods allotted: 9 periods

Competencies

At the end of this sub-unit students will be able to:

- perform the four-fundamental operation on polynomials

Introduction

In this sub-unit, we will combine two or more polynomial functions using the operations addition, subtraction, multiplication and division and discuss on the results obtained by the combination.

Teaching Guide

To combine polynomial functions the knowledge of commutative, associative and distributive laws and like and unlike terms is very important. Students are expected to know these laws and the meaning of like and unlike terms from previous experience. For this, students could do activity 2.3. You could support them while they are doing.

Answer for activity 2.3

1. **a.** Like terms **b.** Like terms

c. Unlike terms **d.** Unlike terms

e. Unlike terms **f.** Like terms

2.

a. True, commutative property of addition.

b. False, subtraction of numbers is not commutative.

c. True, associative property of addition.

d. True, associative property of multiplication.

e. True, distributive property of multiplication over addition.

f. False.

g. True, -1 multiply both b and $-c$.

h. True.

Addition of polynomial function

To combine polynomial function using the operation addition the knowledge of commutative and associative laws is very important. Using these lows, we can group like terms to find their sum. While doing examples given, try to show how and where in the steps these laws are applied. After this give time to students to do exercise 2.4. You could check their work and give support at this time. Finally allow some students to show their work on the board.

Answer for exercise 2.4

a. $f(x) + g(x) = (2x^3 - 3x - 5) + (5x^4 - 7x^2 + 3)$

$= 5x^4 + 2x^3 - 7x^2 - 3x + (-5 + 3)$; grouping like terms

$= 5x^4 + 2x^3 - 7x^2 - 3x - 2$; adding like terms

b. $f(x) + g(x) = (-x^4 + 2x^3 - 3x^2 - 3x + 2) + (5 + 7x - 2x^2 - x^3 + x^4)$

$= (-x^4 + x^4) + (2x^3 - x^3) + (-3x^2 - 2x^2) + (-3x + 7x) + (2 + 5)$; grouping like terms

$= x^3 - 5x^2 + 4x + 7$; adding like terms

c. $f(x) + g(x) = (-2x^5 + 2x^4 - x^3 + 2x^2 + 5x - 1) + (2 + 4x - 5x^5 - 3x^4)$

$= (-2x^5 - 5x^5) + (2x^4 - 3x^4) - x^3 + 2x^2 + (5x + 4x) + (-1 + 2)$;

grouping like terms

$= -7x^5 - x^4 - x^3 + 2x^2 + 9x + 1$; adding like terms

d. $f(x) + g(x) = \left(\sqrt{2}x^4 + 2x^3 - 5x^2 + x - \sqrt{3}\right) + \left(\sqrt{3} - 3x - x^2 + 2x^3 - 2\sqrt{2}x^4\right)$

$= \left(\sqrt{2}x^4 - 2\sqrt{2}x^4\right) + \left(2x^3 + 2x^3\right) + \left(-5x^2 - x^2\right) + (x - 3x) + \left(-\sqrt{3} + \sqrt{3}\right)$;

grouping like terms

$= -\sqrt{2}x^4 + 4x^3 - 6x^2 - 2x$; adding like terms

At this stage you could discuss with students about the degree of the polynomial function obtained by adding two polynomial functions.

Given two polynomial functions f and g , you can ask students about the degree of $f + g$

1. When they have different degrees.
2. When they have same degree.
3. Is $f + g$ a polynomial function?

Finally support them to write their generalization.

ASSESSMENT

You can use any one of assessment techniques such as: class activities, group discussions, homework/assignments and/or tests/quizzes.

Subtraction of Polynomial Functions

We know that subtraction of numbers is not commutative. That is $a - b \neq b - a$. Since $a - b = a + (-b)$, we can use the commutative laws of addition. That is, $a + (-b) = -b + a$. Students are expected to participate while you are doing the given examples. After this give time to students to do exercise 2.5. You could observe their work and give support at this time. Finally allow some students to demonstrate their work on the board.

Answer for Exercise 2.5

a. $f(x) - g(x) = (-x^3 - 5x^2 + 3x + 12) - (9x^3 - x^2 - 6x + 3)$

$= -x^3 - 5x^2 + 3x + 12 - 9x^3 + x^2 + 6x - 3$; removing brackets

$=(-x^3 - 9x^3) + (-5x^2 + x^2) + (3x + 6x) + (12 - 3)$;Grouping like terms

$= -10x^3 - 4x^2 + 9x + 9$; adding like terms

b. $f(x) - g(x) = \left(-\frac{1}{3}x^4 + 3x^3 - 3x^2 - \frac{3}{5}x + 2\right) - (\frac{1}{3}x^4 - x^3 - 2x^2 + \frac{7}{5}x + 5)$

$= -\frac{1}{3}x^4 + 3x^3 - 3x^2 - \frac{3}{5}x + 2 - \frac{1}{3}x^4 + x^3 + 2x^2 - \frac{7}{5}x - 5$; removing brackets

$= \left(-\frac{1}{3}x^4 - \frac{1}{3}x^4\right) + (3x^3 + x^3) + (-3x^2 + 2x^2) + (-\frac{3}{5}x - \frac{7}{5}x) + (2 - 5)$;

Grouping like terms

$= 4x^3 - x^2 - 2x - 3$; adding like terms

c. $f(x) - g(x) = (-5x^5 + 3x^4 - x^3 + 2x^2 + 5x + 1) - (2 + 4x - 5x^5 - 3x^4)$

$= -5x^5 + 3x^4 - x^3 + 2x^2 + 5x + 1 - 2 - 4x + 5x^5 + 3x^4$; removing brackets

$= (-5x^5 + 5x^5) + (3x^4 + 3x^4) - x^3 + 2x^2 + (5x - 4x) + (1 - 2)$;

Grouping like terms

$= 6x^4 - x^3 + 2x^2 + x - 1$; adding like terms

d. $f(x) - g(x) = \left(3\sqrt{3}x^4 + 2x^3 - 5x^2 + x - 5\sqrt{3}\right) - (\sqrt{3} - 3x - x^2 + 2x^3 - 2\sqrt{3}x^4)$

$= 3\sqrt{3}x^4 + 2x^3 - 5x^2 + x - 5\sqrt{3} - \sqrt{3} + 3x + x^2 - 2x^3 + 2\sqrt{3}x^4$; removing brackets

$= \left(3\sqrt{3}x^4 + 2\sqrt{3}x^4\right) + (2x^3 - 2x^3) + (-5x^2 + x^2) + (x + 3x) + (-5\sqrt{3} - \sqrt{3})$;

Grouping like terms

$= 5\sqrt{3}x^4 - 4x^2 + 4x - 6\sqrt{3}$; adding like terms

Students could actively participate and give answer for activity 2.4. The purpose doing the activity to enable students deliver their generalization on the degree of the difference of two polynomial functions.

Answer for activity 2.4

1. True
2. Yes, when $f(x)$ and $g(x)$ have equal degree and equal leading coefficients with opposite sign.
3. Yes

ASSESSMENT

You can use any one of assessment techniques such as: class activities, group discussions, homework/assignments and/or tests/quizzes.

Multiplication of Polynomial Function

Multiplication of two polynomial functions involves the distributive property of multiplication over addition, commutative and associative laws. While doing the examples try to show how and where in the steps these laws are applied. Allow students to actively participate by indicating which property is applied at each step. After this, give time to the students to do exercise 2.6 and exercise 2.7. You could inspect their work and give support at this time. Finally allow some students to show their work on the board.

Answer for exercise 2.6

1.

a. $f(x).g(x) = (3x+1)(2x^2+4x-5)$

$= 3x(2x^2+4x-5)+1(2x^2+4x-5)$

$= (6x^3+12x^2-15x)+(2x^2+4x-5)$

$= 6x^3+(12x^2+2x^2)+(-15x+4x)-5$

$= 6x^3+14x^2-11x-5$

b. $f(x).g(x) = (x^2-2)(3x^2-6x+1)$

$= x^2(3x^2-6x+1)+(-2)(3x^2-6x+1)$

$= (3x^4-6x^3+x^2)+(-6x^2+12x-2)$

$= 3x^4-6x^3+(x^2-6x^2)+12x-2$

$= 3x^4-6x^3-5x^2+12x-2$

c. $f(x).g(x) = (x^3-2x)(x-2x^2)$

$= x^3(x-2x^2)+(-2x)(x-2x^2)$

$= (x^4-2x^5)+(-2x^2+4x^3)$

$= x^4-2x^5-2x^2+4x^3$

$= -2x^5+x^4+4x^3-2x^2$

2. **a.** $-6x^7+3x^6-4x^4+10x^3+10x^2-4x^2$

b. The degree of f is 4, the degree of g is 3 and degree of fg is 7.

c. Yes.

Answer for exercise 2.7

a.

$$\begin{array}{r} 2x^2-2x-1 \\ 3x+5 \\ \hline 10x^2-10x-5 \\ 6x^3-6x^2-3x \quad \\ \hline 6x^3+4x^2-13x-5 \end{array}$$

b.

$$\begin{array}{r} 3x^3-x^2+x-1 \\ -2x^2+5x \\ \hline 15x^4-5x^3+5x^2-5x \\ -6x^5+2x^4-2x^3+2x^2 \quad \\ \hline -6x^5+17x^4-7x^3+7x^2-5x \end{array}$$

c.

$$\begin{array}{r} x^3+2x^2+x-5 \\ -2x^2+5x-3 \\ \hline -3x^3-6x^2-3x+15 \\ 5x^4+10x^3+5x^2-25x \quad \\ -2x^5-4x^4-2x^3+10x^2 \quad \quad \\ \hline -2x^5+x^4+5x^3+9x^2-28x+15 \end{array}$$

At this stage you could ask students about the degree of the product polynomial and support students to summarize the result.

ASSESSMENT

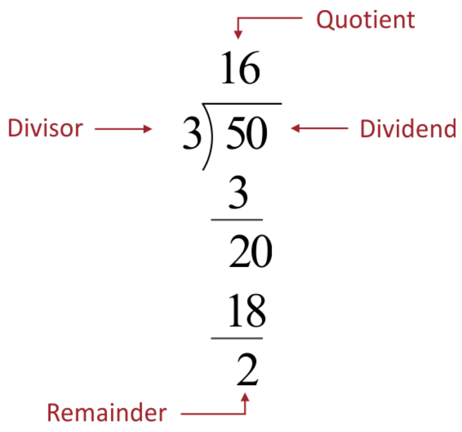
You can use any one of assessment techniques such as: class activities, group discussions, homework/assignments and/or tests/quizzes.

Division of polynomial functions

A number that takes the form $\frac{a}{b}$, were a and b are integers and $b \neq 0$ is called a rational number. If b is positive integer, we can divide a by b to find two other integers q and r with $0 \leq r < b$ such that $\frac{a}{b} = q + \frac{r}{b}$, a is called the dividend, b is called the divisor, q is called the quotient and r is called the remainder.

Support students to divide number given in the example below so that they could have a hint which enable them to divide polynomials. Also support them to do activity 2.5.

For example, to find q and r when 50 is divided by 3, you usually use a process called long division as follow



and $50 \div 3 = \frac{50}{3} = 16 + \frac{2}{3}$.

Answer for activity 2.5

i. $\frac{97}{8} = 12 + \frac{1}{8}$ **ii.** $\frac{168}{5} = 33 + \frac{3}{5}$ **iii.** $\frac{287}{15} = 19 + \frac{2}{15}$ **iv.** $\frac{355}{11} = 32 + \frac{3}{11}$

Steps to divide polynomials using long division

1. Divide the first term of the dividend by the first term of the divisor, this gives the first term of the quotient.
2. Multiply the first term of the quotient by the divisor.
3. Subtract the result obtained in step 2 from the dividend.
4. Consider the result obtained in step 3 as a first new dividend.
5. Divide the first term of the new dividend by the first term of the divisor, this gives the second term of the quotient.
6. Multiply the second term of the quotient by the divisor.
7. Subtract the result obtained in step 6 from the first new dividend.
8. Consider the result obtained in step 7 as a second new dividend.

Repeat these steps until the new divisor is zero or its degree is less than the degree of the divisor. The last new dividend whose degree is less than the degree of the divisor is called the remainder.

You could do the examples for the students by explaining every steps. Allow students to participate in the process. You could form small groups and ask students to do exercise 2.8 and exercise 2.9 in their group. Finally allow students to show their answer on the board by the group representative.

Answer for exercise 2.8

a.

$$\begin{array}{r|l} & x^2 + 2x - 3 \\ \hline x+2 & x^3 + 4x^2 + x + 1 \\ & \underline{x^3 + 2x^2} \\ & \quad 2x^2 + x \\ & \quad \underline{2x^2 + 4x} \\ & \qquad -3x + 1 \\ & \qquad \underline{-3x - 6} \\ & \qquad\qquad 7 \end{array}$$

Therefore, $q(x) = x^2 + 2x - 3$ and $r(x) = 7$

b.

$$\begin{array}{r|l} & 2x^2 + 4x - 2 \\ \hline 2x-1 & 4x^3 + 6x^2 - 8x + 5 \\ & \underline{4x^3 - 2x^2} \\ & \quad 8x^2 - 8x + 5 \\ & \quad \underline{8x^2 - 4x} \\ & \qquad -4x + 5 \\ & \qquad \underline{-4x + 2} \\ & \qquad\qquad 3 \end{array}$$

Therefore, $q(x) = 2x^2 + 4x - 2$ and $r(x) = 3$

c.

$$\begin{array}{r|l} & -x^2 + 2x + 2 \\ \hline -x+1 & x^3 - 3x^2 - 6 \\ & \underline{x^3 - x^2} \\ & \quad -2x^2 - 6 \\ & \quad \underline{-2x^2 + 2x} \\ & \qquad -2x - 6 \\ & \qquad \underline{-2x + 2} \\ & \qquad\qquad -8 \end{array}$$

Therefore, $q(x) = -x^2 + 2x + 2$ and $r(x) = -8$

Answer for exercise 2.9

a.

$$\begin{array}{r|l} & x+2 \\ \hline x^2+x+2 & x^3+3x^2+6x+5 \\ & x^3+x^2+2x \\ \hline & 2x^2+4x+5 \\ & 2x^2+2x+4 \\ \hline & 2x+1 \end{array}$$

Therefore, $\frac{x^3+3x^2+6x+5}{x^2+x+2} = x+2+\frac{2x+1}{x^2+x+2}$

b.

$$\begin{array}{r|l} & x^2+x+2 \\ \hline x^2-1 & x^4+x^3+x^2-6x+7 \\ & x^4-x^2 \\ \hline & x^3+2x^2-6x+7 \\ & x^3-x \\ \hline & 2x^2-5x+7 \\ & 2x^2-2 \\ \hline & -5x+9 \end{array}$$

$$\frac{x^4+x^3+x^2-6x+7}{x^2-1} = x^2+x+2+\frac{-5x+9}{x^2-1}$$

c.

$$\begin{array}{r|l} & x^2+3x-1 \\ \hline 2x^3-x^2+1 & 2x^5+5x^4-5x^3+8x^2+1 \\ & 2x^5-x^4+x^2 \\ \hline & 6x^4-5x^3+7x^2+1 \\ & 6x^4-3x^3+3x \\ \hline & -2x^3+7x^2-3x+1 \\ & -2x^3+x^2-1 \\ \hline & 6x^2-3x+2 \end{array}$$

$$\frac{2x^5+5x^4-5x^3+8x^2+1}{2x^3-x^2+1} = x^2+3x-1+\frac{6x^2-3x+2}{2x^3-x^2+1}$$

Assessment

The purpose of continuous assessment is to gather relevant information about students' performance or progress, or to know how well students are learning or determine student interests to make judgments about their learning process. After receiving this information, you can reflect on each student level of achievement and customize your teaching plans.

You can give class activities, group work, assignments, exercise problems and quiz or test to assess students learning.

To check their understanding and consolidate this subunit, you can ask students the following additional exercise problems.

You could ask students to find $f + hg$, $hf - g$, fg and $\frac{f}{h}$, where $f(x) = 6 + 4x - 2x^2 + 3x^3$, $g(x) = x^4 - 5x^2 + x^3 + 4x^4 - 2$ and $h(x) = x + 2$.

2.3 Theorem on Polynomials

Periods allotted: 5 periods

Competencies

At the end of this sub-unit students will be able to:

- state and prove the polynomial division theorem.
- apply the polynomial division theorem.
- state and prove the factor theorem.
- apply the factor theorem.
- state and prove the remainder theorem.
- apply the remainder theorem.

Key Terms: Factor, Remainder theorem, Factor theorem

Introduction

In the previous subunit we have seen how we divide one polynomial by another polynomial using long division. Based on this, three important theorems called the polynomial division theorem, the remainder theorem and the factor theorem will be treated.

Teaching Guide

You can start the lesson by revising the division al**gorithm** of positive integers. You could ask students to divide 163 by 5 using long division You could also ask students, when they stop performing long division to apply division algorithm.

They could write the result as,

$$\frac{163}{5} = 32 + \frac{3}{5} \quad . \quad . \quad . (*)$$

Let students to think and rewrite (*) in the following equivalent form (By multiplying both sides of (*) by 5).

$$163 = 5 \times 32 + 3$$

Using their concept on division of polynomial function let students divide $f(x) = x^3 + 2x^2 + x - 8$ by $g(x) = x + 3$ to get the quotient $q(x) = x^2 - x - 4$ and the remainder $r(x) = -20$ that satisfy

$x^3 + 2x^2 + x - 8 = (x^2 - x - 4)(x + 3) + (-20)$

That is, $f(x) = q(x)g(x) + r(x)$

Dividend = (Divisor)(Quotient) + Remainder

This could help students to understand the Polynomial Division Theorem.

Polynomial Division Theorem

You can start by encouraging students to do activity 2.6, so that they can recall the division algorithm.

Answer for activity 2.6

i. $q = 17$ and $r = 3$

ii. $q = 50$ and $b = 5$

iii. $q = 20$ and $r = 14$

iv. $q = 4$ and $r = 0$

At this point you can state and prove the polynomial Division Theorem. While proving the theorem you can encourage students to participate by giving reasons for some steps in the proof.

After proving the theorem do the examples in the text so that students can have better understanding of the theorem. After this allow students to do exercise 2.10. You can move among the students to observe their work and give support.

Answer for exercise 2.10

a.

$$\begin{array}{r|l} & 6x+4 \\ \hline x-1 & 6x^2-2x+3 \\ & \underline{6x^2-6x} \\ & \quad 4x+3 \\ & \quad \underline{4x-4} \\ & \qquad 7 \end{array}$$

$6x^2 - 2x + 3 = (x-1)(6x+4) + 7$

b.

$$\begin{array}{r|l} & x+2 \\ \hline x^2+2x-1 & x^3+4x^2+8x+6 \\ & \underline{x^3+2x^2-x} \\ & \quad 2x^2+9x+6 \\ & \quad \underline{2x^2+4x-2} \\ & \qquad 5x+8 \end{array}$$

$x^3 + 4x^2 + 8x + 6 = (x+2)(x^2+2x-1) + (5x+8)$

c.

$$\begin{array}{r|l} & x^2+6x+1 \\ \hline x^2-1 & x^4+6x^3+0x^2-10x+3 \\ & \underline{x^4+0x^3-x^2} \\ & \quad 6x^3+x^2-10x+3 \\ & \quad \underline{6x^3+0x^2-6x} \\ & \qquad x^2-4x+3 \\ & \qquad \underline{x^2+0x-1} \\ & \qquad\quad -4x+4 \end{array}$$

$x^4 + 6x^3 - 10x + 3 = (x^2+6x+1)(x^2-1) + (-4x+4)$

d.

$$\begin{array}{r|l} & -x+5 \\ \hline x^2+x+1 & -x^3+4x^2-x-6 \\ & \underline{-x^3-x^2-x} \\ & \quad 5x^2-6 \\ & \quad \underline{5x^2+5x+5} \\ & \qquad -5x-11 \end{array}$$

$-x^3 + 4x^2 - x - 6 = (x^2+x+1)(-x+5) + (-5x-11)$

e.

$$\begin{array}{r|l} & -x^3+2x^2-4x+8 \\ \hline x+2 & -x^4 \\ & \underline{-x^4-2x^3} \\ & \quad 2x^3 \\ & \quad \underline{2x^3+4x^2} \\ & \qquad -4x^2 \\ & \qquad \underline{-4x^2-8x} \\ & \qquad\qquad 8x \\ & \qquad\qquad \underline{8x+16} \\ & \qquad\qquad\quad -16 \end{array}$$

$$-x^4=(x+2)(-x^3+2x^2-4x+8)+(-16)$$

ASSESSMENT

You can use any one of assessment techniques such as: class activities, group discussions, homework/assignments and/or tests/quizzes.

Remainder Theorem

Let students do activity 2.7, while doing the activity support them to observe that when a polynomial of degree greater or equal to one is divided by $x-c$ the remainder is a constant number and its value is equal to $f(c)$

Answer for activity 2.7

After the student do this activity, tell them to relate the remainder obtained with $f(c)$.

a.

$$\begin{array}{r|l} & x-3 \\ \hline x+2 & x^2-x+3 \\ & \underline{x^2+2x} \\ & \quad -3x+3 \\ & \quad \underline{-3x-6} \\ & \qquad\quad 9 \end{array}$$

$$f(c)=f(-2)=(-2)^2-(-2)+3=9$$

b.

$$\begin{array}{r} x^2+3x+2 \\ x-1\overline{)\,x^3+2x^2-x-5} \\ \underline{x^3-x^2} \\ 3x^2-x-5 \\ \underline{3x^2-3x} \\ 2x-5 \\ \underline{2x-2} \\ -3 \end{array}$$

$f(c)=f(1)=(1)^3+2(1)^2-(1)-5=-3$

At this point you can state and prove the Remainder Theorem. While proving the theorem you can encourage students to participate by giving reasons for some steps in the proof.

After proving the theorem do the examples so that students can have better understanding of the theorem. After this allow students to do exercise 2.11 and exercise 2.12. You can round among the students to observe their work and give support. You can also choose problems from the review exercise and give it as a home work.

Answer for exercise 2.11

a. $c=1$ and the remainder is $f(1)=(1)^3-3(1)^2+4=2$

b. $c=-2$ and the remainder is $f(-2)=-2(-2)^3+4(-2)^2+5(-2)-2=20$

c. $c=1$ and the remainder is $f(1)=(1)^{17}-4(1)^2+7(1)-32=-28$

d. $c=-1$ and the remainder is $f(-1)=(-1)^{16}+8(-1)^3+99=92$

e. $c=\frac{1}{2}$ and the remainder is $f\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^3-2=-\frac{15}{8}$

Answer for exercise 2.12

1. Let $f(x)=5x^3-bx^2+8x-1$.

By the remainder theorem when $f(x)$ is divided by $x+1=x-(-1)$ the remainder is

$f(-1)=5(-1)^3-b(-1)^2+8(-1)-1=-14-b$.

Since the remainder is 15, we have, $-14-b=15$ and solving for b, we have, $b=-29$.

2. Let $f(x)=ax^3-bx^2+5x-2$.

Since $f(x)$ is divided by $x-1$ the remainder,

$f(1)=a(1)^3-b(1)^2+5(1)-2=a-b+3$.

Since the remainder is 4, we have, $a-b+3=4$

$$a - b = 1 \ . \ . \ . \ (*)$$

Since $f(x)$ is divided by $x + 1$ the remainder, $f(-1) = a(-1)^3 - b(-1)^2 + 5(-1) - 2$

Since the remainder is 6, we have, $-a - b - 7 = 6$

$$-a - b = 13 \ \ . \ . \ . \ (**)$$

Solving (*) and (**) simultaneously we get $a = -6$ and $b = -7$.

ASSESSMENT

You can use any one of assessment techniques such as: class activities, group discussions, homework/assignments and/or tests/quizzes.

Factor Theorem

You may start the lesson by considering two polynomials and their product

For example, $(x + 1)(2x - 1) = 2x^2 - x - 1$

The polynomial $2x^2 - x - 1$ is called product or multiple and $(x + 1)$ and $(2x - 1)$ are called factors. Factoring a polynomial means writing it as a product of two or more polynomials. Tell them that the **factor theorem** is very helpful to check whether a linear polynomial is a factor of a given polynomial.

Give time to student do activity 2.8. After this ask them to find out the relation between $f(x)$ and the divisor when the remainder is the zero polynomial. The divisor is a factor of $f(x)$.

Answer for activity 2.8

a. $f(1) = 0$.

b. $q(x) = x^2 + 5x + 6 \ and \ r(x) = 0$.

c. $f(x) = (x - 1)(x^2 + 5x + 6) + 0$.

d. Yes.

At this point you can state and prove the Factor Theorem. While proving the theorem you can encourage students to participate by giving reasons for some steps in the proof.

After proving the theorem do the examples in the textbook so that students can have better understanding of the theorem. After this allow students to do exercise 2.13 and exercise 2.14. You can round among the students to observe their work and give support. You can also choose problems from the review exercise and give it as a home work.

Answer for Exercise 2.13

1. $f(x) = x^3 + 2x^2 - 5x - 6$
   - a. $c = 1$ and $f(1) = 1^3 + 2(1)^2 - 5(1) - 6 = -8 \neq 0$. Therefore, $x - 1$ is not a factor
   - b. $c = 2$ and $f(2) = 2^3 + 2(2)^2 - 5(2) - 6 = 0$. Therefore, $x - 1$ is a factor.
   - c. $c = -1$ and $f(-1) = (-1)^3 + 2(-1)^2 - 5(-1) - 6 = 0$. Therefore, $x - 1$ is a factor
   - d. $c = -2$ and $f(-2) = (-2)^3 + 2(-2)^2 - 5(-2) - 6 = 4 \neq 0$. Therefore, $x - 2$ is not a factor
2. a. $f(1) = (1)^3 - 6(1)^2 + 11(1) - 6 = 0$, $x - c = x - 1$ is a factor of $f(x)$.
   - b. $f\left(-\frac{1}{2}\right) = 2\left(\frac{1}{16}\right) + \frac{1}{8} + 3\left(\frac{1}{4}\right) - 4\left(-\frac{1}{2}\right) - 3 = 0$, $x - c = x + \frac{1}{2}$ is a factor of $f(x)$.
   - c. $f(2) = 2^3 - 3(2^2) + 4(2) - 3 = 1 \neq 0$; $x - c = x - 2$ is not a factor of $f(x)$.

Answer for Exercise 2.14

1. a. Let $f(x) = 2x^3 + kx^2 + 5x - 1$, $x - 2$ is a factor of $f(x)$ implies

$$f(2) = 2(2)^3 + k(2)^2 + 5(2) - 1 = 0$$

$$4k = -25 \text{ and this implies } k = \frac{-25}{4}$$

   b. Let $f(x) = x^4 + 2kx^3 - x^2 - 5kx + 6$, $x + 3$ is a factor of $f(x)$ implies

$$f(-3) = (-3)^4 + 2k(-3)^3 - (-3)^2 - 5k(-3) + 6 = 0$$

$$-39k + 78 = 0 \text{ and this implies } k = 2.$$

2. Let $f(x) = ax^4 + x^3 - 2bx^2 - 11x + 6$,

Since $x + 1$ is a factor of $f(x)$

$$f(-1) = a(-1)^4 + (-1)^3 - 2b(-1)^2 - 11(-1) + 6 = 0$$

$$a - 2b = -16 \quad . . . (*)$$

Since $x - 2$ is a factor of $f(x)$

$$f(2) = a(2)^4 + (2)^3 - 2b(2)^2 - 11(2) + 6 = 0$$

$$2a - b = 1 \quad . . . (**)$$

Solving (*) and (**) simultaneously $a = 6$ *and* $b = 11$

ASSESSMENT

You can use any one of assessment techniques such as: class activities, group discussions, homework/assignments and/or tests/quizzes.

2.4 Zeros of a polynomial function

Periods allotted: 3 periods

Competencies

At the end of this sub-unit students will be able to:

- determine the zero(s) of a given polynomial function.
- state the Location Theorem.
- apply the Location theorem to approximate the zero(s) of a given polynomial function.
- apply the rational root test to determine the zero(s) of a given polynomial function.

Introduction

The zeros of a polynomial $f(x)$ are all the x-values that make the polynomial **equal to** zero. They are interesting to us for many reasons, one of which is that they tell us about the x -intercepts of the polynomial's graph. It is not always possible to find the zeros of a polynomial functions exactly. Sometimes it is important to locate the interval containing the zeros of a polynomial function. In this regard the location theorem is an important tool. The rational root test is used to find all the rational roots of a polynomial function.

Teaching Guide

If c is the zero $f(x)$, then c is the root or solution of the equation $f(x) = 0$. Let students do activity 2.9. You can support them to understand the definition of zero of a polynomial function.

Answer for activity 2.9

a. $x = 9$ **b.** $x = \frac{9}{8}$ **c.** $x = \frac{3}{2}$ or $x = -2$ **d.** $x = 2$ or $x = 3$

e. $x = -1$ **f.** has no solution in the set of real numbers

To find the zeros of some polynomial function whose degree is greater than 2 is treated by the examples given in the students' text. Do this example for students so that they can understand the definition more.

Answer for Exercise 2.15

a. $f(x) = (x-1)(x+5)(3x-2)$

$(x-1)(x+5)(3x-2) = 0$

$x - 1 = 0,\quad x + 5 = 0$ or $3x - 2 = 0$

$x = 1, x = -5\ or\ x = \frac{2}{3}$

Therefore, the zeros of $f(x)$ are $1, -5$ and $\frac{2}{3}$.

b. $f(x) = x^4 - 5x^2 + 4$

$x^4 - 5x^2 + 4 = 0$

$(x^2)^2 - 5x^2 + 4 = 0$, let $y = x^2$

$y^2 - 5y + 4 = 0$

$(y-1)(y-4) = 0$

$y - 1 = 0$ or $y - 4 = 0$

$y = 1$ or $y = 4$

Substituting the value of y, we get $x^2 = 1$ or $x^2 = 4$

$x^2 = 1$ implies $x = -1$ or $x = 1$.

$x^2 = 4$ implies $x = -2$ or $x = 2$.

Therefore, the zeros of $f(x)$ are $-2, -1, 1$ and 2.

c. $f(x) = x^4 - x^2 - 2$

$x^4 - x^2 - 2 = 0$

$(x^2)^2 - x^2 - 2 = 0$, let $y = x^2$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

$y - 2 = 0$ or $y + 1 = 0$

$y = 2$ or $y = -1$

Substituting the value of y, we get $x^2 = 2$ or $x^2 = -1$

$x^2 = 2$ implies $x = -\sqrt{2}$ or $x = \sqrt{2}$.

$x^2 = -1$ has no root in the set of real numbers.

Therefore, the zeros of $f(x)$ are $-\sqrt{2}$ and $\sqrt{2}$.

d. $f(x) = x^3 - x^2 - 10x - 8$

By experiment, $x = -1$ is one zero of $f(x)$ then $x + 1$ is one factor of $f(x)$. Using long division, the other factor is $x^2 - 2x - 8$.

Further factoring $x^2 - 2x - 8 = (x + 2)(x - 4)$. Hence, $x^3 - x^2 - 10x - 8 \ = 0$ is the same as $(x + 1)(x + 2)(x - 4) = 0$.

Therefore, $x = -1$, $x = -2$ and $x = 4$ are the zeros of $f(x)$.

e. $f(x) = 2x^3 - 9x^2 - 5x$

$$2x^3 - 9x^2 - 5x = 0$$

$$x(2x^2 - 9x - 5) = 0$$

$$x(2x^2 + x - 10x - 5) = 0$$

$$x(x(2x + 1) - 5(2x - 1)) = 0$$

$$x(2x + 1)(x - 5) = 0$$

$$x = 0,\ \ 2x + 1 = 0 \ or \ \ x - 5 = 0$$

Therefore, the zeros of $f(x)$ are $0, -\frac{1}{2}$ and 5.

Zeros of a polynomial function and their Multiplicities

The **Multiplicity** of a zero of a polynomial function refers to the number of times that its associated factor appears in the polynomial. Support students to understand the definition of the zero and its multiplicity consider the examples

$f(x) = x^3 - 2x^2 - x - 2 = (x - 1)(x + 1)(x + 2)$ and

$$g(x) = x^3 + 4x^2 - 3x - 18 = (x + 3)^2(x - 2)$$

Indicate the zeros and tell them their multiplicity based on definition 2.9.

After this support students to do exercise 2.16 in group. You can round among them for support.

Answer for exercise 2.16

1 **a.** $x = 1$ and $x = 2$ are the zeros of $f(x)$ with multiplicity 2 and 1, respectively.

b. $t = -\frac{2}{3}$ and $t = 0$ are the zeros of $f(x)$ with multiplicity 3 and 10, respectively.

c. $x = -\sqrt{2}$, -2 and $-\frac{1}{3}$ are the zeros of $f(x)$ with multiplicity 2, 3 and 1, respectively.

d. $x = -1$ is one zero of $h(t)$, we find this zero just by trying. This means $t + 1$ is one factor of $h(t)$. By using long division, we can find the other factors and write $h(t) = (t+1)^2(2t+1)$. Therefore, $t = -1$ and $t = -\frac{1}{2}$ are the zeros with multiplicity 2 and 1, respectively.

2. a. If -2 and 3 are the zeros of $f(x)$, then $(x+2)$ and $(x-3)$ are factors of $f(x)$. Therefore, $f(x) = k(x+2)(x-3)$ for some constant $k \neq 0$

To find the value of k, we use the given condition $f(2) = 6$, that is,

$k(2+2)(2-3) = -4k = 12$. This implies $k = -3$.

Hence, the required polynomial function is,

$f(x) = -3(x+2)(x-3) = -3x^3 + 3x + 18$.

b. If -1, 2 and 1 are the zeros of $f(x)$, then $(x+1), (x-2)$ and $(x-1)$ are factors of $f(x)$. Therefore, $f(x) = k(x+1)(x-2)(x-1)$ for some constant $k \neq 0$

To find the value of k, we use the given condition $f(3) = 16$, that is,

$k(3+1)(3-2)(3-1) = 8k = 16$. This implies $k = 2$.

Hence, the required polynomial function is,

$f(x) = 2(x+1)(x-2)(x-1) = 2x^3 - 4x^2 - 2x + 4$.

c. $x = 2,\ x = -3$ and $x = 0$ are the zeros of multiplicity 3, 2, and 2 respectively implies $(x-2)^3,\ (x+3)^2$ and x^2 are factors of $f(x)$.

Therefore, $f(x) = kx^2(x-2)^3(x+3)^2$ for some constant k. Using the given condition $f(1) = 10$,

$f(1) = k(1)^2(1-2)^3(1+3)^2 = -16k = 48$, this implies $k = -3$

$f(x) = -3x^2(x-2)^3(x+3)^2$

ASSESSMENT

You can use any one of assessment techniques such as: class activities, group discussions, homework/assignments and/or tests/quizzes.

Location Theorem

It is not always possible to find the zero of a polynomial function exactly. Using the continuous property of a polynomial function and if the polynomial satisfies the condition in the location theorem, it is possible to locate the interval containing the real zeros.

Let students do activity 2.10 so that they can recall that there is a polynomial function that has no real zero.

Answer for activity 2.10

a. The zeros of $f(x)$ are $0, -2$ and $\frac{3}{2}$ and they are rational numbers.

b. The zeros of $f(x)$ are $-\sqrt{2}$ and $\sqrt{3}$ and they are irrational numbers.

c. $f(x)$ has no zero in the set of real numbers.

State the location theorem and support it by doing the examples. then let students do exercise 2.17. You may ask students show their work writing on the board.

Answer for exercise 2.17

1. a. $f(-1) = -1$ and $f(1) = 1$, by the location theorem there is a zero of $f(x)$ between -1 and 1.

b. $f(-3) = -20$ and $f(-2) = 5$, by the location theorem there is a zero of $f(x)$ between -3 and -2.

2. a.

x	0	1	2	3	4	5	6
$f(x)$	-14	1	4	1	-2	1	16

The values of $f(x)$ changes sign from negative to positive between 0 and 1, from positive to negative between 3 and 4 and from negative to positive between 4 and 5. Snice $f(x)$ is a polynomial of degree 3, it has at most three zeros and by the location theorem these zeros are located between 0 and 1, between 3 and 4 and between 4 and 5.

b.

x	-3	-2	-1	0	1	2	3
$f(x)$	11	-2	3	2	-5	6	83

The values of $f(x)$ changes sign from positive to negative between -3 and -2, from negative to positive between -2 and -1, from positive to negative between 0 and 1 and from negative to positive between 1 and 2.

Snice $f(x)$ is a polynomial of degree 4, it has at most four zeros and by the location theorem these zeros are located between -3 and -2, between -2 and -1 and between 0 and 1 and between a and 2.

ASSESSMENT

You can use any one of assessment techniques such as: class activities, group discussions, homework/assignments and/or tests/quizzes.

Rational Root Test

For a polynomial function over the set of integers and $a_n \neq 0$ and $a_0 \neq 0$, there is a test to find the rational zeros. This test is called the Rational Root Test.

State the theorem (Rational Root Test) and support it by doing the examples. Allow students to participate whenever necessary.

Let students do exercise 2.18 and you can round for support. You can give exercise 2.16 d. for students as a home work.

Answer for exercise 2.18

a. $f(x) = x^2 - 5x + 4$

$f(x)$ has leading coefficient $a_2 = 1$ and constant term $a_0 = 4$.
Possible values of p are factors of 4. These are $\pm 1, \pm 2$ and ± 4.
Possible values of q are factors of 1. These are ± 1.
The possible rational zeros $\frac{p}{q}$ are $\pm 1, \pm 2$ and ± 4. Since $f(x)$ is a polynomial function of degree 2, it has at most 2 zeros, and from the six possible rational zeros at most 2 can be the zeros of f. Since $f(1) = 0$ and $f(4) = 0$, the zeros of $f(x)$ are 1 and 4.

b. $f(x) = -3x^3 + x^2 - 3x + 1$

$f(x)$ has leading coefficient $a_3 = -3$ and constant term $a_0 = 1$.

Possible values of p are factors of 1. These are ± 1 .

Possible values of q are factors of -3. These are ± 1 and ± 3.

The possible rational zeros $\frac{p}{q}$ are ± 1 and $\pm\frac{1}{3}$. Checking shows that only $f\left(\frac{1}{3}\right) = 0$. So, the given polynomial has $\frac{1}{3}$ as a rational zero.

c. $f(x) = x^3 - 3x^2 - x - 3$

$f(x)$ has leading coefficient $a_3 = 1$ and constant term $a_0 = -3$.

Possible values of p are factors of -3. These are ± 1 and ± 3 .

Possible values of q are factors of 1. These are ± 1.

The possible rational zeros $\frac{p}{q}$ are ± 1 and ± 3.

Checking shows that $f(-1) = f(1) = f(3) = -6 \neq 0$ and $f(-3) = -54 \neq 0$. So, we can conclude that the given polynomial has no rational zero.

d. $f(x) = 10x^3 - 41x^2 + 2x + 8$

$f(x)$ has leading coefficient $a_3 = 10$ and constant term $a_0 = 8$.

Possible values of p are factors of 8. These are $\pm 1, \pm 2, \pm 4$ and ± 8 .

Possible values of q are factors of 10. These are $\pm 1, \pm 2, \pm 5$ and ± 10 .

The possible rational zeros $\frac{p}{q}$ are $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{5}, \pm\frac{1}{10}, \pm 2, \pm\frac{2}{5}, \pm 4, \pm\frac{4}{5}, \pm 8$ and $\pm\frac{8}{5}$.

Checking shows that only $f\left(\frac{1}{2}\right) = f(4) = 0$. So, we can conclude that the given polynomial has $\frac{1}{2}$ and 4 as its rational zeros.

e. $f(x) = 4x^4 + x^3 - 8x^2 - 18x - 4$

$f(x)$ has leading coefficient $a_4 = 4$ and constant term $a_0 = -4$.

Possible values of p are factors of -4. These are $\pm 1, \pm 2, \pm 4$.

Possible values of q are factors of 4. These are $\pm 1, \pm 2, \pm 4$.

The possible rational zeros $\frac{p}{q}$ are $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm 2$ and ± 4.

Checking shows that only $f(2) = f\left(\frac{1}{4}\right) = 0$. So, we can conclude that the given polynomial has $\frac{1}{4}$ and 2 as its rational zeros.

f. $f(x) = -6x^5 + 17x^4 - 14x^3 + 4x - 1$

$f(x)$ has leading coefficient $a_5 = -6$ and constant term $a_0 = -1$.

Possible values of p are factors of -1. These are ± 1.

Possible values of q are factors of -6. These are $\pm 1, \pm 2, \pm 3$ and ± 6.

The possible rational zeros $\frac{p}{q}$ are $\pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}$ and $\pm \frac{1}{6}$.

Checking shows that only $f\left(-\frac{1}{2}\right) = f\left(\frac{1}{3}\right) = f(1) = 0$. So, we can conclude that the given polynomial has $-\frac{1}{2}$, $\frac{1}{3}$ and 1 as its rational zeros.

Assessment

You can give group assignment, individual assignment and test at this time. You can assess their learning by asking them to

find the zero of a polynomial function.

locate the interval containing the zero of a polynomial function.

find the rational zeros of a polynomial function whose coefficients are all integers.

2.5 Graphs of polynomial Function

Periods allotted: 1 period

Competencies

At the end of this sub-unit students will be able to:

- sketch the graph of a given polynomial function.
- describe the properties of the graphs of a given polynomial function.

Introduction

Basically, the graph of a polynomial function is a smooth continuous curve. In this sub-unit, how the sign of the leading coefficient and the degree of a polynomial function can determine the end behavior of the graph will mostly be treated.

Teaching Guide

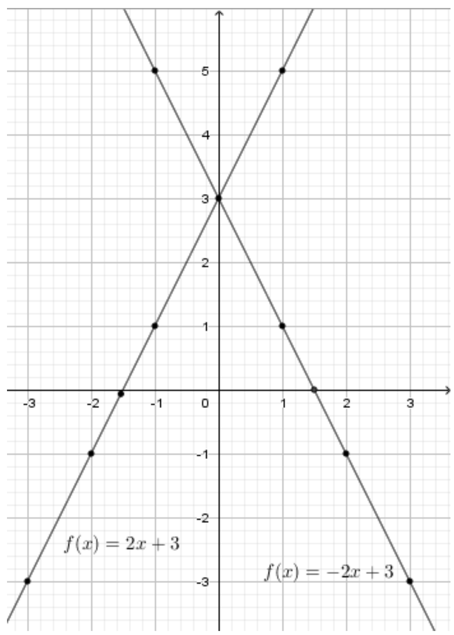
Support students to do activity 2.10 so that they can recall how to sketch the graph of a linear function. In addition to this support them to see how the leading coefficient determine the end behavior of the graph.

Answer for activity 2.11

1.

x	-3	-2	-3/2	-1	0	1
$f(x) = 2x + 3$	-3	-1	0	1	3	5

x	-1	0	1	3/2	2	3
$g(x) = -2x + 3$	5	3	1	0	-1	-3



2. **a.** odd

b. $x -$ intercept $= -\frac{b}{a}$ and $y -$ intercept $= b$.

c. Far to the right the graph moves upward and far to the left the graph moves downward.

d. Far to the right the graph moves downward and far to the left the graph moves upward.

e. Straight line.

f. Both are set of all real numbers.

While doing the examples for the students, allow students to participate in calculating the intercepts and the turning point and find some points that lie on the graph. Here also support students to tell the end behavior of the graph of the function by looking at the sign of the leading coefficient.

Allow students to do exercise 2.19 and exercise 2.20. The exercises enable students to practice sketching graphs of quadratic functions and observe the behavior of the graphs when the leading coefficient is positive and when it is negative.

Answer for exercise 2.19

$f(x) = x^2 + 4x + 3$

a. $f(x) = y = x^2 + 4x + 3 = (x+3)(x+1)$. By making $f(x) = y = 0$ and solving the equation $x^2 + 4x + 3 = (x+3)(x+1) = 0$, we get $x = -3$ or $x = -1$. By making $x = 0$ we have $y = 3$. Thus, $x = -3$ and $x = -1$ are the $x -$ intercepts and $y = 3$ is the y-intercept.

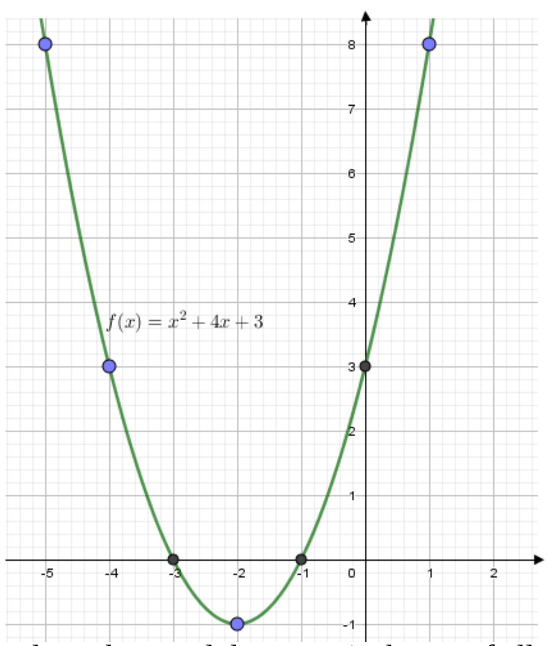
b. $y = x^2 + 4x + 3 = (x^2 + 4x + 4) + 3 - 4 = (x+2)^2 - 1 = -1 + (x+2)^2$

Since $(x+2)^2 \geq 0$ for all real numbers x, $f(x) \geq -1$ for all values of x and -1 is the minimum value of f. This minimum value of f is attained when $x = -2$. **The point $(-2, -1)$ is called turning point or vertex of the graph of f.**

c.

x	-5	-4	-3	-2	-1	0	1
$y = f(x)$	8	3	0	-1	0	3	8

d.



e. The domain is the set of all real numbers and the range is the set of all real numbers greater than or equal to -1.

Answer for exercise 2.20

1.

a. $f(x) = y = -x^2 + 6x - 8 = (x-2)(-x+4)$. By making $f(x) = 0$ and solving the equation $-x^2 + 6x - 8 = (x-2)(-x+4) = 0$, we get $x = 2$ or $x = 4$. By making $x = 0$ we have $y = -8$. Thus, $x = 2$ and $x = 4$ are the $x -$ intercepts and $y = -8$ is the y-intercept.

b. $y = f(x) = -x^2 + 6x - 8 = -(x^2 - 6x) - 8$

$= -(x^2 - 6x + 9) + 9 - 8$

$= -(x-3)^2 + 1$

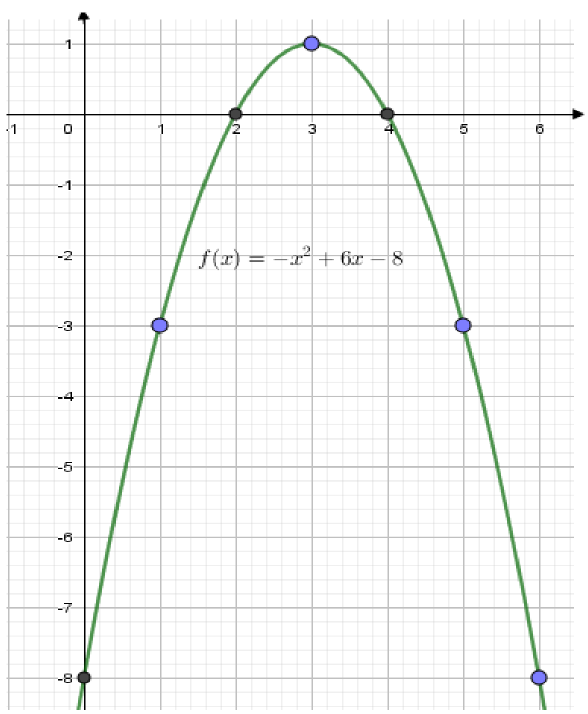
$= 1 - (x-3)^2$

Since $(x-3)^2 \geq 0$, $f(x) \leq 1$ for all values of x and 1 is the maximum value of f. This maximum value of f is attained when $x = 3$. The point $(3, 1)$ is called turning point for the graph of f.

c.

x	0	1	2	3	4	5
$y = f(x)$	-8	-3	0	1	0	-3

d.



2.

- **a.** 2, even
- **b.** 2
- **c.** c
- **d.** The graph goes upward both far to the right and far to the left
- **e.** The graph goes downward both far to the right and far to the left
- **f.** Yes
- **g.** All real numbers
- **h.** No, because both ends of the graph either go upward or downward the graph will turn back at its maximum or minimum value.

Discuss the examples to sketch graphs of a third-degree polynomial with the participation of students. Allow students to participate in calculating the intercepts and finding points that lie on the graph. Finally support students to make observation on the end behavior of the graph for large values of $|x|$.

After discussing the example allow students to do exercise 2.22. They can do this in group so that they can present the observation they have reached to the class by the group representative.

Answer for exercise 2.21

- **a.** 3
- **b.** 1
- **c.** 1
- **d.** 2
- **e.** The graph moves upward far to the right and moves downward far to the left.
- **f.** The graph moves downward far to the right and moves upward far to the left.
- **g.** All real numbers.
- **h.** All real numbers.
- **i.** Yes

At this point it is good to introduce to students a graph calculating software like GeoGebra. You can use the school computer laboratory for this purpose. You can also bring charts showing sample graphs of polynomial functions of higher degree. Using the chart you can show the students the turning points, the local maximum and local minimum points, intercepts, behaviors of the graph far to the right and far to the left of the graph of a polynomial function of different degree.

Answer for exercise 2.22

1.

a	i	Go upwards
	ii	Go upwards
	iii	At most two times
b	i	Go downwards
	ii	Go downwards
	iii	At most 4 times
c	i	Go downwards
	ii	Go upwards
	iii	At most three times
d	i	Go upwards
	ii	Go downwards
	iii	At most five times

2.

	Leading Coefficient	Degree
a	positive	even
b	negative	even
c	positive	odd
d	negative	odd

Assessment

You can ask students to tell the end behaviors of graphs polynomial functions and to identify whether a given graph is graph of a polynomial function or not. you can also ask students to sketch the graph of a polynomial function after listing all its properties down as an assignment.

2.6 Applications

Periods allotted: 2 periods

Competencies

At the end of this sub-unit students will be able to:

- apply polynomial function.

Introduction

In situations where we consider the area of a region and volume of a box, tank or other three-dimensional container, **polynomial functions frequently arise**. To develop a model function that represents a physical situation, we almost always begin by drawing one or more diagrams of the situation and then introduce one or more variables to represent quantities that are changing. From there, we explore relationships that are present and work to express one of the quantities in terms of the other(s).

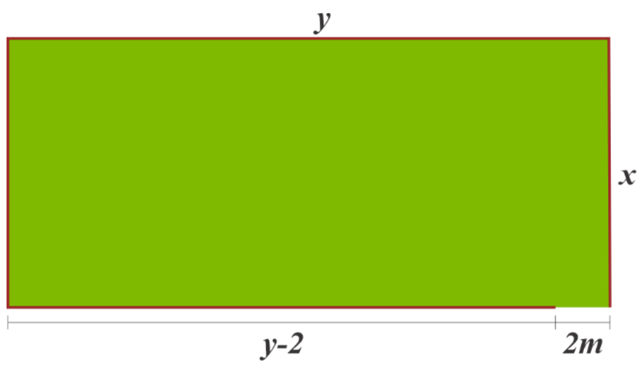
Answer for exercise 2.23

1.a. The perimeter of the rectangular enclosure is $2x + y + (y - 2)$.

Since the length of the fencing material is 100 meters

$$2x + y + (y - 2) = 100 \quad . \quad . \quad . \quad (*)$$

Area of the rectangular garden is:



$$A = xy \quad . \quad . \quad . \quad (**)$$

Solving for y from (*) and substituting the result in (**)

$$y = 51 - x$$

$$A(x) = 51x - x^2$$

b. $A(x) = 51x - x^2 = -x^2 + 51x = -(x^2 - 51x) = -(x^2 - 51x + \left(\frac{51}{2}\right)^2 - \left(\frac{51}{2}\right)^2)$

$$= -\left(x^2 - 51x + \left(\frac{51}{2}\right)^2\right) + \left(\frac{51}{2}\right)^2$$

$$= -\left(x - \frac{51}{2}\right)^2 + \left(\frac{51}{2}\right)^2 = \left(\frac{51}{2}\right)^2 - \left(x - \frac{51}{2}\right)^2$$

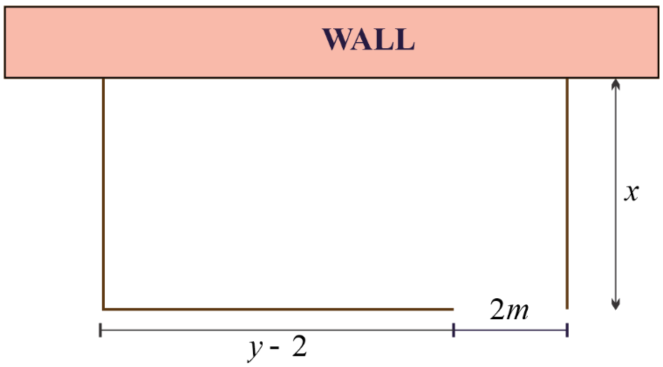
The area is maximum when $x = \frac{51}{2} = 25.5\ m$.

c. The maximum area is $\left(\frac{51}{2}\right)^2 = 650.25\ \text{sq. m}$.

2.a. The perimeter of the rectangular enclosure is $2x + y - 2$.

Since the length of the fencing material is 100 meters

$$2x + y - 2 = 100 \quad . \ . \ . \ (*)$$



Area of the rectangular garden is:

$$A = xy \quad . \ . \ . \ (**)$$

Solving for y from (*) and substituting the result in (**)

$$y = 102 - 2x$$

Hence, $A(x) = 102x - 2x^2$

b. $A(x) = 102x - 2x^2 = -2x^2 + 102x = -2(x^2 - 51x)$

$$= -2(x^2 - 51x + \left(\frac{51}{2}\right)^2 - \left(\frac{51}{2}\right)^2)$$

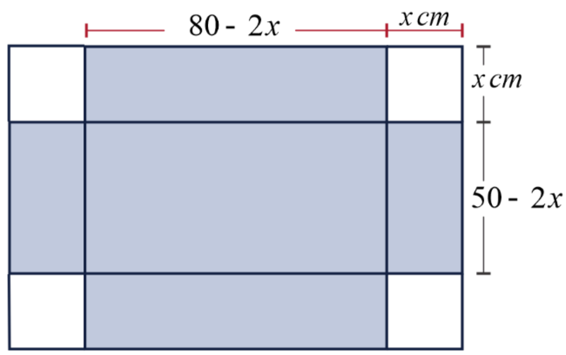
$$= -2\left(x^2 - 51x + \left(\frac{51}{2}\right)^2\right) + 2\left(\frac{51}{2}\right)^2$$

$$= -2\left(x - \frac{51}{2}\right)^2 + 2\left(\frac{51}{2}\right)^2 = 2\left(\frac{51}{2}\right)^2 - 2\left(x - \frac{51}{2}\right)^2$$

The area, A is maximum when $x = \frac{51}{2} = 25.5$ meter.

c. The maximum area is $2\left(\frac{51}{2}\right)^2 = 1300.50\ sq.m.$

3. An open-topped box is to be made by removing squares from each corner of a rectangular piece of card and folding up the sides.



a. The dimensions of the box are:

$$l = 80 - 2x,\ \ w = 50 - 20\ \ and\ h = x$$

The volume V of the box is,

$$V(x) = lwh = (80 - 2x)(50 - 2x)x = 4x^3 - 260x^2 + 4000x.$$

b. $V(20) = 8000\ sq.cm$

c. No, because $V(20) = -6000$ and volume cannot be a negative number.

ASSESSMENT

You can use any one of assessment techniques such as: class activities, group discussions, homework/assignments and/or tests/quizzes.

Answer for review Exercises on Unit 2

1. **a.** Polynomial function with degree 3, leading coefficient -2 and constant term -3.

b. Not polynomial function.

c. Polynomial function with degree 6, leading coefficient 3 and constant term 7.

d. Not polynomial function.

e. Polynomial function with degree 0, constant term $3\pi^2 + 4$.

2. **a.** Polynomial **b.** Polynomial **c.** Polynomial **d.** **Not** Polynomial

e. **Not** Polynomial **f.** **Not** Polynomial **g.** Polynomial

3. **a.** $hf + g = 4x^4 + 5x^3 - 5x^2 + 14x + 10$

b. $f - hg = -x^5 - 3x^4 + 6x^3 + 8x^2 + 6x + 10$

c. $fg = 3x^7 + x^6 - 13x^5 + 20x^4 - 20x^3 - 26x^2 - 8x - 12$

d. $\frac{g}{h} = x^3 - x^2 - 3x + 6 + \frac{-14}{x+2}$

4. **a.** quotient $= 8x - 15$, remainder $= 9$

b. quotient $= 4x - 5$, remainder $= 8x - 22$

c. quotient $= 2x^2 + 2x + 5$, remainder $= 3x - 11$

d. quotient $= x^4 + x^3 - x + 4$, remainder $= -15$

e. quotient $= x^2 + 2x - 1$, remainder $= 4x^2 - 3x - 1$

f. quotient $= x^2 - x + 1.5$, remainder $= -2.5$

5. By the Polynomial Division Theorem there exist the quotient polynomial $q(x)$ and the remainder polynomial $r(x)$ such that $f(x) = (ax + b)q(x) + r(x)$ for all real number x.

$$f(x) = (ax + b)q(x) + r(x)$$

$$f\left(\frac{-b}{a}\right) = \left(a\left(\frac{-b}{a}\right) + b\right)q(x) + r(x) = (-b + b)q(x) + r(x) = r(x)$$

$$r(x) = f\left(\frac{-b}{a}\right)$$

6. **a.** By the remainder theorem the remainder is $f(-1) = (-1)^n + 1 = \begin{cases} 2, & \text{for even } n \\ 0, & \text{for odd } n \end{cases}$

Hence the remainder is zero when n is odd integer.

b. If $f(-1) = 0$ then by the factor theorem $\big(x - (-1)\big) = x + 1$ is a factor of f.

7. **a.** $(x - 5)(x - 1)(x + 2)$ **b.** $(x - 1)^2(2x - 1)(x + 2)$ **c.** $(x + 1)^2(2x^2 + 1)(x - 1)$

8. **a.** $k = -20$ **b.** $k = -6$ **c.** $k = \frac{-7}{3}$

9. $a = \frac{5}{4}$ and $b = -\frac{5}{4}$

10. $p = -8$ and $q = 4$

11. $a = -\frac{1}{2}$ and $k = -\frac{1}{3}$

12. $f(x) = 2x(x + 1)(x + 2) = 2x^3 + 6x^2 + 4x$

13. **a.** $f(x) = \frac{-5}{7}x^3 + \frac{20}{7}x^2 + \frac{55}{7}x - \frac{150}{7}$ **b.** $f(x) = \frac{-3}{4}x^4 + \frac{17}{8}x^3 + \frac{5}{8}x^2 - \frac{3}{4}x$

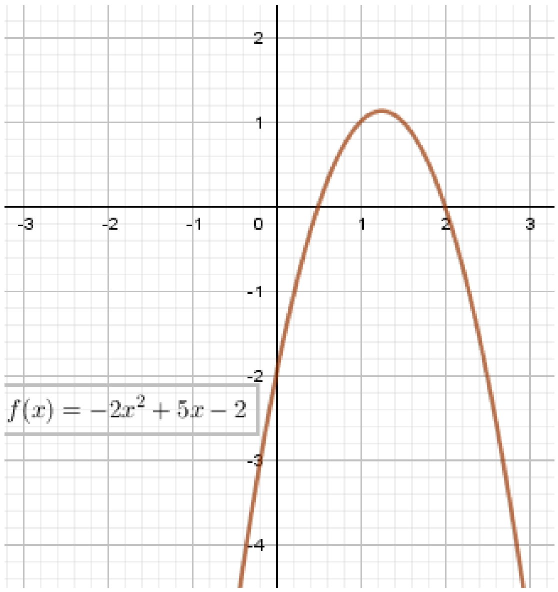
14. **a.** $-\frac{1}{2}, -\frac{1}{3}$ and 2

b. $\frac{1}{3}$ and $\frac{3}{2}$

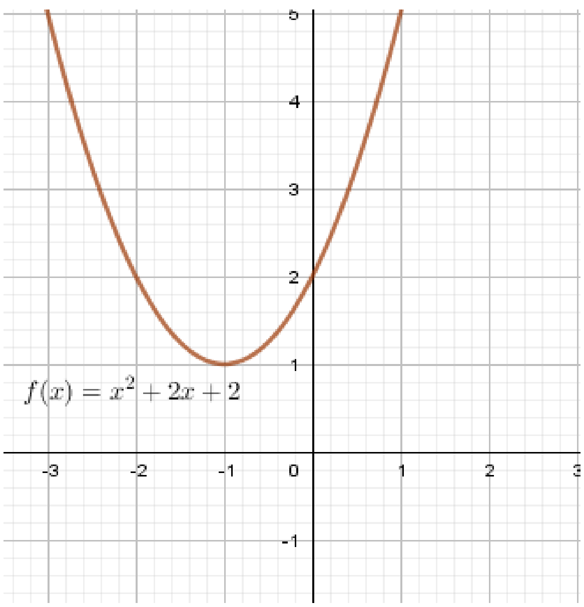
c. $-1, -\frac{2}{3}$ and $\frac{1}{2}$

d. $-2, -\frac{1}{3}$ and $\frac{1}{2}$

15. a



b.



Unit 3

Exponential and Logarithmic Functions

Periods allotted: 30 periods

Introduction

This unit begins by revising the properties of exponents discussed in grade 9. The goal of this unit is to discuss on the concept and properties of exponential and logarithm functions and their graphs. Finally, students are encouraged to solve problems on practical application of exponential and logarithmic functions from different fields such as population growth, compound interest, etc.

Unit outcomes

At the end of this unit the students will be able to:

- apply the laws of exponents for real exponents.
- define exponential and logarithmic functions.
- identify domain and range of exponential and logarithmic functions.
- solve mathematical problems involving exponents and logarithms.

Suggested Teaching Aids in Unit 2

- Graphs of exponential and logarithmic functions drawn on a big drawing paper. You can fix it on the wall so that students can have a better look and able to compare different graphs.
- Enlarged table of logarithm. Fix it to the wall whenever you teach how to use logarithm table.
- Calculators.

3.1 Exponents and logarithms

Periods allotted: 6 periods

Competencies

At the end of this sub-unit students will be able to:

- explain what is meant by exponential expression.

- state and apply the properties of exponents (where the exponents are real numbers).
- express what is meant by logarithmic expression by using the concept of exponential expression.
- solve simple logarithmic equation by using the properties of logarithm.
- recognize the advantage of using logarithm to the base 10 in calculation.
- identify the "characteristics" and "mantissa" of a given common logarithm.
- use the table for finding logarithm of a given positive number and antilogarithm of a number.
- compute using logarithm.

Introduction

This subunit mainly focuses on exponent and logarithms of numbers and their laws. Students are encouraged to apply the laws of exponents and logarithms to simplify exponential and logarithmic expression. Next, the common logarithm and its advantage will be introduced to students. At the end of the subunit, how to use the table of logarithm to find the logarithm and antilogarithm of a given number are discussed.

Teaching Note

3.1.1 Exponents

Introduce to students the meaning of multiplying a number by itself repeatedly. For example,

$2 \times 2 \times 2 \times 2 \times 2 = 32$.

In the above example, we multiply 2 five times and we get the product 32 and we denote it by 2^5.

Therefore $2^5 = 32$.

Example: $3^4 = 3 \times 3 \times 3 \times 3 = 81$

$5^3 = 5 \times 5 \times 5 = 125$

If n is a positive integer, then $a^n = \underbrace{a \times a \times a \times \ldots \times a}_{\text{n factors}}$

At this point let students do activity 3.1 so that they can revise the terms such as power, base and exponent of an exponential expression a^n for a positive integer n.

Answer for activity 3.1

1.

- **a.** The base is 3; the exponent is 4.
- **b.** The base is -3; the exponent is 4.
- **c.** The base is $\frac{3}{5}$; the exponent is 5.
- **d.** The base is -1; the exponent is 9.

2.

- **a.** $(-1)^1 = -1$
- **b.** $(-1)^4 = (-1) \times (-1) \times (-1) \times (-1) = 1$
- **c.** $\left(\frac{3}{5}\right)^1 = \frac{3}{5}$
- **d.** $(-2)^7 = (-2) \times (-2) \times (-2) \times (-2) \times (-2) \times (-2) \times (-2) = -128$
- **e.** $-2^4 = -(2 \times 2 \times 2 \times 2) = -16$
- **f.** $(-2)^4 = (-2) \times (-2) \times (-2) \times (-2) = 16$
- **g.** $\left(-\frac{2}{3}\right)^4 = \left(-\frac{2}{3}\right) \times \left(-\frac{2}{3}\right) \times \left(-\frac{2}{3}\right) \times \left(-\frac{2}{3}\right) = \frac{16}{81}$

After stating definition 3.1, explain the different laws of exponent when the exponent is positive integers. Support the laws with examples. Allow students to participate in the discussion.

Answer for exercise 3.1

1. **a.** $4^2 = 4 \times 4 = 16$ **b.** $-4^2 = -(4 \times 4) = -16$ **c.** $(-4)^2 = -4 \times -4 = 16$

d. $-(-3)^3 = -(-3 \times -3 \times -3) = -(-27) = 27$

2. **a.** $a^3 \times a^2 = a^{3+2} = a^5$ **b.** $\frac{a^5}{a^2} = a^{5-2} = a^3$ **c.** $(a^2)^3 = a^{2\times 3} = a^6$

d. $(2a)^4 = 2^4 a^4 = 16a^4$ **e.** $\left(\frac{a}{2}\right)^2 = \frac{a^2}{2^2} = \frac{a^2}{4}$

The goal of activity 3.2 is to find the value of a^n for $n = 0$ and generalize $a^0 = 1$, for $a \neq 0$.

Answer for activity 3.2

a) $1 = \frac{9}{9} = \frac{3^2}{3^2} = 3^{2-2} = 3^0$. Hence, $(-3)^0 = 1$.

b) $1 = \frac{9}{9} = \frac{(-3)^2}{(-3)^2} = (-3)^{2-2} = (-3)^0$. Hence, $(-3)^0 = 1$.

c) $1 = \frac{0.01}{0.01} = \frac{(0.1)^2}{(0.1)^2} = (0.1)^{2-2} = (0.1)^0$. Hence $(0.1)^0 = 1$.

State definition 3.2, so that students can work with negative exponents.

Answer for exercise 3.2

a. $4^0 = 1$ **b.** $(-11)^0 = 1$ **c.** $\left(\frac{22}{55}\right)^0 = 1$

d. $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ **e.** $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$ **f.** $\left(\frac{3}{4}\right)^{-3} = \frac{1}{\left(\frac{3}{4}\right)^3} = \frac{1}{\left(\frac{27}{64}\right)} = 1 \div \frac{27}{64} = 1 \times \frac{64}{27} = \frac{64}{27}$

g. $\left(\frac{1}{2}\right)^{-5} = \frac{1}{\left(\frac{1}{2}\right)^5} = \frac{1}{\left(\frac{1}{32}\right)} = 1 \div \frac{1}{32} = 1 \times 32 = 32$

At this point you can discuss with students the lows of exponents for integer exponents and discuss the examples in students' text. You can select and give problems from exercise 3.3. You can move among the students to observe their work and give support.

Answer for exercise 3.3

a) $x^{-3} \times x^2 = x^{-3+2} = x^{-1} = \frac{1}{x}$

b) $(4y)^2 \times (8y)^{-3} = (4^2y^2) \times (8^{-3}y^{-3}) = (2^4 \times 2^{-9})(y^2 \times y^{-3}) = 2^{-5}y^{-1} = \frac{1}{32y}$

c) $2^t \times 2^{3t} \times 2^{2t} = (2^t \times 2^{3t}) \times 2^{2t} = 2^{t+3t} \times 2^{2t} = 2^{t+3t+2t} = 2^{6t} = 64^t$

d) $\frac{3^{-5}}{3^{-3}} = 5^{-5+3} = 5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

e) $\frac{(2x)^2}{(2x)^4} = (2x)^{2-4} = (2x)^{-2} = 2^{-2} \times x^{-2} = \frac{1}{2^2} \times \frac{1}{x^2} = \frac{1}{4x^2}$

f) $\frac{(-3x)^2}{(-3x)^4} = (-3x)^{2-4} = (-3x)^{-2} = (-3)^{-2} \times x^{-2} = \frac{1}{(-3)^2} \times \frac{1}{x^2} = \frac{1}{9x^2}$

g) $(3^2)^{2n} = 3^{2\times 2n} = 3^{4n} = 81^n$

h) $(a^y)^{-1} = a^{y\times -1} = a^{-y} = \frac{1}{a^y}$

i) $(a^{3x})^4 = a^{(3x)\times 4} = a^{12x}$

j) $(2a^{-3} \times b^2)^{-2} = (2a^{-3})^{-2} \times (b^2)^{-2} = 2^{-2} \times a^{-3\times -2} \times b^{2\times -2} = \frac{1}{4}a^6b^{-4}$

k) $\frac{(a^2)^{-3} \times (a^3)^4}{a^{10}} = \frac{a^{-6} \times a^{12}}{a^{10}} = \frac{a^{-6+12}}{a^{10}} = \frac{a^6}{a^{10}} = \frac{1}{a^{10-6}} = \frac{1}{a^4}$

l) $\left(\frac{m^{-5}n^2}{n^{-2}m^6}\right)^{-2} == \frac{m^{10}n^{-4}}{n^4m^{-12}} = \frac{m^{10}}{m^{-12}} \times \frac{n^{-4}}{n^4} = \frac{m^{22}}{n^8}$

Next discuss with students how we could extend rule of exponent to rational exponent and state definition 3.3 and definition 3.4.

Answer for exercise 3.4

a. Since $3^4 = 81$, $\sqrt[4]{81} = (81)^{\frac{1}{4}} = 3$

b. Since $2^5 = 32$, $\sqrt[5]{32} = (32)^{\frac{1}{5}} = 2$

c. Since $5^3 = 125$, $\sqrt[3]{125} = (125)^{\frac{1}{3}} = 5$

d. Since $-3^3 = -27$, $\sqrt[3]{-27} = (-27)^{\frac{1}{3}} = -3$

e. Since $(-10)^3 = -1000$, $-\sqrt[3]{1000} = -(-1000)^{\frac{1}{3}} = -(-10) = 10$

f. $\sqrt[4]{-10000} = (-10000)^{\frac{1}{4}}$ is not a real number because there is no number a such that a^4 is -10000

Answer for activity 3.3

a) $2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^{\frac{1}{2}+\frac{1}{2}} = 2^1 = 2$

b) $\sqrt{2} \times \sqrt{2} = 2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^{\frac{1}{2}+\frac{1}{2}} = 2^1 = 2$

c) $2^{\frac{1}{2}} \times 2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}} = 2^{\frac{3}{2}}$

d) $\sqrt{2} \times \sqrt{2} \times \sqrt{2} = 2^{\frac{1}{2}} \times 2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}} = 2^{\frac{3}{2}}$

Once the students have understood the laws of exponents, you can now let them apply the laws of exponent in simplifying exponential expressions. For this let students do exercise 3.5. You can round in the class and give support.

Answer for exercise 3.5

a) $\left(2^{\frac{1}{3}} \times 4^{\frac{4}{3}}\right)^{-3} = (2^{\frac{1}{3}})^{-3} \times (4^{\frac{4}{3}})^{-3} = 2^{-1} \times 4^4 = 2^{-1} \times 2^8 = 2^7$

b) $\frac{16^{\frac{2}{5}}}{8^{\frac{6}{5}}} = \frac{(2^4)^{\left(\frac{2}{5}\right)}}{(2^3)^{\left(\frac{6}{5}\right)}} = \frac{2^{\left(\frac{8}{5}\right)}}{2^{\left(\frac{18}{5}\right)}} = \frac{1}{2^{\left(\frac{10}{5}\right)}} = \frac{1}{2^2} = \frac{1}{4}$

c) $100^{\frac{3}{2}} = 10^{2\left(\frac{3}{2}\right)} = 10^3 = 1000$

d) $\left(a^{\frac{1}{4}} \times 3b^{\frac{5}{2}}\right)^4 = \left(a^{\frac{1}{4}}\right)^4 \times \left(3b^{\frac{5}{2}}\right)^4 = a^{\frac{1}{4}\times 4} \times 3^4 b^{\frac{5}{2}\times 4} = a \times 81b^{10} = 81ab^{10}$

e) $(-27)^{-\frac{2}{3}} = ((-3)^3)^{-\frac{2}{3}} = (-3)^{3\times -\frac{2}{3}} = (-3)^{-2} = \frac{1}{(-3)^2} = \frac{1}{9}$

f) $\left(\frac{x^{\frac{2}{3}}}{y^{-\frac{1}{2}}}\right)^{-6} = \frac{1}{x^4y^3}$

g) $\frac{(a^2)^{-\frac{1}{4}}\times(a^3)^{\frac{2}{9}}}{a^{\frac{1}{2}}} = \frac{a^{-\frac{1}{2}}\times a^{\frac{2}{3}}}{a^{\frac{1}{2}}} = \frac{a^{\frac{1}{6}}}{a^{\frac{1}{2}}} = \frac{1}{a^{\frac{1}{3}}}$

h) $\left(\frac{m^{\frac{1}{8}}n^{-\frac{1}{3}}}{n^{\frac{2}{3}}m^{\frac{1}{4}}}\right)^4 = \frac{1}{m^{\frac{1}{2}}n^4}$

Discuss the relationship between radicals and rational exponents by stating definition 3.5. Explain the meaning of irrational exponents to students. Discuss the examples given in the students' textbook with the students. Allow students to participate in the discussion.

Answer for exercise 3.6

a) $\sqrt[3]{81} = 81^{\frac{1}{3}} = 3^{4\times\frac{1}{3}} = 3^{\frac{4}{3}}$ **b)** $\sqrt[4]{32} = 32^{\frac{1}{4}} = 2^{5\times\frac{1}{4}} = 2^{\frac{5}{4}}$ **c)** $\frac{\sqrt[3]{25}}{\sqrt[3]{5}} = \frac{25^{\frac{1}{3}}}{5^{\frac{1}{3}}} = \left(\frac{25}{5}\right)^{\frac{1}{3}} = 5^{\frac{1}{3}}$

d) $\frac{\sqrt[5]{40}}{\sqrt[5]{5}} = \frac{40^{\frac{1}{5}}}{5^{\frac{1}{5}}} = \left(\frac{40}{5}\right)^{\frac{1}{5}} = 8^{\frac{1}{5}} = 2^{\frac{3}{5}}$ **e)** $\left(\sqrt[4]{27}\right)^6 = \left(27^{\frac{1}{4}}\right)^6 = \left(3^{\frac{3}{4}}\right)^6 = 3^{\frac{9}{2}}$

f) $\left(\sqrt[3]{121}\right)^2 = \left(121^{\frac{1}{3}}\right)^2 = \left(11^{\frac{2}{3}}\right)^2 = 11^{\frac{4}{3}}$ **g)** $\sqrt[3]{\sqrt[2]{\frac{1}{1000000}}} = \left(\left((10^{-6})\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} = 10^{-1}$

Answer for exercise 3.7

a) $2^{\sqrt{3}} \times 2^{\sqrt{3}} = 2^{2\sqrt{3}}$ **b)** $\left(5^{\sqrt{2}}\right)^2 = 5^{2\sqrt{2}}$ **c)** $\left(\sqrt[3]{8}\right)^{-2} = \frac{1}{4}$ **d)** $\sqrt{3}^{\sqrt{2}} \times \sqrt{3}^{\sqrt{8}} = \sqrt{3}^{3\sqrt{2}}$

e) $\frac{3^{\sqrt{2}+3}}{3^{\sqrt{2}-1}} = 3^2 = 9$ **f)** $\left(2^{\sqrt{3}}\right)^{\sqrt{27}} = 2^9 = 512$ **g)** $\frac{3^{\sqrt{2}}\times 9^{\sqrt{8}}}{27^{\sqrt{18}}} = \frac{1}{3^{4\sqrt{2}}} = \frac{1}{81^{\sqrt{2}}}$

h) $\frac{(5^{\sqrt{3}})^2\times 5^{-\sqrt{12}}\times 25^{\sqrt{3}}}{5^{\sqrt{27}}} = \frac{1}{5^{\sqrt{3}}}$

ASSESSMENT

At the end of this lesson, you can give class activities, home-work, assignment, test or quiz to assess their level of understanding. The assessment should include the types of problems that enable you to know slow, medium and fast learners so that you may arrange appropriate support for those in need.

3.1.2 Logarithms

In the exponential equation, $2^3 = 8$, the base is 3 and the exponent is 2. We write this equation in practice converting exponential statements to logarithmic statements and vice versa by practicing on examples given on the students' textbook.

After this, give time to the students to do exercise 3.4. You could look their work and give support at this time. Finally, allow some students to show their work on the board.

Answer for exercise 3.8

1) **a)** $2^{10} = 1{,}024$ if and only if $\log_2 1{,}024 = 10$

b) $2^{-6} = \frac{1}{64}$ if and only if $\log_2 \frac{1}{64} = -6$

c) $\sqrt[3]{125} = 5$ if and only if $\log_{125} 5 = \frac{1}{3}$

d) $27^{-\frac{2}{3}} = \frac{1}{9}$ if and only if $\log_{27} \frac{1}{9} = -\frac{2}{3}$

2) **a)** $\log_{10} 1000 = 3$ if and only if $10^3 = 1{,}000$

b) $\log_8 \sqrt{64} = 1$ if and only if $8^1 = \sqrt{64}$

c) $\log_{10} 0.001 = -3$ if and only if $10^{-3} = 0.001$

d) $\log_3 \frac{1}{27} = -3$ if and only if $3^{-3} = \frac{1}{27}$

3) **a)** $\log_3 27$

Let $\log_3 27 = x$. Then $3^x = 27 = 3^3$ i.e., $3^x = 3^3$ if and only if $x = 3$.

So, $\log_3 27 = 3$.

b) $\log_4 16$

Let $\log_4 16 = y$. Then $4^y = 16 = 4^2$ i.e., $4^y = 4^2$ if and only if $y = 2$.

So, $\log_4 16 = 2$.

c) $\log_{100} 0.001$

Put $t = \log_{100} 0.001$ so that $100^t = 0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$ that is.

$100^t = 10^{2t} = 10^{-3}$ if and only if $2t = -3$ which implies $t = -\frac{3}{2}$.

Hence, $\log_{100} 0.001 = -\frac{3}{2}$.

d) $\log_{\sqrt{49}} 7$

Let $\log_{\sqrt{49}} 7 = m$. Then $\sqrt{49}^m = 7$

But, $\sqrt{49}^m = \left[(49)^{\frac{1}{2}}\right]^m = \left[(7^2)^{\frac{1}{2}}\right]^m = 7^m$

So, $7^m = 7 = 7^1$. i.e., $7^m = 7^1$ if and only if $m = 1$

Consequently, $\log_{\sqrt{49}} 7 = 1$

Based on the Definition 3.6, show to students the following properties of logarithms

1. $\log_a 1 = 0$ because $a^0 = 1$.
2. $\log_a a = 1$ because $a^1 = a$.
3. $\log_a a^x = x$ and $a^{\log_a x} = x...$ Inverse property
4. If $\log_a x = \log_a y$, then $x = y...$ One-to-One property

Discuss the examples with students. Allow students to participate in the discussion.

Answer for exercise 3.9

a) $\log_{12} 144 = \log_{12} 12^2 = 2\log_{12} 12 = 2$

b) $\log_8 8 = 1$

c) $\log_{10} 100{,}000 = \log_{10} 10^5 = 5\log_{10} 10 = 5$

d) $\log_6 216 = \log_6 6^3 = 3\log_6 6 = 3$

e) $\log_2 \sqrt{2} = \log_2 2^{\frac{1}{2}} = \frac{1}{2}\log_2 2 = \frac{1}{2}$

f) $\log_8 2 = \frac{\log_2 2}{\log_2 8} = \frac{1}{\log_2 2^3} = \frac{1}{3\log_2 2} = \frac{1}{3}$

g) $\log_{100} \sqrt[6]{100} = \log_{100} 100^{\frac{1}{6}} = \frac{1}{6}\log_{100} 100 = \frac{1}{6}$

h) $\log_{\frac{1}{3}} 27 = \frac{\log_3 27}{\log_3 \frac{1}{3}} = \frac{\log_3 3^3}{\log_3 3^{-1}} = \frac{3\log_3 3}{-1\log_3 3} = \frac{3}{-1} = -3$

Laws of logarithms

State Theorems 3.1(Lagarithm of product). The proof is based on definition 3.6 and product rule for exponent. Allow students to participate while discussing on the poof by giving reason for each step.

State Theorems 3.2(**Logarithms of powers**). The proof is based on definition 3.6 and power rule for exponents. Allow students to particiapate by indicating where they actually use definition 3.6 and power rule for exponents in proving the theorem.

After proving the theorems discuss on the examples so that students can have better understanding the product and power rules of logarithm.

Answer for exercise 3.10

a) $\log_3 \sqrt{3} = \log_3 3^{\frac{1}{2}} = \frac{1}{2}\log_3 3 = \frac{1}{2}$ **b)** $\log_6 36 = 2$

c) $\log_2\left(\frac{1}{4}\right) = \log_2 2^{-2} = -2$

d) $\log_{\left(\frac{1}{3}\right)}\left(\frac{1}{81}\right) = \log_{\left(\frac{1}{3}\right)}\left(\frac{1}{3}\right)^4 = 4$

e) $\log_{10}\sqrt[3]{\frac{1}{1000}} = \log_{10}\frac{1}{10} = -1$

f) $\log_8 32 + \log_8 2 = \log_8(32 \times 2) = \log_8 64 = 2$

g) $\log_2 6 + \log_2\left(\frac{1}{12}\right) = \log_2(6 \times \frac{1}{12}) = \log_2\left(\frac{1}{2}\right) = -1$

h) $\log_3 10 + \log_3\left(\frac{6}{5}\right) + \log_3\left(\frac{9}{4}\right) = \log_3(10 \times \frac{6}{5} \times \frac{9}{4}) = 3$

i) $\frac{1}{2}\log_4 8 + \log_4 \sqrt{2} = 1$

State Theorems 3.3(**Logarithms of quotients**). The proof is based on definition 3.6 and the quotient rule for exponents. Allow students to particiapate by indicating where they actually use definition 3.6 and the quotient rule for exponents in proving the theorem.

After proving the theorem discuss on the examples so that students can have better understanding the **quotient** rule of logarithm.

Answer for exercise 3.11

a) $\log_5 50 - \log_5 2 = \log_5\left(\frac{50}{2}\right) = \log_5 25 = \log_5 5^2 = 2\log_5 5 = 2 \times 1 = 2$

b) $\log_3 4 - \log_3 108 = \log_3\left(\frac{4}{108}\right) = \log_3\left(\frac{1}{27}\right) = \log_3 3^{-3} = -3 \times \log_3 3 = -3 \times 1 = -3$

c) $\log_{10}\sqrt{2000} - \log_{10}\sqrt{2} = \log_{10}\left(\frac{\sqrt{2000}}{\sqrt{2}}\right) = \log_{10}\sqrt{1000} = \frac{3}{2}$

d) $\log_5 2 + \log_5 50 - \log_5 4 = (\log_5 2 + \log_5 50) - \log_5 4 = \log_5 \frac{2\times 50}{4} = \log_5 25 = 2$

e) $\log_6 9 - \log_6 15 + \log_6 10 = (\log_6 9 - \log_6 15) + \log_6 10 = \log_6\left(\frac{9}{15} \times 10\right) = \log_6 6 = 1$

f) $\log_{10} 24 - 2\log_{10} 6 + \log_{10} 15 = 1$

State Theorems 3.4(change of base). The proof is based on definition 3.6. Allow students to particiapate in proving the theorem. After proving the theorem discuss on the examples so that students can have better understanding change of base rule for logarithm.

After proving the theorem discuss on the examples so that students can have better understanding the **change of base** rule of logarithm.

Answer for exercise 3.12

a) $\log_{\sqrt{2}} 128 = \frac{\log_2 128}{\log_2 \sqrt{2}} = \frac{7\log_2 2}{\frac{1}{2}\log_2 2} = 14$

b) $\log_{\left(\frac{1}{3}\right)} 243 = \frac{\log_3 243}{\log_3\left(\frac{1}{3}\right)} = \frac{5\log_3 3}{-\log_3 3} = -5$

c) $\log_{100} 0.1 = \frac{\log_{10} 0.1}{\log_{10} 100} = \frac{-\log_{10} 10}{2\log_{10} 10} = -\frac{1}{2}$

d) $\log_4\left(\frac{1}{2}\right) = \frac{\log_2\left(\frac{1}{2}\right)}{\log_2 4} = \frac{-\log_2 2}{2\log_2 2} = -\frac{1}{2}$

e) $5^{\log_5 10} = 10$

f) $6^{\log_6 7} = 7$

Tell students to give attenstion for the following

1. $\log_a xy \neq (\log_a x)(\log_a y)$ . . . The logarithm of a product is not the product of the logarithms.
2. $\log_a(x+y) \neq \log_a x + \log_a y$ . . . The logarithm of a sum is not the sum of the logarithms.
3. $\log_a\left(\frac{x}{y}\right) \neq \frac{\log_a x}{\log_a y}$ . . . The logarithm of a quotient is not the quotient of the logarithms.
4. $(\log_a x)^r \neq r\log_a x$ . . . The power of a logarithm is not the exponent times the logarithm.

ASSESSMENT

At the end of this lesson, you can give class activities, home-work, assignment, test or quiz to assess their level of understanding. The assessment should include the types of problems that enable you to know slow, medium and fast learners so that you may arrange appropriate support for those in need.

Logarithm to the base 10 (Common logarithm)

Students should note that the logarithm of any positive number to the base 10 is called a **common logarithm**. They should also note that a common logarithm is witten without indicating its base. For example $\log_{10} x$ is simply denoted by $\log x$.

Let students do activity 3.4 and discuss the examples with students, so that they can evalute the value of the logarithm of a number to the base 10.

Answer for activity 3.4

a) $\log 10 = \log_{10} 10 = 1$

b) $\log_{10} 1000 = 3\log_{10} 10 = 3$

c) $\log_{10} 1 = = \log_{10} 10^0 = 0 \log_{10} 10 = 0$

d) $\log_{10} 0.1 = -1$

Discuss the examples with students so that they could have better understanding of identifying the characterstic and mantissa of a given common logarithm.

Teaching Aid Required

- common logarithim table

Answer for Exercise 3.13

1. **a.** $\log \sqrt[3]{0.1} = \log 10^{\left(-\frac{1}{3}\right)} = -\frac{1}{3}\log 10 = -\frac{1}{3}$

b. $\log_{10}(10\sqrt{10}) = \log_{10} 10^{\frac{3}{2}} = \frac{3}{2}\log_{10} 10 = \frac{3}{2}$

c. $\log\left(\frac{0.01}{\sqrt{1000}}\right) = \log 0.01 - \log\sqrt{1000} = -2 - \frac{3}{2} = -\frac{7}{2}$

d. $\log\left(\frac{1}{\sqrt[5]{10}}\right) = \log 10^{-\frac{1}{5}} = -\frac{1}{5}\log 10 = -\frac{1}{5}$

e. $\log\left(\frac{10^m}{10^n}\right) = \log 10^m - \log 10^n = m\log 10 - n\log 10 = m - n$

2. **a.** $0.00503 = 5.03 \times 10^{-3}$. So, the characterstic is -3 and the mantisa is $\log 5.03$.

b. $0.25 = 2.5 \times 10^{-1}$; the characterstic is -1 and the mantisa is $\log 2.5$.

c. $302 = 3.02 \times 10^2$. So, the characterstic is 2 and the mantisa is $\log 3.02$.

d. $\frac{1}{8} = 0.125 = 1.25 \times 10^{-1}$; the characterstic is -1 and the mantisa is $\log 1.25$.

e. $4.4 = 4.4 \times 10^0$. Therefore, the characterstic is 0 and the mantisa is $\log 4.4$.

f. $9 = 9 \times 10^0$; the characterstic is 0 and the mantisa is $\log 9$.

g. $3280 = 3.28 \times 10^3$; the characterstic is 3 and the mantisa is $\log 3.28$.

h. $53.814 = 5.3814 \times 10^1$; the characterstic is 1 and the mantisa is $\log 5.3814$.

Show students how to use common logarithm table to find the value of a given common logarithm of a number between 1 and 10. You can use a relatively larger logarithmic table, fixing it on the wall, as a teaching aid while you are teaching how to read the logarithm of a number from the common logarithm table.

Answer for Exercise 3.14

a) Reading the number in row 2.1 under column 3, gives 0.3284; $\log 2.13 = 0.3284$.

b) Reading the number in row 2.9 under column 9, gives 0.4757; $\log 2.99 = 0.4757$.

c) Reading the number in row 6.3 under column 0, gives 0.7993; $\log 6.39 = 0.7993$.

d) Read the number at the intersection of row 6.3 and column 4, this gives 0.8021.

From the mean difference part of the common logarithm table, read the number at the intersection of row 6.3 and column 5, this gives 0.0003.

Adding the two results gives 0.8024; $\log 6.345 = 0.8024$.

e) Since, $0.28 = 2.8 \times 10^{-1}$, we have

$\log 0.28 = \log(2.80 \times 10^{-1}) = \log 2.80 + \log 10^{-1} = \log 2.80 - 1$.

Reading the number in row 2.8 under column 0, gives 0.4472; $\log 2.8 = 0.4472$.

So, $\log 0.28 = \log 2.80 - 1 = 0.4472 - 1 = -0.5528$.

f) Reading the number in row 9.9 under column 9, gives 0.9996 ; $\log 9.99 = 0.9996$

g) Since, $0.00008 = 8.00 \times 10^{-5}$, we have

$\log 0.00008 = \log(8.00 \times 10^{-5}) = \log 8.00 + \log 10^{-5} = \log 8.00 - 5$

Reading the number in row 8.0 under column 0, gives 0.9031 ; $\log 8.00 = 0.9031$

So, $\log 0.00008 = \log 8.00 - 5 = 0.9031 - 5 = -4.0969$.

h) Since, $400 = 4.00 \times 10^{2}$, we have

$$\log 400 = \log(4.00 \times 10^{2}) = \log 4.00 + \log 10^{2} = \log 8.00 + 2$$

Reading the number in row 4.0 under column 0, gives 0.6021 ; $\log 4.00 = 0.6021$

So, $\log 400 = \log 4.00 + 2 = 0.6021 + 2 = 2.6021$.

Antilogarithms

Teaching Aid Required

- antilogarithm table

After explaining the meaning of antilogarithm of a number, disuss the examples with students and show them how to find antilogarithm of a given numbe by using the antilogarithm table.

Answer for Exercise 3.15

a) Reading the number at the intersection of row 0.74 and Column 1 from the antilogarithm table gives 5.508. From the mean difference part of the antilogarithm table read the number at the intersection of row 0.74 and column 2 gives 0.003. Adding the two results gives 5.511. So, the $\text{antilog}\, 0.7412 = 5.511$.

b) Reading the number at the intersection of row 0.93 and Column 3 from the antilogarithm table gives 8.57. So, $\text{antilog}\,0.9330 = 8.57$

c) $\text{antilog}\,0.9996 = 9.977 + 0.014 = 9.991$.

d) $\text{antilog}\,0.7 = 5.012$.

e) $\text{antilog}\,1.3010 = \text{antilog}(0.3010 + 1) = \text{antilog}(0.3010) \times 10^1 = 2 \times 10^1 = 20$.

f) $\text{antilog}\,0.9953 = 9.886 + 0.007 = 9.893$

g) $\text{antilog}\,5.721 = \text{antilog}(0.721 + 5) = \text{antilog}(0.721) \times 10^5 = 5.26 \times 10^5 = 526{,}000$

h) $\text{antilog}\,(-0.2) = \text{antilog}\big((1 - 0.2) - 1\big) = \text{antilog}(0.8 - 1) = 6.31 \times 10^{-1} = 0.631$

Computation with Logarithms

Teaching Aid Required

- calculator

In this section you will show to the students how common logarithms are used for computations. Discuss the examples given in the students' textbook. Allow them to participate in the discussion.

Answer for Exercise 3.16

a) Let $x = 4.26 \times 5.73$.

Then $\log x = \log(4.26 \times 5.73) = \log 4.26 + \log 5.73 = 0.6294 + 0.7582 = 1.3876$

So, $x = \text{antilog}(1.3876) = \text{antilog}(0.3876 + 1) = 2.441 \times 10^1 = 24.41$

Therefore, $4.26 \times 5.73 = 24.41$

b) Let $x = \sqrt[5]{25} = 25^{\frac{1}{5}}$.

Then $\log x = \log 25^{\frac{1}{5}} = \frac{1}{5}(\log 25) = \frac{1}{5}(\log 2.5 \times 10^1) = \frac{1}{5}(0.3979 + 1)$

$= \frac{1}{5}(1.3979) = 0.27958$.

Hence, $x = \text{antilog}(0.2796) = 1.904$.

Therefore, $\sqrt[5]{25} \approx 1.904$.

c) Let $x = 3^{1.42}$.

Then $\log x = \log 3^{1.42} = 1.42 \log 3 = 1.42(0.4771) = 0.6775$.

$x = \text{antilog}(0.6775) = 4.758$.

So, $3^{1.42} \approx 4.758$.

d) Let $x = (4.2)^{1.3} \times (0.21)^{4.1}$.

Then $\log x = \log((4.2)^{1.3} \times (0.21)^{4.1}) = \log(4.2)^{1.3} + \log(0.21)^{4.1}$

$= 1.3 \log 4.2 + 4.1 \log 0.21 = 1.3 \times (0.6232) + 4.1 \times (0.3222 - 1)$

$= 0.81016 - 2.7790 = -1.9688 = 2 - 1.9688 - 2 = 0.03118 - 2.$

So, $x = \text{antilog}(0.03118 - 2) = 1.074 \times 10^{-2} = 0.01074.$

Therefore, $(4.2)^{1.3} \times (0.21)^{4.1} \approx 0.01074.$

e) Let $x = \frac{\sqrt{488}}{(2.81)^2}$.

Then $\log x = \log \frac{\sqrt{488}}{(2.81)^2} = \log \sqrt{488} - \log(2.81)^2 = \log 488^{\frac{1}{2}} - \log(2.81)^2 = \frac{1}{2}(\log 4.88 \times 10^2) - 2(\log 2.81) = \frac{1}{2}[0.6884 + 2] - 2[0.4487] = 1.3442 - 0.8974 = 0.4468.$

That is, $\log x = 0.4468$ if and only if $x = \text{antilog}(0.4468) = 2.798.$

Thus, $\frac{\sqrt{488}}{(2.81)^2} \approx 2.798.$

f) Let $x = \sqrt[5]{0.0461} = (0.0461)^{\frac{1}{5}}$.

Then $\log x = \log(0.0461)^{\frac{1}{5}} = \frac{1}{5}(\log(0.0461)) = \frac{1}{5}(\log 4.61 \times 10^{-2}) = \frac{1}{5}(0.6637 - 2)$

$= \frac{1}{5}(-1.3367) = -0.2672 = (1 - 0.2672) - 1 = (0.7327 - 1).$

Hence, $x = \text{antilog}(0.7327 - 1) \approx 5.404 \times 10^{-1} = 0.5404$

Therefore, $\sqrt[5]{0.0461} \approx 0.5404.$

Assessment

At the end of this lesson, you can give class activities, home-work, assignment, test or quiz to assess their level of understanding. The assessment should include the types of problems that enable you to know slow, medium and fast learners so that you may arrange appropriate support for those in need.

3.2 The Exponential Functions and Their Graphs

Periods allotted: 8 periods

Competencies

At the end of this sub-unit students will be able to:

- define an exponential function.
- draw the graph of a given exponential function.

- describe the graphical relationship of exponential functions having bases reciprocal to each other.
- describe the properties of an exponential function by using its graph.

Teaching Aids Required

- Scientific calculator
- Sample graphs drawn on a big graph paper

Introduction

In this sub-unit, definition of exponential function, graphs of exponential function and properties of the graphs are discussed.

Teaching Guide

3.2.1 Exponential functions

Let students begin the lesson by doing activity 3.5. This activity could help students to see how an exponential function is formulated. In this activity, a single bacterium divides itself into two every hour. That means, the number of bacteria created at the first, second, third, fourth, fifth and in general t^{th} hour is given by the table.

Time in hr. (t)	0	1	2	3	4	5	t
No. of bacteria (y)	$1 = 2^0$	$2 = 2^1$	$4 = 2^2$	$8 = 2^3$	$16 = 2^4$	$32 = 2^5$	2^t

The formula which helps to calculate the number of bacteria after t hour is $y = 2^t$. Tell to the students that $y = 2^t$.is an exponential function.

Answer for activity 3.5

a.

- the number of bacteria after one hour is 2
- the number of bacteria after two hours is 4
- the number of bacteria after three hours is 8
- the number of bacteria after four hours is 16
- the number of bacteria after t hours is 2^t

b.

Time in hour(t)	0	1	2	3	4	…	t
Number of bacteria	1	2	4	8	16	…	2^t

c. The formula to calculate the number of bacteria after t hours is 2^t.

Now, you can state definition 3.8. At this point you can ask students to give examples of exponential functions based on the definition like $f(x) = 3^x$, $f(x) = \left(\frac{1}{2}\right)^x$, etc.

Answer for Exercise 3.17

1. **a.** $\frac{1}{16}$ **b.** 16 **c.** $\frac{1}{2}$ **d.** 2

2. **a.** $f(x) = 2^{2x}$ **b.** $f(x) = 3^{\frac{1}{2}x}$ **c.** $f(x) = 3^{-2x}$

d. $f(x) = 2^{\frac{1}{3}x}$ **e.** $f(x) = 3^{-x}$ **f.** $f(x) = 2^{-\frac{4}{3}x}$

3.2.2 Graph of exponential functions

Discuss the examples with students. While doing the example support students to observe the following about the graph of $f(x) = 2^x$.

- ✓ The function $f(x) = 2^x$ is positive for all values of *x* and the graph completely lies above the *x*-axis.
- ✓ As *x* increases, the value of function gets larger and larger and the graph go upward
- ✓ As *x* decreases, the value of the function gets smaller and smaller and approaches zero. And the graph of f approaches the *x*-axis or the line $y = 0$ from above.

Introduce the meaning of Asymptote to students as:

A line that the graph of a function approach but does not intersect is called an asymptote for the graph of a function. And tell them that the x-axis or the line $y = 0$ is an asymptote for the graph of $f(x) = 2^x$.

Use question and answer technique to do activity 2.6. Make Sure that most of the students are participating in doing the activity.

Answer for activity 3.6

1. For positive real numbers *x*.
2. For negative real numbers *x*.
3. Yes
4. Go upward without limit.
5. Approaches the negative *x*-axis from above.
6. Yes. The *x*-axis.

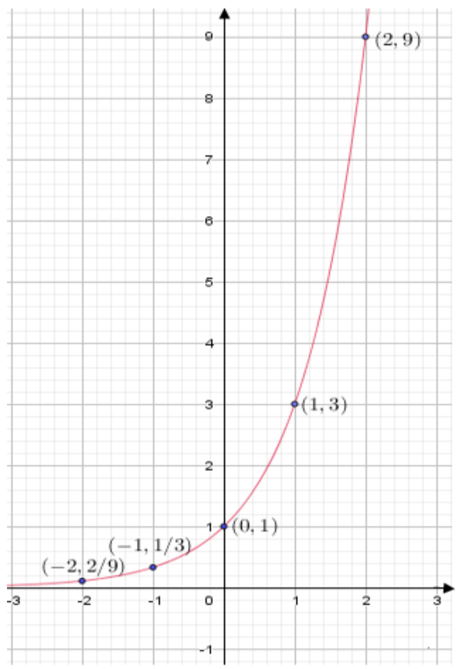
Answer for Exercise 3.18

a)

x	-2	-1	0	1	2
$y = f(x)$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

b) *y*-intercept = 1 and has no *x*-intercept

c)



d) Domain is the set of real numbers and the range is the set of positive real numbers.

Discuss the examples with students. While doing the example support students to observe the following about the graph of $f(x) = \left(\frac{1}{2}\right)^x$.

- ✓ The function $f(x) = \left(\frac{1}{2}\right)^x$ is positive for all values of x and the graph completely lies above the x-axis.
- ✓ As x increases, the value of the function gets smaller and smaller and approaches zero. And the graph of f approaches the x-axis or the line $y = 0$ from above.
- ✓ As x decreases, the value of function gets larger and larger and the graph go upward
- ✓ the x-axis or the line $y = 0$ is an asymptote for the graph of $f(x) = 2^x$.

Answer for Exercise 3.19

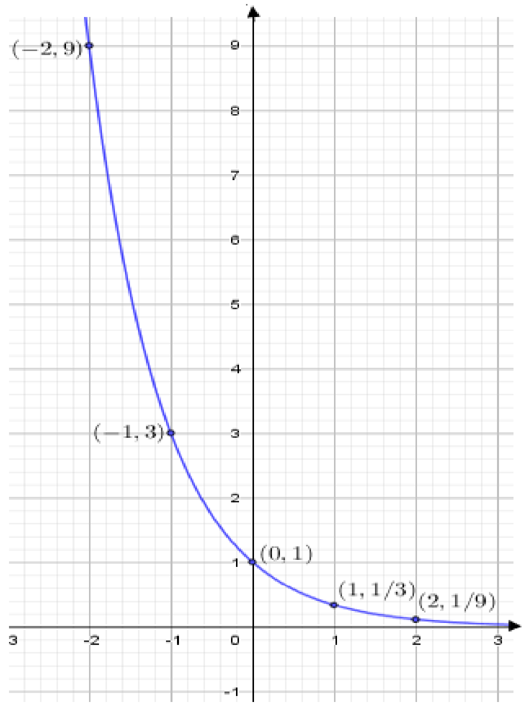
For the function $f(x) = \left(\frac{1}{3}\right)^x$

a) Complete the table of values below

x	-2	-1	0	1	2
$y = f(x)$	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$

b) *y*-intercept = 1 and has no *x*-intercept

c)



d) Domain is the set of real numbers and the range is the set of positive real numbers.

At this point you can support students to observe the behavior and shape of the graph of $f(x) = a^x$ for $0 < a < a$ and $a > 1$ by considering figure 3.3 and the following properties.

Properties of Graph of $f(x) = a^x$, $a > 1$

a) Domain: The set of all real numbers, $R = (-\infty, \infty)$.

b) Range: The set of all positive real numbers, $R^+ = (0, \infty)$.

c) Y-intercept: The point $(0, 1)$.

d) It is increasing on $(-\infty, \infty)$.

e) Horizontal asymptote: The x-axis (the line y=0) is a horizontal asymptote.

properties of Graph of $f(x) = a^x$, $0 < a < 1$.

a) Domain: The set of all real numbers, $R = (-\infty, \infty)$.

b) Range: The set of all positive real numbers, $R^+ = (0, \infty)$.

c) Y-intercept: The point $(0, 1)$.

d) It is decreasing on $(-\infty, \infty)$.

e) Horizontal asymptote: The $x - axis$(the line y=0) is a horizontal asymptote.

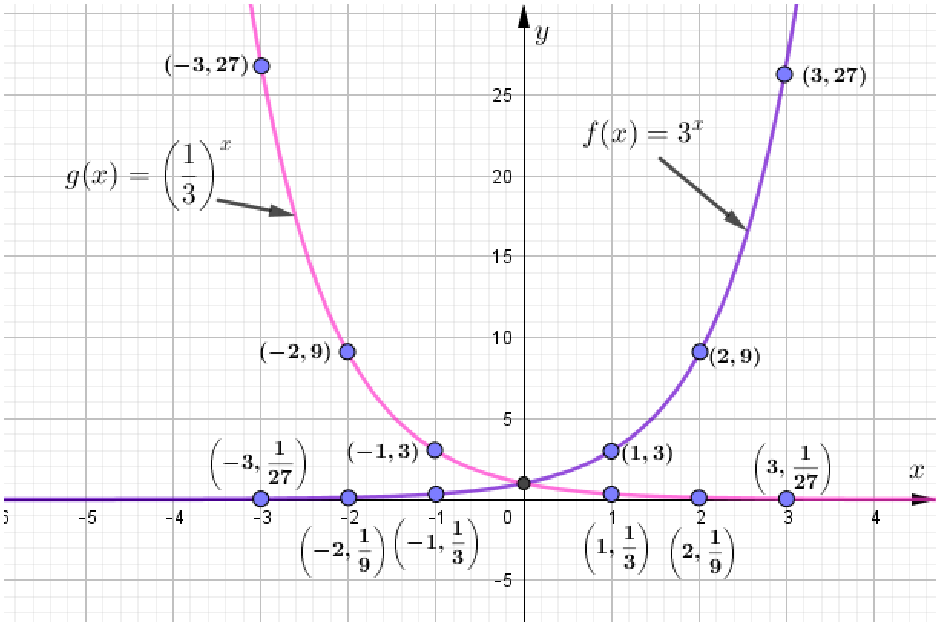
Answer for exercise 3.20

1. If $f(-1) = a^{-1} = 5$ then $a = \frac{1}{5}$. So, $f(x) = \left(\frac{1}{5}\right)^x$ is the required function.

2.

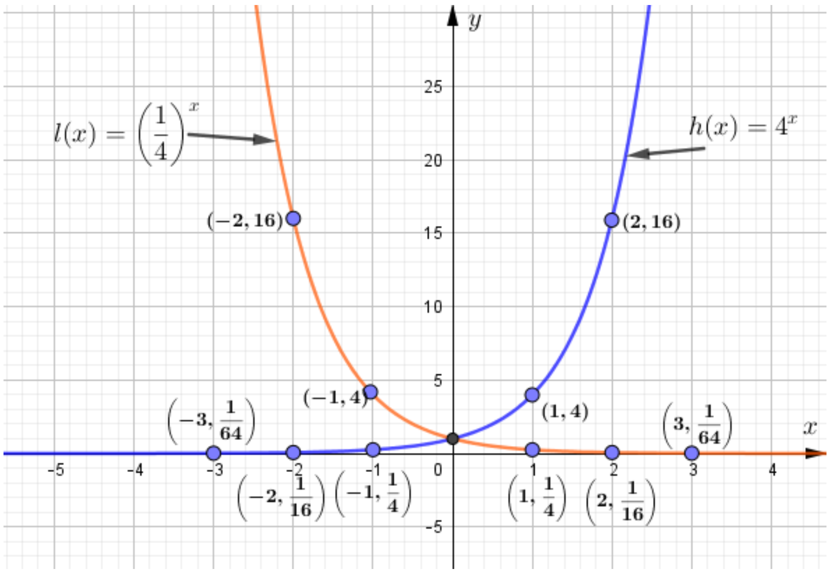
a)

x	-4	-3	-2	-1	0	1	2	3	4
$f(x) = 3^x$	$\frac{1}{81}$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	81
$g(x) = \left(\frac{1}{3}\right)^x$	81	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{81}$



b.

x	-4	-3	-2	-1	0	1	2	3	4
$h(x) = 4^x$	$\frac{1}{256}$	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16	64	256
$l(x) = \left(\frac{1}{4}\right)^x$	256	64	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$



Assessment

At the end of this lesson, you can give class activities, home-work, assignment, test or quiz to assess their level of understanding. The assessment should include the types of problems that enable you to know slow, medium and fast learners so that you may arrange appropriate support for those in need

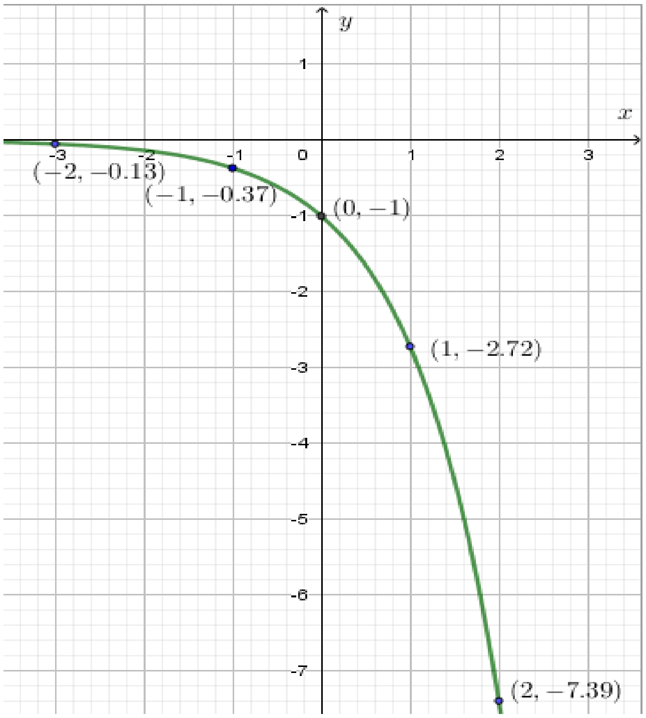
3.2.3 The natural exponential function (The exponential function to the base e)

Introduce the irrational number e and its historical background. Sketch its graph and discuss the properties (domain, range, whether it is increasing or decreasing) of the natural exponential function $f(x) = e^x$. Support students to see the value of e is between 2 and 3. That is $2 < e < 3$. Discuss with students on the graph of $f(x) = e^x$ in relation with the graphs of 2^x and 3^x.

Answer for exercise 3.21

a.

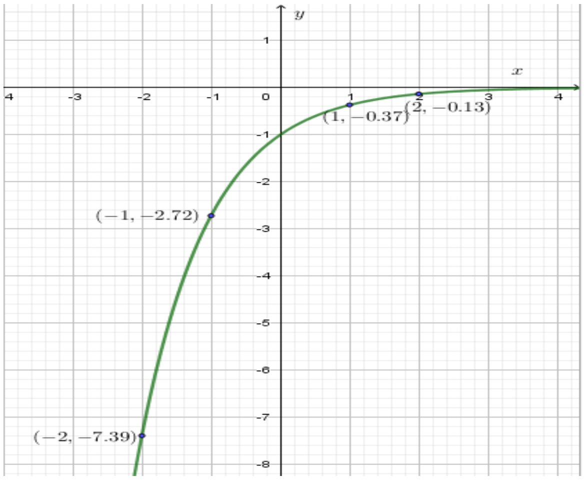
x	-2	-1	0	1	2
$f(x) = -e^x$	-0.13	-0.37	-1	-2.72	-7.39



- Has no x-intercept and the y-intercept is $(0, -1)$.
- The asymptote is the $x-$ axis or the line $y = 0$.
- The domain is the set of all real numbers and the range is the set of negative real numbers

b.

x	$f(x) = -e^{-x}$
-2	-7.39
-1	-2.72
0	-1
1	-0.37
2	-0.13

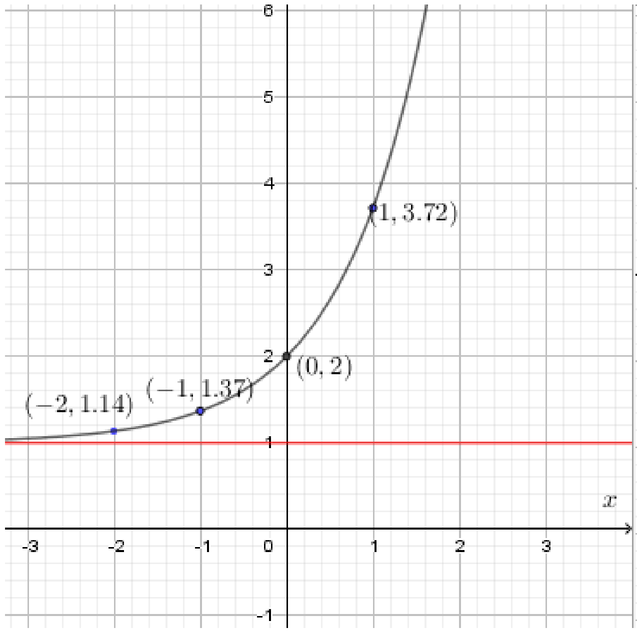


- Has no x-intercept and the y-intercept is $(0, -1)$.

- The asymptote is the $x-$ axis or the line $y = 0$.
- The domain is the set of all real numbers and the range is the set of negative real numbers.

c.

x	-2	-1	0	1	2
$f(x) = 1 + e^x$	1.14	1.37	2	3.72	8.39



- Has no x-intercept and the y-intercept is $(0,2)$.
- The asymptote is the the line $y = 2$.
- The domain is the set of all real numbers and the range is the set of positive real numbers greater than 1.

3.3 The Logarithmic functions and their Graphs.

Periods allotted: 8 periods

Competencies

At the end of this sub-unit students will be able to:

- define a logarithmic function.
- draw the graph of a given logarithmic function.

- describe the properties of a logarithmic function by using its graph.
- describe the graphical relationship of logarithmic function having bases reciprocal to each other.

Introduction

Meaning of logarithmic functions, graphs and properties of logarithmic functions are discussed in this subunit.

Teaching Guide

3.3.1 The Logarithmic functions

Revise the relationship between exponential equation and its corresponding logarithmic equation. And state the formal definition of logarithmic function as:

A Logarithmic function is **the inverses of an exponential function**. The inverse of the exponential function $y = a^x$ is $x = a^y$. The logarithmic function $y = \log_a x$ is defined to be equivalent to the exponential equation $x = a^y$, $y = \log_a x$ only under the following conditions:

$x = a^y$, $a > 0$, and $a \neq 1$.

Let $a > 0$ and $a \neq 1$. The logarithmic function with base a denoted by log_a is defined by

$$\log_a x = y \text{ if and only if } a^y = x.$$

From the definition above, we can see that every logarithmic equation can be written in an equivalent exponential form and every exponential equation can be written in logarithmic form.

When evaluating logarithms, remember that a logarithm is an exponent. That is, $\log_a x$ is the exponent to which a must be raised to obtain x.

Answer for Exercise 3.22

1. **a.** $-\frac{1}{2}$ **b.** does not exist **c.** 2 **d.** $-\frac{1}{4}$

2. **a.** $f(x) = \frac{1}{2}\log_2 x$ **b.** $f(x) = \frac{2}{3}\log_3 x$ **c.** $f(x) = -\frac{1}{2}\log_3 x$

d. $f(x) = 3\log_2 x$

3. **a.** $f(x) = -\frac{1}{5}\log_2\left(-\frac{1}{2}x\right)$ **b.** $f(x) = -\frac{1}{3}\log_3\left(\frac{1}{3}x\right)$

3.3.2 Graphs of logarithmic functions

Discuss the example in the students' text with students, assist them to sketch the graph of $f(x) = \log_2 x$ and find the domain and range of the function from the graph.

Basic characteristic of graph of $f(x) = \log_a x$, $a > 1$.

- ✓ Domain: $(0, \infty)$
- ✓ Range: $R = (-\infty, \infty)$
- ✓ x-intercept: $(1, 0)$
- ✓ Increasing on $(0, \infty)$
- ✓ y -axis is a vertical asymptote
- ✓ Reflection of $g(x) = a^x$, $a > 1$ in the line $y = x$.

Discuss the examples with students, assist them to sketch the graph of $f(x) = \log_2 x$ and find the domain and range of the function from the graph.

Basic characteristic of graph of $f(x) = \log_a x$, $0 < a < 1$.

- ✓ Domain: $(0, \infty)$
- ✓ Range: $R = (-\infty, \infty)$
- ✓ x-intercept: $(1, 0)$
- ✓ decreasing on $(0, \infty)$.
- ✓ y -axis is a vertical asymptote.
- ✓ Reflection of $g(x) = a^x, 0 < a < 1$ in the line $y = x$.

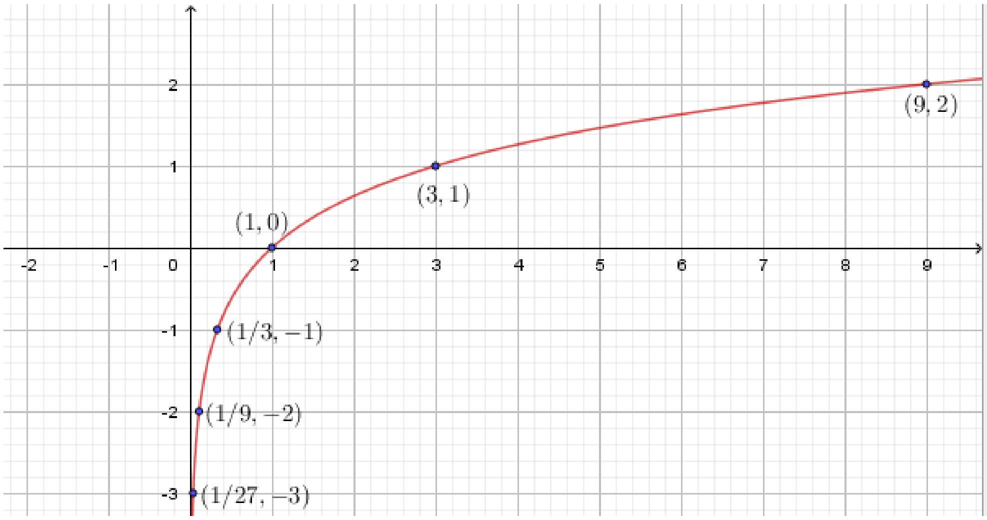
Answer for exercise 3.23

a)

x	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27
$f(x) = \log_3 x$	-3	-2	-1	0	1	2	3

b) Has no y-intercept and the x-intercept is (1,0).

c)



d) Domain = $(0, \infty)$ and Range is the set of all real numbers.

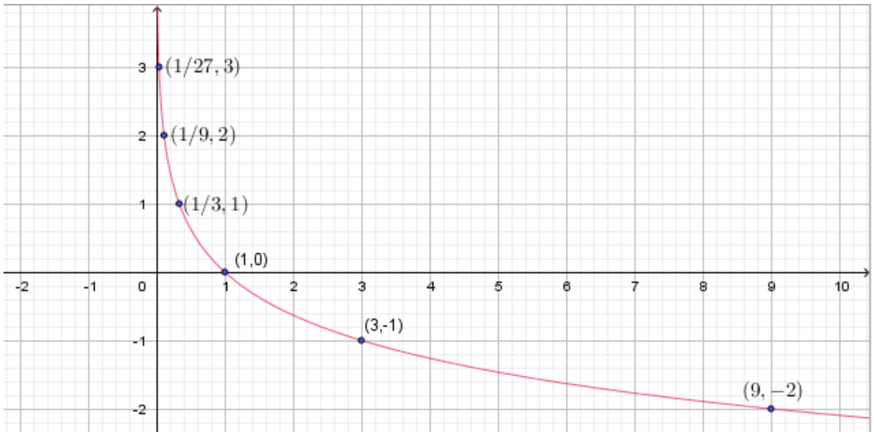
Answer for exercise 3.24

a)

x	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27
$f(x) = \log_{\frac{1}{3}} x$	3	2	1	0	-1	-2	-3

b) Has no y-intercept and the x-intercept is $(1,0)$.

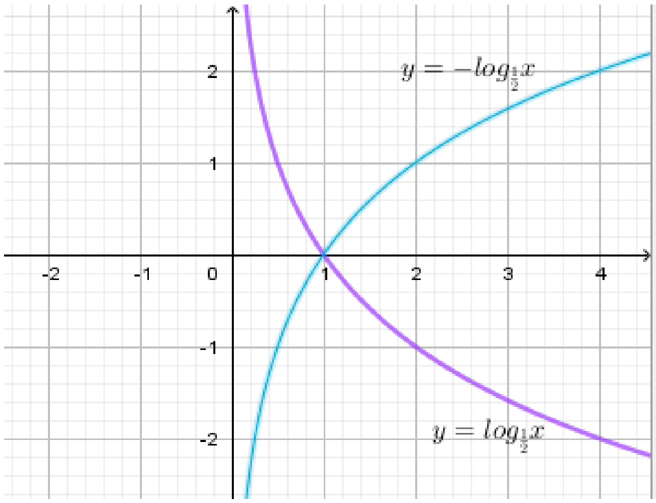
c)



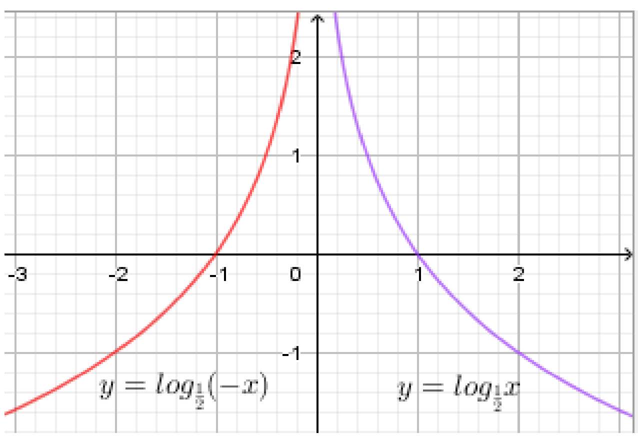
d) Domain = $(0, \infty)$ and Range is the set of all real numbers.

Answer for exercise 3.25

a. Since $g(x) = -\log_{\left(\frac{1}{2}\right)} x = -f(x)$, the graph of $g(x)$ is a reflection of the graph of $f(x)$ along the x-axis.



b. Since $h(x) = \log_{\left(\frac{1}{2}\right)}(-x) = f(-x)$, the graph of $h(x)$ is a reflection of the graph of $f(x)$ along the y-axis.



3.3.3 Natural Logarithms

The logarithm of a number to the base e is called **natural logarithm** and it is written as $\log_e x$ or $\ln x$.

Answer for exercise 3.26

a. $\frac{1}{5}$ **b.** -3 **c.** 5 **d.** $\frac{2}{3}$ **e.** 3 **f.** $x + y$ **g.** $x - y$

Discuss the example in the students text with students. Show them the graph of $y = \ln x$ can be drawn by reflecting the graph of $y = e^x$ along the line $y = x$.

Assessment

At the end of this lesson, you can give class activities, home-work, assignment, test or quiz to assess their level of understanding. The assessment should include the types of problems that enable you to know slow, medium and fast learners so that you may arrange appropriate support for those in need

3.4 Solving exponential and Logarithmic equations

3.4.1 Solving exponential equations

An equation in which the variable occurs in the exponent is called an exponential equation. For instance,

$2^x = 10$ and $3^{5x} = 81$are exponential equations.

To solve exponential equations, we follow the following 3 step procedure.

1) Isolate the exponential expression on one side of the equation.
2) Take the logarithm of both sides, then use the laws of logarithms, (power rule of logarithms) to " bring down the exponent."
3) Solve for the variable.

Moreover, we use the following property:

Base-exponent property

For any real numbers $x, y, a > 0, a \neq 1$,

$$a^x = a^y \text{ if and only if } x = y$$

Answer for exercise 3.27

a. $5^x = 125$,

$5^x = 5^3$,

$x = 3$.

b. $3^{3-x} = 81$,

$3^{3-x} = 3^4$,

$3 - x = 4$. Thus, $x = -1$.

c. $4^{2x-5} = 2^{3x+6}$,

$2^{2(2x-5)} = 2^{3x+6}$,

$4x - 10 = 3x + 6$. Thus, $x = 16$.

d. $\frac{1}{8} = \left(\frac{1}{4}\right)^x$,

$\left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^{2x}$,

$2x = 3$. Thus, $x = \frac{3}{2}$.

e. $2^{-x} = 512$,

$2^{-x} = 2^9$,

$-x = 9$, $x = -9$.

f. $3^{x^2-2} = 9$,

$3^{x^2-2} = 3^2$,

$x^2 - 2 = 2$,

$x^2 = 4$,

$x = \pm 2$.

g. $7^{x^2+x} = 49$, $7^{x^2+x} = 7^2$, $x^2 + x = 2$, $x^2 + x - 2 = 0$, $x = 1, -2$.

h. $3^{3(x+2)} = 9^{x+2}$, $3^{3(x+2)} = 3^{2(x+2)}$, $3(x+2) = 2(x+2)$, $3x + 6 = 2x + 4$,

$x = -2$.

i. $3\left(\frac{27}{8}\right)^{\frac{2}{3}x} = 2\left(\frac{32}{243}\right)^{2x}$, $3\left(\left(\frac{3}{2}\right)^3\right)^{\frac{2}{3}x} = 2\left(\left(\frac{2}{3}\right)^5\right)^{2x}$, $3\left(\frac{3}{2}\right)^{2x} = 2\left(\frac{2}{3}\right)^{10x}$,

$\left(\frac{3}{2}\right)^{2x+1} = \left(\frac{3}{2}\right)^{-10x}$, $2x + 1 = -10x$, $12x = -1$, $x = -\frac{1}{12}$.

Show to the students how common logarithim and the common logarithim table to solve exponetial equations by disussing with students the example in the students' textbook.

Answer for exercise 3.28

a) $3^x = 15$

$\log 3^x = \log 15$

$x \log 3 = \log 15$

$$x = \frac{\log 15}{\log 3}$$

But, $\log 15 = \log(1.5 \times 10) = \log 1.5 + 1 = 1.1761;\ \log 3 = 0.4771$

Therefore, $x = \frac{\log 15}{\log 3} = \frac{1.1761}{0.4771} \approx 2.4651$

b) $10^x = 13.4$

$$10^x = 13.4$$

$$\log 10^x = \log 13.4$$

$$x = \log 13.4$$

But, $\log 13.4 = \log(1.34 \times 10) = \log 1.34 + 1 = 1.1271.$

Therefore, $x = \log 1.34 \approx 1.1271$

c) $10^{2x+1} = 9$

$10^{2x+1} = 92$

$\log 10^{2x+1} = \log 92$

$2x + 1 = \log 92$

But, $\log 92 = \log(9.2 \times 10) = \log 9.2 + 1 = 1.9638.$

So, $2x + 1 = \log 92$implies $2x + 1 = 1.9638$ implies $2x = 0.9638$ implies $x = 0.4819.$

d) $(6)^{3x} = 5$

$6^{3x} = 5$

$\log 6^{3x} = \log 5$

$3x \log 6 = \log 5$

$$x = \frac{\log 5}{3 \log 6}$$

But, $\log 5 = 0.6990; \log 6 = 0.7782.$

So, $x = \frac{\log 5}{3 \log 6} = \frac{0.6990}{3(0.7782)} = \frac{0.6990}{2.3346} \approx 0.2994.$

e) $4^{2x} = 61$

$4^{2x} = 61$

$\log 4^{2x} = \log 61$

$2x \log 4 = \log 61$

$$x = \frac{\log 61}{2 \log 4}$$

But, $\log 61 = \log(6.1 \times 10) = \log 6.1 + 1 = 1.7853; \log 4 = 0.6021.$

So, $x = \frac{\log 61}{2\log 4} = \frac{1.7853}{2(0.6021)} = \frac{1.7853}{1.2042} \approx 1.4826.$

f) $(1.05)^x = 2$

$$(1.05)^x = 2$$

$$\log(1.05)^x = \log 2$$

$$x\log(1.05) = \log 2$$

$$x = \frac{\log 2}{\log(1.05)}$$

But, $\log 2 = 0.3010;\ \log 1.05 = 0.0212$

So, $x = \frac{\log 2}{\log 1.05} = \frac{0.3010}{0.0212} \approx 14.20.$

g) $10^{5x-2} = 348$, $10^{5x-2} = 348$, $\log 10^{5x-2} = \log 348$, $5x - 2 = \log 348$, $x = \frac{\log 348+2}{5}$,

But, $\log 348 = \log(3.48 \times 10^2) = \log 3.48 + 2 = 2.5416.$

Therefore, $x = \frac{\log 348+2}{5} = \frac{4.5416}{5} = 0.90832.$

h) $3^x = 0.283$

$$\log 3^x = \log 0.283$$

$$x\log 3 = \log 0.283$$

$$x = \frac{\log 0.283}{\log 3}$$

But,

$\log 0.283 = \log(2.83 \times 10^{-1}) = \log 2.83 - 1 = 0.4518 - 1 = -0.5482;\ \log 3 = 0.4771$

Therefore, $x = \frac{\log 0.283}{\log 3} = \frac{-0.5482}{0.4771} = -1.1490$

i) $2^x = 0.283$

$$\log 2^x = \log 0.283$$

$$x\log 2 = \log 0.283$$

$$x = \frac{\log 0.283}{\log 2}$$

But,

$\log 0.283 = \log(2.83 \times 10^{-1}) = \log 2.83 - 1 = 0.4518 - 1 = -0.5482;\ \log 3 = 0.4771$

Therefore, $x = \frac{\log 0.283}{\log 2} = \frac{-0.5482}{0.3010} \approx -1.8213$

3.4.2 Solving logarithmic equations

A logarithmic equation is **an equation that involves the logarithm of an expression containing a variable**. For instance, $\log_2(x+3) = 4$ is logarithmic equation.

We use the following property to solve logarithmic equations.

Property of logarithmic equality

For any positive real numbers $x, y, a > 0, a \neq 1, \log_a x = \log_a y$ if and only if $x = y$

We use the following 3 step procedure to solve logarithmic equations.

1. State the universe.
2. Collect the logarithmic term on one side of the equation.
3. Write the equation in exponential form.
4. Solve for the variable.

Answer for exercise 3.29

a) $\log_2(3x-1) = 5,$

$3x - 1 > 0$ implies $x > \frac{1}{3}$. So, the universe or domain is $\left(\frac{1}{3}, \infty\right)$.

$\log_2(3x-1) = 5$ Implies $3x - 1 = 2^5 = 32$ implies $3x = 33$ implies $x = 11$.

b) $\log_{\sqrt{2}} x = 6$

The universe or domain is $(0, \infty)$.

$\log_{\sqrt{2}} x = 6,$

$$\frac{\log x}{\log \sqrt{2}} = 6,$$

$$\frac{\log x}{\log 2^{\frac{1}{2}}} = 6,$$

$$\frac{\log x}{\frac{1}{2}\log 2} = 6,$$

$\log x = 3\log 2$, but $\log 2 = 0.3010$

$\log x = 0.9030$

So, $x = \text{antilog}(0.9030) \approx 8.0$.

c) $x^2 - 3x > 0$ if and only if $x < 0$ or $x > 3$.

So, the universe is $(-\infty, 0) \cup (3, \infty)$.

$\log_2(x^2 - 3x) = 4,$

$x^2 - 3x = 2^4 = 16,$

$x^2 - 3x - 16 = 0,$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3 \pm \sqrt{9+64}}{2} = \frac{3 \pm \sqrt{73}}{2}.$$

Therefore, $x = \frac{3+\sqrt{73}}{2}, \frac{3-\sqrt{73}}{2}$ since both of them are members of the universe.

d) $\log_2(x-1) + \log_2 3 = 3.$

$x - 1 > 0$ if and only if $x > 1$.

So, the universe is $(1, \infty)$.

$\log_2(x-1) + \log_2 3 = 3,$

$\log_2 3(x-1) = 3,$

$3(x-1) = 2^3 = 8,$

$3x - 3 = 8,$

$3x = 11,$

$x = \frac{11}{3}.$

Therefore, $x = \frac{11}{3}$.

e) $\log(x^2 - 121) - \log(x+11) = 1$

$x^2 - 121 > 0$ and $x + 11 > 0$ if and only if $x > 11$.

So, the universe is $(11, \infty)$.

$\log(x^2 - 121) - \log(x+11) = 1,$

$\log\frac{(x^2-121)}{x+11} = 1,$

$\log(x - 11) = 1,$

$x - 11 = 10^1 = 10,$

$x = 21.$

f) $\log_x(x+6) = 2$

$x + 6 > 0$ and $x > 0$ implies $x > 0$.

So, the universe is $(0, \infty)$.

$\log_x(x+6) = 2,$

$x^2 = x + 6,$

$x^2 - x - 6 = 0,$

$x = -2 \text{ } or \text{ } x = 3,$

But, $x = -2$ is not in the universe.

So, $x = 3$.

g) $\log x - \log 3 = \log 4 - \log(x + 4)$

$x > 0 \text{ } and \text{ } x + 4 > 0$ implies $x > 0$.

So, the universe is $(0, \infty)$.

$\log x - \log 3 = \log 4 - \log(x + 4),$

$\log x + \log(x + 4) = \log 4 + \log 3,$

$\log x(x + 4) = \log 3(4),$

$\log x^2 + 4x = \log 12,$

$x^2 + 4x - 12 = 0,$

$x = 2$ or $x = -6$,

But, $x = -6$ is not in the universe.

So, $x = 2$.

h) $\log_2\left(1 + \frac{1}{x}\right) = 3$

$1 + \frac{1}{x} > 0$ implies $\frac{x+1}{x} > 0$ implies $x > 0$ or $x < -1$

So, the universe is $(-\infty, -1) \cup (0, \infty)$.

$\log_2\left(1 + \frac{1}{x}\right) = 3,$

$1 + \frac{1}{x} = 2^3 = 8,$

$\frac{1}{x} = 7,$

$x = \frac{1}{7},$

So, $x = \frac{1}{7}$.

i) $\log_2 2 + \log_2(x + 2) - \log_2(3x - 5) = 3$

$x + 2 > 0 \text{ } and \text{ } 3x - 5 > 0$ implies $x > \frac{5}{3}$.

So, the universe is $\left(\frac{5}{3}, \infty\right)$.

$\log_2 2 + \log_2(x + 2) - \log_2(3x - 5) = 3,$

$1 + \log_2 \frac{(x+2)}{(3x-5)} = 3,$

$\log_2 \frac{(x+2)}{(3x-5)} = 2,$

$\frac{(x+2)}{(3x-5)} = 2^2 = 4,$

$x + 2 = 12x - 20,$

$20 + 2 = 12x - x,$

$x = 2.$

Since $x = 2 \in \left(\frac{5}{3}, \infty\right), x = 2$ is the solution.

Answer for exercise 3.30

a) $3^{x-1} = 2^x$

Taking the common logarithm on both sides we obtain:

$$\log 3^{x-1} = \log 2^x$$

$$(x-1)\log 3 = x \log 2$$

$$x \log 3 - \log 3 = x \log 2$$

$$x \log 3 - x \log 2 = \log 3$$

$$x(\log 3 - \log 2) = \log 3$$

$$x = \frac{\log 3}{\log 3 - \log 2}$$

But, from the common logarithm table, we have

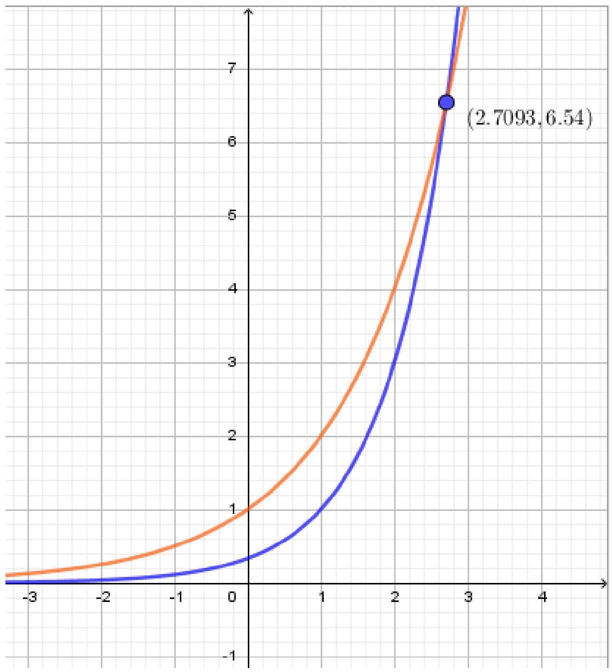
$\log 2 \approx 0.3010\,;\ \log 3 \approx 0.4771.$

Hence, $x \approx \frac{0.4771}{0.4771 - 0.3010}$

$$\approx \frac{0.4771}{0.1761}$$

$$\approx 2.7093$$

You can use graph sketching software like Geogebra to demostrate geometrically that the solution $x = 2.7093$, is the first coordinate of the point of intersection of the graphs $y = 3^{x-2}$ and $y = 2^x$ as shown in the figure bellow.



b) $9^x = 8^{x-1}$

$3^{2x} = 2^{3(x-1)}$

$3^{2x} = 2^{(3x-3)}$

Taking the common logarithm on both sides we obtain:

$$\log 3^{2x} = \log 2^{(3x-3)}$$

$$2x \log 3 = (3x - 3) \log 2$$

$$2x \log 3 - 3x \log 2 = -3 \log 2$$

$$x(2 \log 3 - 3 \log 2) = -3 \log 2$$

$$x = \frac{-3\log 2}{2\log 3 - 3\log 2}$$

But, from the common logarithm table, we have

$$\log 2 \approx 0.3010\,;\ \log 3 \approx 0.4771.$$

Hence, $x \approx \frac{-3(0.3010)}{2(0.4771)-3(0.3010)}$

$$\approx -17.6367$$

Assessment

At the end of this lesson, you can give class activities, home-work, assignment, test or quiz to assess their level of understanding. The assessment should include the types of problems that enable you to know slow, medium and fast learners so that you may arrange appropriate support for those in need

3.5 Relation between exponential and logarithmic functions with the same base

Periods allotted: 4 periods

Competencies

At the end of this sub-unit students will be able to:

- describe how the domains and ranges of y = a^x and $y = \log_a x$are related.
- explain the relationship of the graphs of y = a^x and $y = \log_a x$.

Introduction

In this sub unit the relationship between the graphs, domain and range of the functions $y = a^x$ and $y = \log_a x$ are discussed.

Teaching guide

Assist students to sketch the graphs of $y = 2^x$ and $y = \log_2 x$ on the same coordinate plane by considering the tables of values below. Give them chance to tell the domain range of each.

x	-3	-2	-1	0	1	2	3
$y = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$y = \log_2 x$	-3	-2	-1	0	1	2	3

Support students to reach on the following observation on the relation between the functions $y = a^x$ and $y = \log_a x$, $a > 1$ is shown below.

1. The domain of $y = a^x$is the set of all real number, that is the range of $y = \log_a x$.
2. The range of $y = a^x$is the set of all positive real number, that is the domain of $y = \log_a x$.

- Domain of $y = a^x$ = Range of $y = \log_a x$
- Range of $y = a^x$ = Domain of $y = \log_a x$

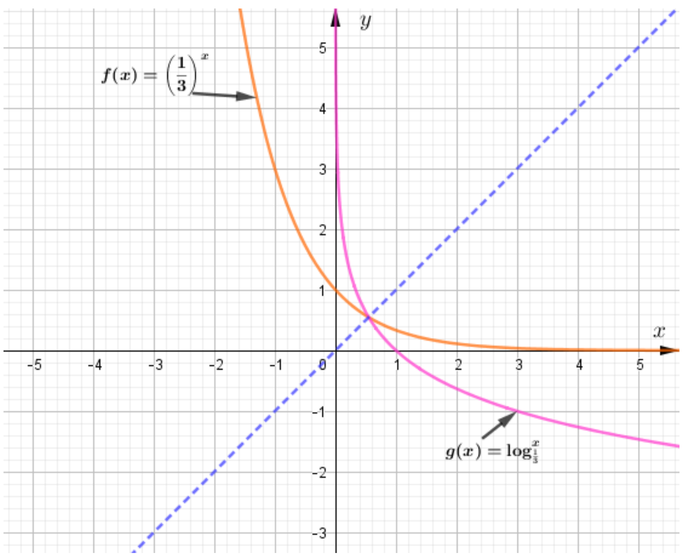
3. The x-axis is the horizontal asymptote of the graph of $y = a^x$; the y-axis is a vertical asymptote of the graph of $y = \log_a x$.
4. The point $(0,1)$ is the y-intercept of the graph of $y = a^x$; the point $(1,0)$ is the x-intercept of the graph of $y = \log_a x$.

Answer for exercise 3.31

1.

x	-3	-2	-1	0	1	2	3
$y = \left(\frac{1}{3}\right)^x$	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$

x	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27
$g(x) = \log_{\left(\frac{1}{3}\right)} x$	3	2	1	0	-1	-2	-3



2. The domain of f is the set of all real numbers; the range of f is the set of all positive real numbers

3. The domain of g is the set of all positive real numbers; the range of g is the set of all real numbers
4. The domain of f is the set of all real numbers which is the range of g.
5. The range of f is the set of all positive real numbers which is the domain of g.

Assessment

At the end of this lesson, you can give class activities, home-work, assignment, test or quiz to assess their level of understanding. The assessment should include the types of problems that enable you to know slow, medium and fast learners so that you may arrange appropriate support for those in need

3.6 Applications

Periods allotted: 4 periods

Competencies

At the end of this sub-unit students will be able to:

- Solve problems involving exponential and logarithmic functions from real life.

Introduction

In this sub unit application problems like compound interest, population growth and pH scale are discussed to show the students how exponential and logarithmic functions are applied in solving real life problem.

Answer for exercise 3.32

a) $A(t) = P(1+r)^t = 10{,}000(1+0.07)^t$

b) at $t = 1,\ A(1) = 10{,}000(1+0.07)^1 = 10{,}700;$

at $t = 5,\ A(5) = 10{,}000(1+0.07)^5 \approx 14{,}025.52;$

at $t = 10,\ A(10) = 10{,}000(1+0.07)^{10} \approx 19{,}671.51;$

at $t = 15,\ A(15) = 10{,}000(1+0.07)^{15} = 27{,}590.32;$

at $t = 20,\ A(20) = 10{,}000(1+0.07)^{20} \approx 38{,}696.84;$

c) $A(t) = 20{,}000$

$A(t) = 20{,}000 = 10{,}000(1+0.07)^t$

$2 = (1+0.07)^t$

$$\log 2 = \log(1 + 0.07)^t$$

$$t = \frac{\log 2}{\log 1.07}$$

But, $\log 2 = 0.3010;\ \log 1.07 = 0.0294$.

So, $t = \frac{\log 2}{\log 1.07} = \frac{0.3010}{0.0294} \approx 10.24$

Therefore, the investement to be doubled takes about 10.24 years

Answer for exercise 3.33

1) **a)** $A(10) = 40{,}000e^{0.06(10)} = 40{,}000\, e^{0.6} \approx 72{,}884.75$

b) $A(10) = 40{,}000e^{0.065(10)} = 40{,}000\, e^{0.65} \approx 76{,}621.63$

c) $A(10) = 40{,}000e^{0.07(10)} = 40{,}000\, e^{0.7} \approx 80{,}550.11$

d) $A(10) = 40{,}000e^{0.075(10)} = 40{,}000\, e^{0.75} \approx 84680.00$

2) At the end of 30th day you will be paid $2^{30} = 1{,}073{,}741{,}824$ cents

In one birr there are 100 cents. Therefore, $1{,}073{,}741{,}824$ cents = 10,737,418.24 birr.

Hence, the methods of payment in described in (b) is more profitable.

Answer for exercise 3.34

a) $P(t) = P_0 e^{kt}$

$$P(1) = 25{,}000 = 10{,}000e^{k}$$

$$e^k = \frac{25000}{10000} = \frac{5}{2}$$

$$\ln e^k = \ln\left(\frac{5}{2}\right) = \ln 2.5$$

$$k = \ln 2.5 \approx 0.9163$$

So, $P(t) = 10{,}000e^{0.9163t}$

$$P(t) = P_0 e^{kt}$$

$$20000 = 10000e^{0.9163t}$$

$$2 = e^{0.9163t}$$

$$0.9163\, t = \ln 2$$

$$t = \frac{\ln 2}{0.9163} = \frac{0.6931}{0.9163} = 0.7565$$

So, the doubling period is about 0.7565 hrs.

b) The number of bacteria after 5 hours, $P(5)$ is about:

$P(5) = 10{,}000e^{0.9163(5)} = 10{,}000e^{4.5815} \approx 976{,}608$

Answer for exercise 3.35

1. a) $PH = -\log[H^+]$

$= -\log[5 \times 10^{-3}]$

$= -[\log(5) + \log 10^{-3}]$ . . . product rule of logarithm

$= -[0.6990 - 3]$ using power rule of logarithms and logarithm table

$= 2.301$

Since $PH = 2.301 < 7$, the Lemon juice is acidic.

b) $PH = -\log[H^+] = -\log[3.2 \times 10^{-4}]$

$= -[\log(3.2) + \log 10^{-4}]$ . . . product rule of logarithm

$= -[0.5051 - 4]$ using power rule of logarithms and logarithm table

$= 3.4949$

Since $PH = 3.4949 < 7$, the Tomato juice is acidic

c) $PH = -\log[H^+]$

$= -\log[5 \times 10^{-9}]$

$= -[\log(5) + \log 10^{-9}]$ . . . product rule of logarithm

$= -[0.6990 - 9]$ using power rule of logarithms and logarithm table

$= 8.301$

Since $PH = 8.301 > 7$, the sea water is basic.

2) a) $PH = -\log[H^+]$

$3 = -\log[H^+]$

$\log[H^+] = -3$ … multiply both sides by -1.

$[H^+] = 1 \times 10^{-3}$

So, the hydrogen ion concentration of the Vinegar is about $1 \times 10^{-3}\ M$.

b) $PH = -\log[H^+]$

$6.5 = -\log[H^+]$

$\log[H^+] = -6.5$ … multiply both sides by -1.

$\log[H^+] = (7 - 6.5) - 7$

$\log[H^+] = 0.5 + (-7)$

$\text{antilog}(\log[H^+]) = antilog(0.5 + (-7))$

$[H^+] = 3.16 \times 10^{-7}$

So, the hydrogen ion concentration of the Milk is about $3.16 \times 10^{-7}\ M$.

Assessment

At the end of this lesson, you can give class activities, home-work, assignment, test or quiz to assess their level of understanding. The assessment should include the types of problems that enable you to know slow, medium and fast learners so that you may arrange appropriate support for those in need

Answer for review Exercises on unit three

1. $5^3 = 125, so\ \log_{\boxed{5}} \boxed{125} = \boxed{3}$.

2. $\log_5 25 = 2, so\ \boxed{5}^{\boxed{2}} = \boxed{25}$.

3. $f(4) = 1$, $f(1) = 0$, $f\left(\frac{1}{4}\right) = -1$, $f(16) = 2$ and $f(2) = \frac{1}{2}$.

4. **a)** 1 **b)** 0 **c)** 4 **d)** $-\frac{1}{3}$ **e)** 8 **f)** -1 **g)** -1

h) $-\frac{3}{2}$ **i)** -2 **j)** -1 **k)** $\frac{1}{3}$ **l)** $\frac{1}{2}$ **m)** 4

5.

Logarithmic form	Exponential form
$\log_7 7 = 1$	$7^1 = 7$
$\log_8 64 = 2$	$8^2 = 64$
$\log_8 4 = \frac{2}{3}$	$8^{\frac{2}{3}} = 4$
$\log_8 512 = 3$	$8^3 = 512$
$\log_8 \left(\frac{1}{8}\right) = -1$	$8^{-1} = \frac{1}{8}$
$\log_8 \left(\frac{1}{64}\right) = -2$	$8^{-2} = \frac{1}{64}$

6.

Logarithmic form	Exponential form
$\log_4 64 = 3$	$4^3 = 64$
$\log_4 2 = \frac{1}{2}$	$4^{\frac{1}{2}} = 2$
$\log_4 8 = \frac{3}{2}$	$4^{\frac{3}{2}} = 8$
$\log_4 \left(\frac{1}{16}\right) = -2$	$4^{-2} = \frac{1}{16}$
$\log_4 \left(\frac{1}{2}\right) = -\frac{1}{2}$	$4^{-\frac{1}{2}} = \frac{1}{2}$
$\log_4 \left(\frac{1}{32}\right) = -\frac{5}{2}$	$4^{\frac{-5}{2}} = \frac{1}{32}$

7. **a)** $\log_5 125 = 3$ if and only if $5^3 = 12$ **b)** $\log_5 1 = 0$ if and only if $5^0 = 1$

c) $\log_{10} 0.1 = -1$ if and only if $10^{-1} = 0.1$ **d)** $\log_8 512 = 3$ if and only if $8^3 = 512$

e) $\log_8 2 = \frac{1}{3}$ if and only if $8^{\frac{1}{3}} = 2$ **f)** $\log_9 3 = \frac{1}{2}$ if and only if $9^{\frac{1}{2}} = 3$

g) $\log_3 81 = 4$ if and only if $3^4 = 81$ **h)** $\log_2 \frac{1}{8} = -3$ if and only if $2^{-3} = \frac{1}{8}$

8. **a)** $\log_3 27 = 3$ **b)** $\log_{10} 0.001 = -3$ **c)** $\log_{10} 1000 = 3$ **d)** $\log_{81} 9 = \frac{1}{2}$

e) $\log_8 \left(\frac{1}{8}\right) = -1$ **f)** $\log_2 \left(\frac{1}{8}\right) = -3$ **g)** $\log_4 0.125 = -\frac{3}{2}$ **h)** $\log_{10} 0.001 = -3$

9. **a)** $x = 8$ **b)** $x = 5.$ **c)** $x = 625$ **d)** $x = -1.$

e) $x = -1$ **f)** $x = 16.$ **g)** $x = 36$ **h)** $x = 27.$

10. Given $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$. Then find the following logarithms.

a) $\log_2 \sqrt{3} = \frac{1}{2} \times \log_2 3 = \frac{\log_{10} 3}{\log_{10} 2} = 0.7925$

b) $\log_2 0.3 = \frac{\log_{10}(3\times10^{-1})}{\log_{10} 2} = \frac{\log_{10} 3+(-1)}{\log_{10} 2} = -1.7370$

c) $\log_3 0.6 = \frac{\log_{10}(2\times3\times10^{-1})}{\log_{10} 3} = -0.4650$

d) $\log_2 108 = \frac{\log_{10}(2^2\times3^2)}{\log_{10} 3} = 6.7549$

e) $\log_3 5 = \frac{\log_{10}\left(\frac{10}{2}\right)}{\log_{10} 3} = 1.4650$

f) $\log_4 75 = \frac{\log_{10}(3\times5^2)}{\log_{10} 2^2} = \frac{\log_{10} 3+2\log_{10}\left(\frac{10}{2}\right)}{2\log_{10} 2} = 0.2822$

11. Match the function with its graph.

a. $f(x) = 4^x$ II

c. $g(x) = \log_4 x$ III

b. $f(x) = \left(\frac{1}{4}\right)^x$ IV

d. $f(x) = \log_{\frac{1}{4}} x$ I

12. a) $\sqrt{5} > 2$ implies $\sqrt{5}^{\frac{1}{4}} > 2^{\frac{1}{4}}$ implies $\sqrt[8]{5} > \sqrt[4]{2}$

b) $3 > \sqrt{3}$ implies $\left(\frac{1}{3}\right)^3 > \left(\frac{1}{3}\right)^{\sqrt{3}}$

c) $\left(\sqrt{0.2}\right)^{-3.5} > \sqrt{0.2}^{\,0}$ implies $\left(\sqrt{0.2}\right)^{-3.5} > 1$

d) $\log_{\left(\frac{1}{2}\right)} x$ id decresing function $\log_{\left(\frac{1}{2}\right)} 20 > \log_{\left(\frac{1}{2}\right)} 50$

e) $\log 5\sqrt{7} > \log(5+\sqrt{7})$ implies $\log 5 + \log\sqrt{7} > \log(5+\sqrt{7})$

f) $\left(2\sqrt[3]{2}\right)^{-6} > 2^{-11}$

13. **a)** $x = \frac{1}{2}$ **b)** $x = \frac{3}{2}$ **c)** $x = -\frac{4}{3}$ **d)** $x = 1,\ x = -2$

e) $x = 0$ **f)** $x_1 = -1,\ x_2 = -\frac{3}{2}$ **g)** $x = 5$ **h)** $x = 1$

14. **a)** $x+2>0$ and $x-1>0$ implies $x>-2$ and $x>1$. So, the universe is $\{x: x>1\}$ and the solution is $x = 2$.

b) The universe is $(-\infty, 0) \cup (8, \infty)$ and the solution is $x_1 = -1;\ x_2 = 9$.

c) The universe is $\left(\frac{3}{5}, \infty\right)$ and the solution is $x = \frac{3}{4}$.

d) The universe is $(0, \infty)$ and the solution is $x = 10^{\left(\frac{13}{6}\right)}$.

e) The universe is $\left(\frac{2}{3}, \infty\right)$ and the solution is $x = 1$.

f) The universe is $(-5, \infty)$ and the solution is $x = -\frac{6}{5}$.

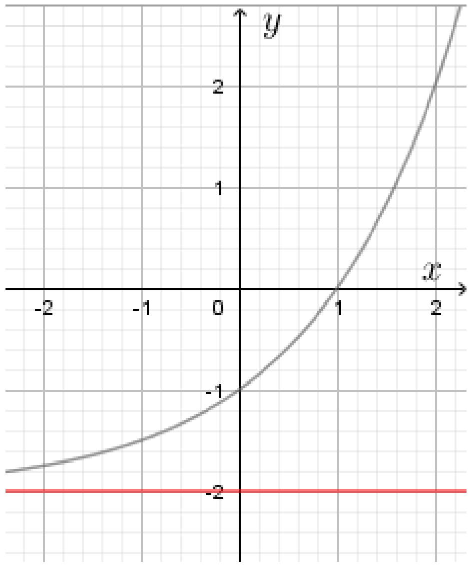
g) The universe is $\left(\frac{1}{3}, \infty\right)$ and the solution is $\emptyset$.

h) The universe is $(\log_2 7, \infty)$ and the solution is $x = 4$.

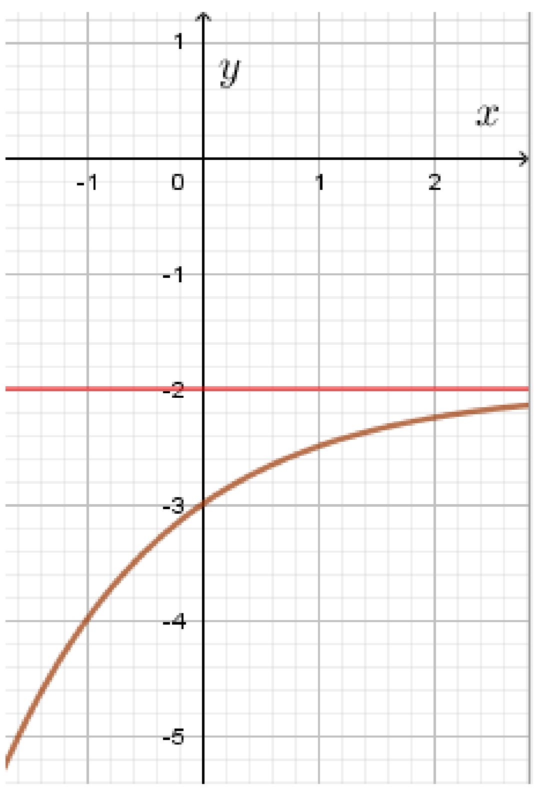
i) The universe is $[1, \infty)$ and the solution is $x_1 = 10;\ x_2 = 10{,}000$.

j) The universe is $(0, \infty)\setminus\{1\}$ and the solution is $x = 4$.

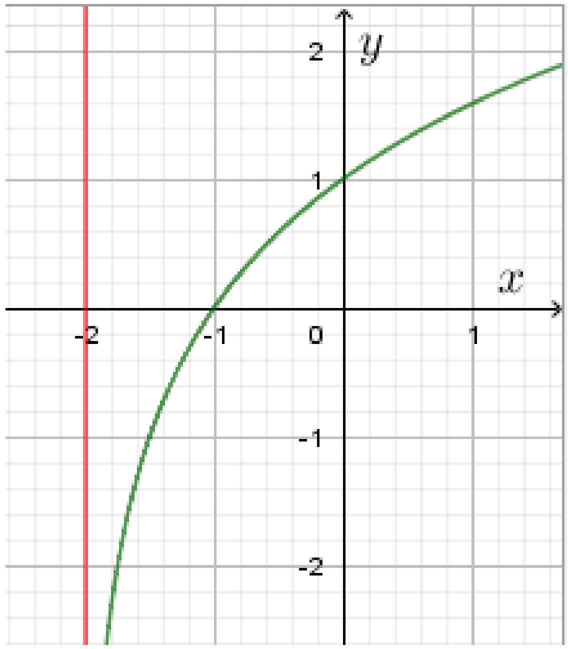
15. a) x-intercept = $(1,0)$, y- intercept = $(0,-1)$, the asympyote is the line $y = -2$, the domain is all real numbers and the range is $\{y: y > -2\}$.



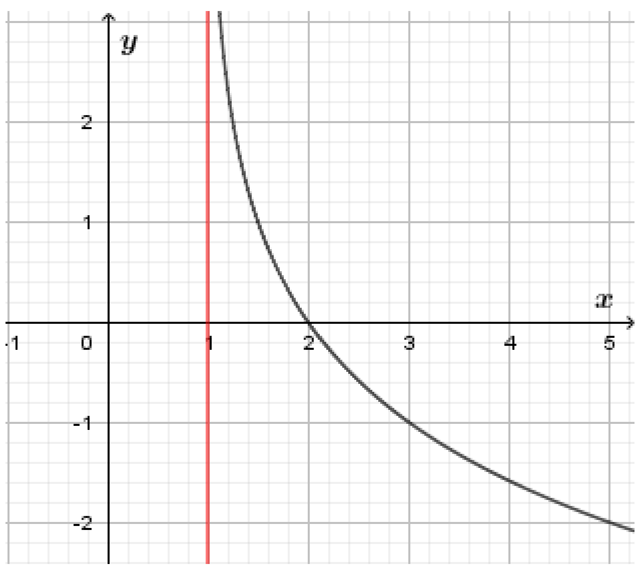
b) has no x -intercept, y- intercept = $(0, -3)$, the asympyote is the line $y = -2$, the domain is all real numbers and the range is $\{y: y < -2\}$.



c) x-intercept = $(-1,0)$, y- intercept = $(0,1)$, the asympyote is the line $x = -2$, the domain is $\{x: x > -2\}$ and the range is all real numbers.



d) x-intercept = $(2, 0)$, has no y- intercept, the asympyote is the line $x = 1$, the domain is $\{x: x > 1\}$ and the range is all real numbers.



16.a) $P(t) = P_0e^{kt}$

$P(1) = 8{,}000 = 1{,}000e^k,$

$e^k = \frac{8000}{1000} = 8,$

$\ln e^k = \ln 8$,

$k = \ln 8 \approx 2.0794$,

So, $P(t) = 1{,}000e^{2.0794t}$ is the function that models the population.

b) $t = 1.5$.

$P(1.5) = 1{,}000e^{2.0794(1.5)} = 1{,}000(\ 22.6274) \approx 22{,}627$,

Therefore, the population will be $22{,}627$ after 1.5 hrs.

c) $p = 15000$

$P(t) = 15{,}000 = 1{,}000e^{2.0794t}$,

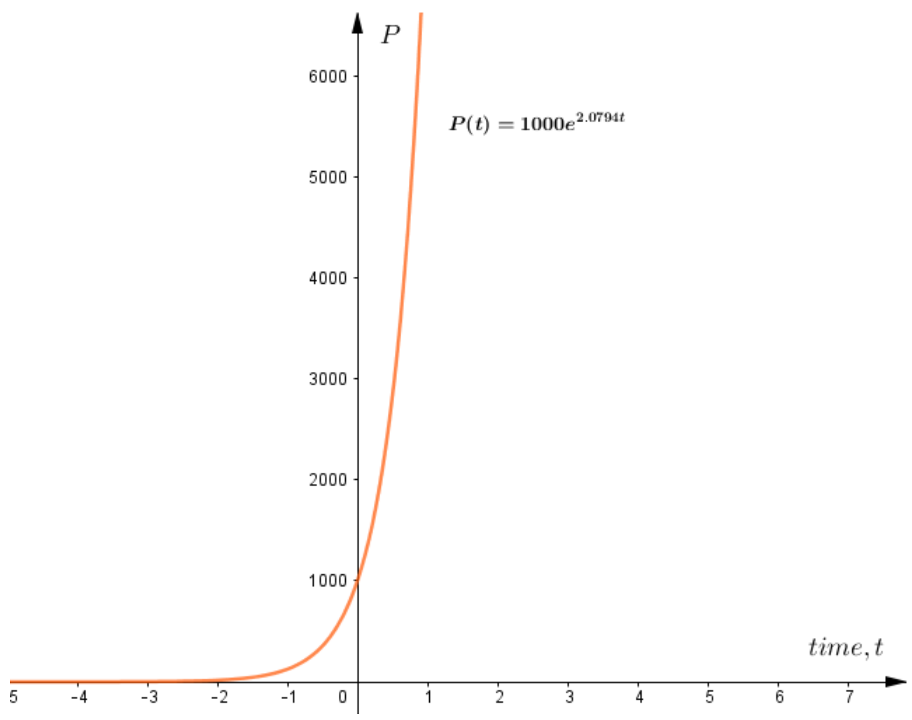
$15 = e^{2.0794t}$,

$2.0794t = \ln 15 \approx 2.7080$,

$t \approx \frac{2.7080}{2.0794} = 1.3$.

So, the population will reach 15,000 almost after 1.3 hours.

d)



The graph of the population function, $P(t) = 1{,}000e^{2.0794t}$.

17.

a) Here P =10,000, r = 7% = 0.07, n = 12,

$A(t) = P\left(1+\frac{r}{n}\right)^{nt} = 10{,}000\left(1+\frac{0.07}{12}\right)^{12t} = 10{,}000(1.0058)^{12t}$ is the formula for the amount in the account after t years.

b) Here $P = 10{,}000$, $r = 7\% = 0.07$, $n = 365$, $t = 5$.

$$A(5) = P\left(1+\frac{r}{n}\right)^{nt} = 10{,}000\left(1+\frac{0.07}{365}\right)^{365(5)}$$

$$= 10{,}000(1.001918)^{1825},$$

$$\approx 10{,}000(33.01617\ ),$$

$$\approx 330{,}161.7,$$

So, the amount in the account after 5 years will be about $330{,}161.7$.

c) Here $P(t) = 25{,}000$, $r = 7\% = 0.07$, $n = 2$,

$$A(t) = P\left(1+\frac{r}{n}\right)^{nt},$$

$$25{,}000 = 10{,}000\left(1+\frac{0.07}{2}\right)^{2t},$$

$$2.5 = (1.035)^{2t},$$

$$\log 2.5 = \log(1.035)^{2t},$$

$$2t = \frac{\log 2.5}{\log 1.035} = \frac{0.3979}{0.0170} = 23.4059,$$

$$t \approx 11.7029.$$

So, the amount in the account to grow to 25,000 takes 11.7029 years.

Unit-4

Trigonometric Functions (25 periods)

Introduction

In Grade 9 you have learnt the trigonometric ratios cosine, sine, and tangent in a right-angled triangle, and have used them to calculate the sides and angles of those triangles. In this unit we review the definition of these trigonometric ratios and extend the concept of cosine, sine and tangent. We define the cosine, sine and tangent as functions of all real numbers. These trigonometric functions are extremely important in science, engineering and mathematics.

We represent an angle as radian measure and convert degrees to radians and radians to degrees. We review the definition of the trigonometric ratios in a right-angled triangle and extend these ideas and define cosine, sine and tangent as functions of real numbers and we will discuss the properties of their graphs. Using trigonometric identities, we will solve real life problems involving trigonometric equations.

Unit outcomes: At the end of this unit, the students will be able to:

- define the basic trigonometric functions.
- sketch graphs of basic trigonometric functions.
- define reciprocals of basic trigonometric functions.
- identify trigonometric identities
- solve some examples on real life problems involving trigonometric equations

4.1 Radian measure of angle: Conversion between radian and degree measures (2 periods)

Competencies

At the end of this subunit, students will be able to:

- describe radian measure of an angle.
- convert radian measure to degree measure and vice versa.

Teaching Materials Required: Calculator, Compass, Ruler, Trigonometric table.

Vocabulary: Angle, Degree, Radian, Trigonometric values.

Introduction

In Mathematics, the radian is the standard unit of angular measure. This sub-topic will define radian and work through some problems involving radians. We will first introduce an entirely new view point related in measuring an angle

that is, the **Radian measure of an angle**. Once this is introduced, the conversion of radian measure to degree measure and vice-versa will be dealt with.

You may start the lesson by introducing the concept of an angle and its measurement. Explain that the degree is used as the unit of measure to indicate the size of an angle.

Discuss with the students that a degree is the measure of an angle formed by $\left(\frac{1}{360}\right)^{th}$ a complete rotation (see the figure).

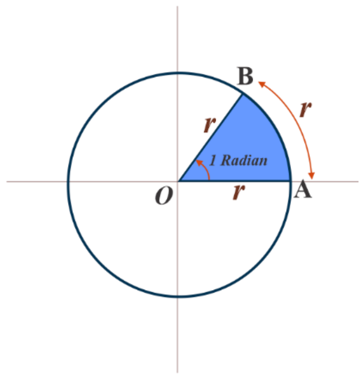
Explain to the students that the other unit of measurement of angle is the radian. Explain the radian in relation to the size of an angle subtended at the Centre by an arc whose length is equal to the length of the radius r.

To determine the number of degrees in 1 radian, we recall the formula from plane geometry that the circumference of a circle is 2π times its radius. This means that the radius "fits into" the circumference 2π times. Hence in a complete rotation an angle of 2π radians is generated.

In using the degree units, a complete rotation represents an angle of 360°. This gives us the following relationships.

1 rotation = 360° = 2π radians

180° = π radians



If a rotation from the initial side to terminal side is $\left(\frac{1}{360}\right)^{th}$ of a revolution, the angle is said to have a measure of one degree, written as 1°. A degree is divided into 60 minutes, and a minute is divided into 60 seconds. One sixtieth of a degree is called a minute, written as $1'$, and one sixtieth of a minute is called a second, written as $1''$. Thus, $1° = 60', \ 1' = 60''$.

Radians are the standard mathematical way to measure angles. One radian is equal to the angle created by taking the radius of a circle and stretching it along the edge of the circle. The radian is a pure mathematical measurement and, therefore, is preferred by mathematicians over degree measures. For use in everyday work, the degree is easier to work with, but for purely mathematical pursuits, the radian gives better results. You probably will never see radian measures used in construction or surveying, but it is a common unit in mathematics and physics.

To describe the measurement of an angle that is most familiar is the degree. To convert radians to degrees or degrees to radian you can refer the student's textbook. To assess students' level of understanding of this concept, you may ask the following. Convert

1. 90°, 60°, 30° to radian;
2. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{3\pi}{2}$ to degree.

Key points

- One radian is the measure of the central angle of a circle such that the length of the arc is equal to the radius of the circle.
- A full revolution of a circle $360°$ equals 2π radians. This means that 1 $radian = \frac{180°}{\pi}$.
- The formula used to convert between radians and degrees is angle in degrees = angle in radians$\cdot\frac{180°}{\pi}$.
- The radian measure of an angle is the ratio of the length of the arc to the radius of the circle $(\theta = sr)$. In other words, if s is the length of an arc of a circle, and r is the radius of the circle, then the central angle containing that arc measures radians.

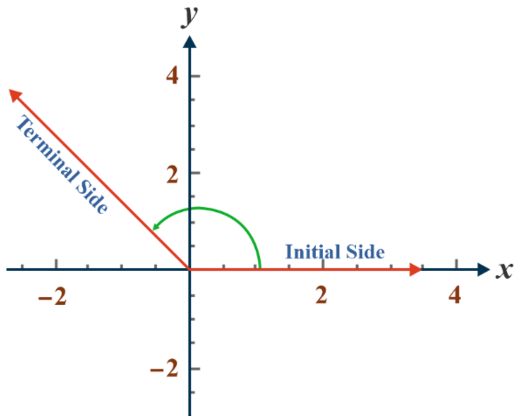
Assessment

Ask the students to describe the measurements of an angle; that is the radian measure and the degree measure. Ask them to convert radian measure to degree measure and the degree measure to radian measure. You can also form groups of students and assign them with some tasks related to angles for the purpose of assessing their understanding. To do this, you can assign Exercise 4.1.

Answers for activity 4.1

1. Angle is a measure of rotation of a given ray about its initial point.
2. An angle is in standard position in the coordinate plane if its vertex is located at the origin and one ray is on the positive x-axis. The ray on the x-axis is called the initial side and the

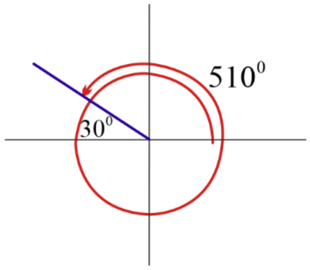
other ray is called the terminal side.



The initial side is where the angle starts and the terminal side is the ray where the measurement of the angle stops, therefore the terminal side defines the angle and if the vertex is at the origin (0, 0) then the angle will be in Standard Position.

3. The **positive angles** on the unit circle are measured with the initial side on the positive *x*-axis and the terminal side moving **counterclockwise** around the origin. The figure shows some positive angles labeled in both degrees and radians.

- positive angles are swept out in a counterclockwise direction; start by going up as shown below.

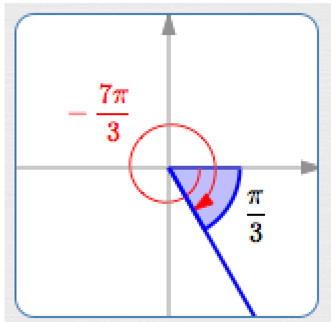


Notice that the terminal sides of the angles measuring 30° and 210°, 60° and 240° , and so on form straight lines. This fact is to be expected because the angles are 180°apart, and a straight angle measure180°. You see the significance of this fact when you deal with the trig functions for these angles.

Negative angles

Recall that graphing a negative angle means rotating clockwise. The graph below shows:

- negative angles are swept out in a clockwise direction; start by going down



If you measure angles clockwise instead of counterclockwise, then the angles have negative measures as shown in the figure above.

Answers for exercise 4.1

1.

a. $765° = 2 \times 360° + 45°$

b. $245° = 1 \times 180° + 65°$

c. $-740° = 2 \times -360° + (-20°)$

2. Help to draw the angles.

Answers for exercise 4.2

1. **a.** $\frac{\pi}{6}$ **b.** $\frac{\pi}{3}$ **c.** $\frac{4\pi}{3}$ **d.** $\frac{3\pi}{2}$ **e.** $\frac{-11\pi}{6}$

2. **a.** $18°$ **b.** $225°$ **c.** $-108°$ **d.** $-15°$ **e.** $132°$

3. **a.** $700°$ **b.** $360°$

Answers for exercise4.3

1. Given, Degree = 220° We know that, Radian = $\frac{degree \times \pi}{180°.} = \frac{220^0 \times \pi}{180°} = \frac{11}{9}\pi$.

2. I can see that I have 43°, but what do I do with the "0.1025" fractional part of a degree? I will treat this fractional portion like a percentage of the sixty minutes in one degree. Using this reasoning, I can then find out how many minutes are in this percentage of a degree:

$\frac{(0.1025\text{degrees})}{1}\frac{(60\text{minutes})}{1\text{minutes}} = 6.15$ minutes or 6 minutes and 0.15 of another minute.

Each minute has sixty seconds. I can apply the same reasoning and method as I did for the fractional portion of a degree to this fractional portion of a minute:

$\frac{(0.15\text{minutes})}{1}\frac{(60\text{secods})}{1\text{minutes}} = 9$ seconds

Then 43.1025° is equal to 43 degrees, 6 minutes, and 9 seconds.

3. **a.** $210°$ **b.** $315°$ **c.** $-630°$,

4. Here, an arc length $l = 37.4$ cm, and $\theta = 60° = \frac{60\pi}{180} = \frac{\pi}{3}$.

Since, $r = \frac{l}{\theta}$, we have radius of the circle,

$r = \frac{37.4 \times 3}{\pi} = \frac{37.4 \times 3 \times 7}{22} = 35.7$ cm.

5. In 60 minutes, the minute hand of a watch completes one revolution. Therefore, in 15 minutes, the minute hand turns through $\frac{1}{4}$ of a revolution.

Therefore, $\theta = \frac{1}{4} \times 360°$ or $\frac{\pi}{2}$.

Hence, the required distance travelled l is calculated as follows:

$$l = r\,\theta = 1.5 \times \frac{\pi}{2} = \frac{3}{4}\pi.$$

Assessment

You can use any one of assessment techniques such as: class activities, group discussions, homework/assignments, and/or tests/quizzes.

4.2 Basic Trigonometric Functions (17 periods)

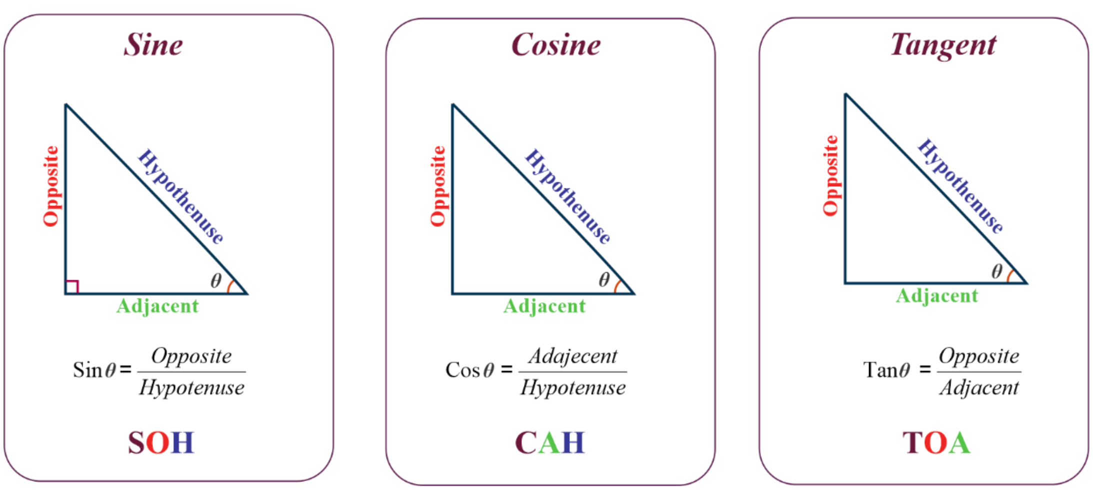
Introduction

There are six functions that are the core of trigonometry. There are three primary ones that you need to understand completely:

The three functions are:

Name	Abbreviation	Relationship to sides of the triangle
sine	sin	sin θ = Opposite/hypotenuse
cosine	cos	cos θ = Adjacent/hypotenuse
tangent	tan	tan θ = Opposite/adjacent

Calculating Sine, Cosine and Tangent



The **trigonometric functions** are real functions which relate an angle of a right-angled triangle to ratios of two side lengths. They are widely used in all sciences that are related to geometry, such as navigation, solid mechanics and many others. They are among the simplest periodic functions. The trigonometric functions most widely used in modern mathematics are the **sine**, the **cosine** and the **tangent**. Their reciprocals are respectively the **cosecant**, the **secant**, and the **cotangent**, which are less used.

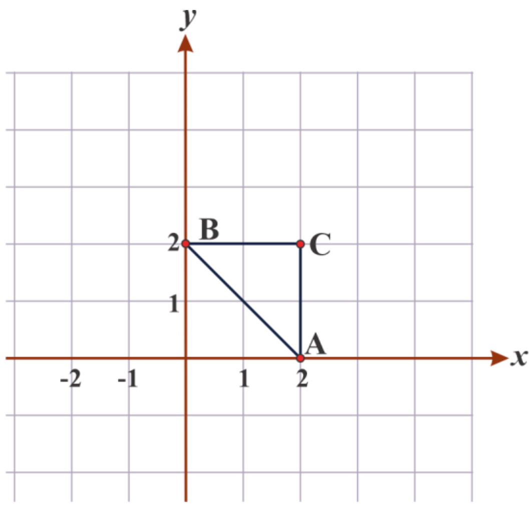
Answers for activity 4.2

1. **a.** $m(B\hat{A}C) = 45°$,

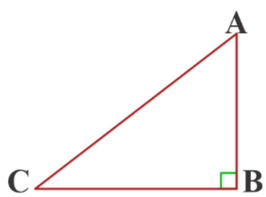
b. $\overline{BC} = 2\text{cm}$, $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$

$\overline{AB} = \sqrt{2^2 + 2^2} = 2\sqrt{2}cm$

c. $\sin 45° = \frac{1}{\sqrt{2}}$, $\cos 45° = \frac{1}{\sqrt{2}}$, and $\tan 45° = 1$



2. A right-angled triangle (also called a right triangle)is a triangle with a right angle (90°in it.

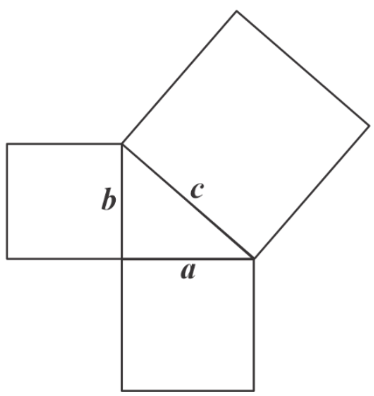


The little square in the corner tells us it is a right angled triangle.

The right angled triangle is one of the most useful shapes in all of mathematics!
(It is used in the Pythagoras Theorem and Sine, Cosine and Tangent for example). There are two types of right angled triangle:

- **Isosceles right-angled triangle**:-One right angle and two other equal angles always of 45°.Two equal sides.
- **Scalene right-angled triangle**:-One right angle and Two other unequal angles. No equal sides.
  Example: the " 3, 4, 5 Triangle" has a right angle in it.

The Pythagorean *(*or Pythagoras'*)* Theorem is the statement that the sum of (the areas of) the two small squares equal (the area of) the big one.



In algebraic terms, $a^2 + b^2 = c^2$ where c is the hypotenuse while a and b are the legs of the triangle.
The theorem is of fundamental importance in Euclidean Geometry where it serves as a basis for the definition of distance between two points. It's so basic and well known that, I believe, anyone who took geometry classes in high school couldn't fail to remember it long after other math notions got thoroughly forgotten.
The Theorem is reversible which means that its converse is also true. The converse states that a triangle whose sides satisfy $a^2 + b^2 = c^2$ is necessarily right angled. Euclid was the first to mention and prove this fact.

4.2.1The Sine, Cosine and Tangent Functions

Competencies

At the end of this subunit, students will be able to:

- use the trigonometric ratios to solve right angled triangles.
- find the trigonometric values of angles from trigonometric table.
- find the angle whose trigonometric value is given (using trigonometric table).

Key terms: Calculator, Compass, Ruler, Trigonometric table.

Key terms: Angle, Degree, Radian, Adjacent side, Opposite side, Hypotenuse, Special angles, Trigonometric values, Pythagoras theorem.

Introduction

This sub-unit is dedicated to discuss sine, cosine and tangent functions. Students need to recognize a right-angled triangle. How to proceed to deal with this sub-unit may differ from one teacher to another but some direction that you can use as a foundation is outlined below.

Sine, cosine, and tangent (abbreviated as sin, cos, and tan) are three primary trigonometric functions, which relate an angle of a right-angled triangle to the ratios of two sides length. The reciprocals of sine, cosine, and tangent are the secant, the cosecant, and the cotangent, respectively.

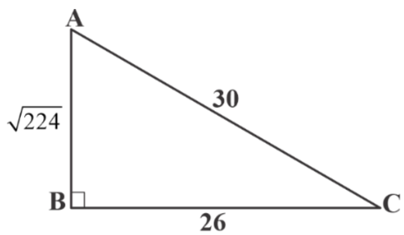
Teaching notes

The triangles which are mostly are in the focus of trigonometry are right-angled triangle, they have two legs and a hypotenuse, the side opposite to the right angle.

They also learnt in Grade 9 the concept of what is meant by Trigonometric ratio. Help the students to recall what they studied in the previous grade about trigonometric ratio. Then given a right-angled triangle, define the sine, cosine and tangent of the acute angles of the right-angled triangle in terms of the lengths of the sides of the triangle. Then after continue by examples.

The functions are a ratio of two side lengths, they always produce the same result for a given angle, regardless of the size of the triangle.

Example1:

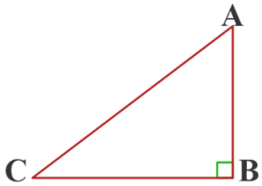


In the figure above, drag the point C. The triangle will adjust to keep the angle C at 30°. Note how the ratio of the opposite side to the hypotenuse does not change, even though their lengths do. Because of that, the sine of 30° does not vary either. It is always 0.5.

Example2: In the given $\Delta\ ABC$, right-angled at B and side $AB = 24\ cm, BC = 7\ cm$. Find

1. sin$A$, cos A
2. sin C, cos C

Solution:



Given $AB = 24cm, BC = 7cm$. That means, AC is hypotenuse.

$$AC^2 = AB^2 + BC^2$$

$$AC = \sqrt{AB^2 + BC^2} = \sqrt{24^2 + 7^2}$$

$$= \sqrt{625} = 25.$$

1. $\sin A = \frac{BC}{AC} = \frac{7}{25}$ and$\cos A = \frac{AB}{AC} = \frac{24}{25}$
2. $\sin C = \frac{AB}{AC} = \frac{24}{25}$ and $\cos C = \frac{BC}{AC} = \frac{7}{25}$

Once students are aware of the ways of determining Trigonometric values or determining measures of angles, they need to see how similar ratios can be determined for triangles with arbitrary lengths of sides. For this purpose, encourage and assist the students to do exercise 4.2. You have to help them discuss and summarize that for any acute angle A.

Assessment

Ask the students to define sine, cosine and tangent. You can also form groups of students and assign them with some tasks related to the three trigonometric ratios; sine, cosine and tangent for the purpose of assessing their understanding.

Answers for exercise 4.4

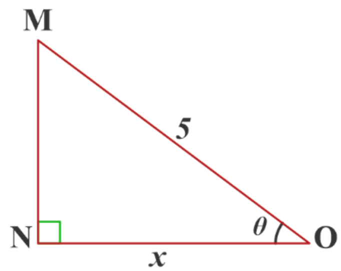
1. **a.** hypotenuse= $\sqrt{2}$, $\sin A = \frac{1}{\sqrt{2}}$, $\cos A = \frac{1}{\sqrt{2}}$ and $\tan A = 1$

b. hypotenuse= 2, $\sin A = \frac{\sqrt{3}}{2}$, $\cos A = \frac{1}{2}$ and $\tan A = \sqrt{3}$

2. Given: $\cos(\theta) = \frac{1}{2}$ and measure of $\overline{MO} = 5$. Then $\cos\theta = \frac{x}{5}$. So, $x = 5\cos\theta$ which implies

$$x = 5(\frac{1}{2}) = 2.5$$

Answers for exercise 4.5



1. $\tan 60° = \frac{15}{x}$ which implies $x = \frac{15}{\tan 60°}$ which implies again $x = \frac{15}{\sqrt{3}} = 5\sqrt{3}$.

2. $\sin 30° = \frac{h}{100}$ which implies $h = 100\sin 30° = 100 \times \frac{1}{2} = 50\text{m}$

In this lesson, three trigonometric ratios (secant, cosecant and cotangent) will be defined and applied. These involve ratios of the lengths of the sides in a right triangle.

Example:

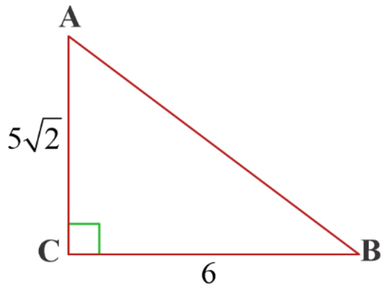
In a right triangle, the legs are 6 and $5\sqrt{2}$. What is the secant of the angle opposite the side of measure 6? What is the cotangent of this angle?

Solution:

$$AB^2 = BC^2 + AC^2$$

$$AB = \sqrt{BC^2 + AC^2} = \sqrt{36 + 50} = \sqrt{86}$$

The angle opposite to the side of measure 6 is A. So, $\sec A = \frac{AC}{AB} = \frac{5\sqrt{2}}{\sqrt{86}}$ and $\cot A = \frac{BC}{AB} = \frac{6}{5\sqrt{2}}$



Assessment

You can use any one of assessment techniques such as: class activities, group discussions, homework/assignments, and/or tests/quizzes.

Answers for activity4.3

a. $\cos A = \frac{4}{5}$ **b.** $\sin A = \frac{3}{5}$

When considering triangles, we are limited to angles less than $90°$. However, trigonometry is equally applicable to all angles, from $0°$ to $360°$. To understand how the trigonometric functions work with angles greater than $90°$, it is helpful to think about triangles constructed within a circle.

Anything to the left of the Centre has an x value of less than 0, or is negative, while anything to the right has appositive value. Similarly, anything below the Centre point has a y value of less than 0, or is negative and any point in the top of the circle has a positive value.

Answers for exercise 4.6

1. **a.** $\sin(0\times 360° + 0°) = \sin 0°, n \in Z = 0$. $\cos(0\times 360° + 0°) = \cos 0° = 1$.

$\tan(0\times 360° + 0°) = \tan 0° = 0$.

b. $\sin(1\times 360° + 90°) = \sin 90° = 1$. $\cos(1\times 360° + 90°) = \cos 90° = 0$.

$\tan(0\times 360° + 90°) = \tan 90°$ =Undefined.

c. $\sin(1\times 360° + 180°) = \sin 180° = 0$. $\cos(1\times 360° + 180°) = \cos 180° = -1$.

$\tan(0\times 360° + 180°) = \tan 180° = 0$.

d. $\sin(1\times 360° + 270°) = \sin 270° = -1$. $\cos(1\times 360° + 270°) = \cos 270° = 0$.

$\tan(1\times 360° + 270°) = \tan 270° = 0$.

2.

degree	$-360°$	$-450°$	$-270°$	$-180°$	$-90°$	$90°$	$180°$	$720°$
$\sin x$	0	1	1	0	-1	1	0	0
$\cos x$	1	0	0	-1	0	0	-1	1
$\tan x$	0	Undefined	Undefined	0	Undefined	Undefined	0	0

Answers for exercise 4.7

1. **a.** Fourth quadrant b. Second quadrant **c.** Fourth quadrant

2.

degree	0°	120°,	135°,	150°	210°	240°	330°
radian	0	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{11}{6}\pi$
$\sin x$	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
$\cos x$	1	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$\tan x$	0	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$-\sqrt{3}$

Answers for exercise 4.8

1. a. $\sec 45° = \sqrt{2}$ **b.** $\sec\frac{2\pi}{3} = -2$ **c.** $\sec\left(-\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}$ **d.** $\csc 30° = 2$ **e.** . $\csc\frac{3\pi}{4} = \sqrt{2}$

f. $\csc(-300°) = \frac{2}{\sqrt{3}}$ **g.** $\cot 60° = \frac{1}{\sqrt{3}}$ **h.** $\cot\frac{5\pi}{6} = -\sqrt{3}$ **i.** $\cot\left(-\frac{5\pi}{4}\right) = -1$

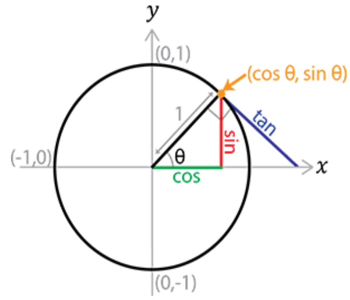
2. a. $\sec\frac{10\pi}{3} = -\sec\frac{\pi}{3} = -\frac{1}{\cos\frac{\pi}{3}} = -2$ and $\csc\left(-\frac{7\pi}{2}\right) = \csc\frac{\pi}{2} = \frac{1}{\sin\frac{\pi}{2}} = 1.$

Therefore, $\sec\frac{10\pi}{3} + \csc\left(-\frac{7\pi}{2}\right) = -2 + 1 = -1.$

b. The related angle of 330° is 30°and the related angle of 480° secant is the reciprocal of cosine, and the cosine is positive in the fourth quadrant. so, the secant is positive in the fourth quadrant. So,

$$\sec 330° = \sec 30° = \frac{1}{\cos 30°} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

cotangent is the reciprocal of tangent, and tangent is negative in the second quadrant so is the cotangent.



$$\cot 480° = -\cot 60° = -\frac{1}{tan 60°} = -\frac{1}{\sqrt{3}}.$$

Therefore, $\sec 330° + \cot 480° = \sec 30° + (-\cot 60°.)$

$$= \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

4.2.2 Trigonometric Values of Angles

Competencies

At the end of this subunit, students will be able to:

- find the trigonometric values of angles from trigonometric table.
- find the angle whose trigonometric value is given (using trigonometric table).

Teaching Materials Required: Calculator, Compass, Ruler, Trigonometric table.

Vocabulary: Angle, Degree, Radian, Adjacent side, Opposite side, Hypotenuse, Special angles, Trigonometric values, Pythagoras theorem.

Introduction

The trigonometric functions are based on the unit circle, that is a circle with radius $r = 1$. Since the circumference of a circle with radius r is $C = 2\pi r$, the unit circle has circumference 2π.

Teaching notes

For any point (x, y) on the unit circle, the associated angle θ can be measured in two different ways: radian and degree measure we already discussed this in the previous sub-sections. Each time we choose an angle , we find a unique point (x, y) on the unit circle. Hence both "x" and "y" can be considered functions of θ. Since these particular functions are of great importance to both pure and applied mathematics, they are given special names and symbols, and are called the trigonometric functions.

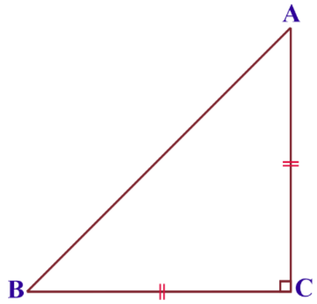
Note that a right-angle triangle is formed, with a hypotenuse of length 1, and two adjacent sides of equal length, that is x=y. Let's denote that length "x". By the Pythagorean theorem, leads to **Activity 4.5.** Once you know how to find the trigonometric functions for the above special angles, it is important to learn how to extend your knowledge to any angle that is based on one of 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$ or $\frac{\pi}{2}$, such as for example $\frac{2\pi}{3}$, $-\frac{\pi}{6}$, $\frac{7\pi}{4}$, $-\frac{5\pi}{2}$, and others.

Answers for activity4.4

ΔABC is isosceles with hypotenuse AB. So, $AC = BC$ and let x be their length.

Using Pythagoras's theorem,

$(AB)^2 = (AC)^2 + (BC)^2$



$= x^2 + x^2 = 2x^2$

$= \sqrt{2}x$

$m(\angle ABC) + m(BAC) = 90°$. . But $\angle ABC \equiv \angle BAC$ base angles of isosceles triangle.

$2m(\angle ABC) = 90°$. So, $m(\angle ABC) = 45°$

Assessment

You can use any one of assessment techniques such as: class activities, group discussions, homework/assignments, and/or tests/quizzes.

If you need to calculate the sine, cosine, or tangent of an angle other than the ones discussed above, you may need to use your calculator. Consult your calculator manual, if necessary, on how to use the trigonometric functions. You should be aware of two facts:

1. In most cases, your calculator will not give you exact answers, but rather decimal approximations. **For example**, your calculator will tell you that the sine of a 45° angle is approximately .70710678, rather than giving you the exact answer is $\frac{\sqrt{2}}{2}$.

2. You need to set your calculator to the appropriate angle measure, degrees or radians. Otherwise, your calculator might return the sine of the angle θ =3.1415... degrees, when you enter "sin (π)" looking for the sine of $\theta = 180°$.

Once students are aware of the ways of determining Trigonometric values or determining measures of angles, they need to see how similar ratios can be determined for triangles with arbitrary lengths of sides. For this purpose, encourage and assist the students to do more activity in group.

Answers for exercise 4.9

1. The **unit circle** is in the xy −plane and it is a circle with a radius of 1, and a center at the origin.
2. **a.** $\sin 30° = \frac{1}{2}$ and $\cos 60° = \frac{1}{2}$ **b.** $\sin 45° = \frac{\sqrt{2}}{2}$ and $\cos 45° = \frac{\sqrt{2}}{2}$.
3. If θ is an acute angle, then $\sin (90 - \theta)° = \cos\theta$

   $\cos 4A = \sin(90° - 4A) = \sin(A - 20°)$ which implies $90° - 4A = A - 20°$ and $A = 22°$.
4. $\cos 59° = 0.5150$
5. $\cos (90° - \theta) = \sin\theta = \frac{3}{5}$

6. $\sin\left(\frac{\pi}{2}-\alpha\right)=\cos\alpha=\frac{4}{5}$
7. $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta=\ k$
8. $\frac{1}{\tan(90°-\beta)}=\frac{1}{\cot\beta}=\tan\beta=\frac{m}{n}$

Reference Angle

The reference angle always has the same trig function values as the original angle. Notice the word values there. The sign may not be the same, but the value always will be. This is useful for common angles like 45° and 60° that we will encounter over and over again. Once we know their sine, cosine, and tangent values, we also know the values for any angle whose reference angle is also $45°\ or\ 60°$. As for the sign, remember that Sine is positive in the $1st$ and $2nd$ quadrant and Cosine is positive in the $1st$ and $4th$ quadrant.

When an angle is greater than 360°, that means it has rotated all the way around the coordinate plane and kept on going. In order to find its reference angle, we first need to find its corresponding angle between 0° and 360°. This is easy to do. We just keep subtracting 360 from it until it's below 360. For instance, if our angle is 544°, we would subtract 360° from it to get 184° (544° – 360° = 184°). Now we would notice that it's in the third quadrant, so we'd subtract 180° from it to find that our reference angle is 4°.

When an angle is negative, we move the other direction to find our terminal side. This means we move clockwise instead of counterclockwise when drawing it. Or we can calculate it by simply adding it to 360°. For instance, if our given angle is(– 110°), then we would add it to 360° to find our positive angle of 250°(– 110° + 360° = 250°). Now we would have to see that we're in the third quadrant and apply that rule to find our reference angle (250° – 180° = 70°).

How to find reference angle in radians?

It's easier than it looks! The procedure is similar to the degree given in the textbook:

1. Choose your angle - for example, $\frac{28\pi}{9}$
2. For angles larger than 2π, subtract the multiples of 2π, until you a left with a value smaller than a full angle, as before. In our case, we're left with $\frac{10\pi}{9}$.
3. Determine the quadrants:

- $0°$to $\frac{\pi}{2}$ - first quadrant, so reference angle=angle,
- $\frac{\pi}{2}$to π - second quadrant, so reference angle= π −angle,
- π to $\frac{3\pi}{2}$ - third quadrant, so reference angle= $angle - \pi$,
- $\frac{3\pi}{2}$to 2π - fourth quadrant, so reference angle= 2π −angle.

$\frac{10\pi}{9}$is a bit more than π.So, it lies in the third quadrant. In this example, the reference angle is

$angle - \pi = \frac{10\pi}{9} - \pi = \frac{\pi}{9}$.

Answers for exercise 4.10

1. **a.** Since $\theta = 109°$is a second quadrant, $\theta_R = 180° - 109° = 71^0$

   b. Since $\theta = 345°$ is a fourth quadrant, $\theta_R = 360° - 345° = 15°$

   c. Since $\theta = 190°$ is a third quadrant, $\theta_R = 190° - 180° = 10°$

   d. Since $\theta = 140°$ is a second quadrant, $\theta_R = 180° - 140° = 40°$

   e. Since $\theta = \frac{5}{3}\pi$ is a fourth quadrant, $\theta_R = 2\pi - \frac{5}{3}\pi = \frac{\pi}{3}$

   f. Since $\theta = \frac{7}{4}\pi$ is a fourth quadrant, $\theta_R = 2\pi - \frac{7}{4}\pi = \frac{\pi}{4}$

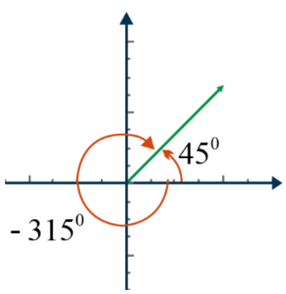
   g. Since $\theta = \frac{14}{3}\pi$ is a second quadrant, $\theta_R = \pi - \frac{2}{3}\pi = \frac{\pi}{3}$

Co-terminal Angle

Answers for activity4.5

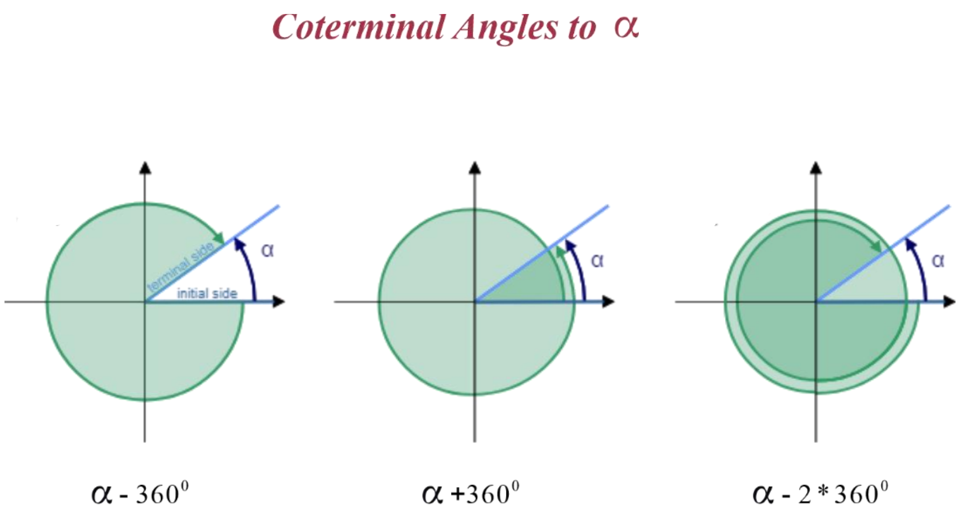
Co-terminal angles are two angles that have the same initial side and the same terminal side, "Co" is the *Latin* prefix that means two things join or come together and "terminal" means to end or conclude. Therefore, Coterminal means two things end or conclude together at the same place!

For example, notice that $45°$ and $-135°$ are coterminal angles



Coterminal angles are those angles that share the terminal side of an angle occupying the standard position. The standard position means that one side of the angle is fixed along the positive x-axis, and the vertex is located at the origin.

In other words, two angles are conterminal when the angles themselves are different, but their sides and vertices are identical. Also, you can remember the conterminal angles definition as angles that differ by a whole number of complete circles**.** Look at the picture below, and everything should be clear!



So, as we said all the conterminal angles start at the same side (initial side) and share the terminal side.

The thing which can sometimes be confusing is the difference between the reference and conterminal angles definitions. Remember that they are not the same thing - the reference angle is the angle between the terminal side of the angle and the x-axis, and it's always in the range of [0°, 90°] or $[0, \frac{\pi}{2}]$.

A useful feature is that in trigonometry, any two conterminal angles have exactly the same trigonometric values. So, if β and α are coterminal, then their sines, cosines and tangents are all equal.

To determine the conterminal angle between 0° and 360°, all you need to do is to use a modulo operation in other words, divide your given angle by the 360° and check what the remainder is.

We'll show you how it works with two examples - covering both positive and negative angles. We want to find a conterminal angle with a measure of θ such that $0° \leq \theta < 360°$, for a given angle equal to:

$420°$

- $420° \text{mod} 360° = 60°$

How to do it manually?

- First, divide one number by the other, rounding down (towards the floor):
  $420/360 = 1$
- Then, multiply the divisor by the obtained number (called the quotient

$$360 \times 1 = 360$$

- Subtract this number from your initial number: $420 - 360 = 60$

Substituting these angles into the conterminal angles formula gives

$$420° = 360° \times 1 + 60°$$

$$-858°$$

$$-858 \text{ mod } 360 = 220°$$

Repeating the steps from above:

- $-858/360 = -3$
- $360 \times (-3) = -1080$
- $-858 + 1080 = 222$

So, the conterminal angles formula, $\beta = \pm 360 \times k$, will look like this for our negative angle example:

$$-858° = 222° - 360° \times 3$$

The same works for the $[0,2\pi)$ range, all you need to change is the divisor - instead of $360°$, use 2π.

Now, check the results with our **coterminal angle calculator** if available. Help the students to do more examples. This should be developed by examples given in the textbook.

Examples

1. Find one negative angle that is conterminal to 30°.

   A negative angle moves in a clockwise direction. In this case, to find the negative conterminal angle, subtract 360° from 30°.

$$30° - 360° = -330°$$

2. Find one negative angle that is conterminal to 415°.

   $415° - 360° = 55°$. Although 55° is a conterminal angle to 415°, this is not a solution to the problem. The problem specifically asked for a negative angle, so the process needs to take place one more time.

$$55° - 360° = -305°$$

Answers for exercise 4.11

1. a. Not supplementary **b.** supplementary **c.** Not supplementary.

2. $55° + 360° = 415°$ is positive conterminal angle with $55°$.

$55° - 360° = -305°$ is negative co-terminal with $55°$.

3. $\frac{\pi}{3} + 2\pi = \frac{7\pi}{3}$ is positive conterminal angle with $\frac{\pi}{3}$. And $\frac{\pi}{3} - 2\pi = -\frac{5\pi}{3}$ is negative conterminal angle with $\frac{\pi}{3}$.

Answers to Exercise4.12

1. a. $\sin 390° = \sin 30° = \frac{1}{2}$

b. $\cos\frac{10\pi}{3} = -\cos\frac{\pi}{3} = -\frac{1}{2}$

c. $\tan(-420°) = -\tan(60°) = -\sqrt{3}$

d $\sin(-660°) = \sin(-300°)^0 = \sin 60° = \frac{\sqrt{3}}{2}$

e. $\cos\left(\frac{41\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

f. $\tan\left(-\frac{19\pi}{3}\right) = \tan(-\frac{\pi}{3}) = -\tan\frac{\pi}{3} = -\sqrt{3}$

g. Find the reference angle for $\theta = 135°$

$\theta_R = 180° - 135° = 45°$. $135°$is a second quadrant angle. In the second quadrant angle, only sine is positive.

Therefore, $4\cos(135°) = 4(-\cos 45°) = 4 \times -\frac{\sqrt{2}}{2} = -2\sqrt{2}$

h. $\theta_R = 360° - 300° = 60°$. So, $\frac{5}{3}\cos 300° = \frac{5}{3}cos 60° = \frac{5}{3} \times \frac{1}{2} = \frac{5}{6}$

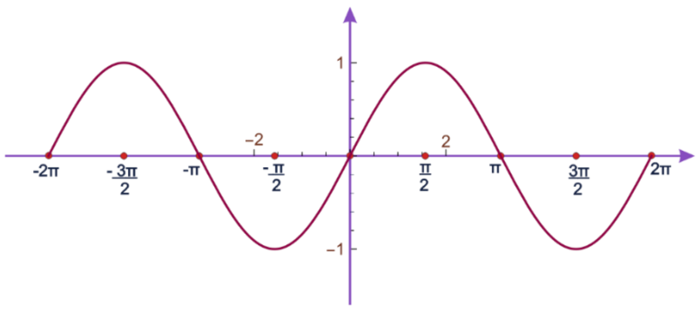
i. $\theta = -150° = 360° - 150° = 210°$ and $\theta R = 210° - 180° = 30°$

$$-2\cos(-150°) = -2(\cos 210°) = -2(-\cos 30°) = -2 \times -\frac{\sqrt{3}}{2} = \sqrt{3}$$

Answers for activity 4.6

1.

θ	$-360°$	$-270°$	$-90°$	$-30°$	$0°$	$30°$	$90°$	$270°$	$360°$
$y = \sin\theta$	0	1	-1	$\frac{-1}{2}$	0	$\frac{1}{2}$	1	-1	0



The graph of $y = \sin\theta$

2 The period of sine function is 360°.

Answers for exercise4.13

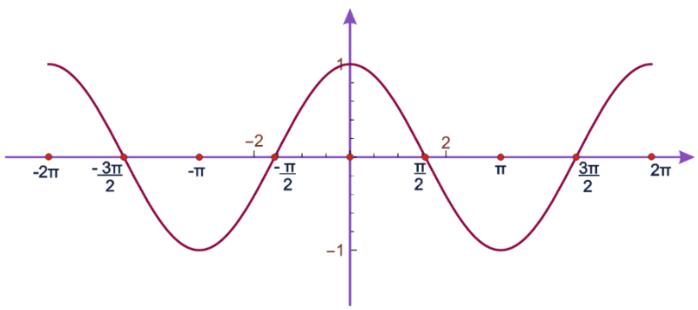
1. It is solved in the textbook.
2. $A = \frac{\pi}{2}$, $B = \frac{\pi}{3}$, $C = -\frac{\pi}{2}$, $D = -\frac{4\pi}{3}$, $E = 2\pi$, $F = -\pi$, $G = \frac{1}{2}$, $H = -\frac{\sqrt{3}}{2}$

Answers for activity 4.7

1.

θ	$-360°$	$-270°$	$-90°$	$-30°$	$0°$	$30°$	$90°$	$270°$	$360°$
$y = \cos\theta$	1	0	0	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	0	0	1

2.



The graph of $y = \cos\theta$

3 The period of cosine function is $360°$.

Answers for exercise4.14

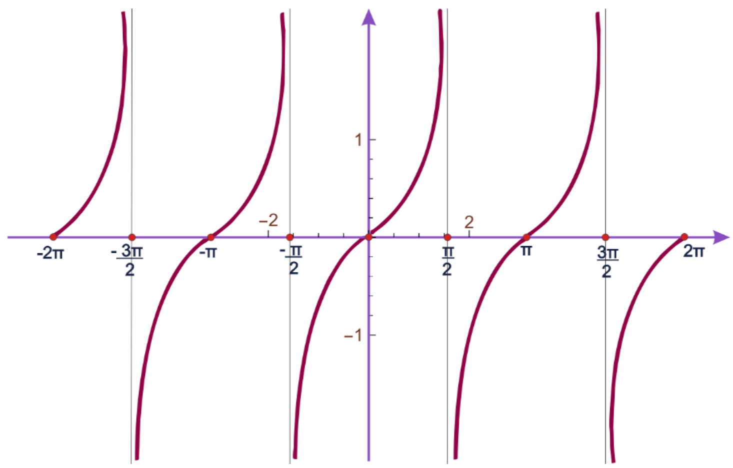
2. It is solved in the textbook.

3. $A = \frac{\pi}{2}$, $B = \frac{\pi}{3}$, $C = -\frac{\pi}{3}$, $D = \pi$, $E = 2\pi$, $F = -\frac{3\pi}{2}$, $G = \frac{\sqrt{3}}{2}$, $H = -\frac{1}{\sqrt{2}}$

Answers for activity 4.8

1.

θ	$-360°$	$-270°$	$-90°$	$-30°$	$0°$	$30°$	$90°.$	$270°.$	$360°.$
$y = \tan\theta$	0	$\nexists$	$\nexists$	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	$\nexists$	$\nexists$	0



The graph of $y = \tan\theta$

2. The period of cosine function is $360°$.

3. For which values of θ is $y = \tan\theta$ not defined?

Answers for exercise4.15

1. It is solved in the textbook.

2. $A = \frac{\pi}{2}$, $B = \frac{\pi}{4}$, $C = \pi$, $D = -\pi$, $E = -\frac{\sqrt{3}}{2}$, $F = \sqrt{3}$, $G = -\frac{1}{\sqrt{3}}$,

4.3 Trigonometric Identities and Equations (10 periods)

Competencies

At the end of this subunit, students will be able to:

- Verify the fundamental trigonometric identities.

- Simplify trigonometric expressions using algebra and the identities.
- Solve linear trigonometric equations in sine and cosine.
- Solve equations involving a single trigonometric function.

Introduction

In this sub-topic, we will discuss trigonometric identities. We have already learned the primary trigonometric ratios sin, cos, and tan. We have also learned that there are three other trigonometric ratios sec, cosec, and cot which are the reciprocal of sin, cos, and tan respectively. How are these trigonometric ratios (sin, cos, tan, sec, csc, and cot) connected with each other? They are connected through trigonometric identities.

Teaching notes

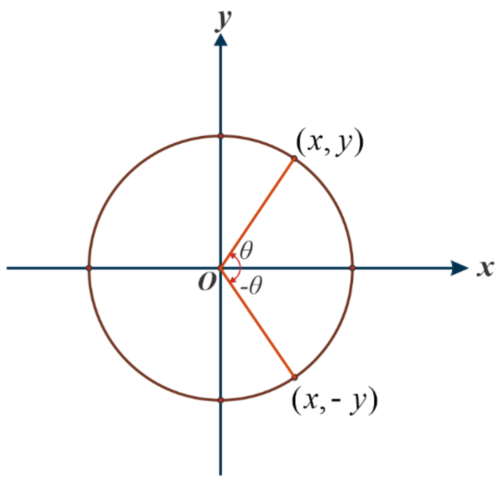
you can start your lesson by asking to prove the identity of $cos(-\theta) = cos\theta$ without graphing.

First, recall that $cos\theta = x$, where (x, y) is the endpoint of the terminal side of θ on the unit circle.

Now, if we have $\cos(-\theta)$, what is its endpoint? Well, the negative sign tells us that the angle is rotated in a clockwise direction, rather than the usual counter-clockwise. If we make this rotation, we see that $cos(-\theta) = x$ as well, as illustrated in the following diagram.

We know that $\cos\theta = x$, so we can set the two expressions equal to one another.

$$\cos\theta = \cos(-\theta)$$



Ask students a question like, without graphing, show that $sin(-\theta) = -sin\theta$.Then continue your discussion with different types identities; the reciprocal, tangent(quotient), Pythagorean identities from the textbook.

Trigonometric identities are true for all replacement values for the variables for which both sides of the equation are defined. Conditional trigonometric equations are true for only some replacement values. Solutions in a specific interval, such as $0 \leq x \leq 2\pi$, are usually called primary solutions. A general solution is a formula that names all possible solutions.

The process of solving general trigonometric equations is not a clear-cut one. No rules exist that will always lead to a solution. The procedure usually involves the use of identities, algebraic manipulation, and trial and error. The following guidelines can help lead to a solution.

If the equation contains more than one trigonometric function, use identities and algebraic manipulation (such as factoring) to rewrite the equation in terms of only one trigonometric function. Look for expressions that are in quadratic form and solve by factoring. Not all equations have solutions, but those they have solution usually can be solved using appropriate identities and algebraic manipulation. Look for patterns.

For example, to find the exact solution of $\cos^2\alpha = -\cos\alpha + \sin^2\alpha,\ 0° \leq \alpha \leq 360°$

First, transform the equation by using the identity

$$sin^2\,\alpha\ +\ cos^2\alpha\ =\ 1. \text{ This implies } sin^{\,2}\,\alpha = 1 - cos^2\alpha$$

$$\cos^2\alpha = -\cos\alpha + 1 - \cos^2\alpha$$

$$2\cos^2\alpha + \cos\alpha - 1 = 0$$

$$(2\cos\alpha - 1)(\cos\alpha + 1) = 0$$

$\cos\alpha + 1 = 0$ or $2\cos\alpha - 1 = 0$ which implies $2\cos\alpha = 1$,

$\cos\alpha = -1$ which implies $\alpha = \cos^{-1}(-1) = 180°$ or $\cos\alpha = \frac{1}{2}$, $\alpha = 60°, 300°$.

Answers for exercise 4.13

1.

a. $\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x = 1 - \left(\frac{2}{3}\right)^2 = 1 - \frac{4}{9} = \frac{5}{9}$

x is the angle of the first quadrant, so $sinx > 0$

Thus $\sin x = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$

b. $\sin 2x = 2\sin x\cos x = 2\cdot\frac{\sqrt{5}}{3}\cdot\left(\frac{2}{3}\right) = -\frac{4\sqrt{5}}{9}$

c. $\cos 2x = \cos^2 x - \sin^2 x = \left(\frac{2}{3}\right)^2 - \left(\frac{\sqrt{5}}{3}\right)^2 = -\frac{1}{9}$

2. $\sec x = -2$ and $\cos x = -\frac{1}{2}$

3. a.

$$sin15° = \pm\sqrt{\frac{1 - cos30°}{2}}$$

$$= \pm\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} \quad \text{(But 15° is in the first quadrant)}$$

$$= \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

b. $sin\frac{\pi}{8} = \pm\sqrt{\frac{1-cos\frac{\pi}{4}}{2}} = \pm\sqrt{\frac{2-\sqrt{2}}{4}} = \pm\frac{\sqrt{2-\sqrt{2}}}{2}$. But $\frac{\pi}{8}$ is in the first quadrant.

So, $sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$

c. $cos\frac{3\pi}{8} = \pm\sqrt{\frac{1+cos\frac{3\pi}{4}}{2}} = \pm\sqrt{\frac{2-\sqrt{2}}{4}} = \pm\frac{\sqrt{2-\sqrt{2}}}{2}$. But $\frac{3\pi}{8}$ is in the first quadrant.

So, $cos\frac{3\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$.

Answers for exercise 4.14

1. This problem asks for the angles whose sine equals $\frac{1}{\sqrt{2}}$. The problems also states that the angle must lie between 0°and360°. So, the angles that satisfy the equation are 45° and135°.

2. $\sqrt{2}cosx = -1$ and $cosx = -1/\sqrt{2}$. The cosine is negative in the second and third quadrant. So, the angles which satisfy the given equation are 135° and 225°.

3. $2 + \sqrt{3}\csc\theta = 0$ (The equation)

$\sqrt{3}\csc\theta = -2$ Added -2 to both sides

$\csc\theta = -\frac{2}{\sqrt{3}}$ Divided by $\sqrt{3}$

If $\csc\theta = -\frac{2}{\sqrt{3}}$, then $\sin\theta = -\frac{\sqrt{3}}{2}$. The $\sin 60° = \frac{\sqrt{3}}{2}$ and the sine is negative in the third and fourth quadrants. So, the angles which satisfy the given equation are 240° and300°.

4. $\sin^2 x + \cos^2 x = 1$ and $\sin^2 x = 1 - \cos^2 x = 1 - \left(\frac{\sqrt{5}}{3}\right)^2 = 1 - \frac{5}{9} = \frac{4}{9}$. But x is the angle of the second quadrant, so $\sin x > 0$.

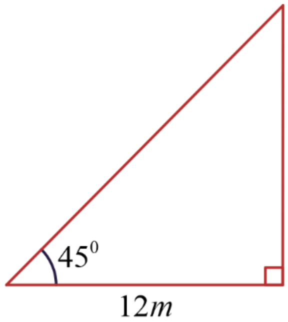
Thus $\sin x = \sqrt{\frac{4}{9}} = \frac{2}{3}$.

4.4 Applications (3 periods)

Answers for activity4.9

1. Let x be the length of the ramp.

$\cos 45° = \frac{12}{x}$ which implies $x = \frac{12}{\frac{1}{\sqrt{2}}} = 12\sqrt{2}$m



2. $\cos\theta = \frac{50}{100} = \frac{1}{2}$ which implies $\theta = 60°$ or $\frac{\pi}{3}$.

Answers for exercise 4.15

1. $\tan 45° = \frac{10}{x}$. Which implies $x = \frac{10}{tan45°} = 10$m.

2. $\tan 30° = \frac{h}{15}$. Which implies $h = 15\tan 30° = 15\left(\frac{1}{\sqrt{3}}\right) = \frac{15}{\sqrt{3}}$m.

Answers for review exercise on Unit 4

1. **a.** $\frac{3\pi}{4}$ **b.** 3π **c.** $\frac{7\pi}{6}$ **d.** $\frac{5\pi}{6}$

2. **a.** $20°$ **b.** $120°$ **c.** $\frac{540°}{7}$ **d.** $40°$

3. **a.** π **b.** -10π **c.** $\frac{11\pi}{6}$ **d.** $\frac{-5\pi}{4}$ **e.** 3π **f.** $\frac{-\pi}{5}$

4. **a.** 0.64279 **b.** 0.83910

5. **a.** $53°$ **b.** $36°$

6. **a.** To find sin 236°, first we consider the quadrant that the angle 236° belongs to. This is done by placing the angle in standard position. We see that the 236° angle lies in quadrant III so that the sine value is negative. The reference angle corresponding to 236° is:

$$\theta_R = 236° - 180° = 56°$$

Using trigonometric table, $\sin 236° = -\sin 56° = -0.8290$

b. To find the value of $\cos 693^0$ first observe that 693° is greater than 360°.

The period of cosine function is 360°. Dividing 693° by 360° we obtain

$$693° = 1 \times 360° + 333°$$

This means that the 693° angle is co terminal with the 333° angle. i.e.,

$$\cos 693° = \cos 333°.$$

Since the terminal side of 333° is in quadrant IV, the reference angle is

$\theta_R = 360° - 333° = 27°$

Cosine is positive in quadrant IV, so $\cos 333° = \cos 27° = 0.8910$.

Hence, $\cos 693° = 0.8910$

7. a. $\frac{5\pi}{4}$ **b.** $\frac{7\pi}{4}$ **c.** $\frac{11\pi}{6}$ **d.** $\frac{7\pi}{3}$ **e.** 5π **f.** $-\frac{4\pi}{3}$

8. a. $-295°$ and $425°$ **b.** $590°$ and $-130°$ **c.** $1150°$ and $430°$

d. $-314°$ and $-1034°$ **e.** $-1185°$ and $-1905°$ **f.** $1700°$ and $2420°$

9. a. $\frac{810°}{7}$ **b.** $-84°$ **c.** $4275°$ **d.** $1260°$

10. a. $\sin 810° = \sin 90° = 1$, $\cos 810° = \cos 90° = 0$, $\tan 810°$ does not exist.

b. $\sin(-450)° = -\sin 90° = -1$, $\cos(-450)° = \cos 90° = 0$

$\tan(-450)°$ does not exist.

c. $\sin(-1080°) = 0, \cos(-1080°) = 1, \tan(-1080°) = 0$

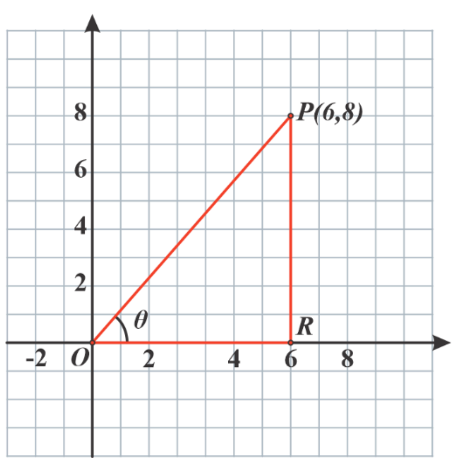
d. $\sin 630° = \sin 270° = -1, \cos 630° = 0, \tan 630°$ does not exist.

e. $\sin 900° = 0$, $\cos 900° = -1, \tan 900° = 0$

11. a. First let's plot the point $P(6,8)$ on the coordinate plane. Before we find the values of the three trig ratios, we need to find the length of the missing side.

We can use $r = f = \sqrt{6^2 + 8^2} = 10$ (from the Pythagorean Theorem). Let B be one of the acute angles. Hence,

$\sin(B) = \frac{g}{r} = \frac{8}{10} = 0.8$, $\cos(B) = \frac{h}{r} = \frac{6}{10} = 0.6$ and $\tan(B) = \frac{g}{h} = \frac{8}{6} = 1.333\ldots$.

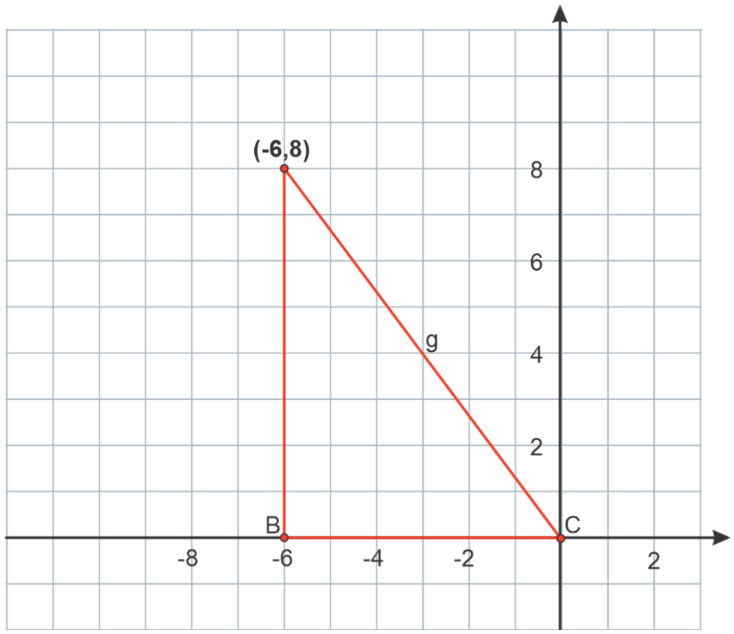


b. Similarly, $g = r\sqrt{(-6)^2 + 8^2} = 10$. Let C be one of the acute angles. Hence,

$$\sin C = \frac{8}{r} = \frac{8}{10} = 0.8,$$

$\cos C = -\frac{6}{10} = -0.6$ and

$\tan C = -\frac{8}{6} = -1.333$ ....

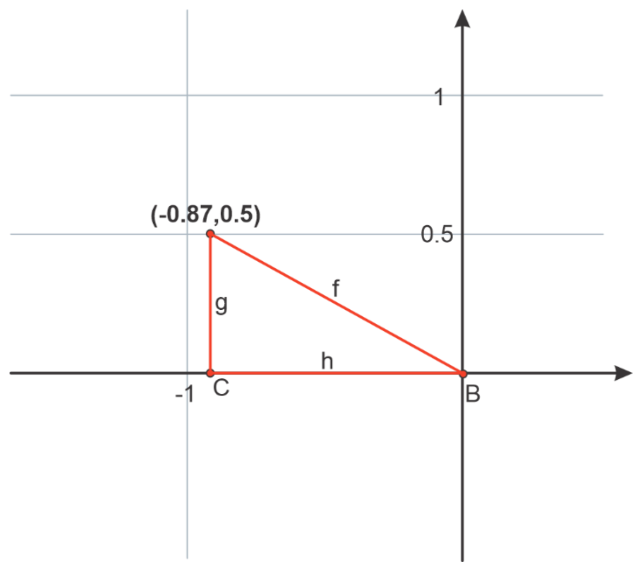


c. Here, $r = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$ and let $B = \theta$.

Then, $\sin B = 0.5$,

$\cos B = -0.87$,

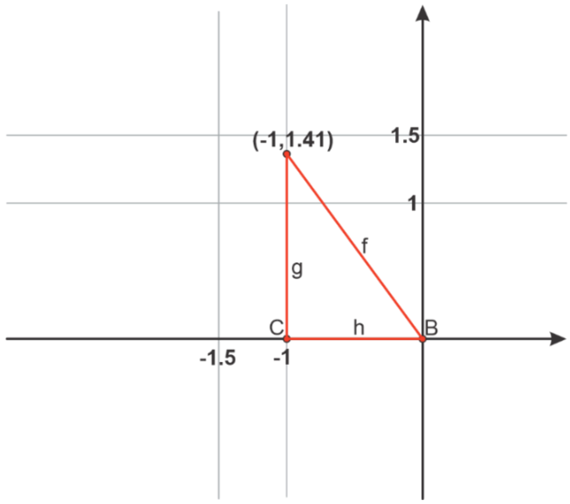
$\tan B = -\frac{0.5}{0.87} = -0.575$.



d. $P(-1, \sqrt{2})$, $r = \sqrt{(-1)^2 + \sqrt{2}^2} = \sqrt{3}$ and if $\theta = B$

$\sin B = \frac{g}{f} = \frac{\sqrt{2}}{\sqrt{3}}$,$\cos B = \frac{h}{r} = \frac{-1}{\sqrt{3}}$ and

$\tan B = \frac{g}{h} = -\frac{\sqrt{2}}{1} = -\sqrt{2}.$



12. a. $\sin\frac{3}{4}\pi = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\cos\frac{3}{4}\pi = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$, $\tan\frac{3}{4}\pi = -\tan\frac{\pi}{4} = -1$

b. $\sin\frac{3}{2}\pi = -1$, $\cos\frac{3}{2}\pi = 0$, $tan\frac{3}{2}\pi$ doesn't exist

c. $\sin(-\frac{7}{4}\pi) = -\sin\frac{7}{4}\pi = -sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$, $\cos -\frac{7}{4}\pi = \cos\frac{7}{4}\pi = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\tan(-\frac{7}{4}\pi) = -tan\frac{7}{4}\pi = -tan\frac{\pi}{4} = -1$

d. $\sin(-\frac{7}{2}\pi) = 1$, $\cos(-\frac{7}{2}\pi) = 0$, $\tan(-\frac{7}{2}\pi)$ doesn't exist

e. $\sin(-\frac{5}{6}\pi) = \sin\frac{\pi}{6} = \frac{1}{2}$, $\cos(-\frac{5}{6}\pi) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$, $\tan(-\frac{5}{6}\pi) = -tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$

13. a. Since 130° is second quadrant, $\theta_R = 180° - 130° = 50°$

b. Since 1030°is fourth quadrant, $\theta_R = 1080° - 1030° = 50°$

c. Since 340° is fourth quadrant, $\theta_R = 360° - 340° = 20°$

d. Since −236°is second quadrant, $\theta_R = 236° - 180° = 56°$

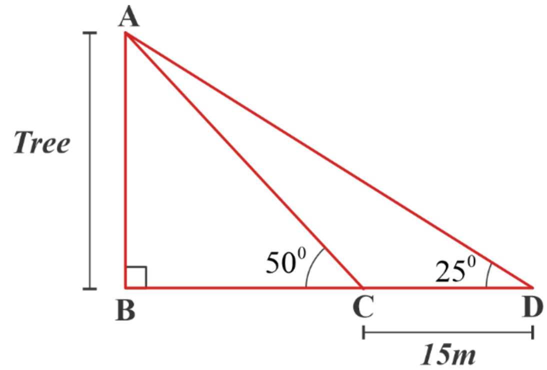
e. Since −720° is a quadrant and lies on the positive x-axis, $\theta_R = 0°$

14. a. $A = 30°$ **b.** $A = 45°$ c. $A = 45°$

15. a. $\cos A = \frac{3}{5}$, **b.** $\tan A = \frac{4}{3}$ **c.** $\csc A = -\frac{5}{4}$ **d.** $\sec A = \frac{5}{3}$

16. Hint: We are given that the angle of elevation of a tree top is 25° which when moved towards the base of the tree changes the angle of elevation to 50° and based on the given information, we have to find the height of the tree. We will make use of the trigonometric function (tangent function) in the two triangles and substitute the required values in either of the expressions obtained from the two triangles and then find the height of the tree.

We will first draw the figure as per what is in the question. We have,



We will use the trigonometric functions to find the height of the tree.

Let the height of the tree be 'h' meters, that is, $AB= h\,meters$

And let the distance BC be 'x' meters. We will first consider the triangle, △ABC

$\tan 50° = \frac{h}{x}$

$x = \frac{h}{\tan 50°}$ ----(1)

Next, we will consider the triangle, △ABD

$\tan 25° = \frac{h}{x+15}$ ----(2)

We will now substitute the value of 'x' from equation (1), we get,

$\tan 25° = \frac{h}{\frac{h}{\tan 50^0}+15}$ ----(3)

We know that, $\tan 25° = 0.4663$ and $\tan 50° = 1.1917$

Substituting the values in the equation (3), we get,

$$0.4663 = \frac{h}{\frac{h}{1.1917} + 15}$$

$$0.4663\left(\frac{h}{1.1917} + 15\right) = h$$

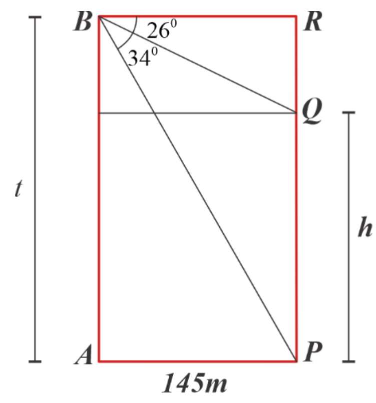
$$0.4663h + 8.3353 = 1.1917h$$

$$0.4663h = 8.3353$$

$h = 11.49$meters

Therefore, the height of the tree is 11.49meters.

17. Angle of elevation is given, on the basis of this we have to draw diagrams



Let $PQ = h$be the height of flagpoles of angles of depression of Q and P are $26°$ and $34°$

as seen from top B of building AB of height be t meters.

given AP= 145m, in $\Delta BRQ, \tan 26° = \frac{t-h}{145}$ and in

$\Delta BRP, \tan 34° = \frac{t}{145}$.

so, $t - h = 145\tan 26°$ and $t = 145\tan 34° = 145 \times 0.6745 = 97.80$

$h = t - 145 \times 0.4877 = 97.80 - 70.72 = 27.28$

therefore, height of the pole 27.28 meters and height of the building 97.80 meters.

18. **i.** $\sin A = \frac{12}{13}$ **ii.** $\tan A = \frac{12}{5}$

19. $2 + \sqrt{3}\sec\theta = 0$ equation

$\sqrt{3}\sec\theta = -2$ added -2 to both sides

$\sec\theta = -\frac{2}{\sqrt{3}}$ divided by $\sqrt{3}$

If $\sec\theta = -\frac{2}{\sqrt{3}}$, then $\cos\theta = -\frac{\sqrt{3}}{2}$. The $\cos 30° = \frac{\sqrt{3}}{2}$ and the cosine is negative in the second and third quadrants. So, the angles which satisfy the given equation are $150°$ and $210°$.

20. We begin by solving the equation for $\csc\theta$.

$3\sqrt{2} + 3\csc\theta = 0$ equation

$3\csc\theta = -3\sqrt{2}$ added $-3\sqrt{2}$ to both sides

$\csc\theta = -\sqrt{2}$ divided by 3

If $\csc\theta = -\sqrt{2}$, then $\sin\theta = -\frac{1}{\sqrt{2}}$. The $\sin 45° = \frac{1}{\sqrt{2}}$ and the sine is negative in the third and fourth quadrants. Thus, the angles that satisfy the given equation is $225°$ and $315°$.

21. Remember that the cosecant function is the reciprocal of the sine,

$\csc\left(\frac{3\pi}{4}\right) = \csc\frac{\pi}{4} = \frac{1}{\sin\frac{\pi}{4}} = \sqrt{2}$ and secant is the reciprocal of the cosine, and the cosine is positive in the fourth quadrant. So, secant is positive in the second quadrant.

$$\sec\left(\frac{7\pi}{4}\right) = \sec\frac{\pi}{4} = \frac{1}{\cos\frac{\pi}{4}} = \sqrt{2}$$

Therefore, $\frac{2}{3}\csc\left(\frac{3\pi}{4}\right) - \sec\frac{11\pi}{4} = \frac{2}{3}(\sqrt{2}) - \sqrt{2} = \frac{-\sqrt{2}}{3}$

22. $\frac{sec^2\theta - tan^2\theta}{1+cot^2\theta} = \frac{\frac{1}{cos^2\theta} - \frac{sin^2\theta}{cos^2\theta}}{csc^2\theta} = \frac{\frac{1-sin^2\theta}{cos^2\theta}}{\frac{1}{sin^2\theta}} = sin^2\theta$

23. $\frac{cosA}{1+sinA} + \frac{1+sinA}{cosA} = \frac{cos^2A+1+2sinA+sin^2A}{(1+sinA)cosA} = \frac{2+2sinA}{(1+sinA)cosA}$ $(\because\ sin^2 + cos^2 = 1)$

$$= \frac{2(1+sinA)}{(1+sinA)cosA}$$

$$= \frac{2}{cosA} = 2\sec A$$

24. $\frac{4\sqrt{2}-\sqrt{3}}{2}$

25. First, we apply the Pythagoras theorem to find the length of the leg opposite to θ. Letting a be the length of this leg, we find

$$a^2 + b^2 = c^2$$

$a^2 + 8^2 = 12^2$, $\quad a^2 = 144 - 64 = 80$, $a = \sqrt{80} = 4\sqrt{5}$

Now using the trigonometric ratios, we obtain:

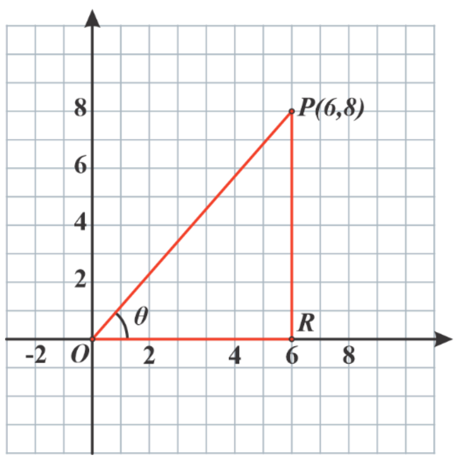
$\sin\theta = \frac{4\sqrt{5}}{12} = \frac{\sqrt{5}}{3} = \frac{1}{\csc\theta}$, $\cos\theta = \frac{8}{12} = \frac{2}{3} = \frac{1}{\sec\theta}$, $\tan\theta = \frac{\sqrt{5}}{2} = \frac{1}{\cot\theta}$,

26. $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$ and $\tan\theta = \frac{3}{4}$.

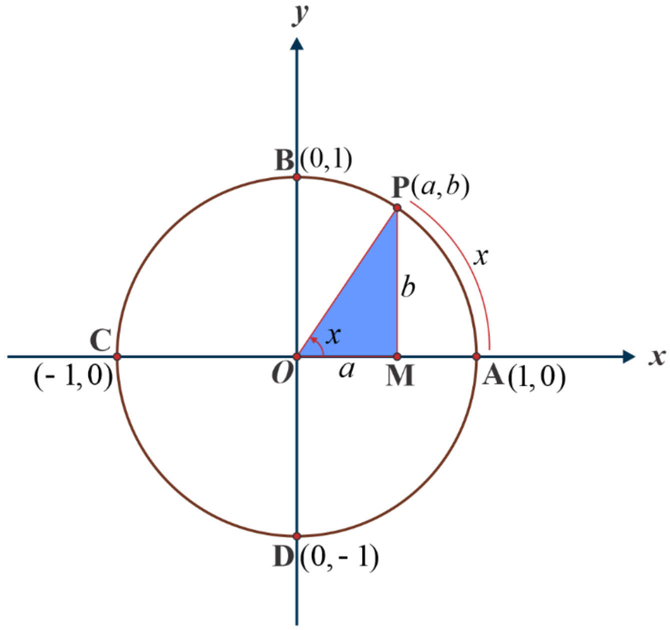
27. $OP = \sqrt{6^2 + 8^2} = 10$

$\sec\theta = \frac{OP}{OR} = \frac{10}{6} = \frac{5}{3}$,$\csc\theta = \frac{OP}{PR} = \frac{10}{8} = \frac{5}{4}$

$\cot\theta = \frac{OR}{PR} = \frac{6}{8} = \frac{3}{4}$.



28. $\csc 45° = \sqrt{2}$, $\sec 45° = \sqrt{2}$ and $\cot 45° = 1$.



29. $\csc(-45°) = -\csc 45° = \sqrt{2}$, $\sec(-45°) = \sec 45° = \sqrt{2}$,

$\cot(-45°) = -\cot 45° = -1$

$\csc(-225°) = \csc 45° = \sqrt{2}$, $sec(-225°) = -sec45° = -\sqrt{2}$

$\cot(-225°) = -\cot 45° = -1$

$\csc(-315°) = csc45° = \sqrt{2}$,$\sec(-315°) = \sec 45° = \sqrt{2}$

$\cot(-315°) = \cot 45° = 1$

$\csc(225°) = -\csc 45° = -\sqrt{2}$, $sec(225°) = -csc45° = -\sqrt{2}$

$\cot(225°) = -cot45° = -1$

$\csc(315°) = -\csc 45° = -\sqrt{2}$, $\sec(315°) = \sec 45° = \sqrt{2}$

$\cot(315°) = -\cot 45° = -1$

30. a $\sec\theta = 13/2$, $\csc\theta = 13/5$ and $\cot\theta = 12/5$

b. $\sec\theta = -\sqrt{89}/8$, $\csc\theta = \sqrt{89}/5$ and $\cot\theta = -8/5$

c. $\sec\theta = 1$, $\csc\theta$ and $\cot\theta$do not exist.

d. $\sec\theta = 5/4$, $\csc\theta = -5/3$ and $\cot\theta = -4/3$

e. $\sec\theta = \sqrt{7}/\sqrt{2}$, $\csc\theta = \sqrt{7}/\sqrt{5}$ and $\cot\theta = \sqrt{2}/\sqrt{5}$

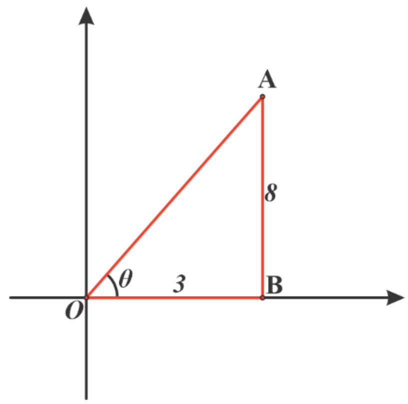
31. given $\cot\theta = \frac{3}{8}$, where $\theta = \mathrm{m}(A\hat{O}B)$

$(AO)^2 = (AB)^2 + (BO)^2$

$AO = \sqrt{8^2 + 3^2} = \sqrt{73}$

$\sin\theta = \frac{8}{\sqrt{73}}$, $\cos\theta = \frac{3}{\sqrt{73}}$, $\tan\theta = \frac{8}{3}$,

$\csc\theta = \frac{\sqrt{73}}{8}$ and $\sec\theta = \frac{\sqrt{73}}{3}$.



Unit 5

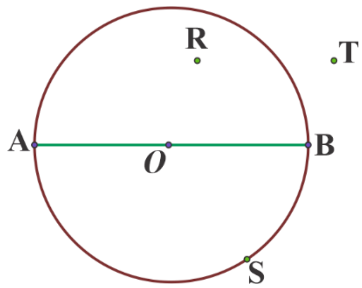
CIRCLE (15 periods)

Introduction

A circle is defined as a set of all points in a given plane which lie at a fixed distance from a fixed point in the plane. This fixed point is called the center of the circle and the fixed distance is called the radius of the circle where point O is the center of the circle and segment OA is known as the radius. The radius is the distance between all points on the circle and O. It follows that if a point R exists such that $m\,(\overline{OA}) > m\,(\overline{OR})$ then R is inside the circle. On the other hand, for a point T if $m(\overline{0T}) > m(\overline{OA})$ then T lies outside the circle, and if $m(\overline{OA}) = m(\overline{OS})$ it can be said that S lies on the circle.

The lines in the plane of the circle are classified into three categories.

- Lines which do not intersect the circle.
- Lines which intersect the circle at only one point is called tangent line.
- Lines which intersect the circle exactly at two points which is called secant line.



In this unit, we consider the symmetry properties of circles. These properties are used to derive relations between circles and lines. Moreover, the theorems we will discuss will deal with the ways in which angles are measured by the arcs that they intercept. Finally, we will consider the concept of arc length, perimeter and area of segments and sectors of a circle.

Unit outcomes: At the end of this unit the students will be able to:

- understand the symmetrical properties of circles.
- use the symmetrical properties of circles to solve related problems.

- understand angle properties of circles in their own words.
- apply angle properties of circles to solve related problems.
- find arc length, perimeters and areas of segments and sectors.

5.1 Symmetrical properties of circles (2periods)

Competencies

At the end of this subunit, students will be able to:

- discover the symmetrical properties of circles.
- use the symmetrical properties of circles to solve related problems.

Vocabulary: symmetry, diameter, radius, angles, tangent, subtend, arc.

Materials required:

Compass, Ruler, Protractor

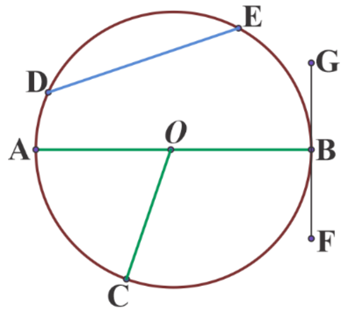
Introduction

In this subunit, we consider the symmetry properties of circles. These properties are used to derive relations between circles and lines. Moreover, the theorems we will discuss will deal with the ways in which angles are measured by the arcs that they intercept.

Teaching Notes

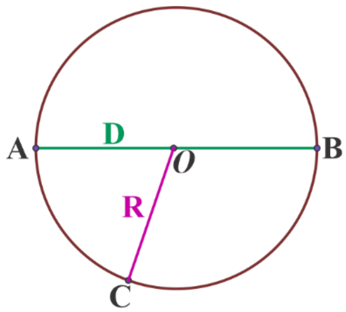
Students are expected to have some background about the circle and its properties. Thus, group the students in pairs, and then, you may start the lesson by asking students to do Activity 5.1. That is, you may ask students to give the definition of a circle, radius of a circle, a diameter and an arc of a circle. You may also ask whether a circle isa symmetrical figure; if so, ask them to indicate the line of symmetry of a circle and the number of lines of symmetries that a circle can have. Let some of the groups present the answers to the class. Discuss their answers and give the correct answers to the questions. Encourage and assist the students to discover that a circle is symmetrical about its diameter. Based on this fact, discuss some properties of a circle that can be proved by using this fact.

In the figure given below, we can notice that, $AO = OB = OC = r$. A chord is a line which joins two points on the circle. In the figure, $\overline{DE}$ is a chord.



Answers for activity 5.1

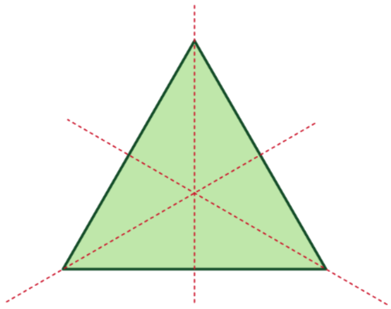
1. A circle is a curve joining a set of points which are at same distance from a fixed point. Here, the fixed point is called the center of the circle and the fixed distance is called the radius of the circle.

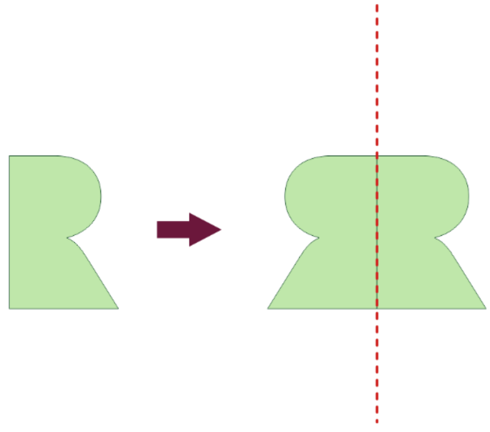


2. O is the center, R is the radius and D is the diameter.

3. The line of symmetry can be defined as the axis or imaginary line that passes through the center of the shape or object and divides it into identical halves.

4. Equilateral triangle has three lines of symmetry.





Answers for exercise 5.1

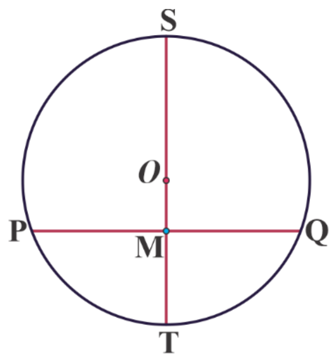
1. An equilateral triangle has three lines of symmetry.
2. Proof: Join $\overline{OP}$ and $\overline{OQ}$.

   i. $\overline{OP} \equiv \overline{OQ}$ radii of circle

   ii. $\overline{PM} \equiv \overline{QM}$ M is midpoint of $\overline{PQ}$

   iii. $\overline{OM} \equiv \overline{OM}$ common side

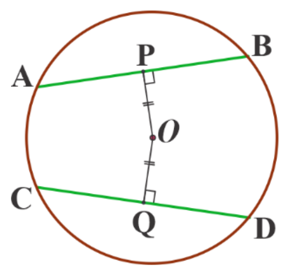
   iv. $\Delta OPM \equiv \Delta OQM$ by SSS-criteria of congruency



Answers for exercise 5.2

Given $\overline{OP} \perp \overline{AB}$ and $\overline{OQ} \perp \overline{CD}$, $m(\overline{OP}) = m(\overline{OQ})$. We want to prove $m(\overline{CD}) = m(\overline{AB})$.

Join $\overline{\mathrm{OA}}$ and $\overline{\mathrm{OC}}$ and consider ΔAPO and ΔCQO.



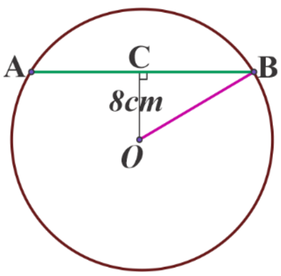
$m(\angle CQO) = \mathrm{m}(\angle APO) = 90^0$	$\overline{OP} \perp \overline{AB}$ and $\overline{OQ} \perp \overline{CD}$
$\overline{OA} \equiv \overline{OC}$	Radii of the same circle
$\overline{\mathrm{OP}} \equiv \overline{\mathrm{OQ}}$	Given
$\Delta\mathrm{OPA} \equiv \Delta\mathrm{OQC}$	RHS postulate of congruency
$\overline{AB} \equiv \overline{CD}$	From statement **5**

Answers for exercise 5.3

1. Given: $AB = 20$cmand $OC = 8$cm. From theorem, $\overline{OC}$ is the perpendicular bisector of $\overline{AB}$ at C. So, ΔBCO is a right-angled. By Pythagoras Theorem

   We observe, $OB = r = \sqrt{OC^2 + CB^2} = \sqrt{8^2 + 10^2}$

   $= \sqrt{164} = 2\sqrt{41}$cm



2. Given:$PR = 10$cm and radius$= OR = 8$cm

$$OM^2 + MR^2 = OR^2$$

$$OM = \sqrt{OR^2 - MR^2}$$

$$= \sqrt{8^2 - 5^2}$$

$$= \sqrt{64 - 25}$$

$$= \sqrt{39}\text{cm.}$$

3. $m(AB) = m(CD) = 16cm$ and $radius = 10cm.$

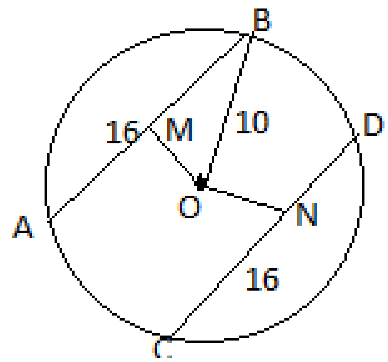
   let M and N be midpoint of AB and CD respectively. So,

   $BM = 8cm.$

   $BM^2 + OM^2 = OB^2$

   $8^2 + OM^2 = 10^2, \quad OM^2 = 100 - 64$

   $OM = \sqrt{36} = 6cm.$



5.2 Angle Properties of circles (4 periods)

Competencies

At the end of this subunit, students will be able to:

- state angle properties of circles in their own words.

- apply angle properties of circles to solve related problems

Vocabulary: Angle, diameter, radius, inscribe, subtend, arc, segment, sector.

Materials required:

Compass, Ruler, Protractor

Introduction

First, we will discuss the five Angle Properties of Circle:

- Angles in the same segment (are equal)
- Angle at Centre is twice angle at the circumference
- Right angle in semi-circle
- Angles in opposite segment (add up to 180 degrees)
- Radius is perpendicular to tangent

Teaching notes

You could start with brainstorming questions like:

- define the terms: chord, diameter, radius, tangent, secant, arc
- what is the relationship between diameter and any chord perpendicular to that diameter?

Do you agree with the following statements?

- in equal circles or in the same circle equal chords are equidistant from the center
- chords which are equidistant from the center are equal
- tangents from an external point are equal in length

Expected Answers

Chord: A line segment joining two points of a circle or a line segment whose end points lie on the circle.

Diameter: The largest chord through the center of a circle.

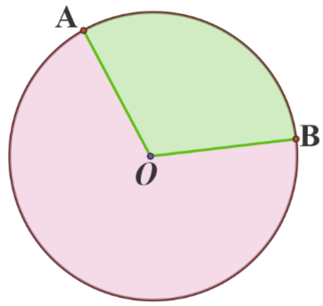
Radius: A line segment drawn from the center to any point on the circle.

Tangent: any line which touches the outer part of the circle at only one point.

Secant: A line that intersects a circle at two points.

A diameter is a perpendicular bisector of a chord.

Yes! All the statements are correct.



Let A and B be two different points on the circle with center O. these two points divide the circle into two opposite arcs. If the chord AB is the diameter, then the chords are semicircles. Otherwise, one arc is longer than the other; the longer arc is called the major arc AB and the shorter arc is called the minor arc AB.

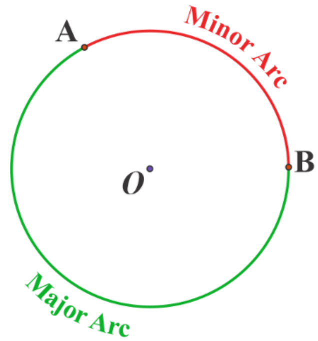
Now join the radii OA and OB. The reflex angle $A\hat{O}B$ is called the angle subtended by the major arc AB. The non-reflex angle $A\hat{O}B$ is called the angle subtended by the minor arc AB, and is also the angle subtended by the chord AB. The radii divide the circle into two sectors, called correspondingly the major sector OAB and the minor sector OAB.

ASSESSMENT

You can use any one of assessment techniques such as: class activities, group discussions, homework/assignments, exercises and/or tests/quizzes.

Answers for activity 5.2

1. The **chord** of a circle can be defined as the line segment joining any two points on the circumference of the circle. The **diameter** is the longest chord in a circle. The distance from the center point to any endpoint on the circle is called the **radius** of a circle. A tangent to a circle is a straight line which touches the circle at only one point. A straight line that intersects a circle at two points is called a secant line. An arc is a smooth curve joining two points. Consequently, on a circle, every pair of distinct points determines two arcs: major arc and minor arc.
2. Two points lying on a circle actually define two arcs. The shortest is called the 'minor arc' and the longer one is called the 'major arc'.



Answers for exercise 5.4

1. Measure of $\angle ABC$ is $37°$.
2. **a.** $x = 70°$ **b.** $x = 240°$

Answers for exercise 5.5

a. $x = 30°$ **b.** $x = 55°$ and $y = 110°$ **c.** $x = 70°$ and $y = 20°$

Answers for exercise 5.6

1. $m(\angle ADC) = 72°$
2. $x = 80°$ and $y = 100°$
3. **a.** Minor arc AB **b.** arc ADB **c.** $\angle BOD$
   d. $\angle BAD$ **e.** arc ABD **f.** $\angle AOB$ and $\angle ADB$
4. **a.** $m(\angle BAD) = 43°$ **b.** $m(\angle AOB) = 94°$ **c.** $m(\angle ABD) = 90°$
   d. $m(\angle ODB) = 47°$ **e.** $m(\angle AEB) = 47°$
5. $m(\angle VZW) = 80°$
6. **a.** $x = 66°$ and $y = 33°$ **b.** $p = 48°$ and $x = 23°$

5.3 Arc length, perimeters and areas of segments and sectors Arc length. (5 periods)

Competencies

At the end of this subunit, students will be able to:

- calculate arc length, perimeter and area of segments.
- find perimeter and area of sectors.

Vocabulary: Angle, area, perimeter, diameter, radius, inscribe, subtend, arc, segment, sector.

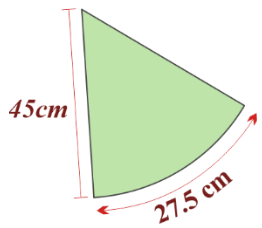
Materials required:

Compass, Ruler, Protractor

Teaching notes

You can start by activity 5.4

Commonly confused with arc measure, **arc length** is the distance between the endpoints along the circle. Arc measure is a degree measurement, equal to the central angle that forms the intercepted arc. Arc length is a fraction of the circumference of the circle and calculated that way: find the circumference of the circle and multiply by the measure of the arc divided by 360.



Perimeter of a sector:

The perimeter is the distance all around the outside of a shape. We can find the perimeter of a sector using what we know about finding the length of an arc.

A sector is formed between two radii and an arc. To find the perimeter, we need to add these values together.

$$\text{Perimeter} = \text{arc length} + 2\text{r}$$

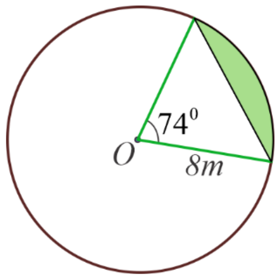
$$= 27.5\text{cm} + 2 \times 45\text{cm} = 27.5\text{cm} + 90\text{cm} = 117.5\text{cm}$$

Find the area and perimeter of the shaded region in the figure.

Area of segment: $A = \frac{\pi r^2 \theta}{360^0} - \frac{1}{2} r^2 \sin\theta$

$$= \frac{\pi \times 8^2 \times 74^\circ}{360^\circ} - \frac{1}{2} \times 8^2 \times \sin 74^\circ$$

$$= 41.31 - 30.76 = 11.99\text{sq. m.}$$



Afterwards, you can assist the students to calculate lengths of arcs, perimeters and areas of segments and sectors that the examples given in their textbook, give class work and give feedback.

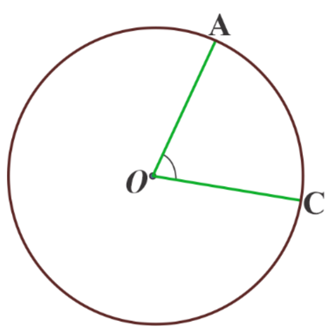
ASSESSMENT

You can use any one of assessment techniques such as: class activities, group discussions, homework/assignments, exercise and/or tests/quizzes.

Answers for activity 5.3

1. **Circumference of the circle or perimeter of the circle** is the measurement of the boundary of the circle. Whereas the area of a circle defines the region occupied by it. If we open a circle and make a straight line out of it, then its length is the circumference. It is usually measured in units, such as cm or unit m.

   When we use the formula to calculate the circumference of the circle, then the radius of the circle is taken into account. Hence, we need to know the value of the radius or the diameter to evaluate the perimeter of the circle. A central angle is an angle $A\hat{O}C$ with endpoints A and C located on a circle's circumference and vertex O is located at the circle's center.



2. A sector is a part of a circle made of the arc of the circle along with its two radii. It is a portion of the circle formed by a portion of the circumference (arc) and radii of the circle at both endpoints of the arc.

Answers for exercise 5.7

1. **a.** arc **b.** chord **c.** sector **d.** segment

2. **a.** arc $AB = \frac{\pi r\theta}{180°} = \frac{\pi(4cm)90°}{180°} = 2\pi cm$ **b.** arc $PQR = \frac{\pi(8cm)225°}{180°} = 10\pi cm$

Answers for exercise 5.8

a. $A_{sector} = \pi r^2\left(\frac{\theta}{360°}\right)$

$= \pi \times (6cm)^2 \times \left(\frac{30°}{360°}\right)$

$= \frac{22}{7} \times 36cm^2 \times \frac{1}{12}\left(\ Use\ for\ \pi = \frac{22}{7}\right)$

$= \frac{66}{7}cm^2.$

b. $A_{sector} = \pi r^2\left(\frac{\theta}{360°}\right)$

$= \pi \times (3cm)^2 \times \left(\frac{240°}{360°}\right)$

$= \frac{22}{7} \times 9cm^2 \times \frac{2}{3}\left(\ Use\ for\ \pi = \frac{22}{7}\right)\ = \frac{132}{7}cm^2.$

5.4 Theorems on angles and arcs determined by lines intersecting inside, on and outside a circle. (4 periods)

Competencies

At the end of this subunit, students will be able to:

- prove theorems.
- apply theorems to solve related problems.

Vocabulary: Angle, area, tangent, secant, diameter, radius, inscribe, subtend, arc.

Materials required:

Compass, Ruler, Protractor

You may start the lesson by revising some important terms of a circle. Using a circle,you can explain to the students what is meant by an arc, minor arc, semi-circle, a central angle, angle inscribed in an arc, and angle subtended at the Centre by an arc. Afterwards, you can assist the students to prove theorems and doing examples that is given in their textbook. After you discuss the examples given in their textbook, give

Exercise 5.2 as class work and home work. Finally give feedback.

Answers for exercise 5.9

1. proved
2. $x = 102°$
3. $m(P\hat{R}H) = \frac{1}{2}(\text{arcHT} - \text{arcTS}) = \frac{1}{2}(116° - 64°) = 26°$

   $m(T\hat{H}S) = m(J\hat{H}S) = 32°$, $m(T\hat{S}H) = m(T\hat{S}H) = 58°$, $m(H\hat{T}S) = m(H\hat{J}S) = 90°$

   minor arc $m(\text{arc}HT) = m(\text{arc}HJ) = 116°$, minor arc $m(\text{arc}TS) = m(\text{arc}JS) = 64°$
4. Proof: Let chord CD and AB of a circle intersect at a point P in the interior of the circle. We connect the endpoints of the chords to formΔAPC and ΔDPB.

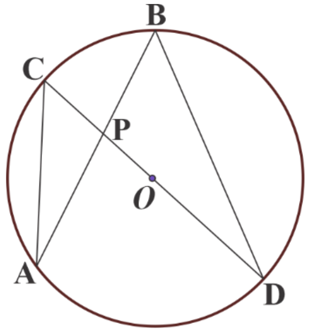
   We want to prove that $(AP)(PB) = (DP)(PC)$

The vertical angles formed by the intersecting chords have equal measures. The inscribed angles at A and D as shown in the figurehave also equal measures because both intercept the same arc BC.

Therefore, ΔAPC≈ ΔDPB by AA-similarity. Hence,

$$\frac{AP}{DP} = \frac{PC}{PB}.$$

Therefore, $(AP)(PB) = (DP)(PC)$.



Answers for exercise 5.10

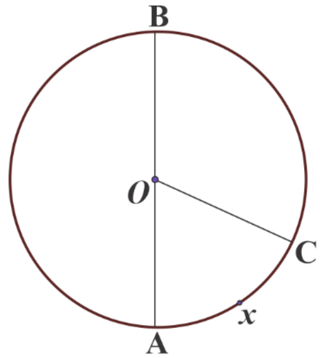
1. proved

2. Proof: Given: circle O with $\hat{B}$ an inscribed angle interceptingarc AC.

To prove: $m(\angle ABC) = \frac{1}{2}m(arcAXC) = \frac{1}{2}m(\angle AOC)$, where x is a point as shown in the figure.

We consider three cases.

Case:1. Suppose that side of $A\hat{B}C$ is a diameter of the circle with Centre O.

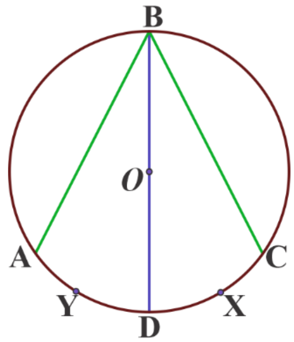


	Statement	Reason
i	$\overline{OA} \equiv \overline{OB} \equiv \overline{OC}$	Radii of the same circle
ii	$\angle OBC \equiv \angle OCB$	Base angles of an isosceles triangle.
iii	$m(\angle AOC) = m(\angle OCB) + m(\angle OBC)$	An exterior angle of a triangle is equal to the sum of the two opposite interior angles.
iv	$m(\angle AOC) = 2m(\angle ABC)$	From statement 2 and 3
V	$m(\angle AOC) = m(arcAXC)$	Central angle
Vi	$m(\angle ABC) = \frac{1}{2}m(\angle AOC) = \frac{1}{2}m(arcAXC)$	substitution

Therefore, $m(\angle ABC) = \frac{1}{2}m(arcAXC)$

Case:2. Suppose A and C on the opposite sides of the diameter

Through B, as shown in the figure at the right.

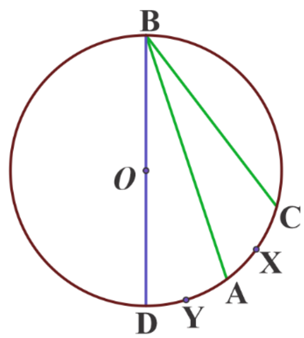


	Statement	Reason
i	$m(\angle ABD) = \frac{1}{2}m(arcAYD)$	Case 1
ii	$m(\angle CBD) = \frac{1}{2}m(arcDXC)$	Case 1
iii	$m(\angle ABD) + m(\angle CBD) = \frac{1}{2}m(arcAYD) + \frac{1}{2}m(arcDXC)$	Addition
iv	$m(\angle ABC) = \frac{1}{2}m(arcAXC)$	substitution

Therefore, $m(\angle ABC) = \frac{1}{2}m(arcAXC)$

Case:3. Suppose A and C on the same sides of the diameter

Through B, as shown in the figure at the right.

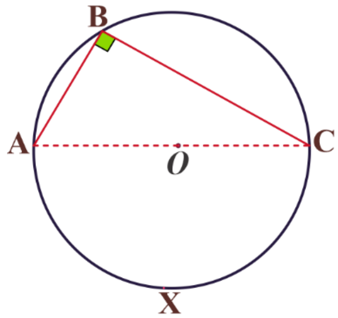


	Statement	Reason
i	$m(\angle DBC) = \frac{1}{2}m(arcDAC)$	Case 1
ii	$m(\angle DBA) = \frac{1}{2}m(arcDYA)$	Case 1
iii	$m(\angle DBC) - m(\angle DBA) = \frac{1}{2}m(arcDAC) - \frac{1}{2}m(arcDYA)$	Addition
iv	$m(\angle ABC) = \frac{1}{2}m(arcAXC)$	substitution

3. Proof: in the figure, $\angle ABC$ is an inscribed angle in a semicircle sinceAC is a diameter. Clearly measure of arc AXC is $180°$.

If we use this fact, immediately we conclude that

$$m(\angle ABC) = \frac{1}{2} m \text{ (arc AXC)} = \frac{1}{2} m \text{ (arc AXC)} = \frac{1}{2}(180°) = 90°.$$



The converse theorem:

The circle whose diameter is the hypotenuse of a right-angled triangle

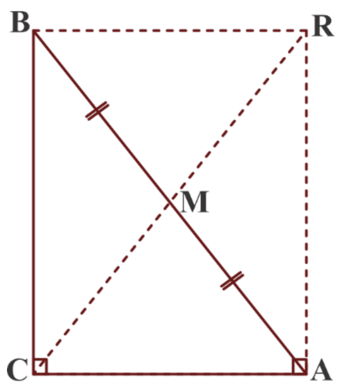
Passes through all three vertices of the triangle

Proof

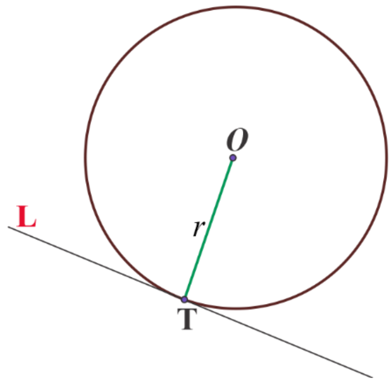
Let ΔABC be right-angled at C, and let M be the midpoint of

the hypotenuse AB.

We need to prove that $MC = MA = MB$. Complete the rectangle ACBR. Because ACBR is a rectangle, its diagonals bisect each other and are equal. Hence M is the midpoint of the other diagonal CR, and $AM = BM = CM = RM$.



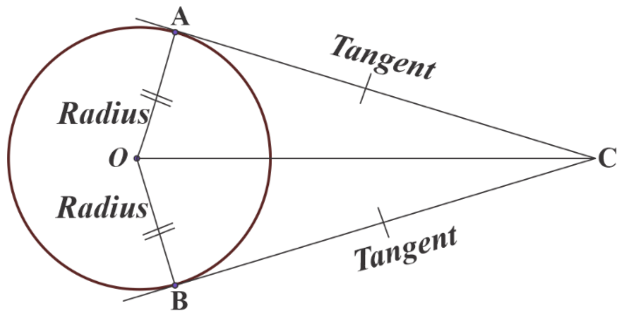
4. A tangent line to a circle is a line that touches the circle at exactly one point, never entering the circle's interior. In the figure, the line L is a tangent line and point T is a point of tangency.



5. Proof: Let O be the center of the circle. We want to prove: $\overline{AC} \equiv \overline{BC}$

Consider ΔAOC and ΔBOC.

1. $\overline{AO} \equiv \overline{BO}$ radii of a circle
2. $m(\angle OAC) = m(\angle OBC) = 90°$ tangent is perpendicular to radius of a circle
3. $\overline{OC} \equiv \overline{OC}$ Common side
4. $\Delta AOC \equiv \Delta BOC$ by RHS congruency
5. $\overline{AC} \equiv \overline{BC}$ Statement iv.



6. $m(\angle SRT) + m(\angle RTO) + m(\angle OSR) + m(\angle TOS) = 360°$

$40° + 90° + 90° + m(T\hat{O}S) = 360°$. So, $m(\angle TOS) = 140° = m(arcST)$,

$$m(\angle TUS) = \frac{1}{2}\, m(arcST) = 70°.$$

7. PQRS is inscribed in a circle, so opposite angles are supplementary by the inscribed Quadrilateral Theorem.

$$m(\hat{R}) + x = 180°$$

$$120^0 + x = 180°$$

$$x = 180° - 120° = 60°$$

$$m(\hat{S}) + y = 180°$$

$$180° + y = 180°$$

$$y = 180° - 80° = 100°$$

Therefore, the value of angle x is $60°$ and the value of angle y is $100°$.

Parts and Terms for Regular Polygons:

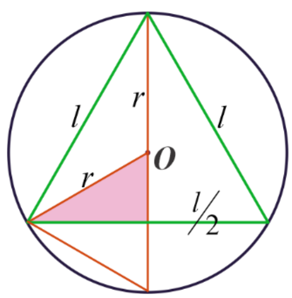
Polygons are regular if all of their sides and angles are equal. This means that all the corners, or vertices, of a regular polygon will lie on a circle. All regular polygons can be inscribed in a circle. The center of an inscribed polygon is also the center of the circumscribed circle. The radius of the inscribed polygon is also the radius of the circumscribed circle.

Side of an Inscribed Equilateral Triangle

By applying the Pythagorean theorem for one of the triangles,
we obtain:

$r^2 = \left(\frac{l}{2}\right)^2 + \left(\frac{r}{2}\right)^2$, where l is the length of sides of the triangle

$\frac{l^2}{4} = r^2 - \frac{r^2}{4}$, $\frac{l^2}{4} = \frac{3r^2}{4}$, $l = \sqrt{3}r$



Example:

Calculate the length of the side of an equilateral triangle inscribed in a circle of 10cm radius.

Solution:

$l = \sqrt{3}r = \sqrt{3} \times 10\text{cm} = 10\sqrt{3}\text{cm}.$

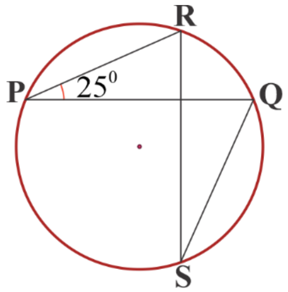
In addition to this example, let the students do and discuss more examples given in the textbook. Furthermore, make sure that students understand and hence can apply the formulas stated. To check this, assign Exercise 5.3 as class work or homework and as an assessment. Let students present their solutions to homework questions to the class.

Answers for review exercise

1. Given: $m(\angle QPR) = 25°$. From the figure,

$\angle QPR$ and $\angle PQS$ are subtended by the same arc.

So, $m(\angle PQS) = 25°$



2. $x = 30°$ and $y = 10°$

3. $y = 84°$

4. $x = 44°$

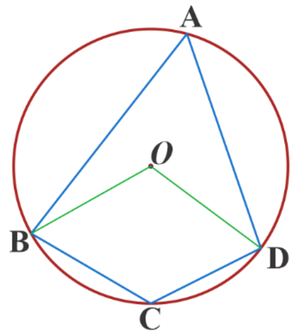
5. $m(\angle MLN) = 49°$

6. **a.** $m(\angle ABC) = 29°$, **b.** $m(\text{arcDG}) = 108°$, **c.** $m(\angle DHG) = 112°$

7. **Proof:** Given: A cyclic quadrilateral $ABCD$ where O is the centre of a circle.

Construction: Join the line segment OB and OD

Since, the angle subtended by an arc at the centre is double the angle on the circle.



Therefore, $m(\angle BAD) = \frac{1}{2}(\text{arc BCD})$ – equation 1

Similarly, $m(\angle BCD) = \frac{1}{2}(\text{arc DAB})$ – equation 2

By adding equation 1 and 2, we get

$$m(\angle BAD) + m(\angle BCD) = \frac{1}{2}(\text{arc } BCD + \text{arc } DAB)$$

$$m(\angle BAD) + m(\angle BCD) = \frac{1}{2}(360°) = 180°$$

Similarly, $m(\angle ABC) + (\angle ADC) = 180°$

Hence proved, that the sum of opposite angles of a cyclic quadrilateral is 180°

8. first we find the area of the sector.

$$A_{sec} = (\pi r^2)\left(\frac{\theta}{2\pi}\right) = 100 \times 0.3 = 30ft^2$$

$sin(0.6) = \frac{h}{r} = \frac{h}{10}$. so, $h = 10sin0.6 = 5.65$

and $a(\Delta AOB) = \frac{hr}{2} = \frac{10\times5.65}{2} = 28.25ft^2$

therefore, the area of the segment is $30ft^2 - 28.25ft^2 = 1.75ft^2$

9. Proof: consider ΔABP and ΔACP

$A\widehat{P}B$ is common angle

$\angle ABC \equiv \angle CAP$ (Intercepted by the same arc or properties of inscribed angle)

$\Delta PAB \approx \Delta PCA$ (By AA-similarity)

Therefore, $\angle ACP \equiv \angle BAP$.

10. 240°.

11. Given: $ABCD$ is a parallelogram. To prove $ABCD$ is a rectangle.

Proof: A rectangle is a parallelogram with one angle is 90°. So, we have to prove angle 90°.

Since $ABCD$ is a parallelogram,

$m(\angle A) = m(\angle C)$ (Opposite angles parallelogram are equal) ………. (1)

$m(\angle A) + m(\angle C) = 180°$ (Sum of opposite angles of a cyclic quadrilateral is 180°)

$m(\angle A) + m(\angle A) = 180°$ from (1)

$2m(\angle A) = 180°$. Therefore, $m(\hat{A}) = 90°$.

12. **a.** $x = 60°$ if $m(\angle BAC) = 30°$ **b.** $x = 153°$ **c.** $x = 115°$

13. Given: central angle AOB whose measure is 80° and $m(\angle POB) = 70°$.

Now, consider ΔPQB;

$m(\angle P) + m(\angle Q) + m(\angle B) = 180°$ (Sum of interior angles of a triangle is 180°)

$40°. + 70°. + m(\angle B) = 180°$ ($\angle AOB$ and $\angle APB$ are intercepted by the same arc)

$m(\angle B) = 180° - 110° = 70°$

therefore, $m(\angle PBO) = 70°$

Unit 6

Solid Figures

Periods allotted: 16 periods

Introduction

Recall that Solid figures are three-dimensional objects, having length, width, and height. Because they have three dimensions, they have depth and take up space in our universe. In this unit students will learn how to drive the surface area formula and volume formula of pyramids and cones. Students will be supported to apply the derived formulae to find the surface area and volumes of pyramids, cones, frustum of cones, frustum of pyramids and composed solids.

You need to state the formula for finding the surface area and volume of sphere. Then support them to apply the formula to calculate surface area and volume of sphere.

Unit Outcomes: At the end of this unit, students will be able to:

- find surface area and volume of pyramids and cones.
- calculate volume of frustum of pyramid and cones.

Suggested teaching aids in unit 2

- Model pictures of solid figures.
- Different Solids whose shape look like a cone, cylinder, pyramids, prisms, spheres and a combination of these. You can make groups and ask each group to bring these solid from their surrounding or by making their own and describe it to the class.

6.1 Revision cylinders and prisms

Periods allotted: 2 periods

Competencies

At the end of this sub-unit students will be able to:

- calculate surface area and volume of prisms and cylinders.

Introduction

In this sub-unit, you will revise on the solid figures cylinders and prisms by doing activity 6.1 and activity 6.2. You will introduce the formula for the lateral surface area (LSA), base area (BA) and total surface area (TSA) of prisms and cylinders.

Teaching Guide

Students are expected to have backgrounds on the definition of cylinders and prisms.

For the purpose of revision about prisms, you could ask students to do activity 6.1. You can also ask students to do exercise 6.1 independently or in groups.

Answers for activity 6.1

1. **a.** ΔABC and ΔDEF
b. Lateral edge
c. right prism
e. Lateral face
f. CG
g. $\overline{EF}$, $\overline{FD}$ and $\overline{DE}$

2.

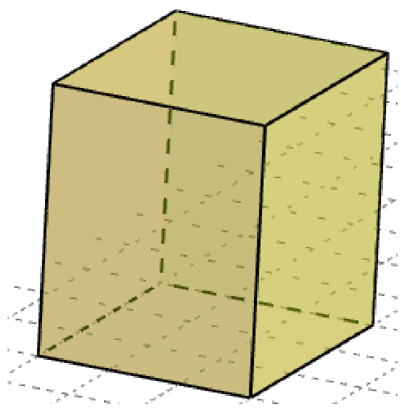


Figure 6.1

Answers for exercise 6.1

1. BA = $\frac{1}{2}absin\theta = \frac{1}{2} \times 4 \times 4sin60° = 4\sqrt{3}cm^2$

LSA = $ph = (4+4+4) \times 10 = 120cm^2$

TSA = LSA + 2BA = $120 + 8\sqrt{3} = 8(15+\sqrt{3})cm^2$

Volume = $BA \times h = 4\sqrt{3} \times 10 = 40\sqrt{3}\ cm^3$

2. **a.** BA = $2 \times 6 = 12cm^2$, LSA = $(1+6+2+6)10 = 160cm^2$,

TSA = $LSA + 2BA = 184cm^2$

There is also another alternative,

BA = $2 \times 10 = 20cm^2$, LSA = $(2 + 10 + 2 + 10)6 = 144cm^2$,

TSA = $LSA + 2BA = 184cm^2$

Volume = $BA \times h = 20 \times 6 = 120cm^3$

b. BA = $\frac{1}{2} \times 5 \times 6 = 15cm^2$, LSA = $\left(5 + 6 + \sqrt{61}\right) \times 3 = 3\left(11 + \sqrt{61}\right)cm^2$,

TSA = $LSA + 2BA = 3\left(11 + \sqrt{61}\right) + 30 = 3\left(21 + \sqrt{61}\right)cm^2$

Volume = $BA \times h = 15 \times 3 = 45cm^3$

c. The base is the trapezium shown by the figure below

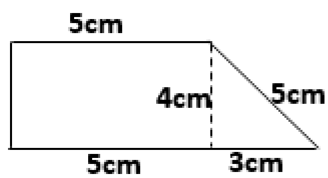


Figure 6.2

BA = $\frac{1}{2}(5 + 8)4 = 26cm^2$, LSA = $ph = (4 + 5 + 5 + 8) \times 11 = 242cm^2$

TSA = LSA + 2BA = $242 + 52 = 294cm^2$

Volume = $BA \times h = 26 \times 11 = 286cm^3$nd

2. LSA = ph

$120 = p \times 5 \ implies \ p = 24cm$

3. A cube is a prism whose all edges have the same length.

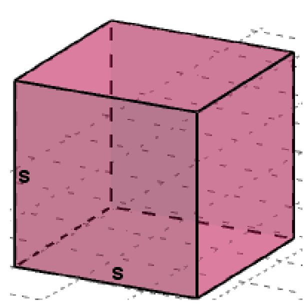


Figure 6.3

If one edge of the cube has edge of length s then

TSA = $6s^2$ and Volume $= s^3$

For the purpose of revision about cylinder, you could ask students to do activity 6.2. You can also ask students to do exercise 6.2 independently or in groups.

Answers for activity 6.2

a. lateral Surface

b. base

c. yes

d. $\overline{LM}$

e. right cylinder

Answers for exercise 6.2

1. $T.S.A = 112\pi\ cm^2$ and $V = 160\pi\ cm^2$.
2. diameter $= 6cm$ implies $r = 3cm,\ T.S.A = L.S.A + 2B.A = 2\pi r(h + r) = 66cm^2$ and $V = \pi r^2 h = 72\pi\ cm^2$.
3. BA $= \pi R^2 - \pi r^2 = \pi(R^2 - r^2) = \pi(7^2 - 5^2) = 24\pi cm^2$.
   LSA = $2\pi h(r + R) = 192\pi cm^2$
   TSA = $L.S.A + 2B.A = 240\pi cm^2$, Volume = $BA.h = 192\pi cm^3$
4. $T.S.A = L.S.A + 2B.A = 2\pi r(h + r) = (2\pi \times 4)(10 + 4) = 112\pi\ cm^2$.
   $V = \pi r^2 h = 160\pi\ cm^2$.

6.2 Pyramids, cones and Spheres

Periods allotted: 6 periods

Competencies

At the end of this sub-unit students will be able to:

- calculate surface areas of a given pyramid and a cone.

Introduction

In this subsection, definition of pyramids, cones and spheres, surface area and volume of pyramids, cones and spheres will be presented.

Teaching Guide

Students are expected to have knowledge of pyramids and cones from former classes. As a revision let them do activity 6.3.

Answers for activity 6.3

1

a. Triangular pyramid

b. Triangular pyramid

c. A regular tetrahedron is a tetrahedron in which alt the four faces are equilateral triangles.

d. Rectangular pyramid

e. Base.

f. Lateral face.

g. Lateral edges.

h. Base edges.

2.

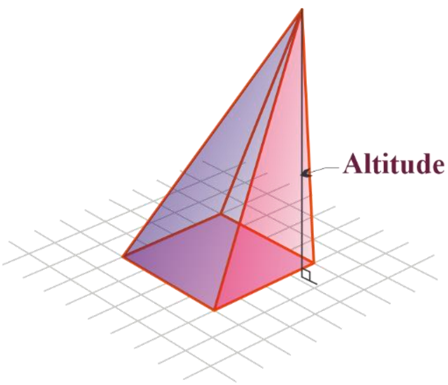


Figure 6.4

3. An altitude of a pyramid is the perpendicular distance from the vertex of the pyramid to the plane containing the base

4. Cone.

5.

Figure 6.5

The objective at this point is to discuss with students on the definition of pyramid and cones. After this present definition 6.1 and 6.2 from the students' text book for students and make discussion with maximum participation of students. Present real samples of pyramids and cone to students. After you discussed with students on the definition of pyramids, cones, altitude, lateral face lateral surface, lateral surface base and total surface, of a pyramid and right pyramid, let them do exercise 6.3 for consolidation.

Answers for exercise 6.3

1. a. Square pyramid or rectangular pyramid.
   b. Triangular pyramid or tetrahedron.
   c. Hexagonal Pyramid.
   d. Pentagonal Pyramid.
2. a. Altitude
   b. Slant height
   c. 5cm

Answers for activity 6.4

1. Triangle
2. 8
3. 10
4. n
5. Determine whether each of the following statements is true or false.
   a. False
   b. True
   c. True
   d. True
   e. False
   f. True

6.2.1 Surface area of Pyramids and cones

The surface area of any given object is **the area or region occupied by the surface of the object.** Sketching the geometric net of a regular pyramid enable students see the surface of the pyramid and calculate surface area. By considering a regular polygon and its net drive the formula for the total surface area of a regular pyramid as:

$$\text{TSA} = BA + LSA = BA + \frac{1}{2}pl.$$

Apply the formula by discussing the examples on students' text with students. Let students do problems 1 and 2 (exercise 6.4) from students' text book, you can round and observe their work and give support.

Answers for Exercise 6.4

1. $h = 6\ cm,\ s = 4\ cm$

For a regular hexagon, $s = r = 4\ cm$.

$BA = \frac{1}{2}nr^2 \sin\left(\frac{360^0}{n}\right) = \frac{1}{2}(6)(16)\sin(60^0) = 24\sqrt{3}\ sq.cm.$

In the figure, ΔAMO and ΔVOM are right angled triangles:

$r^2 = a^2 + \left(\frac{s}{2}\right)^2$, $16 = a^2 + 4$ implies $a = 2\sqrt{3}\ cm.$

$h^2 + a^2 = l^2$, $36 + 12 = l^2$ implies $l = 4\sqrt{3}$ cm.

LSA = $\frac{1}{2}lp = \frac{1}{2}(4\sqrt{3})(4 \times 6) = 48\sqrt{3}\ sq.cm.$

TSA = BA + LSA = $72\sqrt{3} sq.cm.$

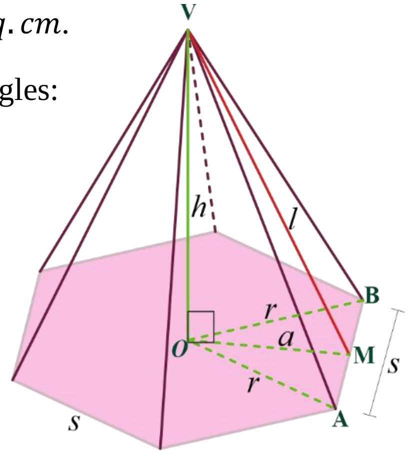


Figure 6.6

2. $h = 3m$, square base with diagonal, $d = 6$ implies

The radius of the square = $r = \frac{d}{2} = 3.$

$s^2 = r^2 + r^2 = 2r^2 = 18$ implies, length of side of the square $= s = 3\sqrt{2}$,

BA = $\frac{1}{2}nr^2 \sin\left(\frac{360^0}{n}\right) = 18\ sq.m$ or BA = $s^2 = \left(3\sqrt{2}\right)^2 = 18\ sq.m.$

$r^2 = a^2 + \left(\frac{s}{2}\right)^2$, $9 = a^2 + \frac{9}{2}$ implies, Apothem $= a = \frac{3\sqrt{2}}{2}.$

$h^2 + a^2 = l^2$, slant height $= l = \frac{3\sqrt{6}}{2}.$

LSA = $\frac{1}{2}pl = \frac{1}{2} \times 12\sqrt{2} \times \frac{3\sqrt{6}}{2} = 18\sqrt{3}\ sq.m.$

The geometric net for a right circular cone of base radius r and slant height l is a sector with radius l and arc length $2\pi r$. Activity 6.5 is to revise the arc length and sector area of a sector.

Answers for activity 6.5

1. A sector of a circle has radius r and central angle θ as shown in figure 6.12 on the students' textbook.

a. Arc length = $\frac{\theta}{360} \times 2\pi r = \frac{\pi r\theta}{180}.$

b. Area of the sector = $\frac{\theta}{360} \times \pi r^2 = \frac{\pi r^2\theta}{360}.$

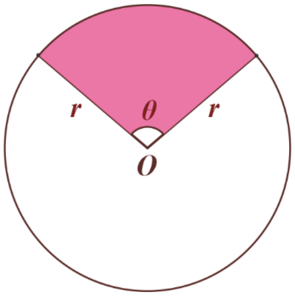


Figure 6.7

2. Arc length = $\frac{\pi r\theta}{180} = \pi\ cm,$

Area of the sector = $\frac{\pi r^2\theta}{360} = 3\pi\ cm.$

At this point drive the formula for the lateral surface area of a right circular cone. Use figure 6.21, on the students' textbook while driving the LSA.

$LSA = \frac{1}{2}pl = \frac{1}{2}(2\pi r)l = \pi rl,$

$l = \sqrt{h^2 + r^2},$

where, l is the slant height, h is the height (altitude) and r is the base radius.

The total surface area (TSA) is the sum of the area of the base and the lateral surface area. That is,

TSA = LSA + BA = $\pi rl + \pi r^2 = \pi r(l + r)$.

Apply the formula by discussing the examples on students' text with students. Let students try problems from exercise 6.5.

Answers for Exercise 6.5

1. $r = 6cm, h = 12cm$

$$l = \sqrt{h^2 + r^2} = \sqrt{144 + 36} = \sqrt{180} = 6\sqrt{5}cm$$

$LSA = \pi rl = \pi \times 6 \times 6\sqrt{5} = 36\sqrt{5}\pi\ sq.cm,\ BA = \pi r^2 = 36\pi\ sq.cm$ and

TSA = $36\sqrt{5}\pi + 36\pi = 36\pi(\sqrt{5} + 1)\ sq.cm.$

2. TSA = 64 sq.cm and $l = 5r$

$TSA = BA + LSA.$

$64 = \pi r^2 + \pi rl = \pi r^2 + \pi r(5r) = 6\pi r^2,$

$r^2 = \frac{64}{6\pi}$ implies $\mathrm{r} = \frac{4\sqrt{6\pi}}{3\pi}\ cm.$

3. $h = 6m,\ r = 8,$

a. $l = \sqrt{h^2 + r^2} = \sqrt{36 + 64} = \sqrt{100} = 10m$

b. Surface area of the tent, $\pi rl = 3.14 \times 8 \times 10 = 251.2\ sq.m$

Since 1 $sq.m$ canvas cost 250 birr, the cost of $251.2\ sq.m$ canvas is $251.2 \times 250 = 62800$ birr.

Assessment

You can give problems as homework. You can also arrange quiz at this point asking them to find the LSA and TSA of pyramids and cones.

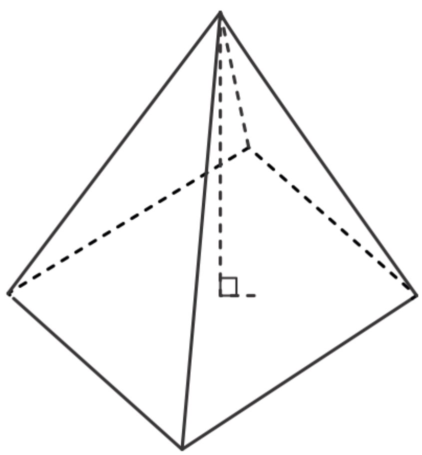
6.2.2 Horizontal cross-section of pyramids and cones

If a pyramid or a cone is cut by a plane parallel to the pane containing the base, the intersection of the plane and the pyramid (or the cone) is called a horizontal cross-section of the pyramid (or the cone). Knowing the cross-section of a pyramid or a cone is a base for next discussion on frustum of pyramids and frustum of cones. The relation between the area of the cross-section, the area of the base, altitude of the pyramid the distance from the vertex to the cross-section will be presented to students. To see this relation state and proof theorems 6.1-6.4 for students, allow students to participate in the discussion.

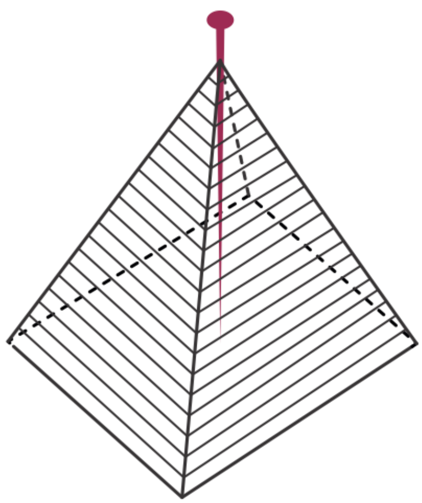
6.2.3 CAVALIERI'S PRINCIPLE

After this state the CAVALIERI'S PRINCIPLE (If two solids of equal height have equal cross-sectional areas at every level parallel to the respective bases, then the two solids have equal volume.) In addition to the ideas presented in the students' text book to demonstrate the principle, the following can also be used to further support students understanding the principle.

First consider a set of square cards cut to the proper size and staked so as to form a square pyramid.



a



b

Figure 6.8

The figure 6.8a represents a square pyramid and the figure6.8b represents the model pyramid made from the cards.

Now stick a pin through the stack from the top to the mid-point of the base of figure 6.8b. Keep the base fixed and tilt the pin in any direction. The shape of the model changes, but the volume remains the same since there are the same number of cards present for any position of the pin figure 6.9.

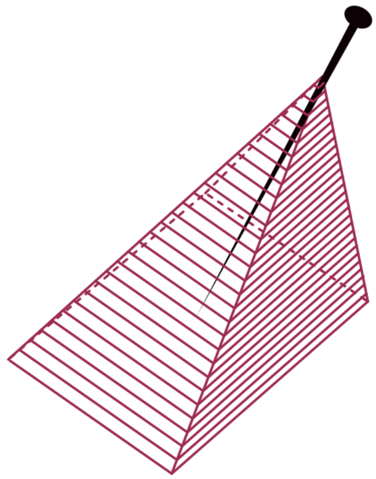


Figure 6.9

The principle is stated by Theorem 6.5 in the students' text and it is the base to drive the volume formula for a pyramid.

Discuss examples in this section with students so that students can understand the relationship between the cross-sectional area, base area and the altitude of the pyramid and the distance from the vertex to the cross section. After this let students do problems 1, 2 and 3 of exercise 6.6 from students' text. You can round for support.

Answers for exercise 6.6

1. BA = A = 160 sq. cm, Cross-section area = A' = 90 sq.cm,

   Distance from the base to the cross-section = 5cm.

   Let the altitude of the pyramid be h,

   The distance from the vertex to the cross-section, $k = h - 5$ and

   $\frac{A\prime}{A} = \frac{k^2}{h^2}$ implies $\frac{90}{160} = \frac{(h-5)^2}{h^2}$ implies $\frac{3}{4} = \frac{h-5}{h}$,

   Hence, $h = 20cm$.

2. Base radius$=r = 3cm$, Cross-section radius $= r' = 2cm$,

   Distance from the base to the cross-section = 6cm.

   Let the altitude of the pyramid be h,

   The distance from the vertex to the cross-section = $k = h - 6$.

   $\frac{A\prime}{A} = \frac{k^2}{h^2}$ implies $\frac{\pi r\prime^2}{\pi r^2} = \frac{(h-6)^2}{h^2}$ implies $h = 18cm$.

3. For a regular hexagon length of side $= s = r =$ radius of the regular hexagon.

   Therefore, $r = 3cm$.

   Base area A = $\frac{1}{2}nr^2 \sin\left(\frac{360^0}{60}\right) = \frac{27\sqrt{3}}{2} sq. cm$.

   $\frac{A\prime}{A} = \frac{k^2}{h^2}$ implies $\frac{A\prime}{\frac{27\sqrt{3}}{2}} = \frac{16}{81}$ implies $A' = \frac{8\sqrt{3}}{3}$ sq.cm.

4. Suppose the base of the pyramid has n sides, the number of triangles that can be made using the diagonals emanating from a vertex are $n-2$.

Let the area of the n-2 triangles be $A_1, A_2, \ . \ . \ . \ , A_{n-2}$ and the areas of the corresponding $n-2$ triangle on the horizontal cross-section be $A_1', A_2', \ . \ . \ . \ , A_{n-2}'$ as shown in the figure.

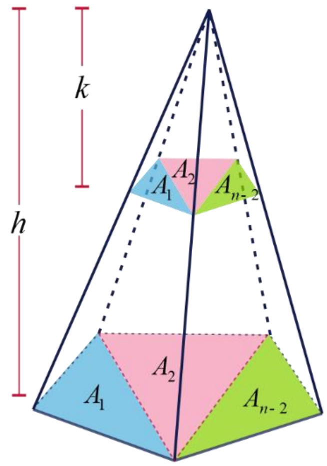


Figure 6.10

Let $A = A_1 + A_2 + \ . \ . \ . + A_{n-2}$ and $A' = A_1' + A_2' + \ . \ . \ . + A_{n-2}'$

By theorem 6.2, $\frac{A_1'}{A_1} = \frac{k^2}{h^2}, \ \frac{A_2'}{A_2} = \frac{k^2}{h^2}, \ \ . \ . \ . \ , \frac{A_{n-2}'}{A_{n-2}} = \frac{k^2}{h^2}$

This implies, $A_1'h^2 = A_1k^2, \ A_2'h^2 = A_2k^2, \ . \ . \ . \ , A_{n-2}'h^2 = A_{n-2}k^2,$

$A_1'h^2 + A_2'h^2 + . \ . \ . + A_{n-2}'h^2 = A_1k^2 + A_2k^2 + . \ . \ . + A_{n-2}k^2,$

$(A_1' + A_2' + . \ . \ . + A_{n-2}')h^2 = (A_1 + A_2 + . \ . \ . + A_{n-2})k^2,$

Hence,

$$\frac{(A_1' + A_2' + . \ . \ . + A_{n-2}')}{(A_1 + A_2 + . \ . \ . + A_{n-2})} = \frac{h^2}{k^2},$$

$$\frac{A'}{A} = \frac{h^2}{k^2}.$$

Assessment

You can give problem 4 of exercise 6.6 as a group assignment.

6.2.4 Volume of Pyramids and Cones

Competencies

At the end of this sub-unit students will be able to:

- calculate the volumes of a given pyramid or a cone.

Introduction

In this subsection, volume formula for a pyramid will be derived and apply the formula to calculate the volume of pyramids and cones.

Teaching Guide

First the volume formula for a triangular pyramid will be presented. The formula is derived by dividing a triangular prism into three triangular pyramids that have equal volumes.

Since students are expected to know the volume formula for a triangular prism as:

$V = (BA)h$, where BA is the base area.

The volume formula for a triangular pyramid is one third of the volume of the triangular prism. That is:

$$V = \frac{1}{3}(BA)h$$

If the number of sides of the base of the pyramid is more than 3, we can divide the base into many triangles. correspondingly, we can divide the pyramid into triangular pyramids and drive the volume formula for any pyramid by taking the sum of the volumes of the triangular pyramids.

Theorem 6.6 and Theorem 6.7 are about the volume of triangular pyramid and any pyramid respectively.

State and prove Theorem 6.6 and Theorem 6.7 for students, allow students to participate while proving the theorems.

We can think a cone as a special pyramid. We defined a cone as a pyramid whose base is a circle. If we inscribe a pyramid in a circular cone (see figure below) as the number of sides of the base of the pyramid increase the base area of the pyramid approaches the base area of the circular cone and the volume of the pyramid approaches to the volume of circular cone.

Therefore, as the number of sides of the base of the pyramid increased the volume formula for the pyramid becomes the volume formula for the circular cone. This can help to explain Theorem 6.8, about the volume of a circular cone.

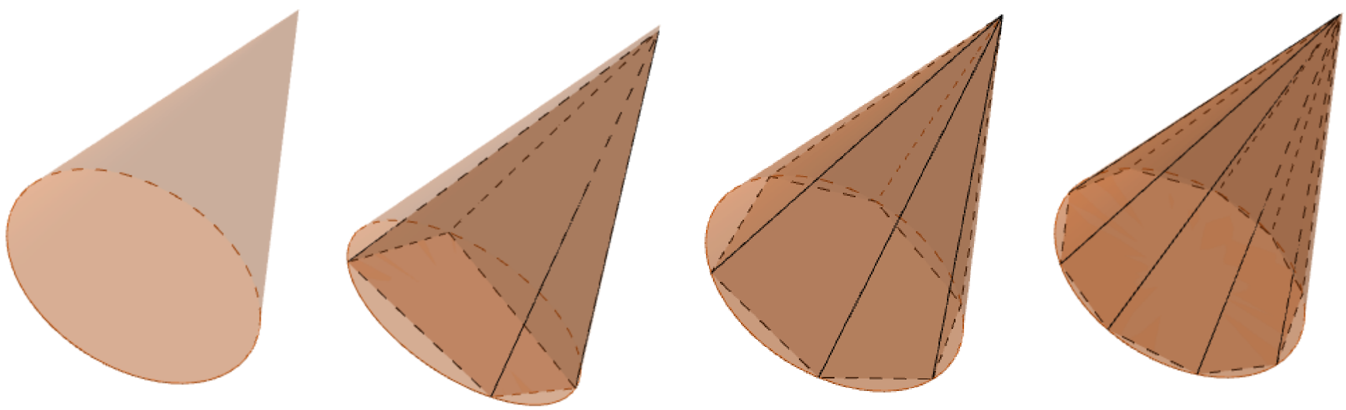


Figure 6.11

Discuss examples in this section with students so that students can understand the volume formula to find the volumes of pyramids and cones.

After this let students do problems 1 and 2 of exercise 6.7 from students' text. You can round for support.

Answers for exercise 6.7

1. The base is a square of side $= s = 4cm$ implies $BA = s^2 = 16sq.cm$

   $h = 6cm.$

   volume, $v = \frac{1}{3}BA \times h = \frac{1}{3} \times 16 \times 6 = 32\ cm^3.$

2. $\mathrm{V} = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(10^2)(12) = 400\pi cm^2$

3. $Circumferance = c = 2\pi r = 12\pi$ implies $r = 6cm,$

   $Volume = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times 36 \times 10 = 120\pi\ cm^3.$

4. All the four faces of a regular tetrahedron are equilateral triangles.

   Area of one face = $\frac{1}{2}s^2 sin60^0 = 9\sqrt{3}.$

   TSA = 4(Area of one face) = $4 \times 9\sqrt{3} = 36\sqrt{3}\ sq.cm.$

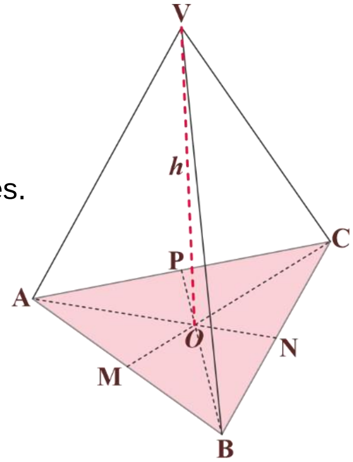


Figure 6.12

The altitude and the median of an equilateral triangle are the same.

The center of an equilateral triangle is the point where its altitudes intersect.

$\overline{AO} = \frac{2}{3}\overline{AN}$ or $\overline{ON} = \frac{1}{3}\overline{AN}$

$$(AN)^2 = (AC)^2 - (CN)^2 = 36 - 9 = 27$$

implies, height of $\Delta ABC = \overline{AN} = 3\sqrt{3}\ cm.$

$\overline{AO} = \frac{2}{3}\overline{AN} = \frac{2}{3} \times 3\sqrt{3} = 2\sqrt{3}\ cm.$

$(VO)^2 + (AO)^2 = (VA)^2,$

$h^2 + \left(2\sqrt{3}\right)^2 = 6^2,$

Altitude of the tetrahedron = $h = 2\sqrt{6}$ cm.

Volume = $\frac{1}{3}BA \times h = \frac{1}{3} \times 9\sqrt{3} \times 2\sqrt{6} = 18\sqrt{2}cm^3.$

5. 1 liter = $1000cm^3$

$A = \pi r^2 = 100\pi cm^2$

vlome of water $= 1000 = \frac{1}{3}\pi r'^2 k$, where r' is the radius of the water level.

$r'^2 = \frac{3000}{\pi k}.$

Area of the water level = $A' = \pi r'^2 = \frac{3000}{k}.$

$\frac{A'}{A} = \frac{k^2}{h^2}$ implies $\frac{\frac{3000}{k}}{100\pi} = \frac{k^2}{400}$ implies $\frac{3000}{100\pi k} = \frac{k^2}{400}$ implies $k^3 = \frac{12000}{\pi}$,

Height of the water = $k = \sqrt[3]{\frac{12000}{\pi}} = 10\sqrt[3]{\frac{12}{\pi}}cm = 15.63$ taking $\pi = 3.14$.

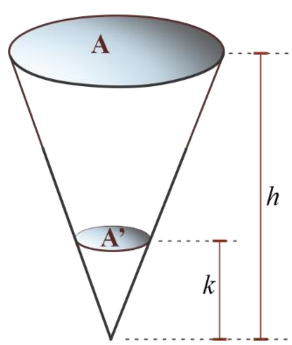


Figure 6.13

6.2.5 Surface area and volume of sphere

Competencies

At the end of this sub-unit students will be able to:

- calculate the surface area of a given sphere.
- calculate the volume of a given sphere.

Introduction

In this subsection, definition of a sphere is stated. Radius, diameter, great circle and small circle of a sphere are also defined. The aim is to state the surface area formula and volume formula of a sphere and apply the formulas to find the surface area and volume of a given sphere

Teaching Guide

Support the instruction by teaching aid. By bringing model pictures of spheres, solid figures that have shape of sphere such as ball and glob. Try to support students understand a hemisphere. Model pictures and half an orange.

After this state the surface area and volume formula and support students to do exercise 6.8 as class work.

Answers for activity 6.6

1. ball, marbles, orange, planets, moon, sun
2. Half of an orange, bowl

Answers for exercise 6.8

1. Surface area = $4\pi r^2 = 400\pi cm^2$, Volume = $\frac{4}{3}\pi r^3 = \frac{4000}{3}\pi cm^3$.
2. $r = \frac{d}{2} = 3cm$, A = $4\pi r^2 = 113.04cm^2$, V = $\frac{4}{3}\pi r^3 = 113.04cm^3$.
3. A = πd^2, V = $\frac{\pi}{6}d^3$.

Assessment

You can arrange a test at this point. The test may include problems to find the volume of pyramids and cones, surface area of sphere and volume of a sphere.

6.3 Frustum of pyramids and cones

Periods allotted: 5 periods

Competencies

At the end of this sub-unit students will be able to:

- define frustums of a pyramid and of a cone.
- calculate the surface areas of frustums of pyramids and of cones.

Introduction

When pyramids or cones are cut by a plane parallel to the base, the solid between the base and the plane is called frustum. The aim in this subsection is to discuss on the lateral surface area, total surface area and volume of frustum of pyramids or cones.

Teaching Guide

The lateral faces of the frustum of a pyramid are trapezium and the lateral face of the frustum of a right circular cone is portion of an annulus between two radii and the arcs. Let students do activity 6.7 just to recall the area of a trapezium and portion of an annulus between two radii and the arcs.

Answers for activity 6.7

1. A quadrilateral with one pair of opposite sides parallel. The parallel sides are called the bases of the trapezium.

2. $A = \frac{1}{2}(b_1 + b_2)h = 96\ cm^2$.

3. $\frac{1}{2}(8 + b)16 = 150,\ b = \frac{43}{4} = 10.75\ cm$.

4. Area of the shaded region = $\frac{\pi r^2\theta}{360} - \frac{\pi r'^2\theta}{360} = \frac{\pi\theta}{360}\left(r^2 - r'^2\right) = \frac{\pi}{12}(40) = \frac{10}{3}\pi\ cm^2$.

Frustum of pyramids and frustum of cones are defined, now let students do activity 6.8 to assess them if they could tell real examples of frustum of cones and pyramids.

Answers for activity 6.8

Figure 6.14

The first example, on the students' text in this section, could be used as an illustrative to generalize the lateral surface area (LSA) of the frustum of a regular pyramid is:

LSA $= \frac{1}{2}l(\mathrm{p} + \mathrm{P}')$ where, l is the slant height and P and P' are the upper and lower bases of the frustum respectively.

Answers for exercise 6.9

a. LSA $= \frac{1}{2}l(p + p') = \frac{1}{2} \times 5(8 + 4) = 30cm^2$.

b. Area of an equilateral triangle of side $s = \frac{1}{2}s^2 sin\theta$.

Area of the upper base $= \frac{1}{2} \times 4^2 sin60^0 = 4\sqrt{3}\ cm^2$.

Area of the upper base $= \frac{1}{2} \times 8^2 sin60^0 = 16\sqrt{3}\ cm^2$.

TA = BA + LSA = $10(3 + 2\sqrt{3})cm^2$.

The volume of a frustum of a pyramid (or a cone) has to be observed as the difference between the original pyramid (or a cone) and the volume of the pyramid (or a cone) that has been cut off to form the frustum. To support this let students do activity 6.9. and discuss on the example.

Answers for activity 6.9

a. Slant height of the bigger cone = $l = \sqrt{r^2 + h^2} = 3\sqrt{5}cm$.

LSA of the bigger cone = $\pi rl = 9\sqrt{5}\pi cm^2$.

b. Slant height of the smaller cone = $l = \sqrt{r^2 + h^2} = \sqrt{5}cm$.

LSA of the smaller cone = $\pi rl = \sqrt{5}\pi cm^2$.

c. L.S.A of the frustum = LSA of the bigger cone – LSA of the smaller cone

$$= (9\sqrt{5} - \sqrt{5})\pi cm^2 = 8\sqrt{5}\pi cm^2.$$

d. The volume of the bigger cone = $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times 9 \times 6 = 18\pi cm^3$.

e. The volume of the smaller cone = $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times 1 \times 2 = \frac{2}{3}\pi cm^2$.

f. Frustum volume = The volume of the bigger cone - The volume of the smaller cone

$$= \frac{52}{3}\pi cm^2.$$

At this point, the formula for the lateral surface area of a right circular cone is derived as

$$LSA = l\pi(r + r'),$$

where r and r' are base radii and l is the altitude of the frustum.

Answers for exercise 6.10

a. $\frac{h'}{\text{h}} = \frac{r'}{r}$.

$\frac{x}{12} = \frac{1}{4}$. This implies $x = 3$. The height of the smaller cone is 3cm.

b. Slant height of the bigger cone = $l = \sqrt{r^2 + h^2} = 4\sqrt{10}cm$.

LSA of the bigger cone = $\pi rl = 16\sqrt{10}\pi cm^2$.

Slant height of the smaller cone = $l = \sqrt{r^2 + h^2} = \sqrt{10}cm$.

LSA of the smaller cone = $\pi rl = \sqrt{10}\pi cm^2$.

L.S.A of the frustum = LSA of the bigger cone – LSA of the smaller cone.

$$= (16\sqrt{10} - \sqrt{10})\pi cm^2 = 15\sqrt{10}\pi cm^2.$$

c. Slant height l of the frustum $=\sqrt{9^2 + 3^2} = 3\sqrt{10}cm$.

LSA of the frustum = $l\pi(r + r') = 3\sqrt{10}\pi(4 + 1) = 15\sqrt{10}\pi cm^2$ and this is the same as the result obtained in (b).

At this point, the volume formula for the frustum of a right circular cone is derived as

$$V = \frac{1}{3}\pi h(r^2 + r'^2 + rr').$$

where r and r' are base radii and h is the altitude of the frustum.

Answers for exercise 6.11

a. The volume of the larger cone = $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times 16 \times 12 = 64\pi cm^3$.

b. The volume of the smaller cone = $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times 1 \times 3 = \pi cm^3$.

c. Find the volume (V) of the frustum V = (volume of the larger cone) – (volume of the smaller cone) = $63\pi cm^3$.

d. V = $\frac{1}{3}\pi h(r^2 + r'^2 + rr') = \frac{1}{3}\pi(9)\left(4^2 + 1^2 + (4)(1)\right) = 63\pi cm^3$.

The volume formula for a frustum of a pyramid is stated as V $= \frac{1}{3}h\left(A + A' + \sqrt{AA'}\right)$, where A and A' are base areas and h the altitude of the frustum.

Discuss the derivations with students and initiate them to participate in the discussion. Examples in this section of the students' textbook are presented to apply the formulas in finding lateral surface area and volume of frustum of right circular cone.

Answers for exercise 6.12

1. Height of the frustum, $h = 16 - 7 = 9m$.

   $V = \frac{1}{3}h\left(A_1 + A_2 + \sqrt{A_1 A_2}\right) = \frac{1}{3} \times 9\left(49 + 81 + \sqrt{49 \times 81}\right) = 579m^3$.

2. **a.** $LSA = \frac{1}{2}l(P + P')$, where P and P' are perimeters of the bases.

   $= \frac{1}{2} \times 8 \times (24 + 16) = 160cm^2$.

   b. BA = $6^2 + 4^2 = 52cm^2$ and TSA = BA + LSA = $212cm^2$.

   c.

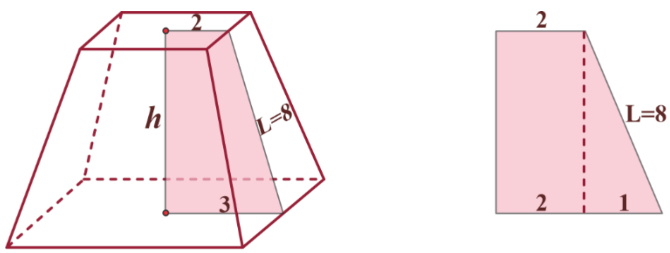


Figure 6.15

$h = \sqrt{8^2 - 1^2} = \sqrt{63}cm$.

$$V = \frac{1}{3}h\left(A_1 + A_2 + \sqrt{A_1A_2}\right) = \frac{1}{3} \times \sqrt{63}\left(36 + 16 + \sqrt{36 \times 16}\right)$$

$$= \frac{76\sqrt{63}}{3} = 76\sqrt{7}cm^3.$$

3. $AC = l = 10cm,$

LSA of the pyramid $= \frac{1}{2}pl = \frac{1}{2} \times 48 \times 10 = 240cm^2.$

$(OB)^2 + (BC)^2 = (OC)^2,$

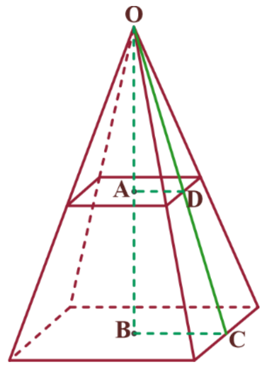


Figure 6.16

implies $(OB)^2 = 100 - 36 = 64.$

height of the pyramid = $h = OB = 8cm.$

$$\frac{\text{cross} - \text{section area}}{Base\ area} = \frac{k^2}{h^2} = \frac{16}{64}$$

$$\frac{\text{cross} - \text{section area}}{144} = \frac{16}{64}$$

$\text{cross} - \text{section area} = 36$, since, $\text{cross} - \text{section area} = s^2 = 36,$

Side of the cross-section has length 6cm.

$(OA)^2 + (AD)^2 = (OD)^2$, implies $4^2 + (3)^2 = (OD)^2,$

implies OD = slant height of the smaller pyramid = $5cm.$

Or, $\Delta OAD \approx \Delta OBC$ implies $\frac{OA}{OB} = \frac{OD}{OC}$ implies $\frac{4}{8} = \frac{OD}{10}$ implies OD = 5cm.

Slant height of the frustum = $10 - 5 = 5cm.$

LSA of the frustum = $\frac{1}{2}l(p + p') = 180cm^2.$

Volume of the frustum = $\frac{1}{3}h\left(A + A' + \sqrt{AA'}\right) = 336cm^3.$

4. Frustum height = $h = 2cm$.

Diagonal of the base square = $d^2 = (3\sqrt{2})^2 + (3\sqrt{2})^2 = 36$,

$d = 6cm$. BC = 3 cm,

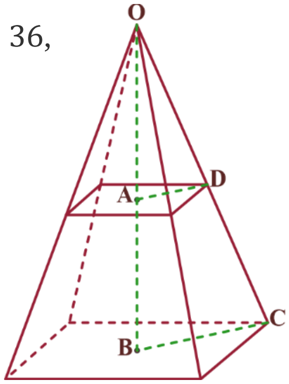
$(OB)^2 + (BC)^2 = (OC)^2$ implies $OB = 3cm$.

implies OA = 1cm.

BA = A = $3\sqrt{2} \times 3\sqrt{2} = 18cm^2$.

$\frac{A'}{A} = \frac{(OA)^2}{(OB)^2}$ implies $\frac{A'}{18} = \frac{1}{9}$,

Area of the cross-section = $A' = 2cm^2$.



Volume of the frustum = $\frac{1}{3}h\left(A + A' + \sqrt{AA'}\right) = \frac{52}{3}cm^2$. Figure 6.17(figure 6.16 Produced)

5. Hight of the frustum = $h = 11 - 6 = 5cm$.

$V = \frac{1}{3}\pi h\left(r^2 + r'^2 + \sqrt{rr'}\right) = \frac{1}{3}\pi \times 5\left(9 + 16 + \sqrt{12}\right) = \frac{5}{3}\pi\left(25 + 2\sqrt{3}\right)cm^3$.

6. $\frac{A'}{A} = \frac{\pi r_1^2}{\pi r_2^2} = \frac{r_1^2}{r_2^2} = \frac{k^2}{h^2} = \frac{8^2}{20^2}$, where r_1 is the radius of the upper base and r_2 is the radius of the lower base.

$\frac{r_1}{r_2} = \frac{2}{5}$ implies $r_1 = \frac{2}{5}r_2$.

The volume of the frustum = $V = 156cm^3$.

Frustum volume = $V = \frac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2)$,

$= \frac{1}{3} \times 12\pi(\frac{4}{25}r_2^2 + r_2^2 + \frac{2}{5}r_2 r_2)$,

$156 = \frac{4\times 39}{25}\pi r_2^2$ implies $r_2 = \frac{5}{\pi}cm$.

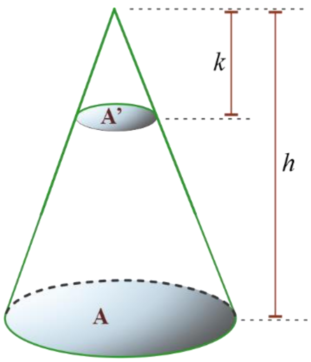


Figure 6.18

7. A' = area of the upper base,

A = area of the lower base,

h = altitude of the frustum

,Volume of the bigger pyramid = $\frac{1}{3}HA$.

Volume of the smaller pyramid = $\frac{1}{3}(H - h)A$.

If we denote the volume of the frustum by V, then

$V = \frac{1}{3}HA - \frac{1}{3}(H - h)A'$,

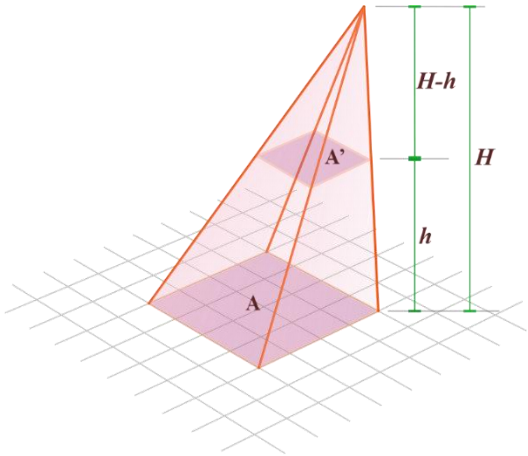


Figure 6.19

$$V = \frac{1}{3}[(A - A')H + A'\,h] \quad . \; . \; . \quad (*)$$

$$\frac{A'}{A} = \frac{(H-h)^2}{H^2} = \left(\frac{H-h}{H}\right)^2$$

$$\sqrt{\frac{A'}{A}} = \frac{H-h}{H} = 1 - \frac{h}{H}$$

$$\frac{\sqrt{A'}}{\sqrt{A}} = 1 - \frac{h}{H}$$

$$\frac{h}{H} = 1 - \frac{\sqrt{A'}}{\sqrt{A}} = \frac{\sqrt{A} - \sqrt{A'}}{\sqrt{A}}$$

$$\frac{H}{h} = \frac{\sqrt{A}}{\sqrt{A} - \sqrt{A'}}$$

$$H = \left(\frac{\sqrt{A}}{\sqrt{A} - \sqrt{A'}}\right)h = \left(\frac{\sqrt{A}}{\sqrt{A} - \sqrt{A'}}\right)\left(\frac{\sqrt{A} + \sqrt{A'}}{\sqrt{A} + \sqrt{A'}}\right)h$$

$$H = \left(\frac{A + \sqrt{AA'}}{A - A'}\right)h \quad . \; . \; . \quad (**)$$

Substituting the value of H in (**) to the value of H in (*), we have,

$$V = \frac{1}{3}\left[(A - A')\left(\frac{A + \sqrt{AA'}}{A - A'}\right)h + A'\,h\right] = \frac{1}{3}[(A + \sqrt{AA'})h + A'\,h\,) = \frac{1}{3}h(A + A' + \sqrt{AA'})$$

$$V = \frac{1}{3}h(A + A' + \sqrt{AA'}).$$

Assessment

You can assess students at this point by giving problems from exercise 6.12 as homework. Let students explain their work by writing it on the in next session board. You can also arrange a test.

6.4 Surface area and volume of composed solids

Periods allotted: 2 periods

Competencies

At the end of this sub-unit students will be able to:

- determine the surface area of simple composed solids.

- calculate volumes of simple composed solids.

Introduction

A composed solid is a solid that is made up of, two or more solids. There are many physical objects that are a combination of prisms, cylinders, cones, pyramids and spheres. In this subsection the surface area and volume of solids composed of such types of solid are examined.

Teaching Guide

The students are already familiar with the surface area and volumes of prism, cylinder, pyramids, cones, spheres, frustums of pyramids and cones.

In order to find the volume and surface area of a composite solid, one needs to identify the different parts it is made of. This decomposition allows working out the volume and surface area of each part independently.

Examples in this section of the students' textbook are presented so that students can practice looking at the different parts a composed solid is made of, find the surface area and volume of the different part and finally the surface area and volume of the composed solid.

Answers for exercise 6.8

1. Find the total surface area and volume of the following

a. The solid is composed of a cylinder and a frustum of a cone

LSA of cylinder = $2\pi rh = 2\pi \times 3 \times 8 = 48\pi\ cm^2$

BA of the cylinder = $\pi r^2 = \pi \times 3^2 = 9\pi\ cm^2$

TSA of the cylinder = $57\pi\ cm^2$

$l = \sqrt{5^2 + 1^2} = \sqrt{26}\ cm^2$

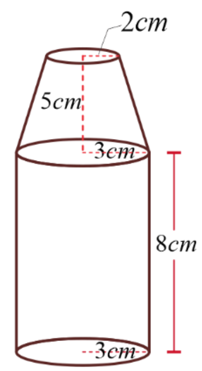


Figure 6.20

LSA of the frustum of the cone = $l\pi(r + r') = \sqrt{26}\pi(3 + 2) = 5\sqrt{26}\pi\ cm^2$

BA of the frustum of the cone = = $\pi r^2 = \pi \times 2^2 = 4\pi\ cm^2$

TSA of the frustum of the cone = $\left(4 + 5\sqrt{26}\right)\pi\ cm^2$

TSA of the solid = TSA of the cylinder + TSA of the frustum of the cone

$=\left(61 + 5\sqrt{26}\right)\pi cm^2$

Volume of the cylinder = $\pi r^2 h = 72\pi\ cm^3$

Volume of the frustum of the cone = $\frac{1}{3}\pi h(r^2 + r'^2 + rr') = \frac{1}{3}\pi(5)\left(3^2 + 2^2 + (3)(2)\right)$

$$= \frac{95}{3}\pi\ cm^3$$

Total volume of the solid = Volume of the cylinder + Volume of the frustum of the cone

$$= \frac{311}{3}\pi\ cm^3$$

b. The solid is composed of two prisms. One is a right rectangular prism and the other is a right prism with a trapezium base.

TSA of the rectangular prism = $(5 \times 6) + 2(3 \times 5) + 2(6 \times 3) = 96\ cm^2$

TSA of the right prism with a trapezium base

$$= (2 \times 6) + 2(6 \times 2.5) + 2\left(\frac{1}{2}(2 + 5) \times 2\right) = 56\ cm^2$$

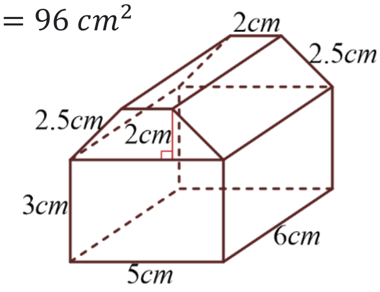


Figure 6.21

Total surface area of the solid = TSA of the rectangular prism + TSA of the right prism with a trapezium base

$= 152\ cm^2$

Volume of the rectangular prism = $(BA)h = (3 \times 5) \times 6 = 90\ cm^3$

Volume of the right prism with trapezium base $(BA)h = \left(\frac{1}{2}(2 + 5) \times 2\right) \times 6 = 42\ cm^3$

Total Volume of the solid = $90\ cm^3 + 42cm^3 = 132\ cm^3$

c. The solid is composed of one right rectangular prism and one right square pyramid.

TSA of the right rectangular prism = $2(4 \times 5) + 2(4 \times 12) + 2(5 \times 12) = 256cm^2$

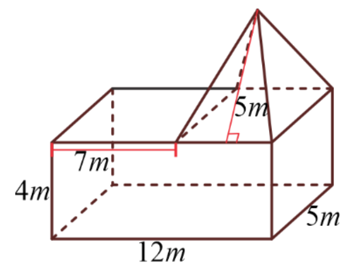


Figure 6.22

TSA of the square pyramid = $4\left(\frac{1}{2}(5 \times 5)\right) = 50\ cm^2$

TSA of the solid =TSA of the right rectangular prism + TSA of the square pyramid

$= 306\ cm^2$

Volume of the right rectangular prism = $(BA)h = 240\ cm^3$

Altitude of the square pyramid = $h = \sqrt{5^2 - 2.5^2} = 4.33\ cm$

Volume of the square pyramid = $\frac{1}{3}(BA)h = \frac{1}{3}(5 \times 5) \times 4.33 = 36.08\ cm^3$

2. Let the radius be r and the height be h.

Volume = Volume of the cylinder – Volume of the cone

$= \pi r^2 h - \frac{1}{3}\pi r^2 h = \frac{2}{3}\pi r^2 h$

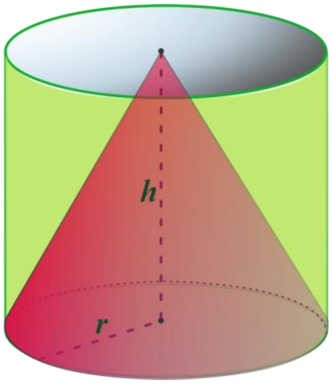


Figure 6.23

3. Volume of the original pencil = $\pi r^2 h = 18\pi\ cm^3$

Volume of the used pencil is composed of cylinder and cone.

Volume cylinder = $\pi r^2 h = 6\pi\ cm^3$

Volume cone = $\frac{1}{3}\pi r^2 h = \frac{2}{3}\pi\ cm^3$

Volume of the used pencil = $\frac{20}{3}\pi\ cm^3$

$$\frac{\text{Volume of the used pencil}}{\text{Volume of the original pencil}} = \frac{\frac{20}{3}\pi}{18\pi} = \frac{10}{27}$$

She used $\frac{10}{27}$ of the original pencil

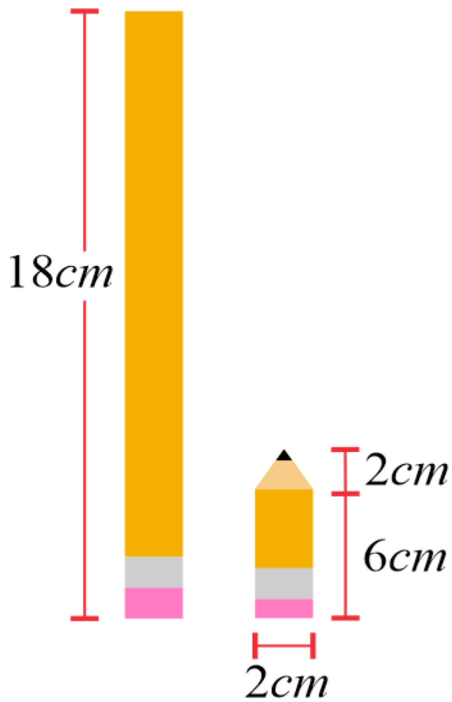


Figure 6.24

4. Volume of the water and the iron ball

= Volume before the iron ball is removed

$= \pi r^2 h = \pi \times 5^2 \times 5 = 125\pi\ cm^3$

where h the height of the water level

Volume of the ball = $\frac{4}{3}\pi r^3 = \frac{24}{3}\pi\ cm^3$

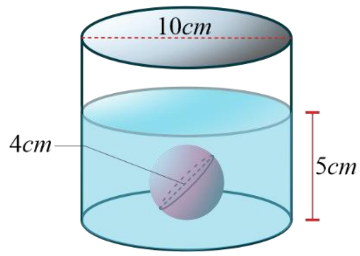


Figure 6.25

where r the radius of the ball.

Volume of water after the iron ball is removed

=Volume before the iron ball is removed - Volume of the ball = $117\pi\ cm^3$

Height of the water level after the iron ball is removed = $\frac{117\pi}{25\pi} = \frac{117}{27}\ cm$

Therefore, the water level drops by = $5 - \frac{117}{27} = \frac{18}{27}\ cm$

5. $height\ of\ th\ cone = radius\ of\ the\ cone = radius\ of\ the\ hemisher$

Volume = Volume of the hemisphere – volume of the cone

$= \frac{2}{3}\pi r^3 - \frac{1}{3}\pi r^2 h = \frac{2}{3}\pi r^3 - \frac{1}{3}\pi r^3 = \frac{1}{3}\pi r^3 = \frac{512}{3}\pi\ cm^3$

TSA = LSA of the hemisphere + LSA of the cone

$= 2\pi r^3 + \pi r l = 2\pi r^3 + \pi r\sqrt{r^2 + h^2}$

$= 1024\pi + 64\sqrt{2}\pi$

$= 64\pi(16 + \sqrt{2})cm$

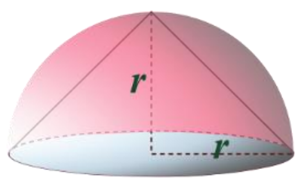


Figure 6.26

6. Volume of frustum = $\frac{1}{3}\pi h(r^2 + r'^2 + rr') = \frac{1040}{3}\pi\ cm^3$

Volume of the cylinder = $\pi r^2 h = 80\pi\ cm^3$

Volume of the remaining solid = $\frac{1040}{3}\pi - 80\pi = \frac{800}{3}\pi\ cm^3$

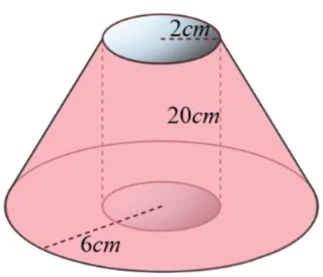


Figure 6.27

LSA of the cylinder = $2\pi rh = 80\pi\ cm^2$

Slant height of the frustum = $l = \sqrt{20^2 + 4^2} = \sqrt{416} = 4\sqrt{26}\ cm$

LSA of the frustum = $l\pi(r + r') = 4\sqrt{26}\pi \times 8 = 32\sqrt{26}\pi\ cm^2$

Base of the frustum is an annulus and its area = $\pi(6^2 - 2^2) = 32\pi\ cm^2$

TSA of the frustum =LSA of the frustum + BA of the frustum = $32\pi\left(\sqrt{26} + 1\right)cm^2$

TSA of the remaining solid = LSA of the cylinder + TSA of the frustum

$$= 80\pi + 32\pi\left(\sqrt{26} + 1\right)$$

$$= 16\pi\left(2\sqrt{26} + 7\right)\ cm^2$$

7. Volume = $\frac{2}{3}\pi(6^3 - 4^3) = \frac{304}{3}\pi\ unit^3$

TSA = $2\pi(6^2) + 2\pi(4^2) + \pi(6^2 - 4^2) = 124\pi\ unit^2$

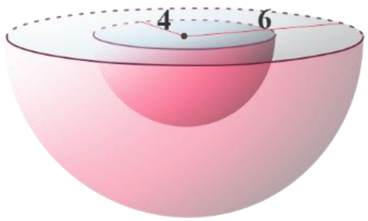


Figure 6.28

8. Volume of the wood scooped = $\frac{1}{3}\pi r^2 h$

Volume of the cylinder = $\pi r^2(2h)$

volume of the wood left = Volume of the cylinder - volume of the wood scooped

$$= 2\pi r^2 h - \frac{1}{3}\pi r^2 h = \frac{5}{3}\pi r^2 h$$

$$\frac{\text{volume of the wood scooped}}{\text{volume of the wood left}} = \frac{\frac{1}{3}\pi r^2 h}{\frac{5}{3}\pi r^2 h} = \frac{1}{5}$$

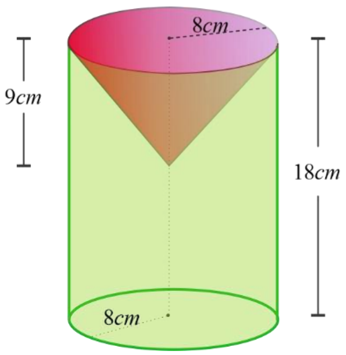


Figure 6.29

Assessment

You can assess students at this point by giving problems from exercise 6.13 as homework and assignment. Let students explain their work by writing it on the board. You can also arrange a test.

6.5 Application

Periods allotted: 1 period

Answers for exercise 6.9

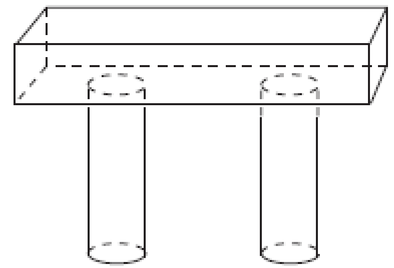


Figure 6.30

1. Volume of the pillars = $2 \times volume\ of\ one\ pilar$

$= 2 \times \pi r^2 h = 2\pi \left(\frac{0.5}{2}\right)^2 (2.5) = 0.98\, m^3$

Volume of the cuboid = $lwh = 0.6 \times 4 \times 0.5 = 1.2\, m^3$

Total volume of concrete needed = $0.98 + 1.2 = 2.18\, m^3$

2. $15\, cm = 0.15\, m\ \ and\ 30\, cm = 0.3\, m$

a. Volume of metal needed to make the pipe = $\pi(r^2 - r'^2)h$

$= \pi(0.3^2 - 0.15^2)50 = 10.6\, m^3$

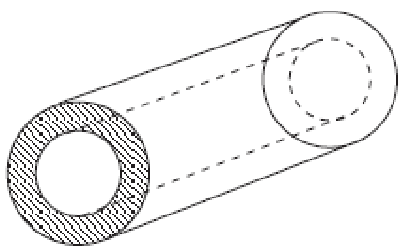


Figure 6.31

b. LSA = $2\pi h(r + r')$

BA = $2\pi(r^2 - r'^2)$

TSA = LSA + BA = $2\pi h(r + r') + 2\pi(r^2 - r'^2)$

= $2\pi(r + r')\big(h + (r - r')\big) = 70.86\, m^2$

3. Volume of the lead bar = 5(Volume of one spherical ornament)

Volume of the lead bar = $lwh = 15 \times 8 \times 5 = 600 cm^3$

5(Volume of one spherical ornament) = $600 cm^3$

$5\left(\frac{4}{3}\pi r^3\right) = 600$ implies $r = \sqrt[3]{\frac{90}{\pi}} = \sqrt[3]{\frac{90}{3.14}} = 3.06\, cm$

Answers for review exercise on unit 6

1. a. Volume = $\frac{1}{3}(BA)h = 8 unit^3$

LSA = $\left(\frac{1}{2} \times 3 \times 4\right) + \left(\frac{1}{2} \times 4 \times 4\right) + \left(\frac{1}{2} \times 3 \times 4\right) + \left(\frac{1}{2} \times 4\sqrt{2} \times \sqrt{17}\right) = 2\left(7 + \sqrt{34}\right)$

b. Volume = $\frac{1}{3}\pi r^2 h = \frac{176}{3}\pi\ unit^3$

LSA = $\pi r l = \pi r\sqrt{r^2+h^2} = 4\pi\sqrt{4^2+11^2} = 4\sqrt{137}\pi\ Unit^2$

c. Volume = $\frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{2}{3}\pi r^3 = \frac{128}{3}\pi\ Unit^2$

LSA = $\frac{1}{2}(4\pi r^2) = 2\pi r^2 = 32\pi\ Unit^2$

d. Volume = $\pi r^2 h = 81\pi\ unit^3$

LSA = $= 2\pi r h = 54\pi\ unit^2$

2. LSA = $hp = 6 \times 36 = 216\ cm^2$

3. TSA = $4\pi r^2$ and Volume = πr^3

4. Apothem = $a = 4\sqrt{3}cm$, slant height = $l = \sqrt{h^2+a^2} = 9\sqrt{2}cm$

LSA = $\frac{1}{2}lp = \frac{1}{2} \times 9\sqrt{2} \times 48 = 216\sqrt{2}\ cm^2$

BA = $\frac{1}{2}nr^2 \sin\left(\frac{360^0}{n}\right) = 96\sqrt{3}\ Cm^2$

TSA = $24(9\sqrt{2}+4\sqrt{3})cm^2$

5. Volume of the stone = $1000\ cm^3$

6. TSA = $LSA + BA = \pi r\sqrt{r^2+h^2} + \pi r^2 = 8\pi(\sqrt{89}+8)cm^2$

V = $\frac{1}{3}\pi r^2 h = \frac{320}{3}\pi\ cm^3$

7. LSA pyramid = $240\ cm^2$

LSA of the frustum = $180\ cm^2$

8. $s = 4\sqrt[3]{\pi}\ m$

9. Volume relation 3:2:1 and surface area relation $1:1:\frac{\sqrt{5}}{4}$

10. Volume = $198\ \pi\ m^3$ and Surface area = $9\pi(14+\sqrt{10})m^2$

Unit 7

Coordinate Geometry (17 periods)

Introduction

In this unit students will develop the ability to use algebra for a better understanding of geometry. It provides a connection between algebra and geometry through graphs of lines and curves Distance between two points, Division of line segment, Equation of a line, Slopes of parallel and perpendicular lines and application of coordinate geometry are discussed in the unit

Unit outcomes:

At the end of this unit the students will be able to:

- find distance between any two given points in the coordinate plane.
- divide a given line segment into a given ratio
- describe equation of a line in different forms.
- relate the slope of parallel and perpendicular lines
- find equations of a circle in different forms.

Suggested teaching aids

A ruler, Mathematical set, Chart representing the Cartesian coordinate plane with the x-axis and the y-axis, Charts representing parallel and perpendicular lines and inclined line.

7.1 Distance between two points. (2 periods)

Competencies

- derive the distance formula (to find distance between two points in the coordinate plane).
- apply the distance formula to solve related problems in the coordinates plane.

Introduction

This subunit is dedicated in computing distance between two points on a plane. In unit one of this textbook, students have discussed about the Cartesian coordinate plane and they have seen that there is one –to-one correspondence between the set of points in the plane and the set of all ordered pairs of real numbers.

In this subunit revision is needed to support students understand Cartesian coordinate plane, Coordinate of a point, the definition of vertical and horizontal lines.

Once the Cartesian coordinate plane is set up, any line that is parallel to the x-axis is a horizontal line and any line parallel to the y-axis is called vertical line.

Teaching Guide

Show the how to set the Cartesian coordinate system. Select points in different quadrants and initiate students to locate the points in the coordinate system.

Support students to sketch graphs of horizontal lines like $y = 2$, $y = -3$ and $y = 0$ and ask students to give name for the line $y = 0$.

Support students to sketch graphs of horizontal lines like $x = 3$, $y = -2$ and $x = 0$ and ask students to give name for the line $x = 0$.

For the purpose of revision, you could ask students to do activity 7.1

Answers for activity 7.1

1. **i)** $P = (-5,0)$ $A = (-3,0)$ $Q = (0,0)$ $B = (4,0)$

ii) distance of $\overline{AB} = |-3-4| = 7$

2. **a.** distance of $\overline{PQ} = |x_2 - x_1|$

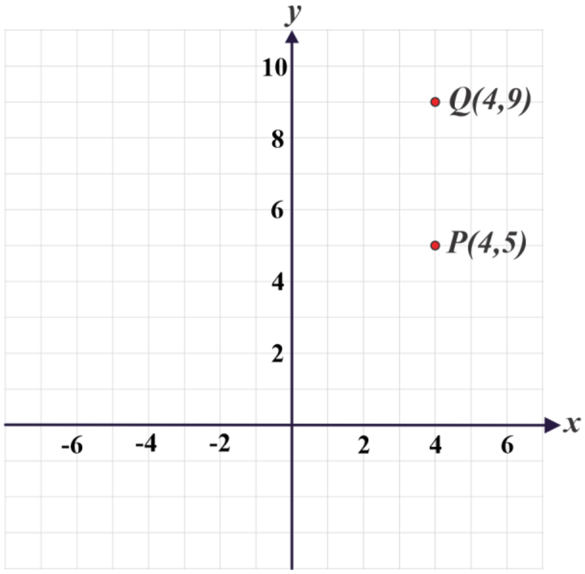
b. distance of $\overline{QP} = |x_1 - x_2|$

c. $distance\ of\ \overline{PQ} = distance\ of\ \overline{QP}$

d. $|x_2 - x_1| = |x_1 - x_2|$

3. We're plotting an ordered pair on the x (horizontal) axis and y (vertical) axis of the coordinate plane. An ordered pair or *coordinate* is simply a set of numbers that define the location of a point on a coordinate grid. For example, if we had $(4, 2)$ our point would be 4 units to the right and 2 units above the origin.

4. **a.**



b. It is a vertical line.

Two points that lie on a horizontal line have the same y-coordinate. The distance between these points is the absolute value of the difference of their x-coordinates. That is, if $A = (x_1, y)$ and $B = (x_2, y)$ then the line passes through these two points is a horizontal line and the distance$d$ between point A and point B is:

$$d = |x_1 - x_2|$$

Two points that lie on a vertical line have the same x-coordinate. The distance between these points is the absolute value of the difference of their y-coordinates. That is, if $A = (x, y_1)$ and $B = (x, y_2)$ then the distance d between point A and point B is:

$$d = |y_1 - y_2|$$

Pythagoras' theorem is used to calculate the distance between two points when the line interval between them is neither vertical nor horizontal. Ask students to state Pythagoras theorem and show to students how to find the distance between two points using Pythagoras theorem. Support it by giving example.

Now, allow students to do exercise 7.1(1a and b) and (2a) as classwork. You can move around to facilitate and support.

Exercise 7.1(1c-f) and (2b-c) can be given to students as a home work. Let students show their work by writing it on the board.

Answers for exercise 7.1.

a. line segment AB is a horizontal line, distance $d = |x_2 - x_1| = |-3 - 2| = 5$.

Or $d = \sqrt{(-3-2)^2 + (-2-(-2))^2} = \sqrt{25+0} = \sqrt{25} = 5$

b. Since PQ is vertical line, distance $d = |y_2 - y_1| = |1-(-2)| = 3$.

Or $d = \sqrt{(4-4)^2 + (1+2)^2} = \sqrt{0+9} = \sqrt{9} = 3$

Answers for exercise 7.2

1. **a.** $d = \sqrt{(7-5)^2 + (2-9)^2} = \sqrt{4+49} = \sqrt{53}$

b. $d = \sqrt{(-3-4)^2 + (5-10)^2} = \sqrt{49+25} = \sqrt{74}$

c. $d = \sqrt{(9-6)^2 + (5+3)^2} = \sqrt{9+64} = \sqrt{73}$

d. $d = \sqrt{(-5-0)^2 + (-2+14)^2} = \sqrt{25+144} = \sqrt{169} = 13$

e. $d = \sqrt{(-4-5)^2 + (-3-7)^2} = \sqrt{81+100} = \sqrt{181}$

f. $d = \sqrt{\left(0-\sqrt{2}\right)^2 + \left(0-\sqrt{2}\right)^2} = \sqrt{2+2} = \sqrt{4} = 2$

7.2 Division of a line segment. (3 periods)

Competencies

- determine the coordinates of points that divide a given line segment in a given ratio.

Introduction

In this sub-unit, the discussion is about a point P that divides a line segment in the ratio $p:q$ internally.

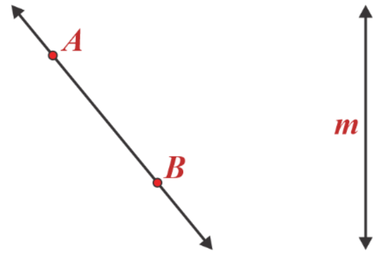
Teaching Note

Participate students in the discussion about line, line segment, mid-point of a line segment and the notion of ratio. For this let students do activity 7.2.

Answers for activity 7.2

a. we all know intuitively what a line is, it is actually difficult to give a good mathematical definition. Roughly, we can say that a line is **an infinitely thin, infinitely long collection of**

points extending in two opposite directions. A line can be named either using two points on the line (for example, $\overleftrightarrow{AB}$) or simply by a letter, usually lowercase (for example, line m).



b. A **line segment** has two endpoints. It contains these endpoints and all the points of the line between them. You can measure the length of a segment, but not of a line. A line segment has two endpoints. A segment is named by its two endpoints

c. A line segment of length 10 cm is divided into two equal parts by using a ruler as follows:

Mark a point 5 cm away from one end, 10 cm is divided into two 5 cm line segments.

d. The midpoint of a line segment is halfway between the two end points of the segment:

Its x value is halfway between the two x values

Its y value is halfway between the two y values

e. A ratio is a comparison of two or more numbers that indicates their sizes in relation to each other.

After students are made to understand how to find the coordinates of that divides the line segment in a given ratio by considering several examples, they should be guided to derive the mid-point formula as a special case.

Allow students do Exercise 7.2 (1-2) as a class-work and Exercise 7.2(2-5) as a homework.

Answers for exercise 7.3

1. a. Given: line segment AB with $A(1, 2)$ and $B(4,\ 5)$.The first point divides the line segment in the ratio $1{:}\,2 = p_1{:}\,p_2$, and hence

$$(x_0, y_0) = \left(\frac{p_1x_2\ + p_2x_1}{p_1 + p_2}, \frac{p_1y_2 + p_2y_1}{p_1 + p_2}\right) = \left(\frac{1 \times 4 + 2 \times 1}{3}, \frac{1 \times 5 + 2 \times 2}{3}\right) = (2, 3)$$

Therefore, the first point is $(2, 3)$

b. Given: line segment AB with $A(2,-3)$ and $B(-1,\ 5)$.The first point divides the line segment in the ratio $3:1 = p_1:p_2$, and hence

$$(x_0, y_0) = \left(\frac{p_1x_2 + p_2x_1}{p_1+p_2}, \frac{p_1y_2+p_2y_1}{p_1+p_2}\right) = \left(\frac{3\times-1+1\times2}{3}, \frac{3\times5+1\times-3}{3}\right) = \left(-\frac{1}{3}, 4\right)$$

Therefore, the first point is $\left(-\frac{1}{3}, 4\right)$.

2. Given: line segment PQ with $P(-1,5)$ and $Q(5,2)$.The first point divides the line segment in the ratio $1:2 = p_1:p_2$, and hence

$$(x_0, y_0) = \left(\frac{p_1x_2 + p_2x_1}{p_1+p_2}, \frac{p_1y_2+p_2y_1}{p_1+p_2}\right) = \left(\frac{1\times5+2\times-1}{3}, \frac{1\times2+2\times5}{3}\right) = (1,4)$$

Therefore, the first point is (1,4)

The second point divides the line segment in the ratio $2:1 = p_1:p_2$, and hence

$$(x_0, y_0) = \left(\frac{p_1x_2 + p_2x_1}{p_1+p_2}, \frac{p_1y_2+p_2y_1}{p_1+p_2}\right) = \left(\frac{2\times5+1\times-1}{3}, \frac{2\times2+1\times5}{3}\right) = (3,3)$$

Therefore, the second point is (3,3).

Answers for exercise 7.4

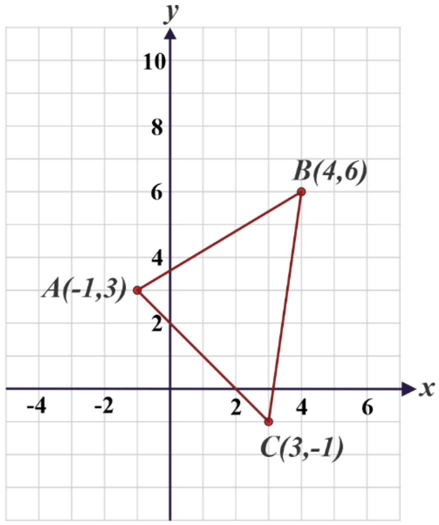
1. a. $\left(\frac{-9+18}{2}, \frac{-3+2}{2}\right) = \left(\frac{9}{2}, \frac{-1}{2}\right)$

b. $\left(\frac{1+4}{2}, \frac{-3+5}{2}\right) = \left(\frac{5}{2}, 1\right)$

2. midpoint of $AB = \left(\frac{-1+4}{2}, \frac{3+6}{2}\right) = \left(\frac{3}{2}, \frac{9}{2}\right)$

Midpoint of $AC = \left(\frac{-1+3}{2}, \frac{3-1}{2}\right) = (1,1)$

Midpoint of $BC = \left(\frac{3+4}{2}, \frac{-1+6}{2}\right) = \left(\frac{7}{2}, \frac{5}{2}\right)$. See figure below.



3. Given: midpoint of $AB = M(4,6)$ and $A = (-3,-2)$. Let the coordinate of B be (a,b), then $M(4,6) = \left(\frac{-3+a}{2}, \frac{-2+b}{2}\right)$.

$4 = \frac{-3+a}{2}$ and $6 = \frac{-2+b}{2}$. So, $a = 11$ and $b = 14$

Therefore, the coordinate of $B = (11,14)$

4. Given coordinates are $A\ (4,-1)$ and $B\ (4,3)$. Let $C(x_0,y_0)$be a point which divides the line segment in the ratio of $3:1$ i.e., $p:q = 3:1$

Now using the formula $C\ (x,y) = \left(\frac{px_2 + qx_1}{p+q}, \frac{py_2+qy_1}{p+q}\right)$ as C is dividing internally.

$$C\ (x,y) = \left(\frac{3\times 4 + 1\times 4}{3+1}, \frac{3\times 3 + 1\times(-1)}{3+1}\right) = \left(\frac{16}{4}, \frac{8}{4}\right) = (4,2).$$

Hence, the coordinates of $C\ (x,y)$ is (4, 2).

7.3 Equation of a line. (8 periods)

Competencies

- define the gradient of a given line.
- determine the gradient of a given line (given two points on the line).
- express the slope of a line in terms of the angle formed by the line and the x-axis.
- determine the equation of a given line.

Introduction

In this sub-unit students should be able to strengthen their knowledge on the concept of gradient(slop), slope of a line in terms of an angle of inclination and different forms of equation of a line. They should be able to use the concept of slope for writing equations of a given line. The main focus is that the students should understand and write the point slop form, slop intercept form, two-point form and the general form of equation of a line supported by different examples and exercise.

Teaching Guide

To practice on the ratio $\frac{y_2-y_1}{x_2-x_1}$, allow students do activity 7.3. Then state the definition of gradient. If $P(x_1,y_1)$ and $Q(x_2,y_2)$ are two points on a line l, then $m = \frac{y_2-y_1}{x_2-x_1}, x_1 \neq x_2$ is the gradient of l. When $y_1 = y_2$, l is horizontal and $m = 0$. Therefore, the slope of a horizontal line is zero. When

$x_1 = x_2$, l is vertical and m is undefined (division by zero). We may say a vertical line has no slop.

Let students do activity 7.4 and give their opinion.

The slope of a given line is unique. The slopes obtained by taking any two points on the same line are equal.

If the slope of a line is positive the line rises from left to right.

If the slope of a line is negative the line rises from right to left.

Answers for activity 7.3

a. **i)** $\frac{y_2-y_1}{x_2-x_1} = \frac{5-2}{7-(-4)} = \frac{3}{11}$ **ii)** $\frac{y_2-y_1}{x_2-x_1} = \frac{8-2}{-3-(-4)} = \frac{6}{1} = 6$

b. no.

Let students do exercise 7.3 to assess their understanding of gradient of a line.

Answers for exercise 7.5

1. **a.** gradient of $PQ = \frac{y_2-y_1}{x_2-x_1} = \frac{-4-(-3)}{7-4} = -\frac{1}{3}$

   b. gradient of $PQ = \frac{y_2-y_1}{x_2-x_1} = \frac{7-3}{1-4} = -\frac{4}{3}$

   c. gradient of $PQ = \frac{y_2-y_1}{x_2-x_1} = \frac{-7-3}{0-0}$ does not exist (Undefine). It has no slope.

   d. gradient of $PQ = \frac{y_2-y_1}{x_2-x_1} = \frac{5-(-3)}{-2-(-6)} = \frac{8}{4} = 2$

2. Gradient of $AB = \frac{y_2-y_1}{x_2-x_1} = \frac{12-6}{-1-(-4)} = 2$,

   Gradient of $AC = \frac{y_2-y_1}{x_2-x_1} = \frac{6-0}{-4-(-7)} = 2$ and gradient of $BC = \frac{y_2-y_1}{x_2-x_1} = \frac{12-0}{-1-(-7)} = 2$. Therefore they are collinear.

3. Since $x_2 = x_1$, then the line is vertical and it has no slope.

4. If $A = (0,4), B = (1,5)\ and\ c = (2,6)$, then

   a. gradient of $AB = \frac{y_2-y_1}{x_2-x_1} = \frac{5-4}{1-0} = 1$

   b. gradient of $AC = \frac{y_2-y_1}{x_2-x_1} = \frac{6-4}{2-0} = 1$

   c. they have the same gradient.

Therefore, the three points A, B, and C have the same gradient that means they lie on the straight line. Hence, they are collinear.

Define an angle of inclination θ of a line l to students. The Angle of inclination θ is the angle measured from the positive x-axis to the line l in counter clockwise direction. Support students to do activity 7.5 in groups and arrive at the formula of slope of a line l in terms of its angle of inclination θ as:

$$slope = m = tan\ \theta$$

If l is a vertical line $= 90°$, $tan\ \theta$ is not defined and a vertical line has no slope. For a horizontal line $= 0$, $tan\ \theta = 0$ and the slope of a horizontal line is zero.

Answers for activity 7.4

1. $\sqrt{5}$
2. $\frac{1}{2}$
3. $\frac{1}{2}$
4. Slope of a line is equal to the tangent of angle of inclination of the line

Assume a line $\boldsymbol{l}$ has slope $\boldsymbol{m}$ and passes through a point $(\boldsymbol{x_1}, \boldsymbol{y_1})$. If $(\boldsymbol{x}, \boldsymbol{y})$ is any point on $\boldsymbol{l}$ we have

$$\boldsymbol{m} = \frac{y-y_1}{x-x_1}.$$

Immediately we observe, $y - y_1 = m(x - x_1)$ which is called point slope form of equation of a line $\boldsymbol{l}$.

If the slope $\boldsymbol{m}$ of a line $\boldsymbol{l}$ and the y-intercept $(\boldsymbol{0}, \boldsymbol{b})$ of $\boldsymbol{l}$ are given the replacing x_1 by 0 and y_1 by $\boldsymbol{b}$ we have $y = mx + b$ and this is called slope intercept form of equation of a line.

If two points (x_1, y_1) and (x_2, y_2) on a line $\boldsymbol{l}$ **are given, by considering any point $(\boldsymbol{x}, \boldsymbol{y})$ on $\boldsymbol{l}$ we can evaluate the slope of $\boldsymbol{l}$ in two different ways as** $\frac{y_2-y_1}{x_2-x_1}$ and $\frac{y-y_1}{x-x_1}$. Since the slope of a line is unique

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x - x_1)$and this is called two-point equations of a line.

The general equation of the line which is linear in both the variables x and y is given by $Ax + By + C = 0$ where A or B is non-zero.

Answers for exercise 7.6

1.

a. Slope, $m = \tan\theta = \tan 60^\circ = \sqrt{3}$

b. $m = \tan\theta = \tan 150^\circ$

$= \tan(180^\circ - 150^\circ)$

$= \tan(-30^\circ) = -\tan 30^\circ = -\frac{1}{\sqrt{3}}.$

Answers for exercise 7.7

1. **a.** $18x + 3y - 5 = 0$

   b. $y = -2$

   c. $9x + 12y - 2 = 0$

2. **a.** $4x - 5y + 20 = 0$

   b. $7x - 3y + 44 = 0$

   c. $y = -4$

   d. $2x + y + 7 = 0$

3. **a.** $3x - 8y + 39 = 0$ or $y = \frac{3}{8}x + \frac{39}{8}$

   b. $5x + 3y - 1 = 0$

   c. $7x + 2y + 55 = 0$

   d. $7x + y - 31 = 0$

   e. $5x + 3y - 29 = 0$

   f. $8x - y - 45 = 0$

4. Given: x-intercept= $(p, 0)$ and y-intercept= $(0, q)$.

   The two-point form of equation of a line

$$y - y_1 = \frac{0 - q}{p - 0}(x - x_1)$$

$$y - 0 = \frac{-q}{p}(x - p)$$

$$py = -qx + pq$$

$$py + qx = pq$$

$$\frac{x}{p}+\frac{y}{q}=1$$

5. **a.** $m=-\frac{5}{2}$, y-intercept $=(0,-5)$

b. $m=\frac{7}{4}$, y-intercept $=(0,-14)$

c. $m=0$, y-intercept $=(0,5)$

d. $m=\frac{5}{4}$, y-intercept $=(0,0)$

e. $m=-\frac{4}{5}$, y-intercept $=\left(0,\frac{3}{5}\right)$

6. **a.** $y-y_1=m(x-x_1)$, $y-5=-\frac{1}{3}(x-2)$. So, $x+3y-17=0$

b. $y=mx+b$, $5=\left(-\frac{1}{3}\right)\times 2+b$

$$5+\frac{2}{3}=b$$

$$b=\frac{17}{3}$$

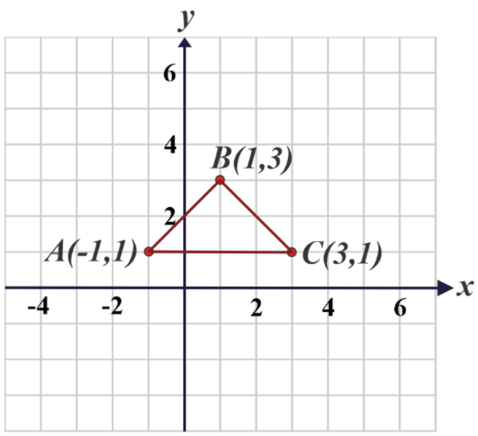
Therefore, $y=-\frac{1}{3}x+\frac{17}{3}$

c. $\frac{y-5}{x-2}=\frac{7-5}{-4-2}$, and the general equation is $x+3y-17=0$

7. a. $AB=\sqrt{(1+1)^2+(3-1)^2}=2\sqrt{2}$, $BC=\sqrt{(1-3)^2+(3-1)^2}=2\sqrt{2}$ and $AC=|-1-3|=4$

b. **Yes**, it is an isosceles right-angled triangle. See the figure below.

Slope of AB is 1 and slope of BC is -1. So, AB and BC are perpendicular. Thus $\angle B$ is right angle and AB is congruent to BC.



Select some problems from exercise 7.4 for students to do as classwork and homework. Initiate students to show their work by writing it on the blackboard.

7.4 Parallel and perpendicular lines. (2 periods)

Competencies

At the end this lesson the students will be able to

- identify whether to lines are parallel or not.
- identify whether two lines are perpendicular or not.
- find equations of lines that are parallel or perpendicular to a given line.

Introduction

In this sub-unit, students use the concept of slope for describing the properties of parallel and perpendicular lines.

Teaching Guide

Discuss activity 7.7 with students so that students can recall the definition of parallel and perpendicular line.

Answers for activity 7.5

1. Parallel lines will never intersect. Notice that they have exactly the same steepness which means their slopes are identical. Unlike parallel lines, **perpendicular lines** do intersect. Their intersection forms a right or 90-degree angle.
2. **a.** Slope of $L_1 = L_2 = 1$

   b. Equation of line $L_1: y = x + 1 and\ L_2: y = x - 1$

   c. They have the same slope and different x and y-intercept. They are parallel.
3. **a.** Slope of $L_1 = 1\ and$ slope of $L_2 = -1$

   b. Equation of line $L_1: y = x + 1$ and equation of line $L_2: y = -x + 1$

   c. The product of their slope is equal to negative one. Therefore, they are perpendicular.

State and prove Theorem 7.1 and Theorem 7.2. Initiate students to participate while proving the theorems. Support students practice the relation between the slopes of parallel line and the relation between the slopes of vertical lines by discussing on the examples given in the student's text.

Answers for exercise 7.8

1. a. slope of L_1 parallel to $y = -7x + 5\ is\ -7$.

2. $\frac{y-1}{x-1} = -\frac{3}{4}$, $y - 1 = -\frac{3}{4}(x-1)$. i.e. $3x + 4y - 7 = 0$

3. Gradient of $AB = \frac{11-4}{7-6} = 7$ and gradient of $PQ = \frac{13-1}{3-1} = 7$. The two lines have the same slope and so are parallel.

Answers for exercise 7.9

1. slope of L_1 perpendicular to $y = -7x + 5$ *is* $\frac{1}{7}$.

2. Given: Point $C(-2,3)$ is equidistant from points $A(3,-1)$ and $B(x,8)$.

$CA = CB$ or $CA^2 = CB^2$ and $CB^2 = (x+2)^2 + (8-3)^2$

$$CA^2 = (3+2)^2 + (-1-3)^2 = 41 \Rightarrow AC = \sqrt{41} = BC$$

$$(x+2)^2 + (8-3)^2 = (3+2)^2 + (-1-3)^2$$

$$x^2 + 4x + 4 + 25 = 25 + 16$$

$$x^2 + 4x + 4 - 16 = 0$$

$$x^2 + 4x + -12 = 0 \Rightarrow (x+6)(x-2) = 0 \Rightarrow x = -6 \text{ or } 2$$

3. Let (a,b)be the other endpoint of the diameter of the circle. Given: one endpoint $(5,6)$ and center $c(-2,1)$.

$(-2,1) = \left(\frac{a+5}{2}, \frac{b+6}{2}\right)$. So, $-2 = \frac{a+5}{2}$ and $1 = \frac{b+6}{2}$

$$a = -9 \text{ and } b = -4$$

4. a. Given: a point $(7,2)$ and slope$= \frac{1}{3}$. Then the equation of the line passing through $(7,2)$ and perpendicular to $y = -3x + 5$ is:

$$\frac{y-2}{x-7} = \frac{1}{3} y - 2 = \frac{1}{3}(x-7)$$

Hence, the equation of the line is $x - 3y - 1 = 0$.

b. $y - 2 = -\frac{1}{4}(x-(-7))$, $y = -\frac{1}{4}x - \frac{7}{4} + 2 = -\frac{1}{4}x + \frac{1}{4}$

Assessment

Select some problems from exercise 7.6 for students to do as classwork and homework. Initiate students to show their work by writing it on the blackboard. You can also adjust quiz or test by selecting problems from review exercise of unit 7. Giving immediate feedback and discussing on the problem which is found difficult for most students together with students is very important.

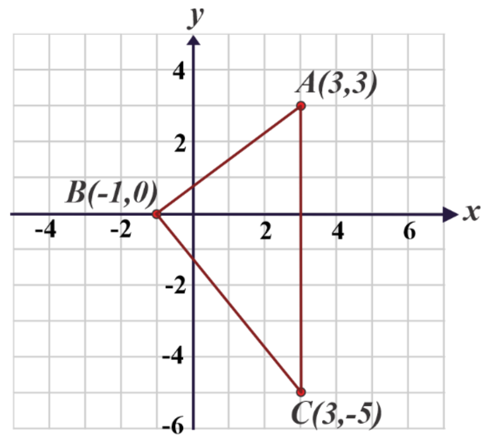
Answers for exercise 7.10

1. Let the coordinates of the point on the x-axis $(x, 0)$.

$25 = (x-6)^2 + (0+3)^2$, $25 = x^2 - 12x + 36 + 9$

$0 = x^2 - 12x + 20$. So, $x = 2 \; or \; x = 10$

Therefore, the coordinates of the points on the x-axis are $(2,0) \; and \; (10,0)$.



2. AC= $\sqrt{(3-3)^2 + (3+5)^2} = 8$ and height= 4

$Area = \frac{1}{2} \times AC \times height = \frac{1}{2} \times 8 \times 4 = 16$ sq.units.

3. $C(a,b) = \left(\frac{-5+3}{2}, \frac{2+(-2)}{2}\right) = (-1, 0)$.

4. **a.** In the figure below,

$AB = \sqrt{(2-5)^2 + (3+1)^2} = \sqrt{9+16} = \sqrt{25} = 5$

$AC = \sqrt{(1+1)^2 + (1-5)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$

$BC = \sqrt{(3-1)^2 + (2-1)^2} = \sqrt{4+1} = \sqrt{5}$

By Pythagoras theorem,

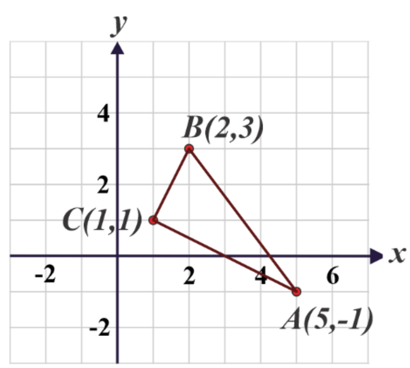
$$AB^2 = AC^2 + BC^2$$

$$AB = \sqrt{AC^2 + BC^2}$$

$$= \sqrt{\sqrt{20}^2 + \sqrt{5}^2}$$

$$= \sqrt{25} = 5$$

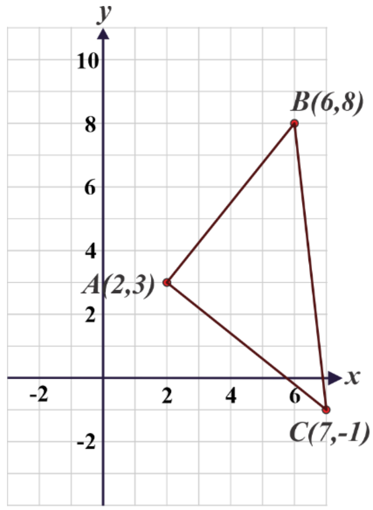
Therefore, ΔABC is a right-angled triangle.



b. In the figure below,

$AB = \sqrt{(6-2)^2 + (8-3)^2} = \sqrt{16+25} = \sqrt{41}$

$AC = \sqrt{(7-2)^2 + (3+1)^2} = \sqrt{255+16} = \sqrt{41}$



Slope of $AB = \frac{5}{4}$ and $AC = -\frac{4}{5}$. their slope product is -1 and $AB = AC$. Therefore, it is isosceles right-triangle.

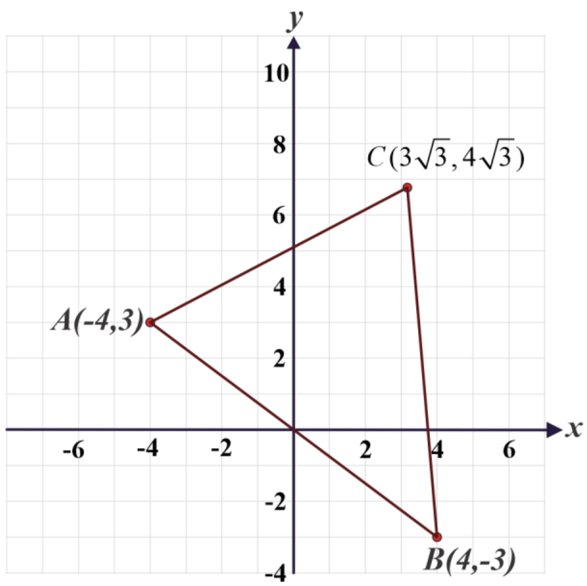
c. In the figure below,

$AB = \sqrt{(4+4)^2 + (-3-3)^2} = \sqrt{100} = 10$

$AC = \sqrt{\left(-4-3\sqrt{3}\right)^2 + \left(3-4\sqrt{3}\right)^2} = \sqrt{100} = 10$

$BC = \sqrt{\left(3\sqrt{3}-4\right)^2 + \left(4\sqrt{3}+3\right)^2} = \sqrt{100} = 10$

Therefore, triangle ABC is an equilateral triangle.



5. Equation of the line in two point form is

$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$. Given vertices are A(3,4), B(2,0) and C(-1, 6).

First find the slopes of the 3 lines: $m = \frac{y_2-y_1}{x_2-x_1}$

1) A(3, 4), B(2, 0) for AB $m = \frac{y_2-y_1}{x_2-x_1} = \frac{2-4}{0-3} = \frac{2}{3}$

To use slope point form choose either point... let's use (3, 4) with slope $\frac{2}{3}$

$y - 4 = \frac{2}{3}(x-3)$

$3y - 12 = 2x - 6$

$\boldsymbol{AB}: \ \mathbf{2x - 3y = -6}$

2) $A(3,4), C(-1,6)$ $for\ AC$

$m = \frac{6-4}{-1-3} = -\frac{1}{2}$

$Again\ let's\ use\ (3,4) with\ slope -\frac{1}{2}$

$y - 4 = -\frac{1}{2}(x-3)$

$2y - 8 = -x + 3$

$\boldsymbol{AC}: \ \mathbf{x + 2y = 11}$

3) B(2, 0), C(-1,6) for BC

$m = \frac{6-0}{-1-2} = -2$ Let's use point (2, 0) with slope -2

$y - 0 = -2(x-2)$

$y = -2X + 4$

$\boldsymbol{BC}: \ \mathbf{2x + y = 4}$

Answers for review exercise on unit 7

1. $y = mx + b$, where m is the slope and b is the y-intercept. Given: $m = -3$ and y-intercept is -7. So, $y = -3x - 7$.

2. $\frac{y-4}{x-5} = \frac{4-2}{5+3}$, $y = \frac{1}{4}(x-5) + 4$

3. $\frac{y-3}{x+2} = -1$, $y = -(x+2) + 3$ implies $y = -x + 1$

4. Slope of the line PQ = $-\frac{1}{2}$

5. **a.** Equation side AB: $\frac{y-1}{x+1} = 1$, $y = x + 2$

Equation side BC: $\frac{y-1}{x-3} = -1$, $y = -x + 4$ and

Equation of side AC: $y = 1$

b. Yes, it is an isosceles right-angled triangle.

c. $(4, 0)$and$(0, 4)$

6. $B = (3, 4)$ $and\ D(-2, -2)$

7. $y = -\frac{1}{2}(x + 2) + 5 = -\frac{1}{2}x + 4$

8. The coordinates of the vertices are $O(0, 0), A(a, 0),$ and $C(c,\ d)$.

a. The coordinates of B is $(a + c,\ d)$.

b. The coordinate of midpoint of $OB = \left(\frac{a+c}{2}, \frac{d}{2}\right)$ and the coordinate of midpoint of $AC = \left(\frac{a+c}{2}, \frac{d}{2}\right)$.

c. The midpoint the diagonals coincided which means the diagonals bisect each other.

9. Place the triangle so that D is at the origin. Then let the coordinates of B and C be (-a, 0) and (a, 0) respectively. Let the coordinates of A be (d, c).

$(AB)^2 = c^2 + (d + a)^2$ and $(AC)^2 = c^2 + (d - a)^2$

$so(AB)^2 + (AC)^2 = 2c^2 + 2d^2 + 2a^2$

$(AD)^2 = c^2 + d^2$ and $(DC)^2 = a^2$

Therefore, $(AB)^2 + (AC)^2 = 2[(AD)^2 + (DC)^2]$.

10. Let $P(x, y)$ a point equidistant from $A(a, b)$ and $C(c, d)$

$PA = PC$, using distance formula

$\sqrt{(x-a)^2 + (y-b)^2} = \sqrt{(x-c)^2 + (y-d)^2}$

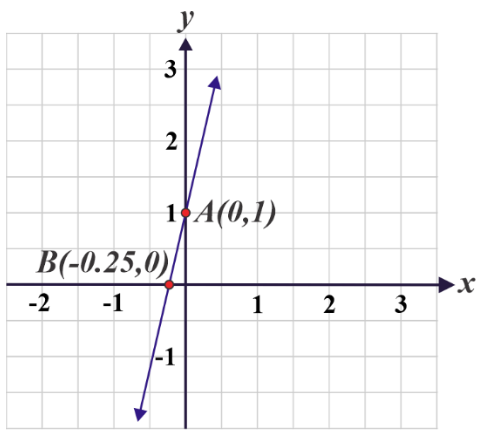
$(x-a)^2 + (y-b)^2 = (x-c)^2 + (y-d)^2$ (Squaring both sides)

$2(c-a)x + 2(d-b)y = c^2 + d^2 - a^2 - b^2$

11. a. $y = 4x + 1$, the x-intercept; y-intercept form of

$y = 4x + 1$:

$\frac{x}{-\frac{1}{4}} + \frac{y}{1} = 1$. So, the x-intercept is $(-\frac{1}{4},\ 0)$ and the y-intercept is $(0, 1)$.



b. $2x + 3y + 6 = 0$.

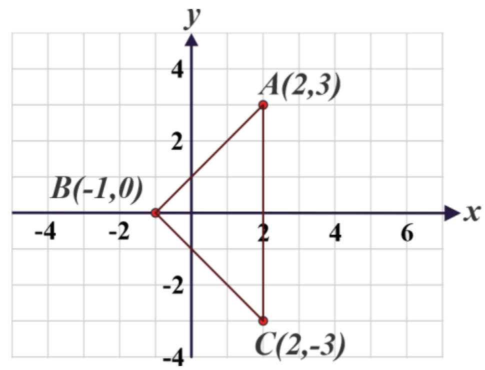
The intercept form: $\frac{x}{-3} + \frac{y}{-2} = 1$.

So, the x-intercept is $(-3, 0)$ and the y-intercept is $(0, -2)$

12. $\boldsymbol{A = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|}$

$\boldsymbol{= \frac{1}{2}|2(0 + 3) - 1(-3 - 3) + 2(3 - 0)|}$

$\boldsymbol{= \frac{1}{2}|6 + 6 + 6| = \frac{18}{2} = 9}$sq. units**.**



$d(AB) = \sqrt{(-3 - 0)^2 + (4 - 1)^2}$

$= \sqrt{18} = 3\sqrt{2}$

$d(BC) = \sqrt{(-3 - 0)^2 + (-2 - 1)^2}$

$= 3\sqrt{2}$

So, $\overline{AB} = \overline{BC}$

And slope of $\overline{AB} = -1$ and slope of$\overline{BC} = 1$

Slope of $\overline{AB} \cdot slope\ of\ \overline{BC} = -1 \times 1 = -1$

Therefore, it is an isosceles right-angled triangle.

13. Let the given points be $A(5, -4)$ and $B(2,3)$ and x-axis intersect the line segment $\overline{AB}$ at P such that $AP: PB = p_1: p_2$. Then the coordinates of P is:

$\left(\frac{p_1x_2 + p_2x_1}{p_1+p_2}, \frac{p_1y_2+p_2y_1}{p_1+p_2}\right) = \left(\frac{p_1\times2+p_2\times5}{p_1+p_2}, \frac{p_1\times3+p_2\times-4}{p_1+p_2}\right)$. Clearly, the point P lies on the x-axis; hence y coordinate of P must be zero.

Therefore, $\frac{p_1\times3+p_2\times-4}{p_1+p_2} = 0.\ 3p_1 - 4p_2 = 0$ and $p_1: p_2 = 4: 3$

$P(x, y) = \left(\frac{4\times2+3\times5}{4+3}, \frac{4\times3+3\times-4}{4+3}\right) = \left(\frac{23}{7}, 0\right)$.

Logarithm table

Logarithm table											Mean Difference								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1.0	0	0.0043	0.0086	0.0128	0.017	0.0212	0.0253	0.0294	0.0334	0.0374	4	8	12	17	21	25	29	33	37
1.1	0.0414	0.0453	0.0492	0.0531	0.0569	0.0607	0.0645	0.0682	0.0719	0.0755	4	8	11	15	19	23	26	30	34
1.2	0.0792	0.0828	0.0864	0.0899	0.0934	0.0969	0.1004	0.1038	0.1072	0.1106	3	7	10	14	17	21	24	28	31
1.3	0.1139	0.1173	0.1206	0.1239	0.1271	0.1303	0.1335	0.1367	0.1399	0.143	3	6	10	13	16	19	23	26	29
1.4	0.1461	0.1492	0.1523	0.1553	0.1584	0.1614	0.1644	0.1673	0.1703	0.1732	3	6	9	12	15	18	21	24	27
1.5	0.1761	0.179	0.1818	0.1847	0.1875	0.1903	0.1931	0.1959	0.1987	0.2014	3	6	8	11	14	17	20	22	25
1.6	0.2041	0.2068	0.2095	0.2122	0.2148	0.2175	0.2201	0.2227	0.2253	0.2279	3	5	8	11	13	16	18	21	24
1.7	0.2304	0.233	0.2355	0.238	0.2405	0.243	0.2455	0.248	0.2504	0.2529	2	5	7	10	12	15	17	20	22
1.8	0.2553	0.2577	0.2601	0.2625	0.2648	0.2672	0.2695	0.2718	0.2742	0.2765	2	5	7	9	12	14	16	19	21
1.9	0.2788	0.281	0.2833	0.2856	0.2878	0.29	0.2923	0.2945	0.2967	0.2989	2	4	7	9	11	13	16	18	20
2.0	0.301	0.3032	0.3054	0.3075	0.3096	0.3118	0.3139	0.316	0.3181	0.3201	2	4	6	8	11	13	15	17	19
2.1	0.3222	0.3243	0.3263	0.3284	0.3304	0.3324	0.3345	0.3365	0.3385	0.3404	2	4	6	8	10	12	14	16	18
2.2	0.3424	0.3444	0.3464	0.3483	0.3502	0.3522	0.3541	0.356	0.3579	0.3598	2	4	6	8	10	12	14	15	17
2.3	0.3617	0.3636	0.3655	0.3674	0.3692	0.3711	0.3729	0.3747	0.3766	0.3784	2	4	6	7	9	11	13	15	17
2.4	0.3802	0.382	0.3838	0.3856	0.3874	0.3892	0.3909	0.3927	0.3945	0.3962	2	4	5	7	9	11	12	14	16
2.5	0.3979	0.3997	0.4014	0.4031	0.4048	0.4065	0.4082	0.4099	0.4116	0.4133	2	3	5	7	9	10	12	14	15
2.6	0.415	0.4166	0.4183	0.42	0.4216	0.4232	0.4249	0.4265	0.4281	0.4298	2	3	5	7	8	10	11	13	15
2.7	0.4314	0.433	0.4346	0.4362	0.4378	0.4393	0.4409	0.4425	0.444	0.4456	2	3	5	6	8	9	11	13	14
2.8	0.4472	0.4487	0.4502	0.4518	0.4533	0.4548	0.4564	0.4579	0.4594	0.4609	2	3	5	6	8	9	11	12	14
2.9	0.4624	0.4639	0.4654	0.4669	0.4683	0.4698	0.4713	0.4728	0.4742	0.4757	1	3	4	6	7	9	10	12	13
3.0	0.4771	0.4786	0.48	0.4814	0.4829	0.4843	0.4857	0.4871	0.4886	0.49	1	3	4	6	7	9	10	11	13
3.1	0.4914	0.4928	0.4942	0.4955	0.4969	0.4983	0.4997	0.5011	0.5024	0.5038	1	3	4	6	7	8	10	11	12
3.2	0.5051	0.5065	0.5079	0.5092	0.5105	0.5119	0.5132	0.5145	0.5159	0.5172	1	3	4	5	7	8	9	11	12
3.3	0.5185	0.5198	0.5211	0.5224	0.5237	0.525	0.5263	0.5276	0.5289	0.5302	1	3	4	5	6	8	9	10	12
3.4	0.5315	0.5328	0.534	0.5353	0.5366	0.5378	0.5391	0.5403	0.5416	0.5428	1	3	4	5	6	8	9	10	11
3.5	0.5441	0.5453	0.5465	0.5478	0.549	0.5502	0.5514	0.5527	0.5539	0.5551	1	2	4	5	6	7	9	10	11
3.6	0.5563	0.5575	0.5587	0.5599	0.5611	0.5623	0.5635	0.5647	0.5658	0.567	1	2	4	5	6	7	8	10	11
3.7	0.5682	0.5694	0.5705	0.5717	0.5729	0.574	0.5752	0.5763	0.5775	0.5786	1	2	3	5	6	7	8	9	10
3.8	0.5798	0.5809	0.5821	0.5832	0.5843	0.5855	0.5866	0.5877	0.5888	0.5899	1	2	3	5	6	7	8	9	10
3.9	0.5911	0.5922	0.5933	0.5944	0.5955	0.5966	0.5977	0.5988	0.5999	0.601	1	2	3	4	5	7	8	9	10
4.0	0.6021	0.6031	0.6042	0.6053	0.6064	0.6075	0.6085	0.6096	0.6107	0.6117	1	2	3	4	5	6	8	9	10
4.1	0.6128	0.6138	0.6149	0.616	0.617	0.618	0.6191	0.6201	0.6212	0.6222	1	2	3	4	5	6	7	8	9
4.2	0.6232	0.6243	0.6253	0.6263	0.6274	0.6284	0.6294	0.6304	0.6314	0.6325	1	2	3	4	5	6	7	8	9
4.3	0.6335	0.6345	0.6355	0.6365	0.6375	0.6385	0.6395	0.6405	0.6415	0.6425	1	2	3	4	5	6	7	8	9
4.4	0.6435	0.6444	0.6454	0.6464	0.6474	0.6484	0.6493	0.6503	0.6513	0.6522	1	2	3	4	5	6	7	8	9
4.5	0.6532	0.6542	0.6551	0.6561	0.6571	0.658	0.659	0.6599	0.6609	0.6618	1	2	3	4	5	6	7	8	9
4.6	0.6628	0.6637	0.6646	0.6656	0.6665	0.6675	0.6684	0.6693	0.6702	0.6712	1	2	3	4	5	6	7	7	8
4.7	0.6721	0.673	0.6739	0.6749	0.6758	0.6767	0.6776	0.6785	0.6794	0.6803	1	2	3	4	5	5	6	7	8
4.8	0.6812	0.6821	0.683	0.6839	0.6848	0.6857	0.6866	0.6875	0.6884	0.6893	1	2	3	4	4	5	6	7	8
4.9	0.6902	0.6911	0.692	0.6928	0.6937	0.6946	0.6955	0.6964	0.6972	0.6981	1	2	3	4	4	5	6	7	8
5.0	0.699	0.6998	0.7007	0.7016	0.7024	0.7033	0.7042	0.705	0.7059	0.7067	1	2	3	3	4	5	6	7	8
5.1	0.7076	0.7084	0.7093	0.7101	0.711	0.7118	0.7126	0.7135	0.7143	0.7152	1	2	3	3	4	5	6	7	8
5.2	0.716	0.7168	0.7177	0.7185	0.7193	0.7202	0.721	0.7218	0.7226	0.7235	1	2	2	3	4	5	6	7	7
5.3	0.7243	0.7251	0.7259	0.7267	0.7275	0.7284	0.7292	0.73	0.7308	0.7316	1	2	2	3	4	5	6	6	7
5.4	0.7324	0.7332	0.734	0.7348	0.7356	0.7364	0.7372	0.738	0.7388	0.7396	1	2	2	3	4	5	6	6	7
5.5	0.7404	0.7412	0.7419	0.7427	0.7435	0.7443	0.7451	0.7459	0.7466	0.7474	1	2	2	3	4	5	5	6	7

Logarithm table

Logarithm table											Mean Difference								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
5.6	0.7482	0.749	0.7497	0.7505	0.7513	0.752	0.7528	0.7536	0.7543	0.7551	1	2	2	3	4	5	5	6	7
5.7	0.7559	0.7566	0.7574	0.7582	0.7589	0.7597	0.7604	0.7612	0.7619	0.7627	1	2	2	3	4	5	5	6	7
5.8	0.7634	0.7642	0.7649	0.7657	0.7664	0.7672	0.7679	0.7686	0.7694	0.7701	1	1	2	3	4	4	5	6	7
5.9	0.7709	0.7716	0.7723	0.7731	0.7738	0.7745	0.7752	0.776	0.7767	0.7774	1	1	2	3	4	4	5	6	7
6.0	0.7782	0.7789	0.7796	0.7803	0.781	0.7818	0.7825	0.7832	0.7839	0.7846	1	1	2	3	4	4	5	6	6
6.1	0.7853	0.786	0.7868	0.7875	0.7882	0.7889	0.7896	0.7903	0.791	0.7917	1	1	2	3	4	4	5	6	6
6.2	0.7924	0.7931	0.7938	0.7945	0.7952	0.7959	0.7966	0.7973	0.798	0.7987	1	1	2	3	3	4	5	6	6
6.3	0.7993	0.8	0.8007	0.8014	0.8021	0.8028	0.8035	0.8041	0.8048	0.8055	1	1	2	3	3	4	5	5	6
6.4	0.8062	0.8069	0.8075	0.8082	0.8089	0.8096	0.8102	0.8109	0.8116	0.8122	1	1	2	3	3	4	5	5	6
6.5	0.8129	0.8136	0.8142	0.8149	0.8156	0.8162	0.8169	0.8176	0.8182	0.8189	1	1	2	3	3	4	5	5	6
6.6	0.8195	0.8202	0.8209	0.8215	0.8222	0.8228	0.8235	0.8241	0.8248	0.8254	1	1	2	3	3	4	5	5	6
6.7	0.8261	0.8267	0.8274	0.828	0.8287	0.8293	0.8299	0.8306	0.8312	0.8319	1	1	2	3	3	4	5	5	6
6.8	0.8325	0.8331	0.8338	0.8344	0.8351	0.8357	0.8363	0.837	0.8376	0.8382	1	1	2	3	3	4	4	5	6
6.9	0.8388	0.8395	0.8401	0.8407	0.8414	0.842	0.8426	0.8432	0.8439	0.8445	1	1	2	2	3	4	4	5	6
7.0	0.8451	0.8457	0.8463	0.847	0.8476	0.8482	0.8488	0.8494	0.85	0.8506	1	1	2	2	3	4	4	5	6
7.1	0.8513	0.8519	0.8525	0.8531	0.8537	0.8543	0.8549	0.8555	0.8561	0.8567	1	1	2	2	3	4	4	5	5
7.2	0.8573	0.8579	0.8585	0.8591	0.8597	0.8603	0.8609	0.8615	0.8621	0.8627	1	1	2	2	3	4	4	5	5
7.3	0.8633	0.8639	0.8645	0.8651	0.8657	0.8663	0.8669	0.8675	0.8681	0.8686	1	1	2	2	3	4	4	5	5
7.4	0.8692	0.8698	0.8704	0.871	0.8716	0.8722	0.8727	0.8733	0.8739	0.8745	1	1	2	2	3	4	4	5	5
7.5	0.8751	0.8756	0.8762	0.8768	0.8774	0.8779	0.8785	0.8791	0.8797	0.8802	1	1	2	2	3	3	4	5	5
7.6	0.8808	0.8814	0.882	0.8825	0.8831	0.8837	0.8842	0.8848	0.8854	0.8859	1	1	2	2	3	3	4	5	5
7.7	0.8865	0.8871	0.8876	0.8882	0.8887	0.8893	0.8899	0.8904	0.891	0.8915	1	1	2	2	3	3	4	4	5
7.8	0.8921	0.8927	0.8932	0.8938	0.8943	0.8949	0.8954	0.896	0.8965	0.8971	1	1	2	2	3	3	4	4	5
7.9	0.8976	0.8982	0.8987	0.8993	0.8998	0.9004	0.9009	0.9015	0.902	0.9025	1	1	2	2	3	3	4	4	5
8.0	0.9031	0.9036	0.9042	0.9047	0.9053	0.9058	0.9063	0.9069	0.9074	0.9079	1	1	2	2	3	3	4	4	5
8.1	0.9085	0.909	0.9096	0.9101	0.9106	0.9112	0.9117	0.9122	0.9128	0.9133	1	1	2	2	3	3	4	4	5
8.2	0.9138	0.9143	0.9149	0.9154	0.9159	0.9165	0.917	0.9175	0.918	0.9186	1	1	2	2	3	3	4	4	5
8.3	0.9191	0.9196	0.9201	0.9206	0.9212	0.9217	0.9222	0.9227	0.9232	0.9238	1	1	2	2	3	3	4	4	5
8.4	0.9243	0.9248	0.9253	0.9258	0.9263	0.9269	0.9274	0.9279	0.9284	0.9289	1	1	2	2	3	3	4	4	5
8.5	0.9294	0.9299	0.9304	0.9309	0.9315	0.932	0.9325	0.933	0.9335	0.934	1	1	2	2	3	3	4	4	5
8.6	0.9345	0.935	0.9355	0.936	0.9365	0.937	0.9375	0.938	0.9385	0.939	1	1	2	2	3	3	4	4	5
8.7	0.9395	0.94	0.9405	0.941	0.9415	0.942	0.9425	0.943	0.9435	0.944	0	1	1	2	2	3	3	4	4
8.8	0.9445	0.945	0.9455	0.946	0.9465	0.9469	0.9474	0.9479	0.9484	0.9489	0	1	1	2	2	3	3	4	4
8.9	0.9494	0.9499	0.9504	0.9509	0.9513	0.9518	0.9523	0.9528	0.9533	0.9538	0	1	1	2	2	3	3	4	4
9.0	0.9542	0.9547	0.9552	0.9557	0.9562	0.9566	0.9571	0.9576	0.9581	0.9586	0	1	1	2	2	3	3	4	4
9.1	0.959	0.9595	0.96	0.9605	0.9609	0.9614	0.9619	0.9624	0.9628	0.9633	0	1	1	2	2	3	3	4	4
9.2	0.9638	0.9643	0.9647	0.9652	0.9657	0.9661	0.9666	0.9671	0.9675	0.968	0	1	1	2	2	3	3	4	4
9.3	0.9685	0.9689	0.9694	0.9699	0.9703	0.9708	0.9713	0.9717	0.9722	0.9727	0	1	1	2	2	3	3	4	4
9.4	0.9731	0.9736	0.9741	0.9745	0.975	0.9754	0.9759	0.9763	0.9768	0.9773	0	1	1	2	2	3	3	4	4
9.5	0.9777	0.9782	0.9786	0.9791	0.9795	0.98	0.9805	0.9809	0.9814	0.9818	0	1	1	2	2	3	3	4	4
9.6	0.9823	0.9827	0.9832	0.9836	0.9841	0.9845	0.985	0.9854	0.9859	0.9863	0	1	1	2	2	3	3	4	4
9.7	0.9868	0.9872	0.9877	0.9881	0.9886	0.989	0.9894	0.9899	0.9903	0.9908	0	1	1	2	2	3	3	4	4
9.8	0.9912	0.9917	0.9921	0.9926	0.993	0.9934	0.9939	0.9943	0.9948	0.9952	0	1	1	2	2	3	3	4	4
9.9	0.9956	0.9961	0.9965	0.9969	0.9974	0.9978	0.9983	0.9987	0.9991	0.9996	0	1	1	2	2	3	3	3	4

	Anti Logarithm table										Mean Difference								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.00	1	1.002	1.005	1.007	1.009	1.012	1.014	1.016	1.019	1.021	0	0	1	1	1	1	2	2	2
0.01	1.023	1.026	1.028	1.03	1.033	1.035	1.038	1.04	1.042	1.045	0	0	1	1	1	1	2	2	2
0.02	1.047	1.05	1.052	1.054	1.057	1.059	1.062	1.064	1.067	1.069	0	0	1	1	1	1	2	2	2
0.03	1.072	1.074	1.076	1.079	1.081	1.084	1.086	1.089	1.091	1.094	0	0	1	1	1	1	2	2	2
0.04	1.096	1.099	1.102	1.104	1.107	1.109	1.112	1.114	1.117	1.119	0	1	1	1	1	2	2	2	2
0.05	1.122	1.125	1.127	1.13	1.132	1.135	1.138	1.14	1.143	1.146	0	1	1	1	1	2	2	2	2
0.06	1.148	1.151	1.153	1.156	1.159	1.161	1.164	1.167	1.169	1.172	0	1	1	1	1	2	2	2	2
0.07	1.175	1.178	1.18	1.183	1.186	1.189	1.191	1.194	1.197	1.199	0	1	1	1	1	2	2	2	2
0.08	1.202	1.205	1.208	1.211	1.213	1.216	1.219	1.222	1.225	1.227	0	1	1	1	1	2	2	2	3
0.09	1.23	1.233	1.236	1.239	1.242	1.245	1.247	1.25	1.253	1.256	0	1	1	1	1	2	2	2	3
0.10	1.259	1.262	1.265	1.268	1.271	1.274	1.276	1.279	1.282	1.285	0	1	1	1	1	2	2	2	3
0.11	1.288	1.291	1.294	1.297	1.3	1.303	1.306	1.309	1.312	1.315	0	1	1	1	2	2	2	2	3
0.12	1.318	1.321	1.324	1.327	1.33	1.334	1.337	1.34	1.343	1.346	0	1	1	1	2	2	2	2	3
0.13	1.349	1.352	1.355	1.358	1.361	1.365	1.368	1.371	1.374	1.377	0	1	1	1	2	2	2	3	3
0.14	1.38	1.384	1.387	1.39	1.393	1.396	1.4	1.403	1.406	1.409	0	1	1	1	2	2	2	3	3
0.15	1.413	1.416	1.419	1.422	1.426	1.429	1.432	1.435	1.439	1.442	0	1	1	1	2	2	2	3	3
0.16	1.445	1.449	1.452	1.455	1.459	1.462	1.466	1.469	1.472	1.476	0	1	1	1	2	2	2	3	3
0.17	1.479	1.483	1.486	1.489	1.493	1.496	1.5	1.503	1.507	1.51	0	1	1	1	2	2	2	3	3
0.18	1.514	1.517	1.521	1.524	1.528	1.531	1.535	1.538	1.542	1.545	0	1	1	1	2	2	2	3	3
0.19	1.549	1.552	1.556	1.56	1.563	1.567	1.57	1.574	1.578	1.581	0	1	1	1	2	2	3	3	3
0.20	1.585	1.589	1.592	1.596	1.6	1.603	1.607	1.611	1.614	1.618	0	1	1	1	2	2	3	3	3
0.21	1.622	1.626	1.629	1.633	1.637	1.641	1.644	1.648	1.652	1.656	0	1	1	2	2	2	3	3	3
0.22	1.66	1.663	1.667	1.671	1.675	1.679	1.683	1.687	1.69	1.694	0	1	1	2	2	2	3	3	3
0.23	1.698	1.702	1.706	1.71	1.714	1.718	1.722	1.726	1.73	1.734	0	1	1	2	2	2	3	3	4
0.24	1.738	1.742	1.746	1.75	1.754	1.758	1.762	1.766	1.77	1.774	0	1	1	2	2	2	3	3	4
0.25	1.778	1.782	1.786	1.791	1.795	1.799	1.803	1.807	1.811	1.816	0	1	1	2	2	2	3	3	4
0.26	1.82	1.824	1.828	1.832	1.837	1.841	1.845	1.849	1.854	1.858	0	1	1	2	2	3	3	3	4
0.27	1.862	1.866	1.871	1.875	1.879	1.884	1.888	1.892	1.897	1.901	0	1	1	2	2	3	3	3	4
0.28	1.905	1.91	1.914	1.919	1.923	1.928	1.932	1.936	1.941	1.945	0	1	1	2	2	3	3	4	4
0.29	1.95	1.954	1.959	1.963	1.968	1.972	1.977	1.982	1.986	1.991	0	1	1	2	2	3	3	4	4
0.30	1.995	2	2.004	2.009	2.014	2.018	2.023	2.028	2.032	2.037	0	1	1	2	2	3	3	4	4
0.31	2.042	2.046	2.051	2.056	2.061	2.065	2.07	2.075	2.08	2.084	0	1	1	2	2	3	3	4	4
0.32	2.089	2.094	2.099	2.104	2.109	2.113	2.118	2.123	2.128	2.133	0	1	1	2	2	3	3	4	4
0.33	2.138	2.143	2.148	2.153	2.158	2.163	2.168	2.173	2.178	2.183	0	1	1	2	2	3	3	4	4
0.34	2.188	2.193	2.198	2.203	2.208	2.213	2.218	2.223	2.228	2.234	1	1	2	2	3	3	4	4	5
0.35	2.239	2.244	2.249	2.254	2.259	2.265	2.27	2.275	2.28	2.286	1	1	2	2	3	3	4	4	5
0.36	2.291	2.296	2.301	2.307	2.312	2.317	2.323	2.328	2.333	2.339	1	1	2	2	3	3	4	4	5
0.37	2.344	2.35	2.355	2.36	2.366	2.371	2.377	2.382	2.388	2.393	1	1	2	2	3	3	4	4	5
0.38	2.399	2.404	2.41	2.415	2.421	2.427	2.432	2.438	2.443	2.449	1	1	2	2	3	3	4	4	5
0.39	2.455	2.46	2.466	2.472	2.477	2.483	2.489	2.495	2.5	2.506	1	1	2	2	3	3	4	5	5
0.40	2.512	2.518	2.523	2.529	2.535	2.541	2.547	2.553	2.559	2.564	1	1	2	2	3	4	4	5	5
0.41	2.57	2.576	2.582	2.588	2.594	2.6	2.606	2.612	2.618	2.624	1	1	2	2	3	4	4	5	5
0.42	2.63	2.636	2.642	2.649	2.655	2.661	2.667	2.673	2.679	2.685	1	1	2	2	3	4	4	5	6
0.43	2.692	2.698	2.704	2.71	2.716	2.723	2.729	2.735	2.742	2.748	1	1	2	3	3	4	4	5	6
0.44	2.754	2.761	2.767	2.773	2.78	2.786	2.793	2.799	2.805	2.812	1	1	2	3	3	4	4	5	6
0.45	2.818	2.825	2.831	2.838	2.844	2.851	2.858	2.864	2.871	2.877	1	1	2	3	3	4	5	5	6
0.46	2.884	2.891	2.897	2.904	2.911	2.917	2.924	2.931	2.938	2.944	1	1	2	3	3	4	5	5	6
0.47	2.951	2.958	2.965	2.972	2.979	2.985	2.992	2.999	3.006	3.013	1	1	2	3	3	4	5	5	6
0.48	3.02	3.027	3.034	3.041	3.048	3.055	3.062	3.069	3.076	3.083	1	1	2	3	4	4	5	6	6
0.49	3.09	3.097	3.105	3.112	3.119	3.126	3.133	3.141	3.148	3.155	1	1	2	3	4	4	5	6	6
0.50	3.162	3.17	3.177	3.184	3.192	3.199	3.206	3.214	3.221	3.228	1	1	2	3	4	4	5	6	7

	Anti Logarithm table										Mean Difference								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.51	3.236	3.243	3.251	3.258	3.266	3.273	3.281	3.289	3.296	3.304	1	2	2	3	4	5	5	6	7
0.52	3.311	3.319	3.327	3.334	3.342	3.35	3.357	3.365	3.373	3.381	1	2	2	3	4	5	5	6	7
0.53	3.388	3.396	3.404	3.412	3.42	3.428	3.436	3.443	3.451	3.459	1	2	2	3	4	5	6	6	7
0.54	3.467	3.475	3.483	3.491	3.499	3.508	3.516	3.524	3.532	3.54	1	2	2	3	4	5	6	6	7
0.55	3.548	3.556	3.565	3.573	3.581	3.589	3.597	3.606	3.614	3.622	1	2	2	3	4	5	6	7	7
0.56	3.631	3.639	3.648	3.656	3.664	3.673	3.681	3.69	3.698	3.707	1	2	3	3	4	5	6	7	8
0.57	3.715	3.724	3.733	3.741	3.75	3.758	3.767	3.776	3.784	3.793	1	2	3	3	4	5	6	7	8
0.58	3.802	3.811	3.819	3.828	3.837	3.846	3.855	3.864	3.873	3.882	1	2	3	4	4	5	6	7	8
0.59	3.89	3.899	3.908	3.917	3.926	3.936	3.945	3.954	3.963	3.972	1	2	3	4	5	5	6	7	8
0.60	3.981	3.99	3.999	4.009	4.018	4.027	4.036	4.046	4.055	4.064	1	2	3	4	5	6	6	7	8
0.61	4.074	4.083	4.093	4.102	4.111	4.121	4.13	4.14	4.15	4.159	1	2	3	4	5	6	7	8	9
0.62	4.169	4.178	4.188	4.198	4.207	4.217	4.227	4.236	4.246	4.256	1	2	3	4	5	6	7	8	9
0.63	4.266	4.276	4.285	4.295	4.305	4.315	4.325	4.335	4.345	4.355	1	2	3	4	5	6	7	8	9
0.64	4.365	4.375	4.385	4.395	4.406	4.416	4.426	4.436	4.446	4.457	1	2	3	4	5	6	7	8	9
0.65	4.467	4.477	4.487	4.498	4.508	4.519	4.529	4.539	4.55	4.56	1	2	3	4	5	6	7	8	9
0.66	4.571	4.581	4.592	4.603	4.613	4.624	4.634	4.645	4.656	4.667	1	2	3	4	5	6	7	9	10
0.67	4.677	4.688	4.699	4.71	4.721	4.732	4.742	4.753	4.764	4.775	1	2	3	4	5	7	8	9	10
0.68	4.786	4.797	4.808	4.819	4.831	4.842	4.853	4.864	4.875	4.887	1	2	3	4	6	7	8	9	10
0.69	4.898	4.909	4.92	4.932	4.943	4.955	4.966	4.977	4.989	5	1	2	3	5	6	7	8	9	10
0.70	5.012	5.023	5.035	5.047	5.058	5.07	5.082	5.093	5.105	5.117	1	2	4	5	6	7	8	9	11
0.71	5.129	5.14	5.152	5.164	5.176	5.188	5.2	5.212	5.224	5.236	1	2	4	5	6	7	8	10	11
0.72	5.248	5.26	5.272	5.284	5.297	5.309	5.321	5.333	5.346	5.358	1	2	4	5	6	7	9	10	11
0.73	5.37	5.383	5.395	5.408	5.42	5.433	5.445	5.458	5.47	5.483	1	3	4	5	6	8	9	10	11
0.74	5.495	5.508	5.521	5.534	5.546	5.559	5.572	5.585	5.598	5.61	1	3	4	5	6	8	9	10	12
0.75	5.623	5.636	5.649	5.662	5.675	5.689	5.702	5.715	5.728	5.741	1	3	4	5	7	8	9	10	12
0.76	5.754	5.768	5.781	5.794	5.808	5.821	5.834	5.848	5.861	5.875	1	3	4	5	7	8	9	11	12
0.77	5.888	5.902	5.916	5.929	5.943	5.957	5.97	5.984	5.998	6.012	1	3	4	5	7	8	10	11	12
0.78	6.026	6.039	6.053	6.067	6.081	6.095	6.109	6.124	6.138	6.152	1	3	4	6	7	8	10	11	13
0.79	6.166	6.18	6.194	6.209	6.223	6.237	6.252	6.266	6.281	6.295	1	3	4	6	7	9	10	11	13
0.80	6.31	6.324	6.339	6.353	6.368	6.383	6.397	6.412	6.427	6.442	1	3	4	6	7	9	10	12	13
0.81	6.457	6.471	6.486	6.501	6.516	6.531	6.546	6.561	6.577	6.592	2	3	5	6	8	9	11	12	14
0.82	6.607	6.622	6.637	6.653	6.668	6.683	6.699	6.714	6.73	6.745	2	3	5	6	8	9	11	12	14
0.83	6.761	6.776	6.792	6.808	6.823	6.839	6.855	6.871	6.887	6.902	2	3	5	6	8	9	11	13	14
0.84	6.918	6.934	6.95	6.966	6.982	6.998	7.015	7.031	7.047	7.063	2	3	5	6	8	10	11	13	15
0.85	7.079	7.096	7.112	7.129	7.145	7.161	7.178	7.194	7.211	7.228	2	3	5	7	8	10	12	13	15
0.86	7.244	7.261	7.278	7.295	7.311	7.328	7.345	7.362	7.379	7.396	2	3	5	7	8	10	12	13	15
0.87	7.413	7.43	7.447	7.464	7.482	7.499	7.516	7.534	7.551	7.568	2	3	5	7	9	10	12	14	16
0.88	7.586	7.603	7.621	7.638	7.656	7.674	7.691	7.709	7.727	7.745	2	4	5	7	9	11	12	14	16
0.89	7.762	7.78	7.798	7.816	7.834	7.852	7.87	7.889	7.907	7.925	2	4	5	7	9	11	13	14	16
0.90	7.943	7.962	7.98	7.998	8.017	8.035	8.054	8.072	8.091	8.11	2	4	6	7	9	11	13	15	17
0.91	8.128	8.147	8.166	8.185	8.204	8.222	8.241	8.26	8.279	8.299	2	4	6	8	9	11	13	15	17
0.92	8.318	8.337	8.356	8.375	8.395	8.414	8.433	8.453	8.472	8.492	2	4	6	8	10	12	14	15	17
0.93	8.511	8.531	8.551	8.57	8.59	8.61	8.63	8.65	8.67	8.69	2	4	6	8	10	12	14	16	18
0.94	8.71	8.73	8.75	8.77	8.79	8.81	8.831	8.851	8.872	8.892	2	4	6	8	10	12	14	16	18
0.95	8.913	8.933	8.954	8.974	8.995	9.016	9.036	9.057	9.078	9.099	2	4	6	8	10	12	15	17	19
0.96	9.12	9.141	9.162	9.183	9.204	9.226	9.247	9.268	9.29	9.311	2	4	6	8	11	13	15	17	19
0.97	9.333	9.354	9.376	9.397	9.419	9.441	9.462	9.484	9.506	9.528	2	4	7	9	11	13	15	17	20
0.98	9.55	9.572	9.594	9.616	9.638	9.661	9.683	9.705	9.727	9.75	2	4	7	9	11	13	16	18	20
0.99	9.772	9.795	9.817	9.84	9.863	9.886	9.908	9.931	9.954	9.977	2	5	7	9	11	14	16	18	20

Trigonometric table

	sin	cos	tan	cot	sec	csc	
0	0	1	0		1		90
1	0.017452	0.999848	0.017455	57.2900	1.000152	57.29874	89
2	0.03490	0.999391	0.034921	28.63628	1.00061	28.65373	88
3	0.052336	0.99863	0.052408	19.08115	1.001372	19.10734	87
4	0.069756	0.997564	0.069927	14.30068	1.002442	14.3356	86
5	0.087156	0.996195	0.087489	11.43006	1.00382	11.47372	85
6	0.104528	0.994522	0.10510	9.514373	1.005508	9.56678	84
7	0.121869	0.992546	0.122784	8.144353	1.00751	8.205516	83
8	0.139173	0.990268	0.140541	7.115376	1.009828	7.18530	82
9	0.156434	0.987688	0.158384	6.313757	1.012465	6.392459	81
10	0.173648	0.984808	0.176327	5.671287	1.015427	5.758775	80
11	0.190809	0.981627	0.19438	5.144558	1.018717	5.240847	79
12	0.207912	0.978148	0.212556	4.704634	1.022341	4.809738	78
13	0.224951	0.97437	0.230868	4.33148	1.026304	4.445415	77
14	0.241922	0.97030	0.249328	4.010784	1.030614	4.133569	76
15	0.258819	0.965926	0.267949	3.732054	1.035276	3.863706	75
16	0.275637	0.961262	0.286745	3.487418	1.040299	3.627958	74
17	0.292371	0.95630	0.30573	3.270856	1.045692	3.420306	73
18	0.309017	0.951057	0.324919	3.077686	1.051462	3.236071	72
19	0.325568	0.945519	0.344327	2.904214	1.057621	3.071556	71
20	0.34202	0.939693	0.36397	2.74748	1.064178	2.923807	70
21	0.358368	0.933581	0.383864	2.605091	1.071145	2.79043	69
22	0.374606	0.927184	0.404026	2.475089	1.078535	2.669469	68
23	0.390731	0.920505	0.424474	2.355855	1.08636	2.559307	67
24	0.406736	0.913546	0.445228	2.246039	1.094636	2.45860	66
25	0.422618	0.906308	0.466307	2.144509	1.103378	2.36620	65
26	0.438371	0.898794	0.487732	2.050306	1.112602	2.281174	64
27	0.45399	0.891007	0.509525	1.962612	1.122326	2.202691	63
28	0.469471	0.882948	0.531709	1.880728	1.13257	2.130056	62
29	0.484809	0.87462	0.554308	1.80405	1.143354	2.062667	61
30	0.5000	0.866026	0.57735	1.732053	1.1547	2.00000	60
31	0.515038	0.857168	0.60086	1.664281	1.166633	1.941606	59
32	0.529919	0.848048	0.624869	1.600336	1.179178	1.887081	58
33	0.544639	0.838671	0.649407	1.539867	1.192363	1.83608	57
34	0.559192	0.829038	0.674508	1.482563	1.206218	1.788293	56
35	0.573576	0.819152	0.700207	1.42815	1.220774	1.743448	55
36	0.587785	0.809017	0.726542	1.376383	1.236068	1.70130	54
37	0.601815	0.798636	0.753553	1.327046	1.252135	1.661641	53
38	0.615661	0.788011	0.781285	1.279943	1.269018	1.62427	52
39	0.62932	0.777146	0.809783	1.23490	1.286759	1.589017	51
40	0.642787	0.766045	0.83910	1.191755	1.305407	1.555725	50
41	0.656059	0.75471	0.869286	1.15037	1.325012	1.524254	49
42	0.66913	0.743145	0.90040	1.110614	1.345632	1.494478	48
43	0.68200	0.731354	0.932514	1.07237	1.367327	1.46628	47
44	0.694658	0.71934	0.965688	1.035532	1.390163	1.439558	46
45	0.707106	0.707107	1.00000	1.00000	1.414213	1.414215	45
	cos	sin	cot	tan	csc	sec	