Chapter 3

Fuzzy Membership Function Formulation and Parameterization

The membership function of a fuzzy set is a generalization of the indicator function in classical sets. In fuzzy logic, it represents the degree of truth as an extension of valuation. Degrees of truth are often confused with probabilities, although they are conceptually distinct, because fuzzy truth represents membership in vaguely defined sets, not likelihood of some event or condition. Membership functions were introduced by Zadeh in the first paper on fuzzy sets (1965).

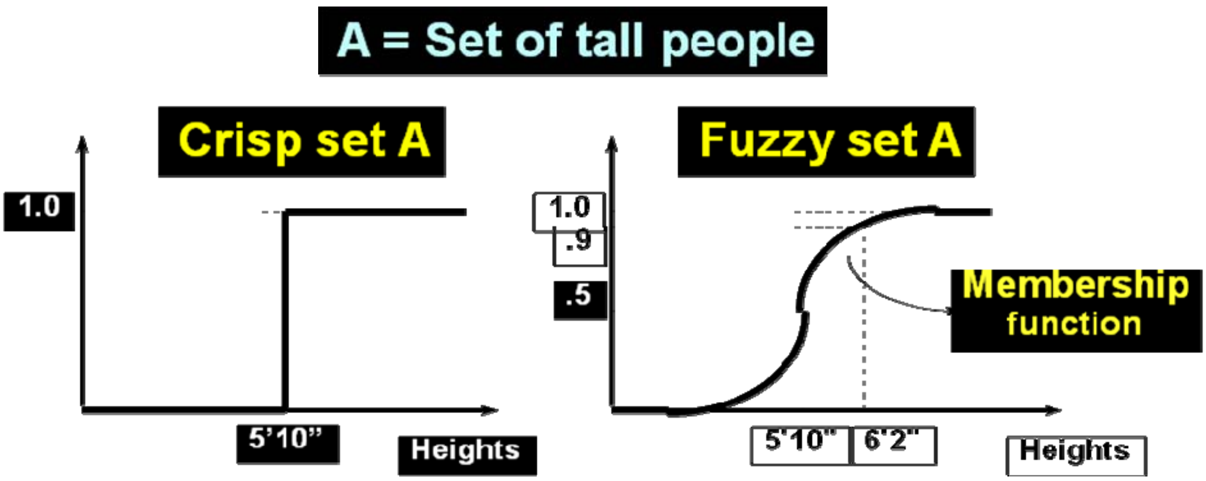


Figure 3.1: Membership functions in Fuzzy vs. crisp sets

Formal Definition of membership function

Let us consider fuzzy set A, A = {(x, μA(x))| x Є X} where μA(x) is called the membership function for the fuzzy set A. X is referred to as the universe of discourse. The membership function associates each element x Є X with a value in the interval [0, 1].

In fuzzy sets, each elements is mapped to [0,1] by membership function. That is, $\mu_A : X$ €[0, 1], where [0,1] means real numbers between 0 and 1 (including 0,1). Consequently, fuzzy set is with 'vague boundary set' comparing with crisp set.

The fuzzy set A can be alternatively denoted as follows:

If X is discrete then $A = \Sigma\ \mu_A(xi) / xi$

If X is continuous then $A = \int \mu_A(x) / x$

Here, μ_A (x) is the "membership function". Value of this function is between 0 and 1. This value represents the "degree of membership" (membership value) of element x in set A. The members of a fuzzy set are members to some degree, known as a membership grade or degree of membership. The membership grade is the degree of belonging to the fuzzy set. The larger the number (in [0, 1]) the more the degree of belonging. (N.B. This is not a probability).The translation from x to $\mu_A(x)$ is known as fuzzification.

In the fuzzy theory, fuzzy set A of universe X is defined by function $\mu_A(x)$ called the membership function of set A. We already discussed this point.

$\mu A(x): X \rightarrow [0, 1]$, where $\mu_A(x) = 1$ if x is totally in A;

$\mu_A(x) = 0$ if x is not in A;

$0 < \mu_A(x) < 1$ if x is partly in A.

This set allows a continuum of possible choices. For any element x of universe X, membership function $\mu_A(x)$ equals the degree to which x is an element of set A. This degree, a value between 0 and 1, represents the degree of membership, also called membership value, of element x in set A.

Basics on Fuzzy Membership Functions

Support: The support of fuzzy set A is the set of all point $x \in X$ such that $\mu_A(x) > 0$
Mathematically we can express Support (A) = $\{(x,\mu_A(x))/ \mu_A(x)>0)\}$

Core: The core of a fuzzy set A is the set of all $x \in X$ such that $\mu_A(x) = 1$.
Mathematically we can express core (A) = $\{x \in X \mid \mu_A(x) = 1\}$

Crossover: A crossover point of a fuzzy set 'A' is a point $x \in X$ at which $\mu_A(x) = 0.5$
Mathematically we can express Crossover (A) = $\{x \in X \mid \mu_A(x) = 0.5\}$

Normality: A fuzzy set 'A' is a normal if its core is non-empty, i.e. core (A) $\neq \emptyset \Rightarrow$ A is a normal fuzzy set.
Mathematically we can express Normality (A) =1 if $\mu_A(x) = 1$, for all $x \in X$ and $(x, \mu_A(x)) \in A$

Fuzzy singleton: A fuzzy set 'A' whose support is single point in x with $\mu_A(x) = 1$ is called fuzzy singleton.

$|A| = |\{(x, \mu_A(x)) | \mu_A(x) = 1\}|$

α - cut: $A\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$

Strong α - cut: $A\alpha = \{x \in X \mid \mu_A(x) > \alpha\}$. In this case 'A' is defined as Crisp set.

Convexity of Fuzzy Sets: A fuzzy set *A* is convex if and only if for any x1, x2 ∈X and there exists x = [0, 1] such that

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

Alternatively, *A* is convex if all its α-cuts are convex.

Bandwidths: for a normal & convex set, the bandwidth is the distance between two unique crossover points Bandwidth (A) = |x2 – x1| where $\mu_A(x1) = \mu_A(x2) = 0.5$.

Symmetry: A fuzzy set A is symmetric if it's MF around a certain point x = c, satisfies the following criteria i.e. $\mu_A(x + c) = \mu_A(c - x) \quad \forall x \in X$.

Open left, Open Right and Closed:

A fuzzy set 'A' is open left $A \Leftrightarrow \lim_{x\to-\infty} \mu_A(x) = 1$ and $\lim_{x\to+\infty} \mu_A(x) = 0$

A fuzzy set 'A' is open right $A \Leftrightarrow \lim_{x\to-\infty} \mu_A(x) = 0$ and $\lim_{x\to+\infty} \mu_A(x) = 1$

A fuzzy set 'A' is closed $A \Leftrightarrow \lim_{x\to-\infty} \mu_A(x) = \lim_{x\to+\infty} \mu_A(x) = 0$

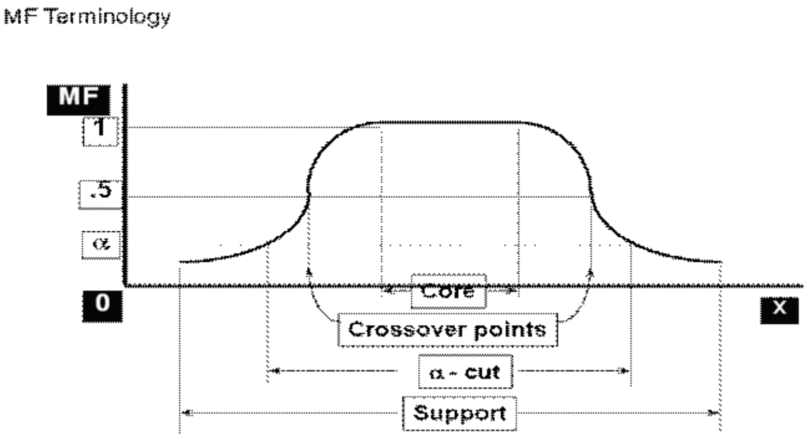


Figure 3.2: Illustration of terminologies on Fuzzy Membership function

Membership functions: Parameterization and Formulation

1) Triangular Membership function

2) Trapezoidal MF
3) Gaussian MF
4) Generalized bell MF
5) Sigmoid membership function

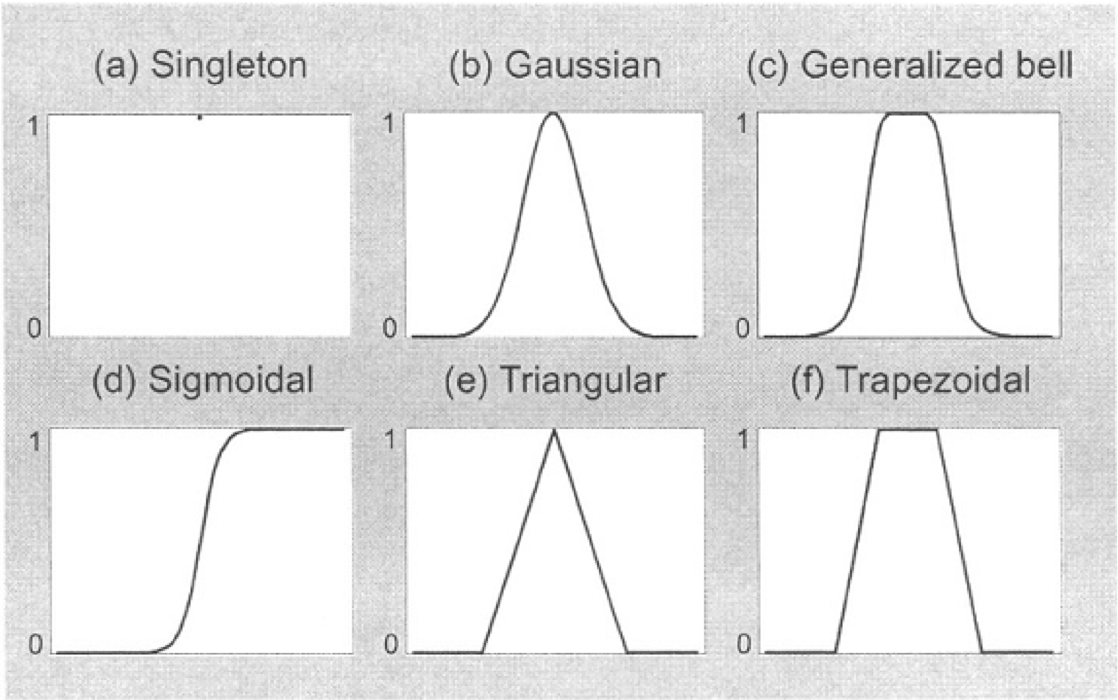
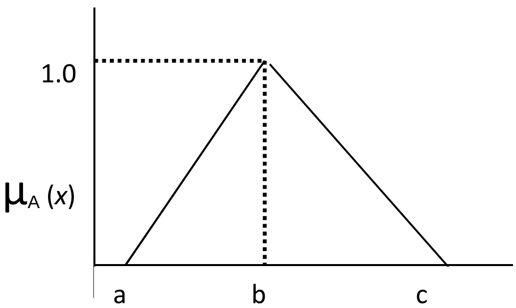


Figure 3.3: Various type of Fuzzy membership functions

Triangular Membership function:

Let a, b and c represent the x coordinates of the three vertices of $\mu_A(x)$ in a fuzzy set A (a: lower boundary and c: upper boundary where membership degree is zero, b: the centre where membership degree is 1).



$$\mu_A(x) = \begin{cases} 0 & if \ x \le a \\ \dfrac{x-a}{b-a} & if \ a \le x \le b \\ \dfrac{c-x}{c-b} & if \ b \le x \le c \\ 0 & if \ x \ge c \end{cases}$$

Trapezoidal membership function:

Let a, b, c and d represents the x coordinates of the membership function. then

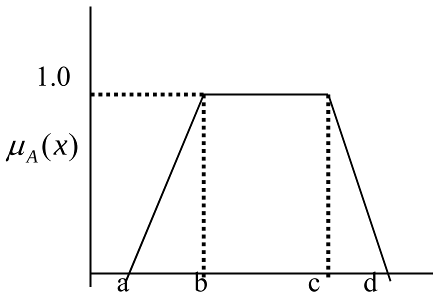
Trapezoid(x; a, b, c, d) = 0 if x ≤ a;

= (x-a)/ (b-a) if a ≤ x ≤ b

= 1 if b ≤ x ≤ c;

= (d-x)/ (d-c) 0 if c ≤ x ≤ d;

= 0, if d ≤ x.



$$\mu_{\text{trapezoid}} = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$

Gaussian membership function:

The Gaussian membership function is usually represented as Gaussian(x:c,s) where c, s represents the mean and standard deviation.

$$\mu_A(x, c, s, m) = \exp\left[-\frac{1}{2}\left|\frac{x-c}{s}\right|^m\right]$$

Here c represents centre, s represents width and m represents fuzzification factor.

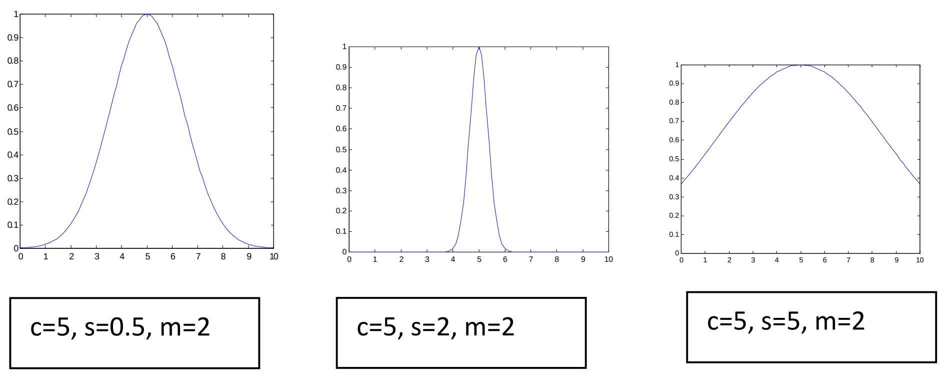
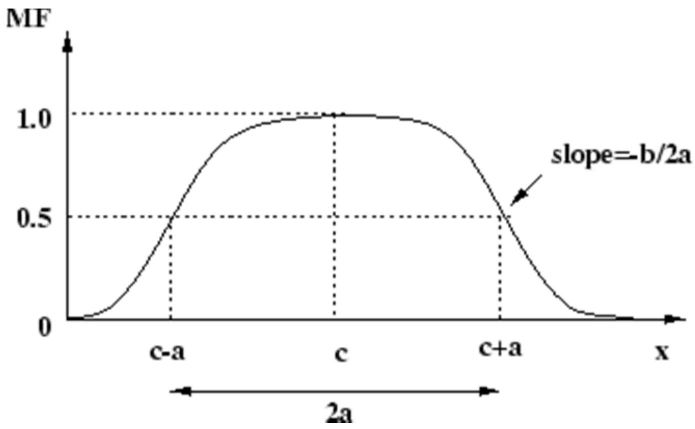


Figure 3.4: Different shapes of Gaussian MFs with different values of s and m.

Generalized Bell membership function:

A generalized bell membership function has three parameters: a –responsible for its width, c –responsible for its center and b –responsible for its slopes. Mathematically,

$$gbellmf(x;a,b,c) = \frac{1}{1 + \left| \frac{x-c}{b} \right|^{2b}}$$



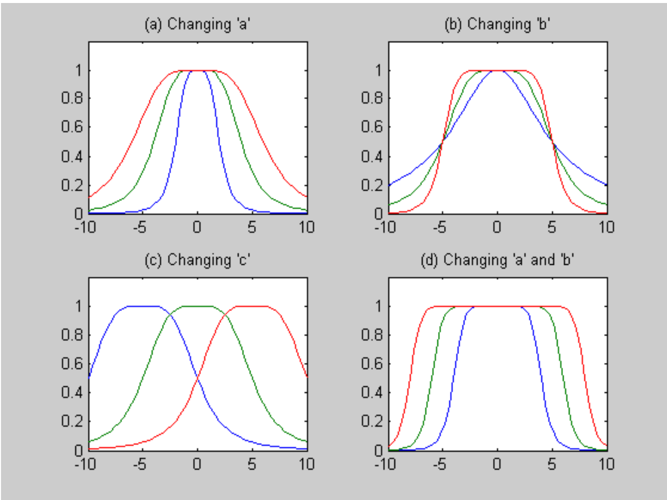


Figure 3.5: Different shapes of Gaussian MFs with different values of s and m.

Sigmoid Membership function:

A sigmoidal membership function has two parameters: a responsible for its slope at the crossover point x = c. The membership function of the sigmoid function can be represented as Sigmf (x:a, c) and it is

$$sigmf(x;a,b,c) = \frac{1}{1 + e^{-a(x-c)}}$$

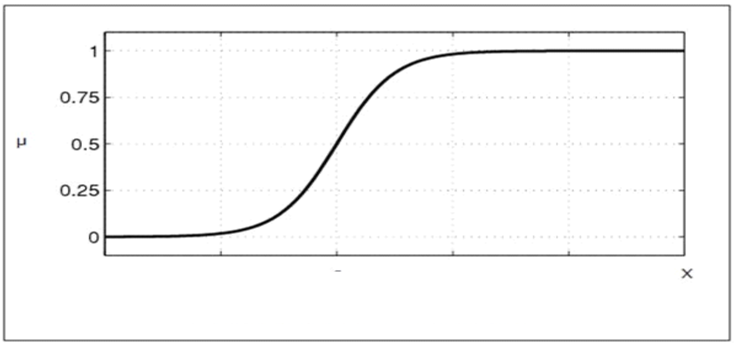


Figure 3.6: A general structures of sigmoid MF.

A sigmoidal MF is inherently open right or left & thus, it is appropriate for representing concepts such as “very large” or “very negative”. Sigmoidal MF mostly used as activation function of artificial neural networks (NN).A NN should synthesize a close MF in order to simulate the behavior of a fuzzy inference system.

Left –Right (LR) MF

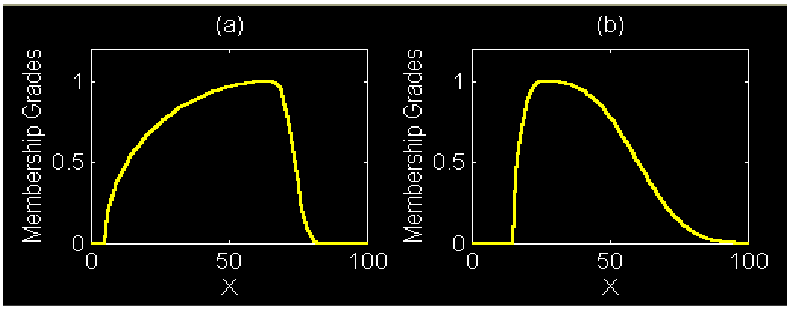
$$LR(x;c,\alpha,\beta)=\begin{cases}F_L\left(\dfrac{c-x}{\alpha}\right), x<c\\ F_R\left(\dfrac{x-c}{\beta}\right), x\geq c\end{cases}$$

Example:

$$F_L(x)=\sqrt{\max(0,1-x^2)}\ ,\ F_R(x)=\exp(-|x|^3)$$

c=65

a=60



c=25

a=10

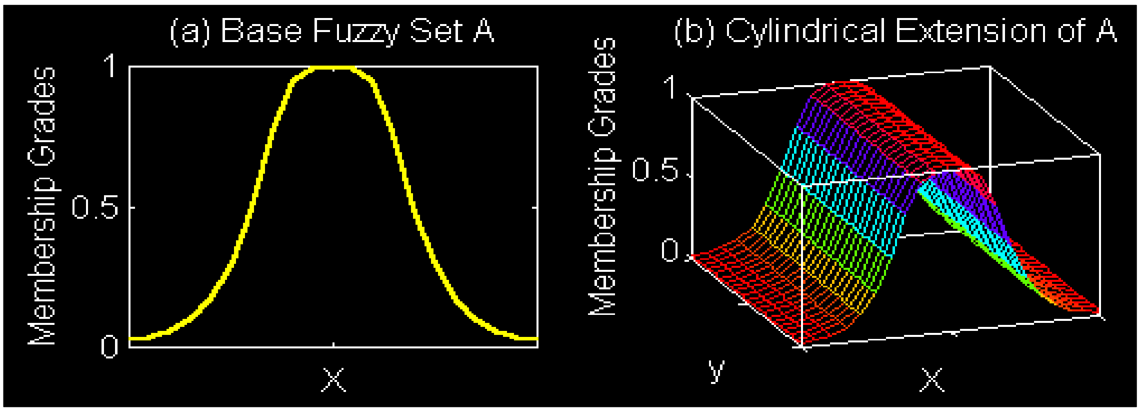
Figure 3.7: Examples of L-R MFs

2-D membership function

In this case, there are two inputs assigned to an MF: this MF is a two dimensional MF. A one input MF is called ordinary MF.

Extension of a one-dimensional MF to a two-dimensional MF via cylindrical extensions is shown in below fig. If A is a fuzzy set in X, then its cylindrical extension in X*Y is a fuzzy set C(A) defined by: C(A) can be viewed as a two-dimensional fuzzy set.

$$C(A) = \int_{X*Y} \mu_A(x) \,|\, (x, y)$$



Base set Cylindrical extension

Figure 3.8: Example of a 2D MF

Projection of fuzzy sets (decrease dimension): Let R be a two-dimensional fuzzy set on X*Y. Then the projections of R onto X and Y are defined as:

$$R_X = \int_X \left[\max_y \mu_R(x, y) \right] | \, x \qquad R_Y = \int_Y \left[\max_x \mu_R(x, y) \right] | \, y$$

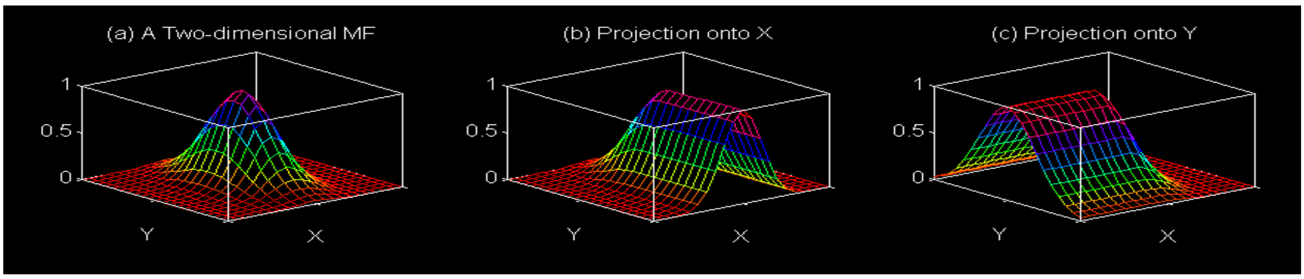


Figure 3.9: Different views of 2D MFs

Composite anf non-composite MFs

Suppose that the fuzzy A = “(x,y) is near (3,4)” is defined by:

$$\mu_A(x,y) = \exp\left[-\left(\frac{x-3}{2}\right)^2 - (y-4)^2\right]$$
$$= \exp\left[-\left(\frac{x-3}{2}\right)^2\right]\exp\left[-\left(\frac{y-4}{1}\right)^2\right]$$
$$= G(x;3,2) * G(y;4,1)$$

This two-dimensional MF is composite, the fuzzy set A is composed of two statements:

“x is near 3” and “y is near 4”

These two statements are respectively defined as: $\mu_{\text{near } 3}$ (x) = G(x;3,2) & $\mu_{\text{near } 4}$ (x) = G(y;4,1)

If a fuzzy set is defined by:

$$\mu_A(x,y) = \frac{1}{1+|x-3||y-4|^{2.5}}$$,it is non-composite.

A composite two-dimensional MF is usually the result of two statements joined by the AND or OR connectives.

Composite two-dimensional MFs based on min & max operations

Let trap(x) = trapezoid (x;-6,-2,2,6) ,trap(y) = trapezoid (y;-6,-2,2,6) be two trapezoidal MFs on X and Y respectively. By applying the min and max operators, we obtain two-dimensional MFs on X*Y.

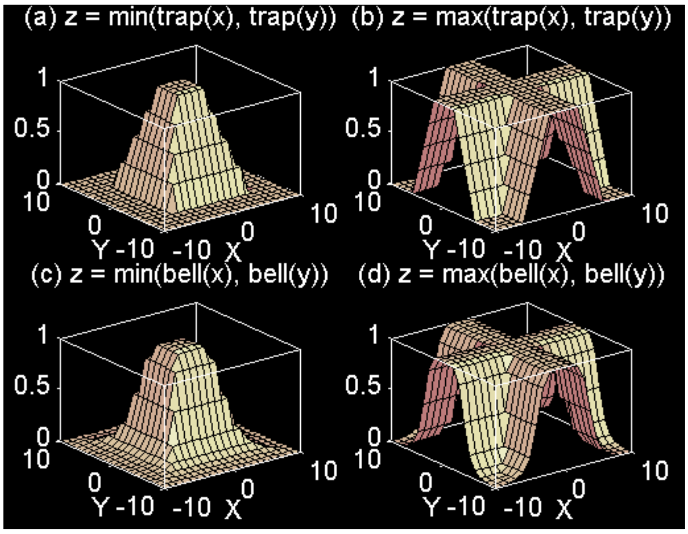


Figure 3.9: Two dimensional MFs defined by the min and max operators

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