

Motion in one dimension (1D)

In this chapter, we study speed, velocity, and acceleration for motion in one-dimension. One dimensional motion is motion along a straight line, like the motion of a glider on an airtrack.

speed and velocity

$$\text{speed} = \frac{\text{distance traveled}}{\text{time elapsed}}, \quad s = \frac{d}{t}, \quad \text{units are m/s or mph or km/hr or...}$$

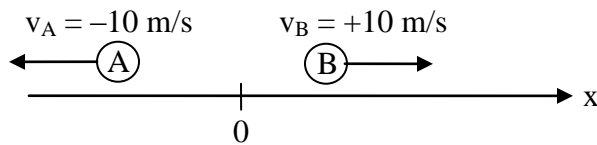
speed s and distance d are both always positive quantities, by definition.

velocity = speed + direction of motion

Things that have both a magnitude and a direction are called vectors. More on vectors in Ch.3.

For 1D motion (motion along a straight line, like on an air track), we can represent the direction of motion with a +/- sign

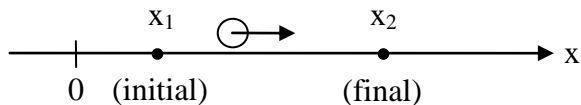
$$\left. \begin{array}{l} + = \text{going right } \rightarrow \\ - = \text{going left } \leftarrow \end{array} \right\} \text{always!}$$



Objects A and B have the same speed $s = |v| = +10$ m/s, but they have different velocities.

If the velocity of an object varies over time, then we must distinguish between the *average* velocity during a time interval and the *instantaneous* velocity at a particular time.

Definition: average velocity = $\bar{v} \equiv \frac{\text{change in position}}{\text{change in time}} = \frac{\Delta x}{\Delta t}$



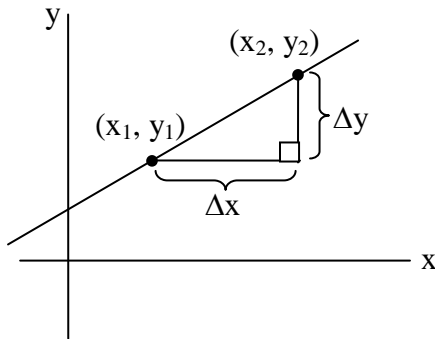
$$\bar{v} = \frac{x_f - x_i}{t_f - t_i} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t}$$

$$\Delta x = x_{\text{final}} - x_{\text{initial}} = \text{displacement (can be + or -)}$$

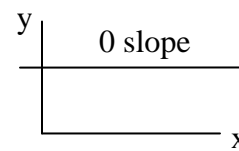
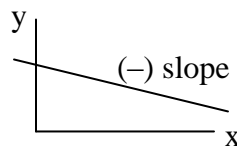
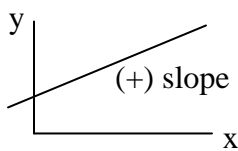
Notice that Δ (delta) always means "final minus initial".

$\bar{v} = \frac{\Delta x}{\Delta t}$ is the *slope* of a graph of x vs. t

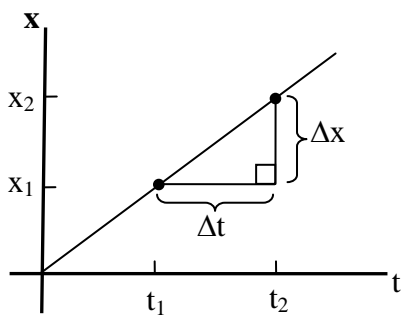
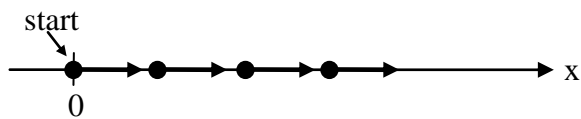
Review: Slope of a line



$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



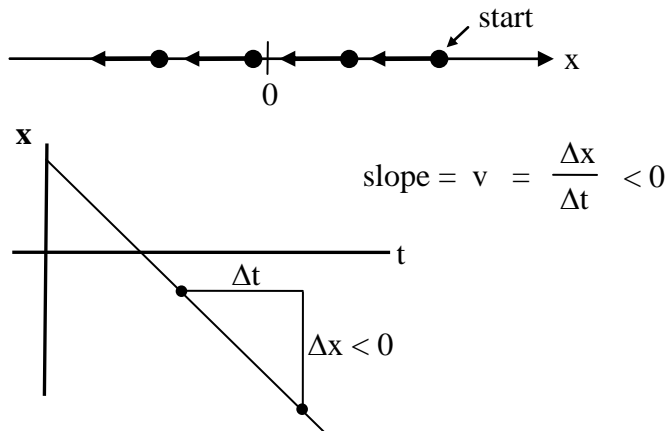
Suppose we travel along the x -axis, in the positive direction, at constant velocity v :



$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta x}{\Delta t} = v$$

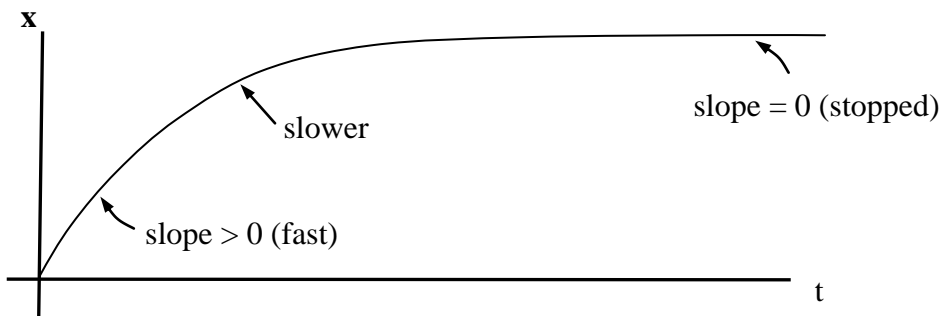
y -axis is x , x -axis is t .

Now, let us travel in the negative direction, to the left, at constant velocity.

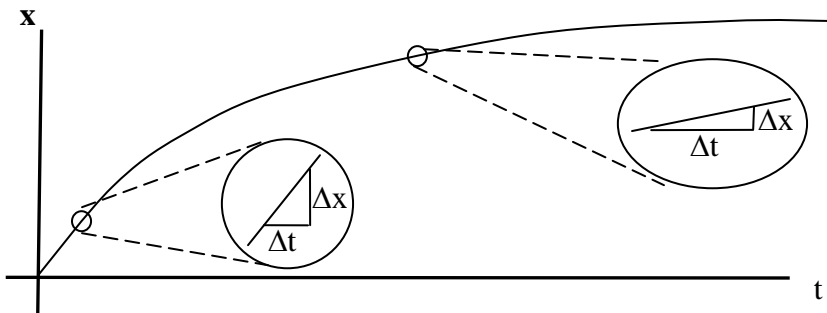


Note that $v = \text{constant} \Rightarrow \text{slope of } x \text{ vs. } t = \text{constant} \Rightarrow \text{graph of } x \text{ vs. } t \text{ is a straight line}$

But what if $v \neq \text{constant}$? If an object starts out going fast, but then slows down and stops...

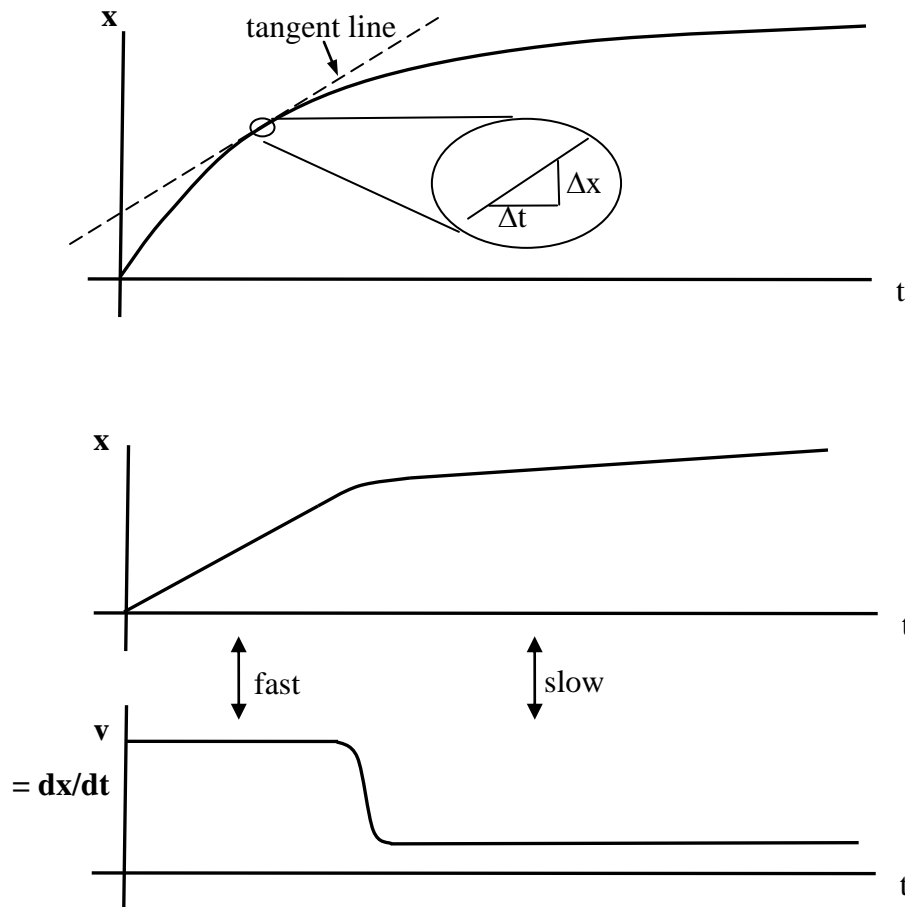


The slope *at a point* on the x vs. t curve is the **instantaneous velocity** at that point.



Definition: instantaneous velocity = velocity averaged over a very, very short (infinitesimal) time interval

$v \equiv \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} \equiv \frac{dx}{dt} = \text{slope of } \textit{tangent line}$. In Calculus class, we would say that the velocity is the *derivative* of the position with respect to time. The derivative of a function $x(t)$ is defined as the slope of the tangent line: $\frac{dx}{dt} \equiv \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t}$.



Acceleration

If the velocity is *changing*, then there is non-zero acceleration.

Definition: acceleration = time rate of change of velocity = derivative of velocity with respect to time

In 1D: **instantaneous acceleration** $a \equiv \lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$

average acceleration over a non-infinitesimal time interval Δt : $\bar{a} \equiv \frac{\Delta v}{\Delta t}$

units of $a = [a] = \frac{\text{m/s}}{\text{s}} = \frac{\text{m}}{\text{s}^2}$

Sometimes I will be a bit sloppy and just write $a = \frac{\Delta v}{\Delta t}$, where it is understood that Δt is either a infinitesimal time interval in the case of instantaneous a or Δt is a large time interval in the case of average a .

$$a = \frac{dv}{dt} \quad \square \quad \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i} = \frac{v_2 - v_1}{t_2 - t_1}$$

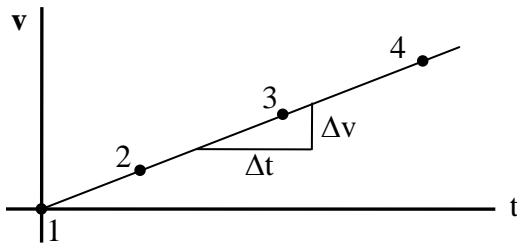
- $v = \text{constant} \Rightarrow \Delta v = 0 \Rightarrow a = 0$
- v increasing (becoming more positive) $\Rightarrow a > 0$
- v decreasing (becoming more negative) $\Rightarrow a < 0$

In 1D, acceleration a is the slope of the graph of v vs. t (just like $v = \text{slope of } x \text{ vs. } t$)

Examples of constant acceleration in 1D on next page...

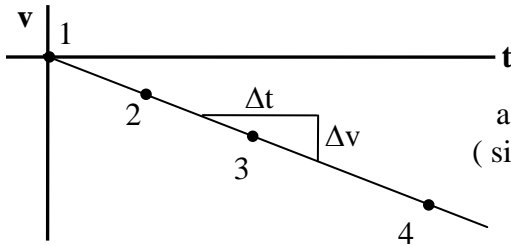
Examples of constant acceleration in 1D

Situation I An object starts at rest, then moves to the right (+ direction) with constant acceleration, going faster and faster.



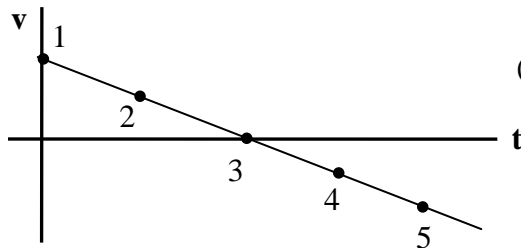
$a > 0$, $a = \text{constant}$
 (a constant, since v vs. t is straight)

Situation II An object starts at rest, then moves to the left (– direction) with constant acceleration, going faster and faster.



$a < 0$, $a = \text{constant}$
 (since v vs. t has constant, negative slope)

Situation III



$a < 0$, $a = \text{constant} !!$
 (since v vs. t has constant, negative slope)

The direction of the acceleration

For 1D motion, the acceleration, like the velocity, has a sign (+ or -). Just as with velocity, we say that positive acceleration is acceleration to the right, and negative acceleration is acceleration to the left. But what is it, exactly, that is pointing right or left when we talk about the direction of the acceleration?

Acceleration and velocity are both examples of **vector** quantities. They are mathematical objects that have both a magnitude (size) and a **direction**. We often represent vector quantities by putting a little arrow over the symbol, like \vec{v} or \vec{a} .

direction of \vec{a} \neq direction of \vec{v}

direction of \vec{a} = the direction toward which the velocity is tending \neq direction of \vec{v}

Reconsider Situation I (previous page)

1 \longrightarrow 2 \longrightarrow 1 is an earlier time, 2 is a later time

\longrightarrow $\vec{v}_1 =$ velocity at time 1 = \vec{v}_{init}

\longrightarrow $\vec{v}_2 =$ velocity at time 2 = \vec{v}_{final}

$\vec{\Delta v} =$ "change vector" = how \vec{v}_1 must be "stretched" to change it into \vec{v}_2

\vec{v}_1 $\xrightarrow{\vec{\Delta v}}$ \vec{v}_2 direction of $\vec{a} =$ direction of $\vec{\Delta v}$

Situation II: $\vec{\Delta v}$ \longleftarrow \vec{v}_1 \longleftarrow \vec{v}_2

Situation III: \vec{v}_1 \longrightarrow \vec{v}_2 \longleftarrow $\vec{\Delta v}$

In both situations II and III, Δv is to the left, so acceleration a is to the left

(This has been a preview of Chapter 3, $\vec{a} = \frac{d\vec{v}}{dt}$)

Our mantra:

" Acceleration is not velocity, velocity is not acceleration."

Constant acceleration formulas (1D)

In the special case of constant acceleration ($a = \text{constant}$), there are a set of formulas that relate position x , velocity v , and time t to acceleration a .

	formula	relates
(a)	$v = v_o + at$	(v, t)
(b)	$x = x_o + v_o t + (1/2)at^2$	(x, t)
(c)	$v^2 = v_o^2 + 2a(x - x_o)$	(v, x)
(d)	$\bar{v} = \frac{v_o + v}{2}$	

$x_o, v_o =$ initial position, initial velocity $x, v =$ position, velocity at time t

Reminder: all of these formulas are only valid if $a = \text{constant}$, so these are special case formulas. They are not laws. (Laws are *always* true.)

Proof of formula (a) $v = v_o + at$. Start with definition $a = \frac{dv}{dt}$.

In the case of constant acceleration, $a = \bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1}$

Since $a = \text{constant}$, there is no difference between average acceleration \bar{a} and instantaneous acceleration at any time.

$$\left. \begin{array}{l} v_1 = v_o, \quad v_2 = v \\ t_1 = 0, \quad t_2 = t \end{array} \right\} a = \frac{v - v_o}{t} \Rightarrow v = v_o + at$$

(See the appendix or your text for proofs of the remaining formulas.)

Example: Braking car. A car is moving to the right with initial velocity $v_o = +21 \text{ m/s}$. The brakes are applied and the car slows to a stop in $t = 3 \text{ s}$ with constant acceleration. What is the acceleration of the car during braking?

$$a = ? \quad a = \bar{a} = \frac{\Delta v}{\Delta t} = \frac{v - v_o}{t} = \frac{0 - 21 \text{ m/s}}{3 \text{ s}} = -7 \text{ m/s}^2$$

(Do you understand why we have set $v = 0$ in this problem?)

Negative acceleration means that the acceleration is to the left.

Let's stare at the formula $x = x_0 + v_0 t + (1/2) a t^2$ until it starts to make sense. You should always stare at new formulas, turning them over in your mind, until they start to make a little sense.

$$\underbrace{x - x_0}_{\text{how far you travel}} = \underbrace{v_0 t}_{\substack{\text{how far you would travel if} \\ v = \text{constant, } a = 0}} + \underbrace{(1/2) a t^2}_{\substack{\text{how much more (} a > 0 \text{) or less (} a < 0 \text{)} \\ \text{you travel compared to how far you would} \\ \text{have gotten if } a = 0}}$$

Gravitational acceleration

Experimental fact: In free-fall, near the surface of the earth, all objects have a constant downward acceleration with magnitude $g = +9.8 \text{ m/s}^2$. ($g > 0$ by definition)

The term *free-fall* means that the only force acting on the object is gravity — no other forces are acting, no air resistance, just gravity. A falling object is in free-fall only if air resistance is small enough to ignore. (Later, when we study gravity, we will find out *why* $g = \text{constant} = 9.8 \text{ m/s}^2$ for all objects, regardless of mass. For now, we simply accept this as an experimental fact.)

Things to notice:

- The acceleration during free-fall is always straight down, even though the velocity might be upward. Repeat after me: "Acceleration is not velocity, velocity is not acceleration."
- All objects, regardless of mass, have the same-size acceleration during free-fall. Heavy objects and light objects all fall with the same acceleration (so long as air resistance is negligible).

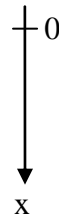
Example: Object dropped from rest. What is the position, velocity, and acceleration at 1 s intervals as the object falls?

Choose downward as the (+) direction, so that $a = +g$. If we instead chose upward as the positive direction, then the acceleration would be in the negative direction, $a = -g$. Remember, the symbol g is defined as the magnitude of the acceleration of gravity. $g > 0$ always, by definition.

Often, we call the vertical axis the y-axis, but let's call it the x-axis here:

$$x_0 = 0, v_0 = 0 \Rightarrow x = (1/2) a t^2, \quad v = a t \quad (\text{from constant acceleration formulas})$$

$$g \cong 10 \text{ m/s}^2 \Rightarrow x = 5 t^2, \quad v = 10 t$$



t (s)	x (m)	v (m/s)	a (m/s ²)
0	0	0	10
1	5	10	10
2	20	20	10
3	45	30	10

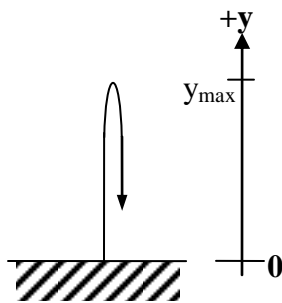
Notice that you can compute the acceleration a by taking any pair of (t, v) values and computing

$$a = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i}. \quad \text{You always get } a = 10 \text{ m/s}^2.$$

Example: Projectile Motion: A projectile is fired straight up from the ground with an initial speed of $|v_0| = 10 \text{ m/s}$.

◆ Describe the velocity vs. time. (Assume negligible air resistance.)

Choose upward as the (+) direction and set the ground at $y = 0$.

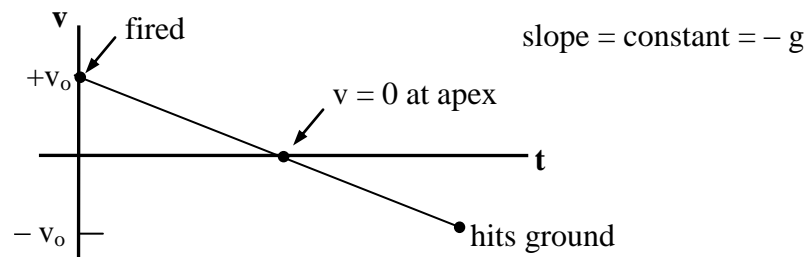


$$y_0 = 0, \quad v_0 = +10 \text{ m/s}$$

$$a = -g = -9.8 \text{ m/s}^2$$

$$v = v_0 + a t = v_0 - g t$$

Graph of v vs. t :



◆ What is time to reach maximum height, y_{\max} ?

$$\text{At the maximum height, } v = 0 \Rightarrow 0 = v_0 - g t, \quad t = \frac{v_0}{g} = \frac{10 \text{ m/s}}{10 \text{ m/s}^2} = 1.0 \text{ s}$$

◆ What is y_{\max} ?

$$\text{Method I: } y(t) = y_0 + v_0 t + \frac{1}{2} a t^2 = v_0 t - \frac{1}{2} g t^2$$

$$\text{At } t = 1 \text{ s, } y = y_{\max} = v_0 t - \frac{1}{2} g t^2 = 10(1) - (0.5)(10)(1)^2 = 10 - 5 = 5 \text{ m}$$

Method II: Use $v^2 = v_o^2 + 2a(y-y_o)$.

At apex, $v = 0$, $a = -g$, and $(y - y_o) = (y_{\max} - 0) = y_{\max}$, so we have

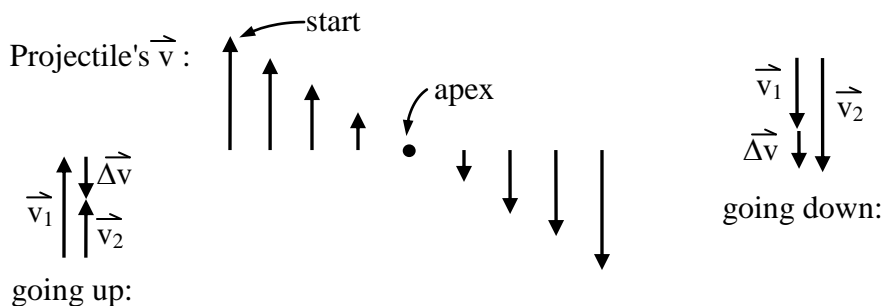
$$0 = v_o^2 - 2gy_{\max}, \quad y_{\max} = \frac{v_o^2}{2g} = \frac{10^2}{2(10)} = 5 \text{ m}$$

Comments about projectile motion:

◆ The acceleration is constant (straight down, magnitude g), only if we can ignore air resistance. Real projectiles, like cannonballs moving through air, are strongly affected by air resistance. Play with a simulation of projectile motion with and without air resistance. Go to <http://phet.colorado.edu> and find the simulation called Projectile Motion.

◆ The formula $y = y_o + v_o t - \frac{1}{2}gt^2$ is a quadratic equation, so there are two solutions, that is, two values of t for a given value of y . These two times correspond to on the way up and on the way down.

◆ Recall that the direction of $\vec{a} = \text{direction of } \Delta\vec{v}$ (not the direction of \vec{v}). How does this square with vertical projectile motion?



Notice that "delta-v" is always downward, regardless of the direction of the velocity.

Qualitative comments about acceleration

◆ You can feel acceleration, but you cannot feel constant velocity. If you are in an airplane traveling at constant velocity (heading NW at 600 mph, say) and the ride is smooth, then you can eat dinner, juggle, fall asleep, ... exactly as if you were at rest. In fact, if the ride is perfectly smooth, there is no way to tell that you are moving relative to the ground, except by looking out the window. Prof. Einstein says that it makes just as much sense to say the you and the airplane are at rest, and the ground is moving backwards. All that can be said is that the airplane and the ground are in relative motion. Which object is at rest depends on your frame of reference.

But you can tell right away if you are accelerating. If you are accelerating forward, you feel yourself being pushed back into the seat. (Actually, the seat is pushing you forward; it just feels like you are being pushed back — more on that later).

◆ If you are in a car, there are two different ways that you can be accelerating forward:

- 1) Start at rest, and then floor it!
- 2) Move in reverse at high constant velocity, and then apply brakes!

In both these cases, you are accelerating in the forward direction. In both these cases, you feel exactly the same thing: you feel yourself being pressed back into the seat. Puzzle for later: when you are in a chair that is accelerating forward, why does it feel like there is a force pushing you backwards.

◆ Acceleration and velocity are completely different things. Acceleration is the time rate of change of velocity. Be aware that...

the rate of change of something \neq the something

$$(\vec{a} = \text{rate of change of } \vec{v} \neq \vec{v})$$

Example: The radio weatherman says,
"The temperature is 48° and the temperature is falling at 10° per hour."

$$T = 48^\circ, \quad \frac{dT}{dt} = -10^\circ/\text{hr} = -10^\circ \text{ hr}^{-1}$$

If all you know is $\frac{dT}{dt}$, then you know nothing at all about T. And if all you know is T, you know nothing at all about $\frac{dT}{dt}$. They are completely different; knowledge of one tells you nothing about the other.

Important Math Appendices:

Proving the constant acceleration formulas:

We can derive the constant acceleration formulas on page 8, without using calculus. Let's derive $x = x_0 + v_0 t + (1/2)at^2$.

When $a = \text{constant}$, a plot of v vs. t is a straight line (since a is the slope of v vs. t). In this case, the average velocity is half-way

between the velocities at the start and finish: $\bar{v} = \frac{v + v_0}{2}$

We can also write the average velocity using the definition

$$\bar{v} \equiv \frac{\Delta x}{\Delta t} = \frac{x - x_0}{t - 0}.$$

Some rearranging and substituting gives

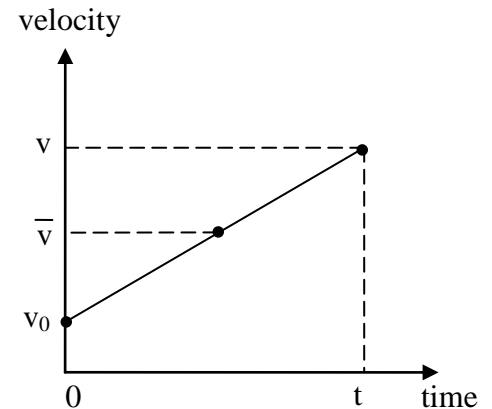
$$x = x_0 + \bar{v}t = x_0 + \frac{1}{2}(v_0 + v)t.$$

Now substitute $v = v_0 + at$ (proven on p.8): $x = x_0 + \frac{1}{2}(v_0 + v_0 + at)t = x_0 + v_0 t + \frac{1}{2}at^2$.

Done!

To prove $v^2 = v_0^2 + 2a(x - x_0)$, combine the formulas $v = v_0 + at$ and

$x = x_0 + v_0 t + (1/2)at^2$, eliminating time t . (I'll let you do that.)



Just enough about derivatives

You don't have to know a lot about derivatives in this course. But what you do have to know, you have to know extremely well. So please study this appendix carefully.

The derivative of a function $f(x)$ is another function $f'(x) = df/dx$, defined by $\frac{df}{dx} \equiv \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x}$.

The derivative of $f(x)$ is the slope of the tangent line to the curve $f(x)$ vs. x .

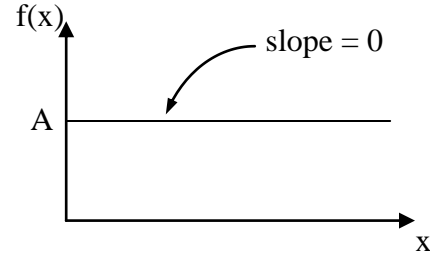
In this chapter, we have been considering functions of time: $x = x(t)$ and $v = v(t)$. Position x and velocity v are related by $v = \frac{dx}{dt}$. Velocity v and acceleration a are related by $a = \frac{dv}{dt}$.

There are 4 important theorems about derivatives that we will need again and again in this course.

Theorem 1. The derivative of a constant is zero.

Proof 1: Function $f(x) = A$, where A is a constant. The plot $f(x)$ vs. x is a straight line with zero slope.

Remember that the derivative is the slope of the tangent line to the curve. The slope is zero, so the derivative is zero.



Proof 2: Start with the definition of derivative:

$$\frac{df}{dx} \equiv \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{A - A}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = 0.$$

Theorem 2: If $f(x) = Ax^n$, where A and n are constants, then $\frac{df}{dx} = Anx^{n-1}$.

Let's prove this for the special case $n = 2$: $f(x) = Ax^2 \Rightarrow \frac{df}{dx} = 2Ax$

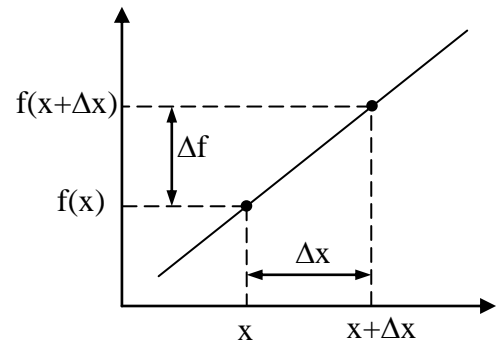
Again, we start with the definition of derivative. (In any proof, you have to start with something you know to be true. Definitions are always true, by definition.)

$$\frac{df}{dx} \equiv \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

$$f(x) = Ax^2, \quad f(x + \Delta x) = A(x + \Delta x)^2,$$

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{A(x + \Delta x)^2 - Ax^2}{\Delta x}$$

$$= \frac{A(x^2 + 2x\Delta x + \Delta x^2) - Ax^2}{\Delta x} = 2Ax + A\Delta x$$



Taking the limit $\Delta x \rightarrow 0$, we have $\frac{df}{dx} = 2Ax$. Done.

Theorem 3: The derivative of a sum is the sum of the derivatives:

$$f(x) = g(x) + h(x) \Rightarrow \frac{df(x)}{dx} = \frac{dg(x)}{dx} + \frac{dh(x)}{dx}$$

Proof:

$$\begin{aligned} \frac{df(x)}{dx} &= \frac{d[g(x)+h(x)]}{dx} \equiv \lim_{\Delta x} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x} \frac{[g(x+\Delta x)+h(x+\Delta x)]-[g(x)+h(x)]}{\Delta x} \\ &= \lim_{\Delta x} \left\{ \frac{[g(x+\Delta x)-g(x)]}{\Delta x} + \frac{[h(x+\Delta x)-h(x)]}{\Delta x} \right\} = \frac{dg}{dx} + \frac{dh}{dx} \end{aligned}$$

Theorem 4:

The derivative of a constant times a function is the constant times the derivative of the function

$$g(x) = Af(x), \text{ where } A = \text{constant} \Rightarrow \frac{dg}{dx} = \frac{d(Af(x))}{dx} = A \frac{df(x)}{dx}$$

Proof:

$$\begin{aligned} \frac{d(Af)}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{Af(x+\Delta x) - Af(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{A[f(x+\Delta x) - f(x)]}{\Delta x} \\ &= A \cdot \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = A \frac{df}{dx} \end{aligned}$$

Exercise. Starting with the constant acceleration formula $x(t) = x_0 + v_0 t + (1/2)at^2$, use your knowledge of calculus to prove that $v = v_0 + at$. (Hint: take the derivative dx/dt and use the theorems above.)

Example: A rocket in space has position as a function of time given by $x = x_0 + At^3$, where A is a constant. What is the velocity and acceleration of the rocket?

Hey! You should try to work this out yourself, before looking at the solution below.

Solution: $x(t) = x_0 + At^3$

Take the first derivative to get velocity

$$v = \frac{dx}{dt} = \frac{d(x_0 + At^3)}{dt} = \frac{dx_0}{dt} + \frac{d(At^3)}{dt} = 0 + 3At^2$$

(Notice how we needed the theorems .)

Now take another derivative (the second derivative) to get the acceleration.

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \frac{d(3At^2)}{dt} = 3 \cdot 2At = 6At.$$

Notice that this is a case of non-constant acceleration, so none of our constant acceleration formulas applies here.

Comment about notation: The acceleration is the second derivative of position w.r.t time:

$$a = \frac{dv}{dt} = \frac{d}{dt} \left(\frac{dx}{dt} \right). \text{ We usually write the 2}^{\text{nd}} \text{ derivative like this: } a = \frac{d^2x}{dt^2}.$$

The third derivative $\frac{d}{dt} \left(\frac{d}{dt} \left(\frac{dx}{dt} \right) \right)$ is written $\frac{d^3x}{dt^3}$.