ISYE 7201: Production & Service Systems
Fall 2011
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Midterm Exam
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Name: SOLUTIONS

Problem 1 (20 points): Let $P^{(1)}$ and $P^{(2)}$ denote transition probability matrices for ergodic Markov chains having the same state space $\{1, 2, \ldots, K\}$. Also, let $\pi^{(1)}$ and $\pi^{(2)}$ denote the stationary (limiting) probability vectors for the two chains. Consider the process defined as follows:

$X_0 = 1$. Then, a coin is flipped, and if it comes up heads, the remaining states $X_1, \ldots$ are obtained from the transition probability matrix $P^{(1)}$, while in the opposite case, the remaining states are obtained from the matrix $P^{(2)}$.

Answer the following questions:

i. Is $\{X_n, n \geq 0\}$ a Markov chain?

ii. If $p =$ Prob(coin comes up heads), what is $\lim_{n\to\infty} P(X_n = i)$?

(i) This is not a Markov chain since, at every state X_n, the transition of the process does not depend only on the state X_n, but also on the outcome of the coin toss that took place at the very beginning of the process evolution.

(ii) From the above description, it is clear that the considered process will evolve according to the dynamics that are defined by the transition prob. matrix $P^{(1)}$ with probability p, and according to the dynamics defined by the matrix $P^{(2)}$ with the remaining probability. Hence,

$$\lim_{n\to\infty} P(X_n = i) = p \cdot \pi_i^{(1)} + (1-p)\,\pi_i^{(2)}$$

Problem 2 (25 points): In a certain system, a customer must first be served by server 1 and then by server 2. The service times at server i are exponential with rate μ_i, $i = 1, 2$. An arrival finding server 1 busy waits in line for that server. Upon completion of service at server 1, a customer either enters service with server 2 if that server is free, or else remains with server 1 (blocking any other customer from entering service) until server 2 is free. Customers depart the system after being served by server 2. Suppose that when you arrive, there is one customer in the system and that customer is being served by server 1. What is the expected total time you spend in the system?

One way to compute this quantity is through the following breakdown:

Expected total time you spend in system =

= Expected ~~total~~ waiting time until server 1 is cleared by the previous customer +

+ Expected time until the next completion either in server 1 (by you) or in server 2 (by the other customer) +

+ Expected remaining time in system after the ~~next~~ completion mentioned in the second term above.

The exponential nature of the proc. times at each server further implies that:

- 1st term above $= 1/\mu_1$
- 2nd " " $= 1/(\mu_1+\mu_2)$

The third term in the above sum can be computed by conditioning on the completion event that was mentioned in the second term. Hence,

3rd term = (Expected remaining time in system given that you finished first) ×
(Prob. that you finished first) +
(Exp. remaining time in system given that the other customer finished first) ×
(Prob that other customer finished first)

$$= \left(\frac{1}{\mu_2} + \frac{1}{\mu_2}\right) \times \frac{\mu_1}{\mu_1+\mu_2} +$$

$$\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right) \times \frac{\mu_2}{\mu_1+\mu_2}$$

Putting everything together, ~~we~~ get:

Expected total time you spend in system =

$$= \frac{1}{\mu_1} + \frac{1}{\mu_1+\mu_2} + \frac{2}{\mu_2}\,\frac{\mu_1}{\mu_1+\mu_2} + \frac{\cancel{\mu_1+\mu_2}}{\mu_1\cancel{\mu_2}}\,\frac{\cancel{\mu_2}}{\cancel{\mu_1+\mu_2}} =$$

$$= \frac{2}{\mu_1} + \frac{1}{\mu_1+\mu_2} + \frac{2\mu_1}{\mu_2(\mu_1+\mu_2)}$$

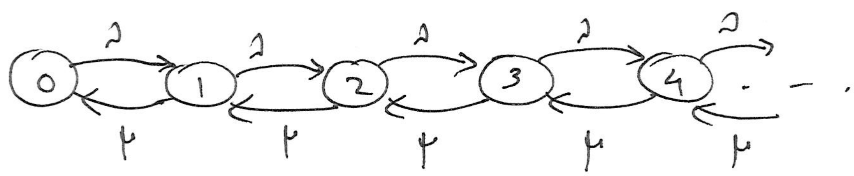
Problem 3 (30 points): Consider a taxi station where taxis and customers arrive in accordance with Poisson processes with respective rates of one and two per minute. A taxi will wait no matter how many other taxis are present. However, an arriving customer that does not find a taxi waiting leaves. Answer the following questions:

i. What is the average number of taxis waiting?

ii. What is the proportion of arriving customers that get taxis?

iii. Provide a CTMC that will describe the dynamics of the aforementioned station if an arriving customer will leave only if the number of waiting customers exceeds a certain number N.

Defining as $\{X_t, t \geq 0\}$ the stochastic process that traces the number of taxis waiting at the station, it should be clear from the above description that this process is a CTMC with the following structure:



That transition rates λ and μ that appear in the above diagram are:

* λ = taxi arrival rate = 1/min
* μ = customer " " = 2/min

Also, it is worth-noticing that the above CTMC has the same structure with the CTMC that models the dynamics of an M/M/1 queue with arrival rate λ and proc. rate μ. Hence, the same line of analysis will give the following results:

a) $\rho = \lambda/\mu = 1/2 < 1 \Rightarrow$ CTMC is ergodic.

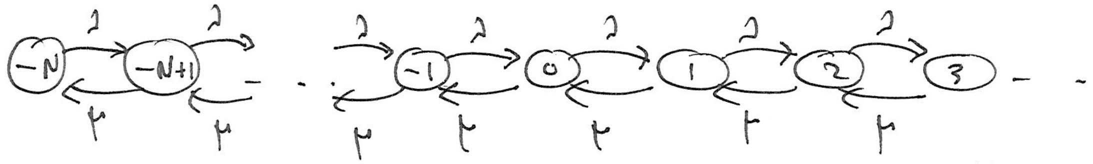
and $\pi_i = (1-\rho)\rho^i$, $i = 0, 1, 2, \ldots$, where π_i define the limiting distribution.

b) Average # of taxis waiting $= \sum_{i=0}^{\infty} i\pi_i = \ldots$

$$\ldots = \frac{\rho}{1-\rho} = \frac{1/2}{1-1/2} = 1 \quad \text{(this answers (i))}$$

(c-ii) An arriving customer will get a taxi as long as the considered CTMC is in a state $X_t \geq 1$. But the ~~......~~ total probability for this class of states is $\rho = 0.5$.

(iii) In this case, we need also to trace the evolution of the customer queue, in our state definition. Recognizing, however, that only one of the two queues (i.e., ~~taxi~~ for customers) can be non-zero at any time point, the necessary extension of the CTMC structure can be done elegantly as follows:



In the above state space, a negative ~~s~~state models the accumulation of customers and a positive one models the accumulation of taxis.

Problem 4 (25 points): Consider two *independent* exponentially distributed random variables X_1 and X_2 with corresponding rates μ_1 and μ_2. Compute $E[X_2|X_2 > X_1]$.

Let $F_{X_2>X_1}(x) = P[X_2 \le x \mid X_2 > X_1] =$

$$= \frac{P[X_1 < X_2 \le x]}{P[X_2 > X_1]} =$$

$$= \frac{\mu_1+\mu_2}{\mu_1} \cdot \int_0^x P[X_1 < X_2 \le x \mid X_2 = y]\, dP(X_2 = y) =$$

$$= \frac{\mu_1+\mu_2}{\mu_1} \int_0^x P[X_1 < y]\, f_{X_2}(y)\, dy$$

where $f_{X_2}(y)$ is the pdf of X_2 (exponential dist. with rate μ_2)

Then,

$$f_{X_2>X_1}(x) = \frac{d F_{X_2>X_1}(x)}{dx} = \frac{\mu_1+\mu_2}{\mu_1} P[X_1 < x]\, f_{X_2}(x) =$$

$$= \frac{\mu_1+\mu_2}{\mu_1} (1 - e^{-\mu_1 x})\, \mu_2 e^{-\mu_2 x}$$

and

$$E[X_2 \mid X_2 > X_1] = \int_0^\infty x f_{X_2>X_1}(x)\, dx = \frac{\mu_2(\mu_1+\mu_2)}{\mu_1} \int_0^\infty x e^{-\mu_2 x}(1 - e^{-\mu_1 x})\, dx$$

$$= \frac{\mu_1+\mu_2}{\mu_1} \int_0^\infty x \mu_2 e^{-\mu_2 x}\, dx - \frac{\mu_2}{\mu_1} \int_0^\infty x(\mu_1+\mu_2) e^{-(\mu_1+\mu_2)x}\, dx$$

$$= \frac{\mu_1+\mu_2}{\mu_1} \frac{1}{\mu_2} - \frac{\mu_2}{\mu_1} \frac{1}{\mu_1+\mu_2} = \frac{1}{\mu_2} + \frac{\mu_2}{\mu_1}\left(\frac{1}{\mu_2} - \frac{1}{\mu_1+\mu_2}\right) =$$

$$= \frac{1}{\mu_2} + \frac{1}{\mu_1}\frac{\mu_1}{\mu_1+\mu_2} = \frac{1}{\mu_2} + \frac{1}{\mu_2+\mu_1}$$

Notice that the above result makes perfect sense, since the second term (i.e., $\frac{1}{\mu_2+\mu_1}$) is the expected time until X_1 expires (this is an "exponential race" between the two exponentials) and the first term (i.e., $\frac{1}{\mu_2}$) is the expected remaining time to the expiration of X_2 (due to its memoryless nature). So, with these insights, the above result could have also been derived while arguing from the fundamental properties of the exponential distribution.