Final Exam Math 468/568 Spring 2016 Name: Solutions

1. In a test paper, the true-false questions are arranged so that 3/4th of the time a True answer is followed by a True, while 2/3th of the time a False answer is followed by a False.

(a) Formulate as a discrete time Markov chain.

(b) If the exam has 100 questions, roughly what fraction of the answers will be True?

(a)
$$\begin{array}{c|cc} & T & F \\ \hline T & \multicolumn{2}{c}{\begin{bmatrix} 3/4 & 1/4 \\ 1/3 & 2/3 \end{bmatrix}} \\ F & & \end{array}$$

(b) $100 \cdot \pi(T) = 100 \cdot 4/7$

$$\tfrac{3}{4}\pi(T) + \tfrac{1}{3}\pi(F) = \pi(T)$$

$$\Rightarrow \tfrac{1}{3}\pi(F) = \tfrac{1}{4}\pi(T).$$

$$\Rightarrow \pi(T) = \tfrac{4}{3}\pi(F), \quad \tfrac{7}{3}\pi(F) = 1 \Rightarrow \begin{array}{l}\pi(F) = 3/7 \\ \pi(T) = 4/7\end{array}$$

2. Consider a discrete time Markov chain with transition matrix with states $\{1, 2, 3\}$:

$$P = \begin{bmatrix} 2/5 & 2/5 & 1/5 \\ 1/8 & 3/4 & 1/8 \\ 1/10 & 1/2 & 2/5 \end{bmatrix}.$$

Starting from state 3, what is the probability of hitting state 1 before state 2?
Hint: Use first-step analysis.

want $P_3(T_1 < T_2) = \frac{1}{10} + \frac{2}{5} P_3(T_1 < T_2)$

$$\Rightarrow \tfrac{3}{5} P_3(T_1 < T_2) = \tfrac{1}{10} \Rightarrow P_3(T_1 < T_2) = \tfrac{5}{3}\cdot\tfrac{1}{10} = \tfrac{1}{6}$$

3. A person catches fish at times of a Poisson process with rate 2 per hour. Forty percent of the fish are salmon, while sixty percent are trout.

What is the chance that exactly one salmon and two trout are caught in 2.5 hours?

Salmon and Trout processes, by Thinning, are independent Poisson processes.

$$P(N_S = 1, N_T = 2) = \left[(2.5)(.4)(2)\, e^{-(2.5)(.4)(2)}\right]$$

$$\times \left[\frac{((2.5)(.6)(2))^2}{2}\, e^{-(2.5)(.6)(2)}\right]$$

4. A person waits at the bus stop but has forgotten the bus schedule. The person guesses that the wait until the next bus is a uniform on $(0, 1)$ hours. Cars drive by the bus stop according to a Poisson process at a rate of 6 per hour. Each will be willing to pick up the person at the bus stop.

What is the probability that the person will take the bus instead of being picked up?

Let T be uniform $(0,1)$ } independent.
X be exponential 6 }

want $P(T < X) = \int_0^1 P(X > s)\, ds$

$$= \int_0^1 e^{-s \cdot 6}\, ds = \frac{-1}{6} e^{-6s} \Big|_0^1$$

$$= \boxed{\frac{1}{6} - \frac{1}{6} e^{-6}}$$

5. Consider two machines maintained by a single repairman. Machine i functions for an exponentially distributed amount of time with rate λ_i. The repair times for each unit are exponential with rate μ_i. They are repaired in the order in which they fail.

(a) Formulate as a continuous time Markov chain with state space $\{0, 1, 2, 12, 21\}$ where the states of the system refer to what has exactly failed and in what order; for instance, 1 means that only machine 1 has failed; also, for instance, 12 means machines 1 and 2 are not working and machine 1 failed first.

(b) Find, in terms of the parameters, $E_{12}[T_0]$ and $E_{21}[T_0]$. Hint: One can solve this as a 2×2 system using jump probabilities and waiting times.

(a)

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 12 & 21 \\ \hline 0 & -(\lambda_1+\lambda_2) & \lambda_1 & \lambda_2 & 0 & 0 \\ 1 & \mu_1 & -(\mu_1+\lambda_2) & 0 & \lambda_2 & 0 \\ 2 & \mu_2 & 0 & -(\mu_2+\lambda_1) & 0 & \lambda_1 \\ 12 & 0 & 0 & \mu_1 & -\mu_1 & 0 \\ 21 & 0 & \mu_2 & 0 & 0 & -\mu_2 \end{array}$$

$$E_{21}[T_0] = \frac{1}{\mu_2} + E_1[T_0] = \frac{1}{\mu_2} + \frac{1}{\mu_1+\lambda_2} + \frac{\lambda_2}{\mu_1+\lambda_2} E_{12}[T_0]$$

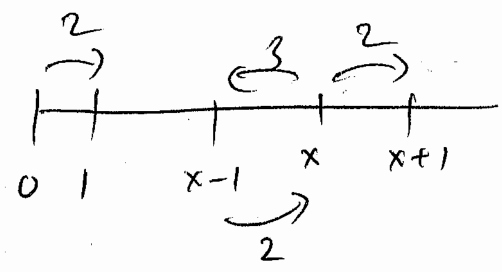
$$E_{12}[T_0] = \frac{1}{\mu_1} + E_2[T_0] = \frac{1}{\mu_1} + \frac{1}{\mu_2+\lambda_1} + \frac{\lambda_1}{\mu_2+\lambda_1} E_{21}[T_0]$$

$$\Rightarrow E_{21}[T_0] = E_{12}[T_0] = 4$$

6. Consider a taxi station at an airport where taxis and customers arrive at times of Poisson processes with rates 2 and 3 per minute. Suppose that a taxi will wait no matter how many other taxis are present. When a customer arrives, the customer departs in the taxi at the front of the queue. However, if an arriving person does not find a taxi waiting, the person will leave to find alternative transportation.

(a) Formulate the queue of taxis as a Birth-Death process.

(b) Find the proportion of arriving customers that get taxis.



$$3\,\pi(x) = 2\,\pi(x-1)$$

$$\Rightarrow \pi(x) = \left(\frac{2}{3}\right)^x \pi(0) \qquad x \geq 0.$$

$$\Rightarrow \pi(0) = \left(\sum_{x=0}^{\infty} \left(\frac{2}{3}\right)^x\right)^{-1} = \left(\frac{1}{1-2/3}\right)^{-1} = \frac{1}{3}.$$

$\Rightarrow$ proportion of arriving customers who see a queue with $x \geq 1$ taxis is $1 - \pi(0) = 2/3$

7. Let $B(\cdot)$ be standard Brownian motion.

Find the distribution of $B(1) + B(2) + B(3)$. Hint: One can use the independent increments property, and that sum of independent Normals is a Normal.

$$B(3) + B(2) + B(1)$$

$$= \underbrace{B(3) - B(2)} + \underbrace{2B(2) - 2B(1)} + \underbrace{3B(1)}$$

Indep Normals!

$$\sim \boxed{N(0, 1+4+9)}$$