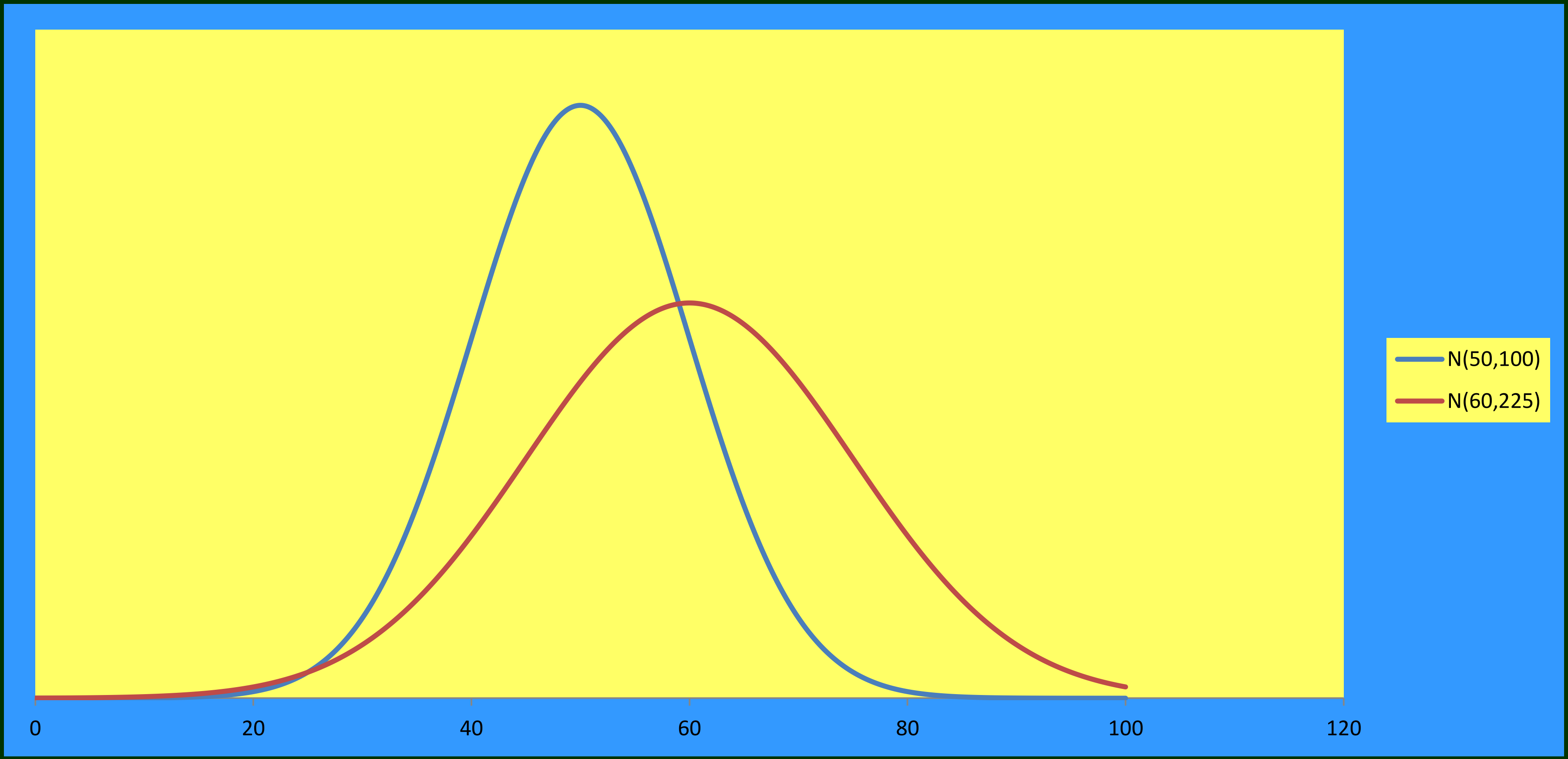
Satterthwaite's Approximation for Degrees of Freedom

Art Instruction Effect on Reading Development

J.C. Mills (1973), "The Effect of Art Instruction Upon a Reading Development Test: An Experimental Study with Rural Appalachian Children," *Studies in Art Education*, Vol. 14, #3, pp.4-8

Setting

- Comparing Means from 2 Normal Distributions
- Small Samples (Computer Packages Solve for any sizes)
- Distributions have Possibly Different Variances



Case 1 – Variances are Equal

$$Y_{11},\ldots,Y_{1n_1} \sim N\left(\mu_1,\sigma_1^2\right) \quad Y_{21},\ldots,Y_{2n_2} \sim N\left(\mu_2,\sigma_2^2\right) \quad \sigma_1^2=\sigma_2^2=\sigma^2 \quad \{Y_1\}\perp\{Y_2\}$$

$$\Rightarrow \overline{Y}_i = \frac{\sum_{j=1}^{n_i} Y_{ij}}{n_i} \quad S_i^2 = \frac{\sum_{j=1}^{n_i}\left(Y_{ij}-\overline{Y}_i\right)^2}{n_i-1} \quad i=1,2 \qquad S_p^2 = \frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}$$

$$\overline{Y}_i \sim N\left(\mu_i,\frac{\sigma^2}{n_i}\right) \qquad \frac{(n_i-1)S_i^2}{\sigma^2} \sim \chi^2_{n_i-1}$$

$$Z = \frac{\left(\overline{Y}_1-\overline{Y}_2\right)-(\mu_1-\mu_2)}{\sqrt{\sigma^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} \sim N(0,1) \qquad W = \frac{(n_1+n_2-2)S_p^2}{\sigma^2} \sim \chi^2_{n_1+n_2-2} \quad Z\perp W$$

$$\sqrt{\frac{W}{df_W}} = \sqrt{\frac{\left[\frac{(n_1+n_2-2)S_p^2}{\sigma^2}\right]}{n_1+n_2-2}} = \sqrt{\frac{S_p^2}{\sigma^2}} \qquad \frac{Z}{\sqrt{\frac{W}{df_W}}} = \frac{\left[\frac{\left(\overline{Y}_1-\overline{Y}_2\right)-(\mu_1-\mu_2)}{\sqrt{\sigma^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}\right]}{\sqrt{\frac{S_p^2}{\sigma^2}}} = \frac{\left(\overline{Y}_1-\overline{Y}_2\right)-(\mu_1-\mu_2)}{\sqrt{S_p^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} \sim t_{n_1+n_2-2}$$

$$\Rightarrow P\left(-t_{\alpha/2,n_1+n_2-2} \le \frac{\left(\overline{Y}_1-\overline{Y}_2\right)-(\mu_1-\mu_2)}{\sqrt{S_p^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} \le t_{\alpha/2,n_1+n_2-2}\right) = 1-\alpha$$

Case 2 – Variances are Unequal - I

$$Y_{11},\ldots,Y_{1n_1} \sim N\left(\mu_1,\sigma_1^2\right) \quad Y_{21},\ldots,Y_{2n_2} \sim N\left(\mu_2,\sigma_2^2\right) \quad \sigma_1^2 \neq \sigma_2^2 \quad \{Y_1\} \perp \{Y_2\}$$

$$\frac{(n_i-1)S_i^2}{\sigma_i^2} \sim \chi^2_{n_i-1} \quad \Rightarrow \quad E\left[\frac{(n_i-1)S_i^2}{\sigma_i^2}\right] = n_i - 1, \quad V\left[\frac{(n_i-1)S_i^2}{\sigma_i^2}\right] = 2(n_i-1) \quad \Rightarrow \quad E\left(S_i^2\right) = \sigma_i^2, V\left(S_i^2\right) = \frac{2\sigma_i^4}{n_i-1}$$

$$Z = \frac{\left(\overline{Y}_1 - \overline{Y}_2\right) - (\mu_1 - \mu_2)}{\sqrt{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)}} \sim N(0,1)$$

Problem: Replacing Denominator with estimated variances, consider:

$$\frac{W^*}{df_{W^*}} = \frac{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)}{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)}$$ which is NOT a chi-square divided by its degrees of freedom.

Aside: $W \sim \chi^2_\nu \quad \Rightarrow \quad E(W) = \nu, V(W) = 2\nu \quad \Rightarrow \quad E\left(\frac{W}{\nu}\right) = 1, V\left(\frac{W}{\nu}\right) = \frac{2}{\nu}$

$$E\left(\frac{W^*}{df_{W^*}}\right) = E\left(\frac{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)}{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)}\right) = \frac{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)}{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)} = 1 \quad V\left(\frac{W^*}{df_{W^*}}\right) = \frac{1}{\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)^2}\left\{\frac{1}{n_1^2}\frac{2\sigma_1^4}{n_1-1} + \frac{1}{n_2^2}\frac{2\sigma_2^4}{n_2-1}\right\} = \frac{2}{\nu^*}$$

Case 2 – Variances are Unequal - II

$$V\left(\frac{W^*}{df_{W^*}}\right)=\frac{1}{\left(\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}\right)^2}\left\{\frac{1}{n_1^2}\frac{2\sigma_1^4}{n_1-1}+\frac{1}{n_2^2}\frac{2\sigma_2^4}{n_2-1}\right\}=\frac{2}{\nu^*}\quad\Rightarrow$$

$$\frac{2}{\nu^*}=\frac{2\left(\frac{\sigma_1^4}{n_1^2(n_1-1)}+\frac{\sigma_2^4}{n_2^2(n_2-1)}\right)}{\left(\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}\right)^2}\quad\Rightarrow\quad\nu^*=\frac{\left(\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}\right)^2}{\left(\frac{\left(\sigma_1^2/n_1\right)^2}{n_1-1}+\frac{\left(\sigma_2^2/n_2\right)^2}{n_2-1}\right)}$$

Replacing the unknown variances with their estimates:

$$\hat{\nu}^*=\frac{\left(\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}\right)^2}{\left(\frac{\left(S_1^2/n_1\right)^2}{n_1-1}+\frac{\left(S_2^2/n_2\right)^2}{n_2-1}\right)}=\frac{\left(\sum_{i=1}^{2}g_iMS_i\right)^2}{\sum_{i=1}^{2}\frac{\left(g_iMS_i\right)^2}{\nu_i}}\quad\text{where: } g_i=\frac{1}{n_i}\quad MS_i=S_i^2\quad \nu_i=n_i-1$$

So, we have the approximate degrees of freedom if our denominator were the square root of the ratio of a chi-square to its degrees of freedom

Example – Art Instruction Effect on Reading

- Experiment to Determine Effect of Art Instruction on a Reading Development Test
  - 52 Children Given Baseline Reading Test
  - 26 Received Art Instruction (Trt), 26 Did not (Control)
  - Y=Post-Test – Pre-Test Score

$$\overline{Y}_T = 7.77 \quad S_T^2 = 70.49 \quad n_T = 26$$

$$\overline{Y}_C = -1.58 \quad S_C^2 = 26.00 \quad n_C = 26$$

$$H_0: \sigma_T^2 = \sigma_C^2 \quad H_A: \sigma_T^2 \neq \sigma_C^2 \quad T.S.: F_{obs} = \frac{S_T^2}{S_C^2} = 2.71 \quad RR: \max\left(F_{obs}, 1/F_{obs}\right) \geq F_{.025,25,25} = 2.23$$

$$H_0: \mu_T = \mu_C \quad H_A: \mu_T \neq \mu_C \quad T.S.: t_{obs} = \frac{\overline{Y}_T - \overline{Y}_C}{\sqrt{\frac{S_T^2}{n_T} + \frac{S_C^2}{n_C}}} = \frac{7.77-(-1.58)}{\sqrt{\frac{70.49}{26} + \frac{26.00}{26}}} = 4.85$$

$$\hat{\nu}^* = \frac{\left(\frac{70.49}{26} + \frac{26}{26}\right)^2}{\left(\frac{(70.49/26)^2}{25} + \frac{(26.00/26)^2}{25}\right)} = \frac{13.77}{0.33} = 41.23 \qquad RR: |t_{obs}| \geq t_{.025,41.23} = 2.020$$