The t Test

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Again, we begin with independent normal observations $X_1 \ldots, X_n$ with unknown mean μ and unknown variance σ^2. The likelihood function

$$L(\mu, \sigma^2|\mathbf{x}) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp -\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu)^2$$

$$\ln L(\mu, \sigma^2|\mathbf{x}) = -\frac{n}{2}(\ln 2\pi + \ln \sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu)^2$$

$$\frac{\partial}{\partial \mu} \ln L(\mu, \sigma^2|\mathbf{x}) = -\frac{1}{\sigma^2}\sum_{i=1}^n (x_i - \mu)$$

Thus, $\hat{\mu} = \bar{x}$.

$$\frac{\partial}{\partial \sigma^2} \ln L(\mu, \sigma^2|\mathbf{x}) = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^n (x_i - \mu)^2.$$

Thus,

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2.$$

For the hypothesis

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0,$$

the **likelihood ratio test**

$$\Lambda(x) = \frac{L(\mu_0, \hat{\sigma}_0^2|\mathbf{x})}{L(\hat{\mu}, \hat{\sigma}^2|\mathbf{x})}$$

where the value

$$\hat{\sigma}_0^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \mu_0)^2$$

gives the maximum likelihood on the set $\mu = \mu_0$.

$$L(\mu_0, \hat{\sigma}_0^2|\mathbf{x}) = \frac{1}{(2\pi\hat{\sigma}_0^2)^{n/2}} \exp -\frac{1}{2\hat{\sigma}_0^2}\sum_{i=1}^n (x_i - \mu_0)^2 = \frac{1}{(2\pi\hat{\sigma}_0^2)^{n/2}} \exp -\frac{2}{n},$$

$$L(\hat{\mu}, \hat{\sigma}^2|\mathbf{x})) = \frac{1}{(2\pi\hat{\sigma}^2)^{n/2}} \exp -\frac{1}{2\hat{\sigma}^2}\sum_{i=1}^n (x_i - \bar{x})^2 = \frac{1}{(2\pi\hat{\sigma}^2)^{n/2}}, \exp -\frac{2}{n},$$

and

$$\Lambda(x) = \left(\frac{\hat{\sigma}^2}{\hat{\sigma}_0^2}\right)^{n/2} = \left(\frac{\sum_{i=1}^n (x_i - \mu_0)^2}{\sum_{i=1}^n (x_i - \bar{x})^2}\right)^{-n/2}$$

The critical region $\lambda)\mathbf{x}) \leq \lambda_0$ is equivalent to

$$c \leq \frac{\sum_{i=1}^n (x_i - \mu_0)^2}{\sum_{i=1}^n (x_i - \bar{x})^2} = 1 + \frac{n(\bar{x} - \mu_0)^2}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

or

$$T(x)^2 \geq (c-1)(n-1)$$

where

$$T(x) = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

and we write s for the square root of the *unbiased* estimator of the variance

$$s^2 = \frac{1}{n-1}\sum_{i=1}^n (x_i - \bar{x})^2.$$

1 Deriving the t distribution

For the hypothesis test, our next goal is to understand the distribution of $T(X)$.

Step 1. $\sqrt{n}(\bar{X} - \mu_0)/\sigma$ is a standard normal random variable.

For this, notice that $\bar{X}$ is a normal random variable with mean μ_0 and standard deviation $\sigma/\sqrt{n}$

Step 2. For each i, $X_i - \bar{X}$ and $\bar{X}$ are independent.

For normal random variables, uncorrelated random variables are independent. Thus, it suffices to show that the covariance is 0. To that end, note that

$$\text{Cov}(X_i - \bar{X}, \bar{X}) = \text{Cov}(X_i, \bar{X}) - \text{Cov}(\bar{X}, \bar{X}).$$

For the first term, use the fact that $\text{Cov}(X_1, X_j) = 0$ if $i \neq j$ and $\text{Cov}(X_1, X_i) = \text{Var}(X_i) = \sigma^2$. Then,

$$\text{Cov}(X_i, \bar{X}) = \frac{1}{n}\sum_{j=1}^n \text{Cov}(X_i, X_j) = \frac{1}{n}\sigma^2.$$

From Step 1, we know that

$$\text{Cov}(\bar{X}, \bar{X}) = \text{Var}(\bar{X}) = \frac{1}{n}\sigma^2.$$

Now combine to see that $\text{Cov}(X_i - \bar{X}, \bar{X}) = 0$.

Step 3. $\sum_{i=1}^n (X_i - \bar{X})^2/\sigma^2$ is a χ-square random variable with $n-1$ degrees of freedom.

Let $Z_i = (X_i - \mu)/\sigma$ and $\bar{Z}$ be the average of the Z_i. Then Z_i are independent standard normal random variables.

$$\frac{1}{\sigma^2}\sum_{i=1}^n (X_i - \bar{X})^2 = \sum_{i=1}^n (Z_i - \bar{Z})^2 = \sum_{i=1}^n Z_i^2 - n\bar{Z}^2$$

or

$$\sum_{i=1}^n Z_i^2 = \sum_{i=1}^n (Z_i - \bar{Z})^2 + n\bar{Z}^2.$$

Let's write this

$$Y = U + V.$$

By step 2, the sum is of independent random variables. So, if we use the properties of moment generating functions

$$M_Y(r) = M_U(r) \cdot M_V(r).$$

Now Y is a χ^2_n random variable. So, $M_Y(r) = (1-2r)^{-n/2}$. V is a χ^2_1 random variable. So, $M_Y(r) = (1-2r)^{1/2}$. Consequently,

$$M_U(r) = \frac{M_Y(r)}{M_V(r)} = \frac{(1-2r)^{-n/2}}{(1-2r)^{-1/2}} = (1-2r)^{-(n-1)/2}$$

and U is a χ^2_{n-1} random variable.

In summary, we can write

$$T = \frac{Z}{\sqrt{U/(n-1)}}$$

where Z is a standard random variable, U is a χ^2_{n-1} random variable, and Z and U are independent. Consequently, their densities are

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad \text{and} \quad f_U(u) = \frac{u^{n/2-3/2}}{2^{(n-1)/2}\Gamma((n-1)/2)} e^{-u/2}.$$

Step 4. Finding the density of T, $f_T(t)$.

Z and U have joint density

$$f_{Z,U}(z,u) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \frac{u^{n/2-3/2}}{2^{(n-1)/2}\Gamma((n-1)/2)} e^{-u/2}.$$

Define the one to one transformation

$$t = \frac{z}{\sqrt{u/(n-1)}} \quad \text{and} \quad v = u.$$

Then, the inverse transformation

$$z = \frac{t\sqrt{v}}{\sqrt{n-1}} \quad \text{and} \quad u = v.$$

the joint density

$$f_{T,V}(t,v) = f_{Z,U}(\frac{t\sqrt{v}}{\sqrt{n-1}}, v)|J(t,v)|.$$

where $|J(z,u)|$ is the absolute value of the Jacobian of the inverse transformation.

In this case,

$$J(t,v) = \det \begin{bmatrix} \partial z/\partial t & \partial z/\partial v \\ \partial u/\partial t & \partial u/\partial v \end{bmatrix} = \det \begin{bmatrix} \sqrt{v}/\sqrt{n-1} & t/(2\sqrt{v(n-1)} \\ 0 & 1 \end{bmatrix} = \frac{\sqrt{v}}{\sqrt{n-1}}.$$

Then,

$$f_{T,V}(t,v) = \frac{1}{\sqrt{2\pi}2^{(n-1)/2}\Gamma((n-1)/2)} v^{n/2-3/2} \exp\left(-\frac{v}{2}\left(1+\frac{t^2}{n-1}\right)\right) \frac{\sqrt{v}}{\sqrt{n-1}}$$

Finally, to find the marginal density for T, we integrate with respect to v to obtain

$$f_T(t) = \frac{1}{\sqrt{2\pi}2^{(n-1)/2}\Gamma((n-1)/2)\sqrt{n-1}} \int_0^\infty v^{n/2-1} \exp\left(-\frac{v}{2}\left(1+\frac{t^2}{n-1}\right)\right)\, dv.$$

Change variables by setting $w = v(1+t^2/(n-1))/2$.

$$\begin{aligned} f_T(t) &= \frac{1}{\sqrt{2\pi}2^{(n-1)/2}\Gamma((n-1)/2)\sqrt{n-1}} \int_0^\infty \left(\frac{2w}{1+t^2/(n-1)}\right)^{n/2-1} e^{-w} \left(\frac{2}{1+t^2/(n-1)}\right)\, dw \\ &= \frac{1}{\sqrt{\pi(n-1)}\Gamma((n-1)/2)} \left(1+\frac{t^2}{n-1}\right)^{-n/2} \int_0^\infty w^{n/2-1}e^{-w}\, dw \\ &= \frac{\Gamma(n/2)}{\sqrt{\pi(n-1)}\Gamma((n-1)/2)} \left(1+\frac{t^2}{n-1}\right)^{-n/2}. \end{aligned}$$

Note that

$$\left(1+\frac{t^2}{n-1}\right)^{-n/2} \to \exp -\frac{t^2}{2}$$

and $n \to \infty$. Thus, for large n, the t density is very close to the density of a standard normal.

2 Tests and confidence intervals using t distribution

We can now return to our original hypothesis. The critical region for an alpha level test is determined by the extreme values of the t statistic.

$$C = \{\mathbf{x}; |T(\mathbf{x})| \geq t_{\alpha/2}\}$$

where

$$P\{T > t_{\alpha/2}\} = \frac{\alpha}{2}.$$

Correspondingly, the γ-level confidence interval is obtained by choosing $\alpha = 1-\gamma$ and setting

$$\bar{x} \pm \frac{s}{\sqrt{n}} t_{\alpha/2}.$$

For a one-sided hypothesis

$$H_0 : \mu \leq \mu_0 \quad \text{versus} \quad H_1 : \mu \geq \mu_0,$$

the critical region becomes

$$C = \{\mathbf{x}; T(\mathbf{x}) \geq t_\alpha\}$$

3 Two sample t test

We now have a three dimensional parameter space $\Theta = \{(\mu_1, \mu_2, \sigma^2); \mu_1 \in \mathbb{R}, \mu_2 \in \mathbb{R}, \sigma^2 \in \mathbb{R}\}$..

The two-sided test is

$$H_0 : \mu_1 = \mu_2 \quad \text{versus} \quad H_1 : \mu_1 \neq \mu_2,$$

The data $X_{1,j}, \ldots, X_{n_j,j}$ are independent $N(\mu_j, \sigma^2)$ random variables, $j = 1, 2$.

The likelihood function is

$$L(\mu_1, \mu_2, \sigma^2|\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{(2\pi\sigma^2)^{(n_1+n_2)/2}} \exp -\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n_1}(x_{i,1} - \mu_1)^2 + \sum_{i=1}^{n_2}(x_{i,2} - \mu_2)^2\right).$$

Then, the likelihood ratio,

$$\Lambda(\mathbf{x}_1, \mathbf{x}_2)) = \frac{L(\hat{\mu}, \hat{\mu}, \hat{\sigma}^2_{12}|\mathbf{x}_1, \mathbf{x}_2)}{L(\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}^2|\mathbf{x}_1, \mathbf{x}_2)}$$

Write the unbiased estimators for the mean and the variance

$$\bar{x}_j = \frac{1}{n_j}\sum_{i=1}^{n_j} x_{i,j} \quad \text{and} \quad s_j^2 = \frac{1}{n_j - 1}\sum_{i=1}^{n_j}(x_{i,j} - \bar{x}_j)^2, \quad j = 1, 2.$$

Then, the overall maximum likelihood estimators are

$$\hat{\mu}_1 = \bar{x}_1 \quad \hat{\mu}_2 = \bar{x}_2, \quad \hat{\sigma}^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2}.$$

The maximum likelihood estimators in the numerator takes place on the set $\mu_1 = \mu_2$.

$$\hat{\mu} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}, \quad \hat{\sigma}^2_{12} = \frac{1}{n_1 + n_2}\left(\sum_{i=1}^{n_1}(x_{i,1} - \hat{\mu})^2 + \sum_{i=1}^{n_2}(x_{i,2} - \hat{\mu})^2\right).$$

The yields the test statistic

$$T(\mathbf{x}_1, \mathbf{x}_2) = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\left(\frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}\right)}}$$

from the fact that the likelihood ratio

$$\Lambda(\mathbf{x}_1, \mathbf{x}_2) = \left(\frac{n_1 + n_2 - 2}{n_1 + n_2 - 2 + T(\mathbf{x}_1, \mathbf{x}_2)^2}\right)^{2/(n_1+n_2)}.$$

Under the null hypothesis, $T(\mathbf{x}_1, \mathbf{x}_2)$ has a t distribution with $n_1 + n_2 - 2$ degrees of freedom. Now, we can proceed as before in the single sample t test in designing the test and constructing the confidence interval. The case in which the variances are equal is called the **pooled two-sample t test.**.

If the relationship between the variance from the two samples is unknown, then the commonly used approach is to set

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}.$$

and use the formula

$$df = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)}{\frac{1}{n_1-1}\left(\frac{s_1^2}{n_1}\right)^2 + \frac{1}{n_2-1}\left(\frac{s_2^2}{n_2}\right)^2}$$

for the degrees of freedom.