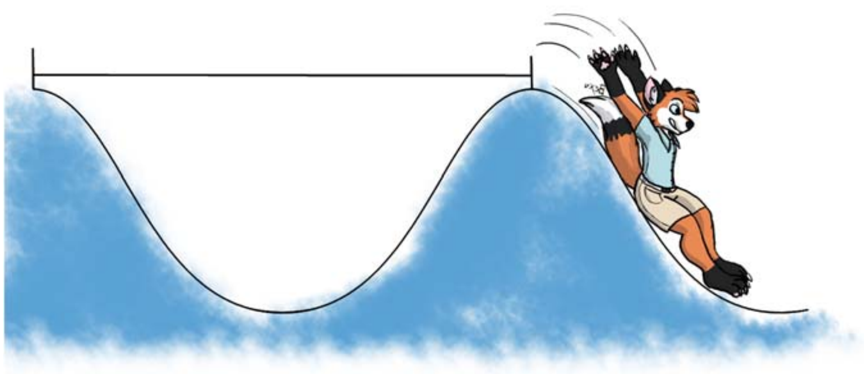
Math Handbook

of Formulas, Processes and Tricks

(www.mathguy.us)

Trigonometry



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Note to Students

This Trigonometry Handbook was developed primarily through work with a number of High School and College Trigonometry classes. In addition, a number of more advanced topics have been added to the handbook to whet the student's appetite for higher level study.

One of the main reasons why I wrote this handbook was to encourage the student to wonder; to ask "what about …" or "what if …". I find that students are so busy today that they don't have the time, or don't take the time, to seek out the beauty and majesty that exists in Mathematics. And, it is there, just below the surface. So be curious and go find it.

The answers to most of the questions below are inside this handbook, but are seldom taught.

- Is there a method I can learn that will help me recall the key points on a unit circle without memorizing the unit circle?
- What's the fastest way to graph a Trig function?
- Can I convert the sum of two trig functions to a product of trig functions? How about the other way around, changing a product to a sum?
- Is there an easy way to calculate the area of a triangle if I am given its vertices as points on a Cartesian plane?
- Don't some of the Polar graphs in Chapter 9 look like they have been drawn with a Spirograph? Why is that?
- A cycloid is both a *brachistochrone* and a *tautochrone*. What are these and why are they important? (you will have to look this one up, but it is well worth your time)
- What is a vector cross product and how is it used?
- How do the properties of vectors extend to 3 dimensions, where they really matter?

Additionally, ask yourself:

- What trig identities can I create that I have not yet seen?
- What Polar graphs can I create by messing with trig functions? What makes a pretty graph instead of one that just looks messed up?
- Can I come up with a simpler method of doing things than I am being taught?
- What problems can I come up with to stump my friends?

Those who approach math in this manner will be tomorrow's leaders. Are you one of them?

Please feel free to contact me at mathguy.us@gmail.com if you have any questions or comments.

Thank you and best wishes!
Earl

Cover art by Rebecca Williams,
Twitter handle: @jolteonkitty

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Mathguy.us – Developed specifically for math students from Middle School to College, based on the author's extensive experience in professional mathematics in a business setting and in math tutoring. Contains free downloadable handbooks, PC Apps, sample tests, and more.

www.mathguy.us

Wolfram Math World – Perhaps the premier site for mathematics on the Web. This site contains definitions, explanations and examples for elementary and advanced math topics.

mathworld.wolfram.com

Khan Academy – Supplies a free online collection of thousands of micro lectures via YouTube on numerous topics. It's math and science libraries are extensive.

www.khanacademy.org

Analyze Math Trigonometry – Contains free Trigonometry tutorials and problems. Uses Java applets to explore important topics interactively.

www.analyzemath.com/Trigonometry.html

Schaum's Outline

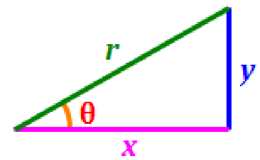
An important student resource for any high school or college math student is a Schaum's Outline. Each book in this series provides explanations of the various topics in the course and a substantial number of problems for the student to try. Many of the problems are worked out in the book, so the student can see examples of how they should be solved.

Schaum's Outlines are available at Amazon.com, Barnes & Noble and other booksellers.

Introduction

What is Trigonometry?

The word "Trigonometry" comes from the Greek "trigonon" (meaning triangle) and "metron" (meaning measure). So, simply put, Trigonometry is the study of the measures of triangles. This includes the lengths of the sides, the measures of the angles and the relationships between the sides and angles.

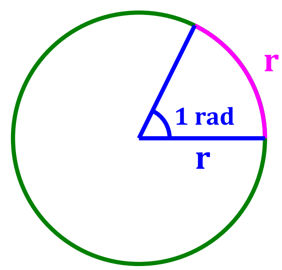


The modern approach to Trigonometry also deals with how right triangles interact with circles, especially the Unit Circle, i.e., a circle of radius 1. Although the basic concepts are simple, the applications of Trigonometry are far reaching, from cutting the required angles in kitchen tiles to determining the optimal trajectory for a rocket to reach the outer planets.

Radians and Degrees

Angles in Trigonometry can be measured in either radians or degrees:

- There are 360 degrees (i.e., $360°$) in one rotation around a circle. Although there are various accounts of how a circle came to have 360 degrees, most of these are based on the fact that early civilizations considered a complete year to have 360 days.
- There are 2π (~ 6.283) radians in one rotation around a circle. The ancient Greeks defined π to be the ratio of the circumference of a circle to its diameter (i.e., $\pi = \frac{C}{d}$). Since the diameter is double the radius, the circumference is 2π times the radius (i.e., $C = 2\pi r$). One radian is the measure of the angle made from wrapping the radius of a circle along the circle's exterior.

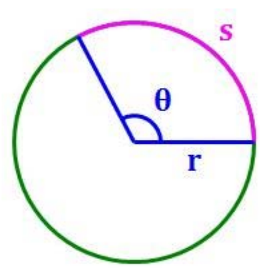


Measure of an Arc

One of the simplest and most basic formulas in Trigonometry provides the measure of an arc in terms of the radius of the circle, r, and the arc's central angle θ, expressed in radians. The formula is easily derived from the portion of the circumference subtended by θ.

Since there are 2π radians in one full rotation around the circle, the measure of an arc with central angle θ, expressed in radians, is:

$$S = C \cdot \left(\frac{\theta}{2\pi}\right) = 2\pi r \cdot \left(\frac{\theta}{2\pi}\right) = r\theta \qquad \text{so} \qquad \boldsymbol{S = r\theta}$$

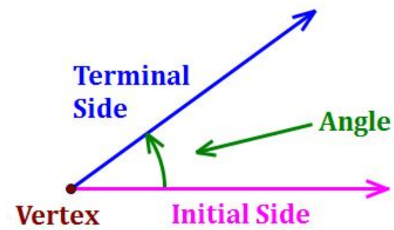


Angle Definitions

Basic Definitions

A few definitions relating to angles are useful when beginning the study of Trigonometry.

Angle: A measure of the space between rays with a common endpoint. An angle is typically measured by the amount of rotation required to get from its initial side to its terminal side.



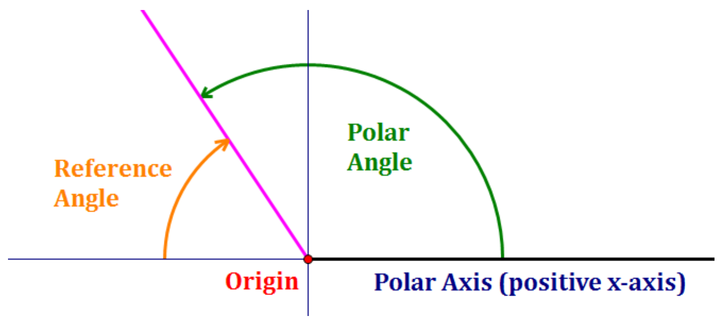
Initial Side: The side of an angle from which its rotational measure begins.

Terminal Side: The side of an angle at which its rotational measure ends.

Vertex: The vertex of an angle is the common endpoint of the two rays that define the angle.

Definitions in the Cartesian (xy) Plane

When angles are graphed on a coordinate system (Rectangular or Polar), a number of additional terms are useful.



Standard Position: An angle is in standard position if its vertex is the origin (i.e., the point $(0, 0)$) and its initial side is the positive x-axis.

Polar Axis: The Polar Axis is the positive x-axis. It is the initial side of all angles in standard position.

Polar Angle: For an angle in standard position, its polar angle is the angle measured from the polar axis to its terminal side. If measured in a counter-clockwise direction, the polar angle is positive; if measured in a clockwise direction, the polar angle is negative.

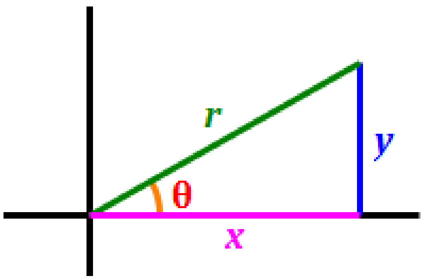
Reference Angle: For an angle in standard position, its reference angle is the angle between $0°$ and $90°$ measured from the x-axis (positive or negative) to its terminal side. The reference angle can be $0°$; it can be $90°$; it is never negative.

Coterminal Angle: Two angles are coterminal if they are in standard position and have the same terminal side. For example, angles of measure $50°$ and $410°$ are coterminal because $410°$ is one full rotation around the circle (i.e., $360°$), plus $50°$, so they have the same terminal side.

Quadrantal Angle: An angle in standard position is a quadrantal angle if its terminal side lies on either the x-axis or the y-axis.

Trigonometric Functions

Trigonometric Functions (on the x- and y-axes)



$$\sin\theta = \frac{y}{r} \qquad \sin\theta = \frac{1}{\csc\theta}$$

$$\cos\theta = \frac{x}{r} \qquad \cos\theta = \frac{1}{\sec\theta}$$

$$\tan\theta = \frac{y}{x} \qquad \tan\theta = \frac{1}{\cot\theta} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{x}{y} \qquad \cot\theta = \frac{1}{\tan\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{r}{x} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\csc\theta = \frac{r}{y} \qquad \csc\theta = \frac{1}{\sin\theta}$$

Pythagorean Identities

(for any angle θ)

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sec^2\theta = 1 + \tan^2\theta$$

$$\csc^2\theta = 1 + \cot^2\theta$$

Sine-Cosine Relationship

$$\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$$

$$\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$$

Key Angles

($180° = \pi$ radians)

$$0° = 0 \text{ radians}$$

$$30° = \frac{\pi}{6} \text{ radians}$$

$$45° = \frac{\pi}{4} \text{ radians}$$

$$60° = \frac{\pi}{3} \text{ radians}$$

$$90° = \frac{\pi}{2} \text{ radians}$$

Cofunctions (in Quadrant I)

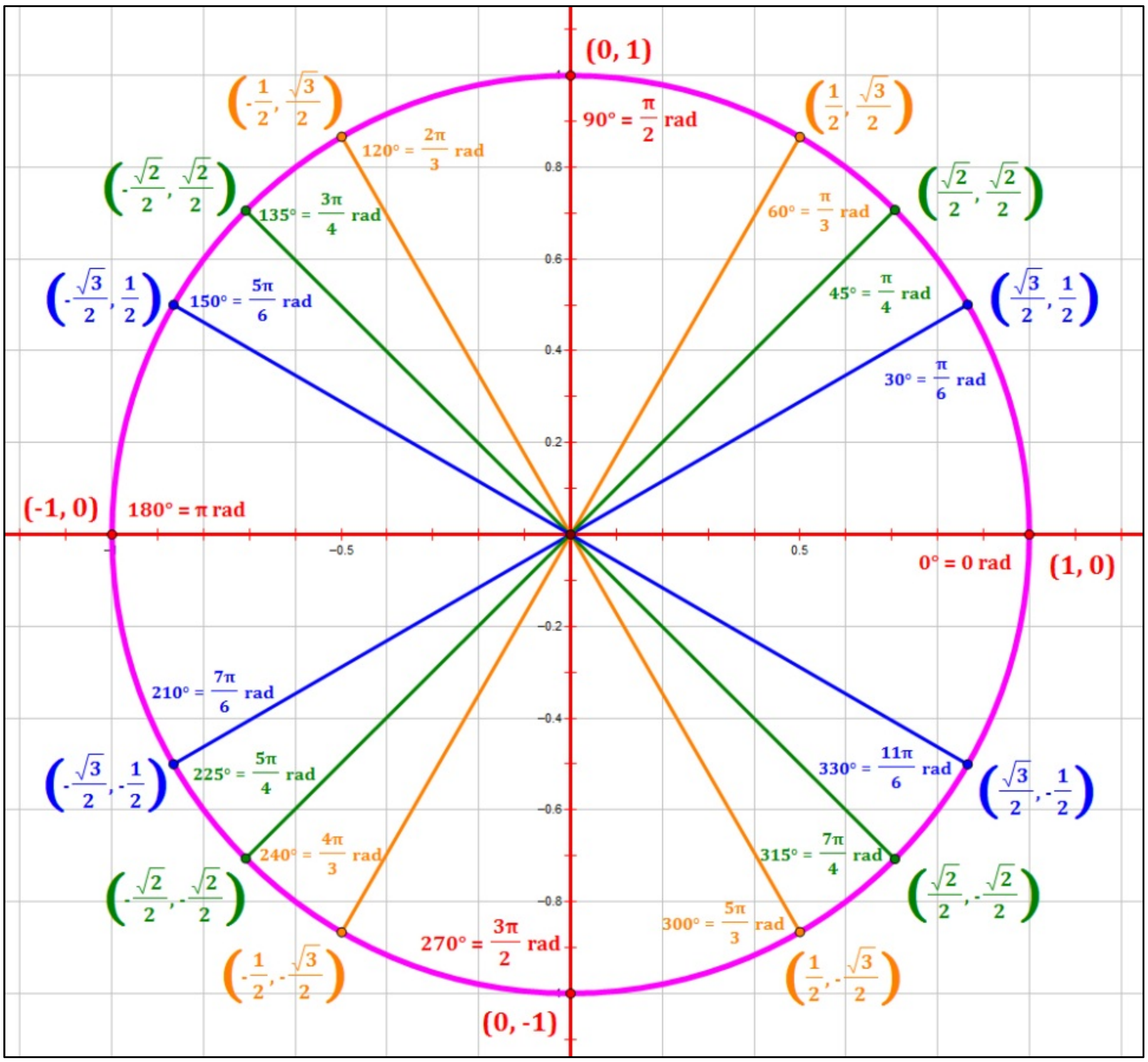
$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right) \quad \Leftrightarrow \quad \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right) \quad \Leftrightarrow \quad \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$

$$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right) \quad \Leftrightarrow \quad \csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$$

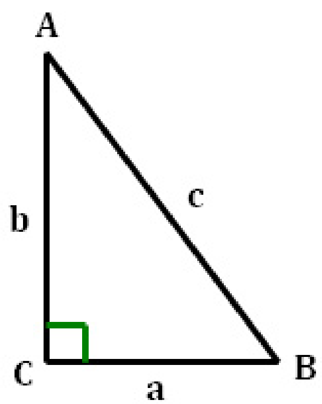
The Unit Circle

The Unit Circle diagram below provides x- and y-values on a circle of radius 1 at key angles. At any point on the unit circle, the x-coordinate is equal to the cosine of the angle and the y-coordinate is equal to the sine of the angle. Using this diagram, it is easy to identify the sines and cosines of angles that recur frequently in the study of Trigonometry.



Trigonometric Functions and Special Angles

Trigonometric Functions (Right Triangle)



SOH-CAH-TOA

$\sin = \dfrac{opposite}{hypoteneuse}$ $\qquad \sin A = \frac{a}{c}$ $\qquad \sin B = \frac{b}{c}$

$\cos = \dfrac{adjacent}{hypoteneuse}$ $\qquad \cos A = \frac{b}{c}$ $\qquad \cos B = \frac{a}{c}$

$\tan = \dfrac{opposite}{adjacent}$ $\qquad \tan A = \frac{a}{b}$ $\qquad \tan B = \frac{b}{a}$

Special Angles

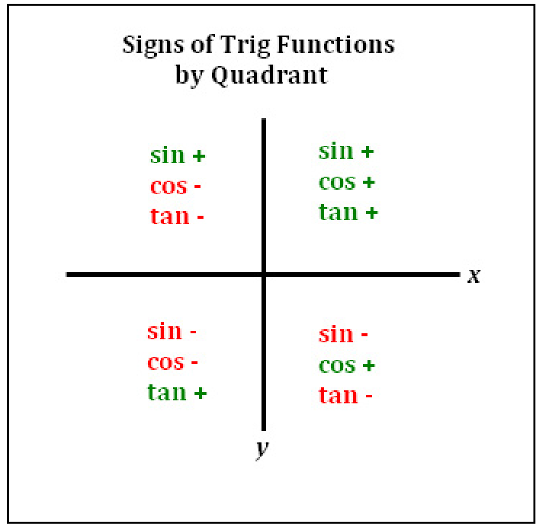
Trig Functions of Special Angles (θ)				
Radians	**Degrees**	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	**0°**	$\frac{\sqrt{0}}{2}=0$	$\frac{\sqrt{4}}{2}=1$	$\frac{\sqrt{0}}{\sqrt{4}}=0$
$\pi/6$	**30°**	$\frac{\sqrt{1}}{2}=\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{\sqrt{3}}=\frac{\sqrt{3}}{3}$
$\pi/4$	**45°**	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{\sqrt{2}}=1$
$\pi/3$	**60°**	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{2}=\frac{1}{2}$	$\frac{\sqrt{3}}{\sqrt{1}}=\sqrt{3}$
$\pi/2$	**90°**	$\frac{\sqrt{4}}{2}=1$	$\frac{\sqrt{0}}{2}=0$	***undefined***

Note the patterns in the above table: In the sine column, the numbers 0 to 4 occur in sequence under the radical! The cosine column is the sine column reversed. Tangent = sine ÷ cosine.

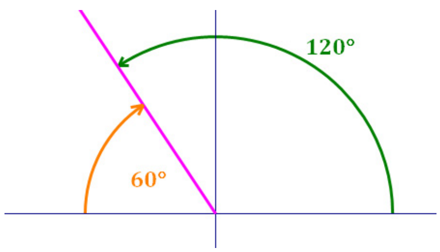
Trigonometric Function Values in Quadrants II, III, and IV

In quadrants other than Quadrant I, trigonometric values for angles are calculated in the following manner:

- Draw the angle $\boldsymbol{\theta}$ on the Cartesian Plane.
- Calculate the measure of the reference angle from the x-axis to $\boldsymbol{\theta}$.
- Find the value of the trigonometric function of the angle in the previous step.
- Assign a "+" or "–" sign to the trigonometric value based on the function used and the quadrant $\boldsymbol{\theta}$ is in (from the table at right).



Examples:



***θ* in Quadrant II** – Calculate: $(180^0 - m\angle\theta)$

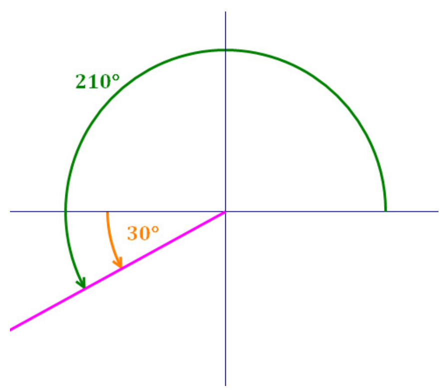
For $\theta = 120^0$, the reference angle is $180° - 120° = 60°$

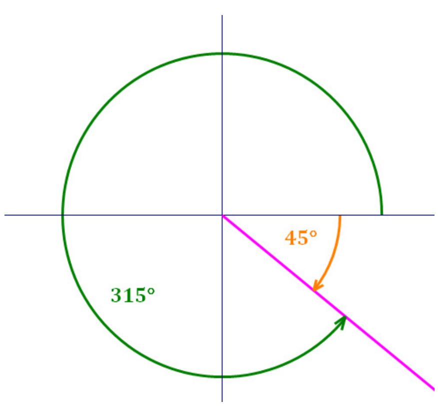
$\sin 60° = \frac{\sqrt{3}}{2}$, so: $\mathbf{\sin 120° = \frac{\sqrt{3}}{2}}$

***θ* in Quadrant III** – Calculate: $(m\angle\theta - 180^0)$

For $\theta = 210^0$, the reference angle is $210° - 180° = 30°$

$\cos 30° = \frac{\sqrt{3}}{2}$, so: $\mathbf{\cos 210° = -\frac{\sqrt{3}}{2}}$





***θ* in Quadrant IV** – Calculate: $(360^0 - m\angle\theta)$

For $\theta = 315^0$, the reference angle is $360° - 315° = 45°$

$\tan 45° = 1$, so: $\mathbf{\tan 315° = -1}$

Problems Involving Trig Function Values in Quadrants II, III, and IV

A typical problem in Trigonometry is to find the value of one or more Trig functions based on a set of constraints. Often, the constraints involve the value of another Trig function and the sign of yet a third Trig Function. The key to solving this type of problem is to draw the correct triangle in the correct quadrant.

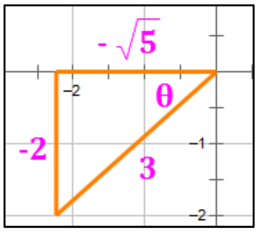
A couple of examples will illustrate this process.

Example 1.1: $\sin\theta = -\frac{2}{3}$, $\tan\theta > 0$. Find the values of $\sec\theta$ and $\cot\theta$.

Notice that $\sin\theta < 0$, $\tan\theta > 0$. Therefore, θ is in $Q3$, so we draw the angle in that quadrant.

In $Q3$, y is negative; r is always positive. Since $\sin\theta = \frac{y}{r} = -\frac{2}{3}$, we let $y = -2$, $r = 3$.

Using the Pythagorean Theorem, we calculate the length of the horizontal leg of the triangle: $\sqrt{3^2 - (-2)^2} = \sqrt{5}$. Since the angle is in $Q3$, x is negative, so we must have $x = -\sqrt{5}$.



Then, $\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x} = \frac{3}{-\sqrt{5}} = -\frac{3\sqrt{5}}{5}$

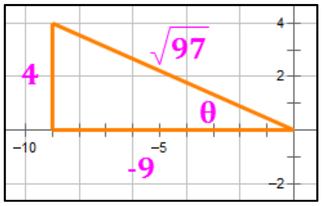
And, $\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y} = \frac{-\sqrt{5}}{-2} = \frac{\sqrt{5}}{2}$

Example 1.2: $\cot\theta = -\frac{9}{4}$, $\cos\theta < 0$. Find the value of $\csc\theta$ and $\cos\theta$.

Notice that $\cot\theta < 0$, $\cos\theta < 0$. Therefore, θ is in $Q2$, so we draw the angle in that quadrant.

In $Q2$, x is negative, and y is positive. Since $\cot\theta = \frac{x}{y} = -\frac{9}{4}$, we let $x = -9$, $y = 4$.

Using the Pythagorean Theorem, we can calculate the length of the hypotenuse of the triangle: $r = \sqrt{(-9)^2 + 4^2} = \sqrt{97}$.



Then, $\csc\theta = \frac{1}{\sin\theta} = \frac{r}{y} = \frac{\sqrt{97}}{4}$

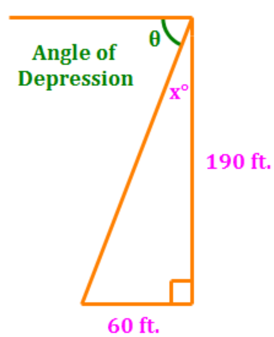
And, $\cos\theta = \frac{x}{r} = \frac{-9}{\sqrt{97}} = \frac{-9\sqrt{97}}{97}$

Problems Involving Angles of Depression and Inclination

A common problem in Trigonometry deals with angles of depression or inclination. An angle of depression is an angle below the horizontal at which an observer must look to see an object. An angle of inclination is an angle above the horizontal at which an observer must look to see an object.

Example 1.3: A building 185 feet tall casts a 60 foot long shadow. If a person looks down from the top of the building, what is the measure of the angle of depression? Assume the person's eyes are 5 feet above the top of the building.

The total height from which the person looks down upon the shadow is: $185 + 5 = 190$ ft. We begin by drawing the diagram below, then consider the trigonometry involved.



$$\tan x° = \frac{60}{190} = 0.3158$$

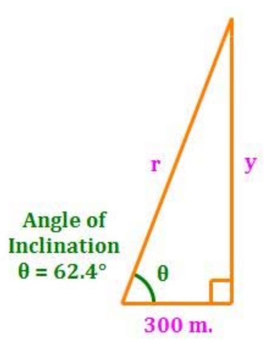
$$x = \tan^{-1} 0.3158 = 17.5°$$

The angle of depression is the complement of $x°$.

$$\theta = 90° - 17.5° = 72.5°$$

Example 1.4: A ship is 300 meters from a vertical cliff. The navigator uses a sextant to determine the angle of inclination from the deck of the ship to the top of the cliff to be 62.4°. How far above the deck of the ship is the top of the cliff? What is the distance from the deck to the top of the cliff?

We begin by drawing the diagram below, then consider the trigonometry involved.



a) To find how far above the deck the top of the cliff is (y):

$$\tan 62.4° = \frac{y}{300}$$

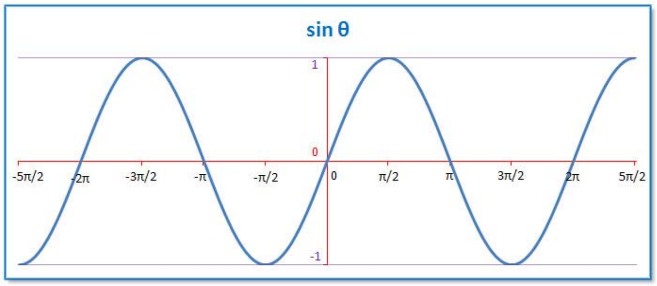
$$y = 300 \tan 62.4° = 573.8 \text{ meters}$$

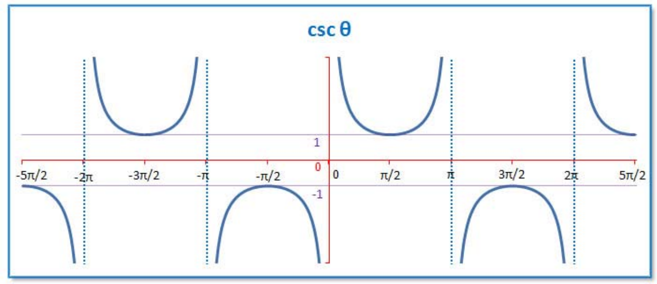
b) To find the distance from the deck to the top of the cliff (r):

$$\cos 62.4° = \frac{300}{r}$$

$$r = \frac{300}{\cos 62.4°} = 647.5 \text{ meters}$$

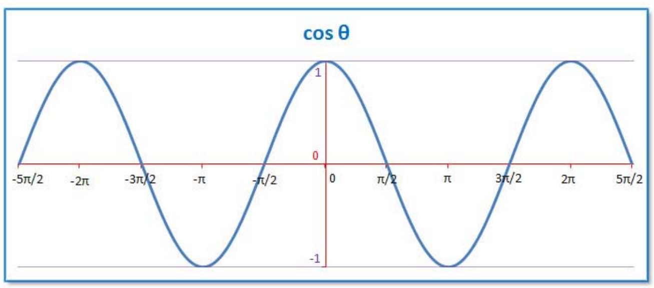
Graphs of Basic (Parent) Trigonometric Functions

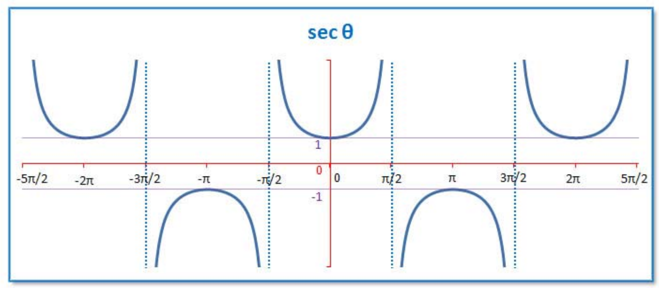




The sine and cosecant functions are reciprocals. So:

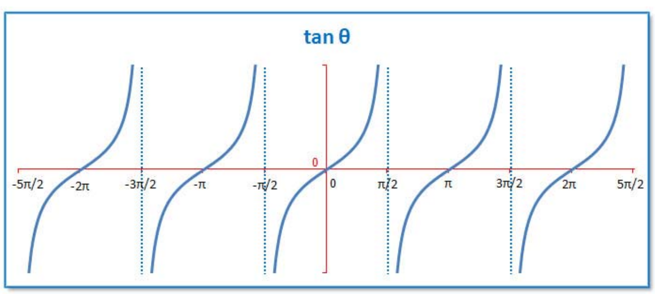
$$\sin\theta = \frac{1}{\csc\theta} \quad \text{and} \quad \csc\theta = \frac{1}{\sin\theta}$$

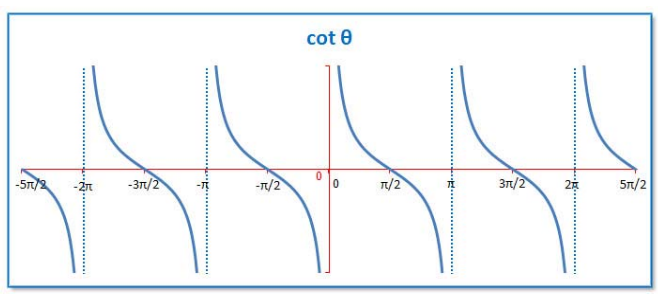




The cosine and secant functions are reciprocals. So:

$$\cos\theta = \frac{1}{\sec\theta} \quad \text{and} \quad \sec\theta = \frac{1}{\cos\theta}$$



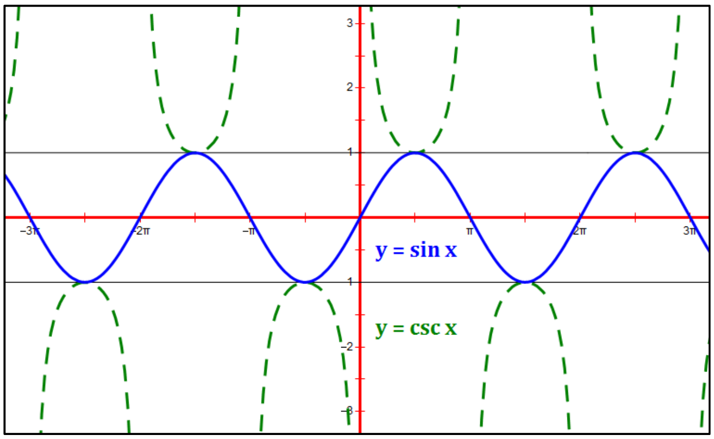


The tangent and cotangent functions are reciprocals. So:

$$\tan\theta = \frac{1}{\cot\theta} \quad \text{and} \quad \cot\theta = \frac{1}{\tan\theta}$$

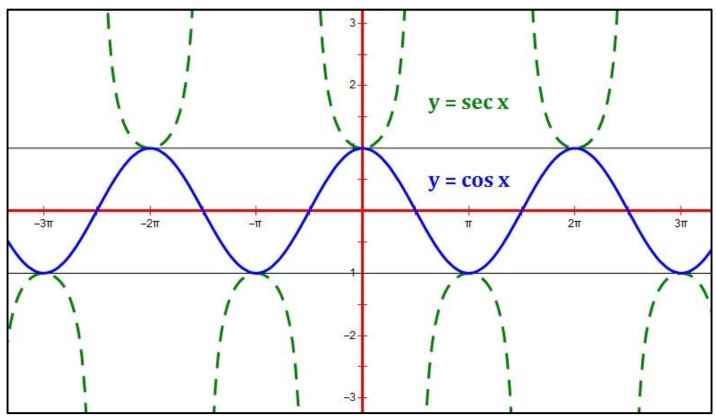
Graphs of Basic (Parent) Trigonometric Functions

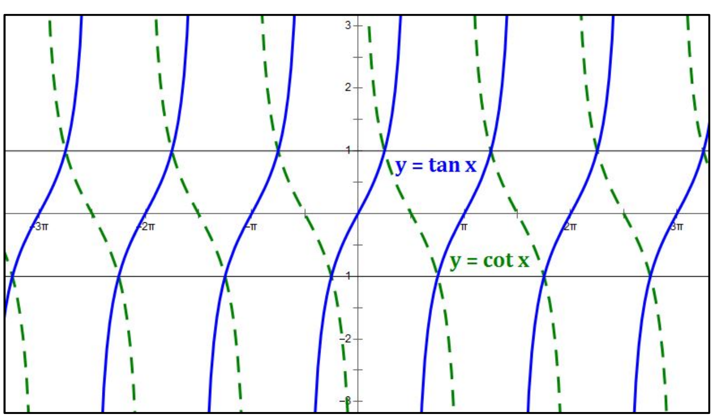
It is instructive to view the parent trigonometric functions on the same axes as their reciprocals. Identifying patterns between the two functions can be helpful in graphing them.



Looking at the **sine** and **cosecant** functions, we see that they intersect at their maximum and minimum values (i.e., when $y = 1$). The vertical asymptotes (not shown) of the cosecant function occur when the sine function is zero.

Looking at the **cosine** and **secant** functions, we see that they intersect at their maximum and minimum values (i.e., when $y = 1$). The vertical asymptotes (not shown) of the secant function occur when the cosine function is zero.





Looking at the **tangent** and **cotangent** functions, we see that they intersect when $\sin x = \cos x$ (i.e., at $x = \frac{\pi}{4} + n\pi$, n an integer). The vertical asymptotes (not shown) of the each function occur when the other function is zero.

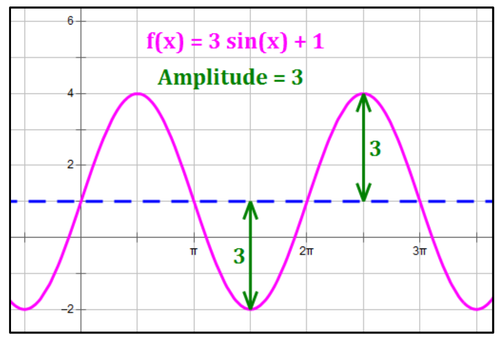
Characteristics of Trigonometric Function Graphs

All trigonometric functions are periodic, meaning that they repeat the pattern of the curve (called a cycle) on a regular basis. The key characteristics of each curve, along with knowledge of the parent curves are sufficient to graph many trigonometric functions. Let's consider the general function:

$$f(x) = \mathrm{A} \cdot trig(\mathrm{B}x - \mathrm{C}) + \mathrm{D}$$

where A, B, C and D are constants and "$trig$" is any of the six trigonometric functions (sine, cosine, tangent, cotangent, secant, cosecant).

Amplitude



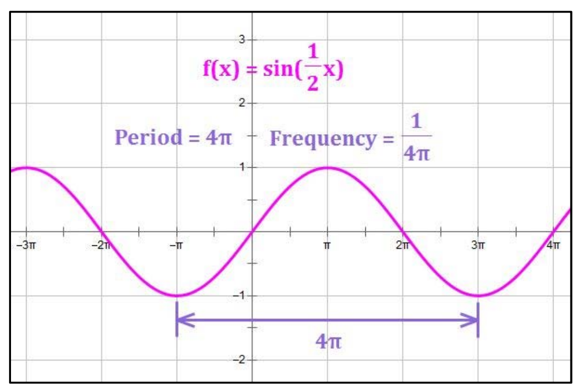
Amplitude is the measure of the distance of peaks and troughs from the midline (i.e., center) of a *sine or cosine function*; amplitude is always positive. The other four functions do not have peaks and troughs, so they do not have amplitudes. For the general function, $f(x)$, defined above, amplitude $= |\mathrm{A}|$.

Period

Period is the horizontal width of a single cycle or wave, i.e., the distance it travels before it repeats. Every trigonometric function has a period. The periods of the *parent functions* are as follows: for sine, cosine, secant and cosecant, period $= 2\pi$; for tangent and cotangent, period $= \pi$.

For the general function, $f(x)$, defined above,

$$\text{period} = \frac{\text{parent function period}}{\mathrm{B}}.$$



Frequency

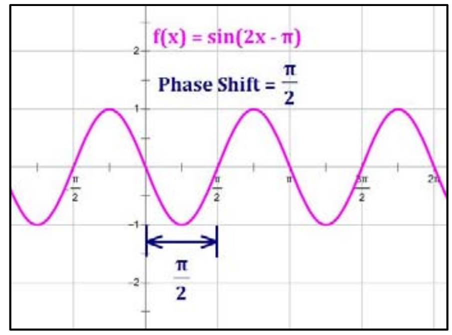
Frequency is most useful when used with the sine and cosine functions. It is the reciprocal of the period, i.e.,

$$\text{frequency} = \frac{1}{\text{period}}.$$

Frequency is typically discussed in relation to the sine and cosine functions when considering harmonic motion or waves. In Physics, frequency is typically measured in Hertz, i.e., cycles per second. 1 Hz = 1 cycle per second.

For the general sine or cosine function, $f(x)$, defined above, frequency $= \dfrac{\mathrm{B}}{2\pi}$.

Phase Shift



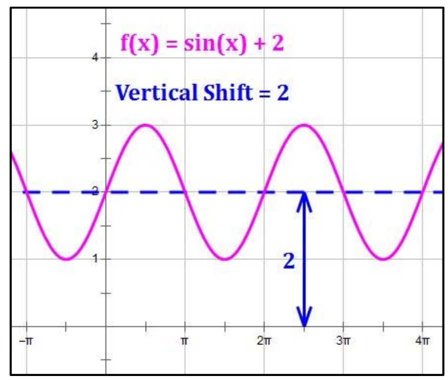
Phase shift is how far has the function been shifted horizontally (left or right) from its parent function. For the general function, $f(x)$, defined above,

$$\text{phase shift } = \frac{C}{B}.$$

A positive phase shift indicates a shift to the right relative to the graph of the parent function; a negative phase shift indicates a shift to the left relative to the graph of the parent function.

A trick for calculating the phase shift is to set the argument of the trigonometric function equal to zero: $(Bx - C) = 0$, and solve for x. The resulting value of x is the phase shift of the function.

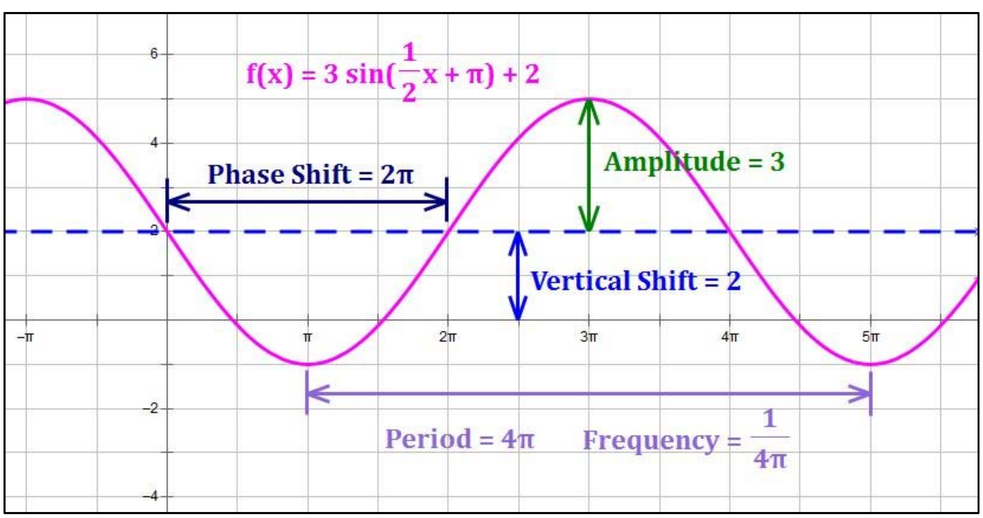
Vertical Shift



Vertical shift is the vertical distance that the midline of a curve lies above or below the midline of its parent function (i.e., the x-axis). For the general function, $f(x)$, defined above, $\text{vertical shift} = D.$ The value of D may be positive, indicating a shift upward, or negative, indicating a shift downward relative to the graph of the parent function.

Putting it All Together

The illustration below shows how all of the items described above combine in a single graph.



Summary of Characteristics and Key Points – Trigonometric Function Graphs

Function:	**Sine**	**Cosine**	**Tangent**	**Cotangent**	**Secant**	**Cosecant**
Parent Function	$y = \sin(x)$	$y = \cos(x)$	$y = \tan(x)$	$y = \cot(x)$	$y = \sec(x)$	$y = \csc(x)$
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$	$(-\infty, \infty)$ except $\frac{n\pi}{2}$, where n is odd	$(-\infty, \infty)$ except $n\pi$, where n is an Integer	$(-\infty, \infty)$ except $\frac{n\pi}{2}$, where n is odd	$(-\infty, \infty)$ except $n\pi$, where n is an Integer
Vertical Asymptotes	none	none	$x = \frac{n\pi}{2}$, where n is odd	$x = n\pi$, where n is an Integer	$x = \frac{n\pi}{2}$, where n is odd	$x = n\pi$, where n is an Integer
Range	$[-1, 1]$	$[-1, 1]$	$(-\infty, \infty)$	$(-\infty, \infty)$	$(-\infty, -1] \cup [1, \infty)$	$(-\infty, -1] \cup [1, \infty)$
Period	2π	2π	π	π	2π	2π
x-intercepts	$n\pi$, where n is an Integer	$\frac{n\pi}{2}$, where n is odd	midway between asymptotes	midway between asymptotes	none	none
Odd or Even Function (1)	Odd Function	Even Function	Odd Function	Odd Function	Even Function	Odd Function

General Form	$y = A\sin(Bx - C) + D$	$y = A\cos(Bx - C) + D$	$y = A\tan(Bx - C) + D$	$y = A\cot(Bx - C) + D$	$y = A\sec(Bx - C) + D$	$y = A\csc(Bx - C) + D$
Amplitude/Stretch, Period, Phase Shift, Vertical Shift	$\lvert A\rvert, \frac{2\pi}{B}, \frac{C}{B}, D$	$\lvert A\rvert, \frac{2\pi}{B}, \frac{C}{B}, D$	$\lvert A\rvert, \frac{\pi}{B}, \frac{C}{B}, D$	$\lvert A\rvert, \frac{\pi}{B}, \frac{C}{B}, D$	$\lvert A\rvert, \frac{2\pi}{B}, \frac{C}{B}, D$	$\lvert A\rvert, \frac{2\pi}{B}, \frac{C}{B}, D$
$f(x)$ when $x = PS$ (2)	D	$A + D$	D	vertical asymptote	$A + D$	vertical asymptote
$f(x)$ when $x = PS + \frac{1}{4}P$	$A + D$	D	$A + D$	$A + D$	vertical asymptote	$A + D$
$f(x)$ when $x = PS + \frac{1}{2}P$	D	$-A + D$	vertical asymptote	D	$-A + D$	vertical asymptote
$f(x)$ when $x = PS + \frac{3}{4}P$	$-A + D$	D	$-A + D$	$-A + D$	vertical asymptote	$-A + D$
$f(x)$ when $x = PS + P$	D	$A + D$	D	vertical asymptote	$A + D$	vertical asymptote

Notes:

(1) An odd function is symmetric about the origin, i.e. $f(-x) = -f(x)$. An even function is symmetric about the y-axis, i.e., $f(-x) = f(x)$.

(2) All Phase Shifts are defined to occur relative to a starting point of the y-axis (i.e., the vertical line $x = 0$).

Graph of a General Sine Function

General Form

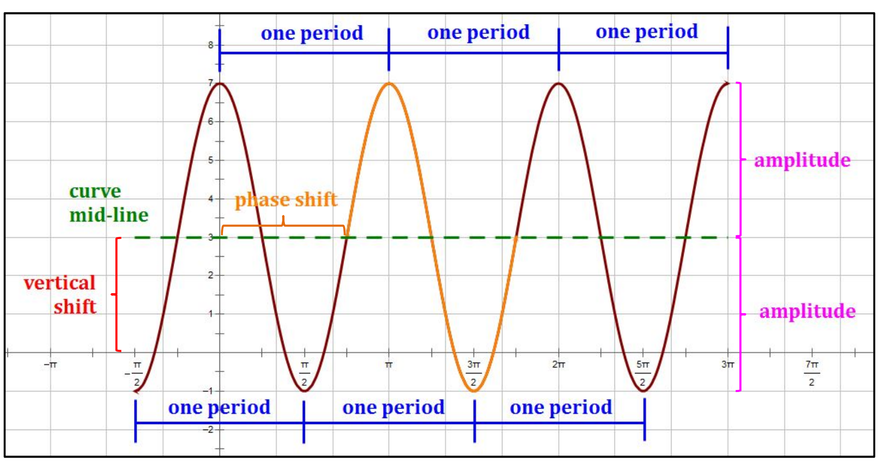
The general form of a sine function is: $\boldsymbol{y = A\ \sin(Bx - C) + D}$.

In this equation, we find several parameters of the function which will help us graph it. In particular:

- Amplitude: $\boldsymbol{Amp = |A|}$. The amplitude is the magnitude of the stretch or compression of the function from its parent function: $y = \sin x$.
- Period: $\boldsymbol{P = \frac{2\pi}{B}}$. The period of a trigonometric function is the horizontal distance over which the curve travels before it begins to repeat itself (i.e., begins a new cycle). For a sine or cosine function, this is the length of one complete wave; it can be measured from peak to peak or from trough to trough. Note that 2π is the period of $y = \sin x$.
- Phase Shift: $\boldsymbol{PS = \frac{C}{B}}$. The phase shift is the distance of the horizontal translation of the function. Note that the value of $\boldsymbol{C}$ in the general form has a minus sign in front of it, just like $\boldsymbol{h}$ does in the vertex form of a quadratic equation: $\boldsymbol{y = (x - h)^2 + k}$. So,
    - o A minus sign in front of the $\boldsymbol{C}$ implies a translation to the right, and
    - o A plus sign in front of the $\boldsymbol{C}$ implies a implies a translation to the left.
- Vertical Shift: $\boldsymbol{VS = D}$. This is the distance of the vertical translation of the function. This is equivalent to $\boldsymbol{k}$ in the vertex form of a quadratic equation: $\boldsymbol{y = (x - h)^2 + k}$.

Example 2.1: $\boldsymbol{y = 4\sin\left(2x - \frac{3}{2}\pi\right) + 3}$

The midline has the equation $\mathrm{y = D}$. In this example, the midline is: $\mathrm{y = 3}$. One wave, shifted to the right, is shown in orange below.



For this example:

$\boldsymbol{A = 4;\ \ B = 2;\ \ C = \frac{3}{2}\pi;\ \ D = 3}$

Amplitude: $\boldsymbol{Amp = |A| = |4| = 4}$

Period: $\boldsymbol{P = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi}$

Phase Shift: $\boldsymbol{PS = \frac{C}{B} = \frac{\frac{3}{2}\pi}{2} = \frac{3}{4}\pi}$

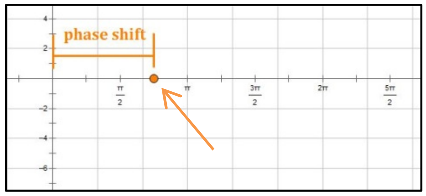
Vertical Shift: $\boldsymbol{VS = D = 3}$

Graphing a Sine Function with No Vertical Shift: $y = A\sin(Bx - C)$

A wave (cycle) of **the sine function** has three zero points (points on the x-axis) – at the beginning of the period, at the end of the period, and halfway in-between.

Example: $y = 4\sin\left(2x - \frac{3}{2}\pi\right)$.

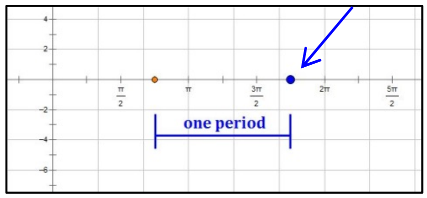
Step 1: Phase Shift: $PS = \frac{C}{B}$. The first wave begins at the point PS **units to the right of the Origin**.



$PS = \frac{C}{B} = \frac{\frac{3}{2}\pi}{2} = \frac{3}{4}\pi$.

The point is: $\left(\frac{3}{4}\pi, 0\right)$

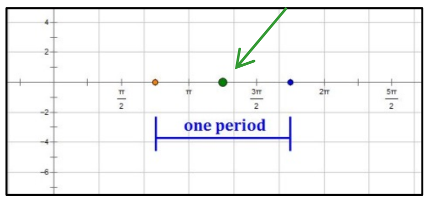
Step 2: Period: $P = \frac{2\pi}{B}$. The first wave ends at the point P **units to the right of where the wave begins**.



$P = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$. The first wave ends at the point:

$$\left(\frac{3}{4}\pi + \pi, 0\right) = \left(\frac{7}{4}\pi, 0\right)$$

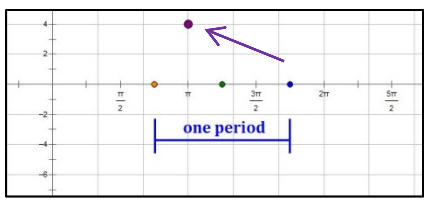
Step 3: The third zero point is located halfway between the first two.



The point is:

$$\left(\frac{\frac{3}{4}\pi + \frac{7}{4}\pi}{2}, 0\right) = \left(\frac{5}{4}\pi, 0\right)$$

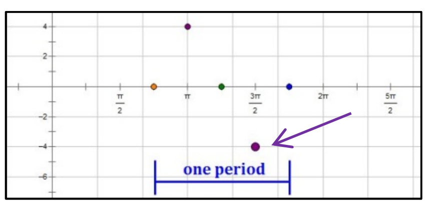
Step 4: The y-value of the point halfway between the left and center zero points is "A".



The point is:

$$\left(\frac{\frac{3}{4}\pi + \frac{5}{4}\pi}{2}, 4\right) = (\pi, 4)$$

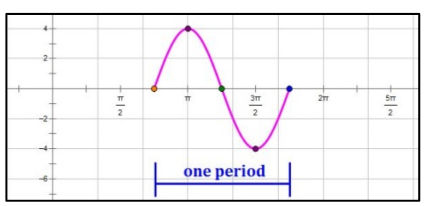
Step 5: The y-value of the point halfway between the center and right zero points is "$-A$".



The point is:

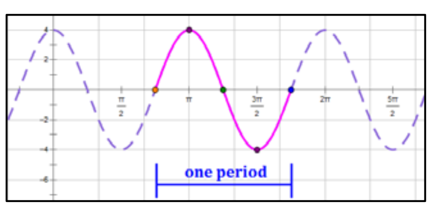
$$\left(\frac{\frac{5}{4}\pi + \frac{7}{4}\pi}{2}, -4\right) = \left(\frac{3}{2}\pi, -4\right)$$

Step 6: Draw a smooth curve through the five key points.



This will produce the graph of one wave of the function.

Step 7: Duplicate the wave to the left and right as desired.



Note: If $D \neq 0$, all points on the curve are shifted vertically by D units.

Graph of a General Cosine Function

General Form

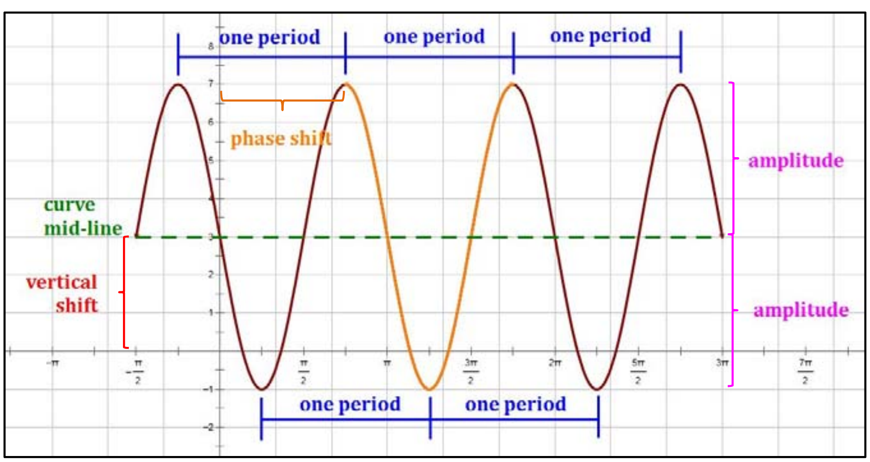
The general form of a cosine function is: $y = A\ \cos(Bx - C) + D$.

In this equation, we find several parameters of the function which will help us graph it. In particular:

- Amplitude: $Amp = |A|$. The amplitude is the magnitude of the stretch or compression of the function from its parent function: $y = \cos x$.
- Period: $P = \frac{2\pi}{B}$. The period of a trigonometric function is the horizontal distance over which the curve travels before it begins to repeat itself (i.e., begins a new cycle). For a sine or cosine function, this is the length of one complete wave; it can be measured from peak to peak or from trough to trough. Note that 2π is the period of $y = \cos x$.
- Phase Shift: $PS = \frac{C}{B}$. The phase shift is the distance of the horizontal translation of the function. Note that the value of C in the general form has a minus sign in front of it, just like h does in the vertex form of a quadratic equation: $y = (x - h)^2 + k$. So,
  - A minus sign in front of the C implies a translation to the right, and
  - A plus sign in front of the C implies a implies a translation to the left.
- Vertical Shift: $VS = D$. This is the distance of the vertical translation of the function. This is equivalent to k in the vertex form of a quadratic equation: $y = (x - h)^2 + k$.

Example 2.2: $y = 4\cos\left(2x - \frac{3}{2}\pi\right) + 3$

The midline has the equation $y = D$. In this example, the midline is: $y = 3$. One wave, shifted to the right, is shown in orange below.



For this example:

$A = 4;\ \ B = 2;\ \ C = \frac{3}{2}\pi;\ \ D = 3$

Amplitude: $Amp = |A| = |4| = 4$

Period: $P = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$

Phase Shift: $PS = \frac{C}{B} = \frac{\frac{3}{2}\pi}{2} = \frac{3}{4}\pi$

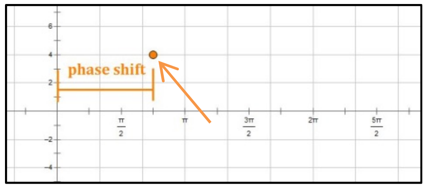
Vertical Shift: $VS = D = 3$

Graphing a Cosine Function with No Vertical Shift: $y = A\cos(Bx - C)$

A wave (cycle) of **the cosine function** has two maxima (or minima if $A < 0$) – one at the beginning of the period and one at the end of the period – and a minimum (or maximum if $A < 0$) halfway in-between.

Example: $y = 4\cos\left(2x - \frac{3}{2}\pi\right)$.

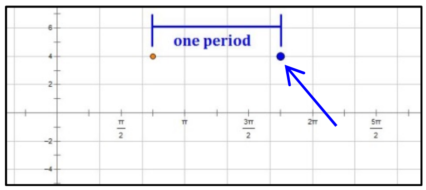
Step 1: Phase Shift: $PS = \frac{C}{B}$. The first wave begins at the point **PS units to the right of the point $(0, A)$.**



$PS = \frac{C}{B} = \frac{\frac{3}{2}\pi}{2} = \frac{3}{4}\pi$, $A = 4$

The point is: $\left(\frac{3}{4}\pi, 4\right)$

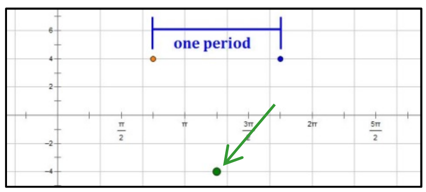
Step 2: Period: $P = \frac{2\pi}{B}$. The first wave ends at the point **P units to the right of where the wave begins**.



$P = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$. The first wave ends at the point:

$\left(\frac{3}{4}\pi + \pi, 4\right) = \left(\frac{7}{4}\pi, 4\right)$

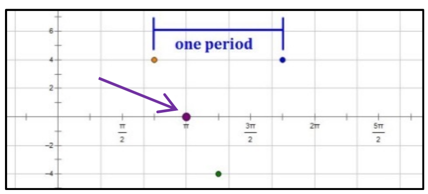
Step 3: The y-value of the point halfway between those in the two steps above is "$-A$".



The point is:

$\left(\frac{\frac{3}{4}\pi + \frac{7}{4}\pi}{2}, -4\right) = \left(\frac{5}{4}\pi, -4\right)$

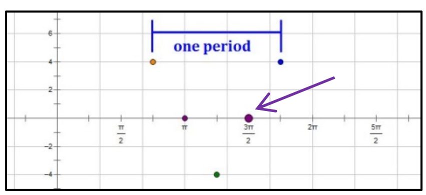
Step 4: The y-value of the point halfway between the left and center extrema is "0".



The point is:

$\left(\frac{\frac{3}{4}\pi + \frac{5}{4}\pi}{2}, 0\right) = (\pi, 0)$

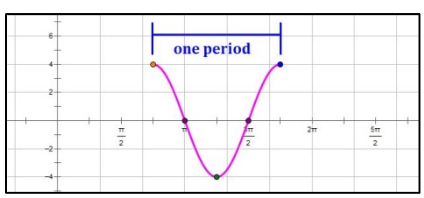
Step 5: The y-value of the point halfway between the center and right extrema is "0".



The point is:

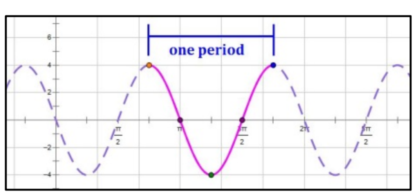
$\left(\frac{\frac{5}{4}\pi + \frac{7}{4}\pi}{2}, 0\right) = \left(\frac{3}{2}\pi, 0\right)$

Step 6: Draw a smooth curve through the five key points.



This will produce the graph of one wave of the function.

Step 7: Duplicate the wave to the left and right as desired.



Note: If $D \neq 0$, all points on the curve are shifted vertically by D units.

Graph of a General Tangent Function

General Form

The general form of a tangent function is: $y = A\ \tan(Bx - C) + D$.

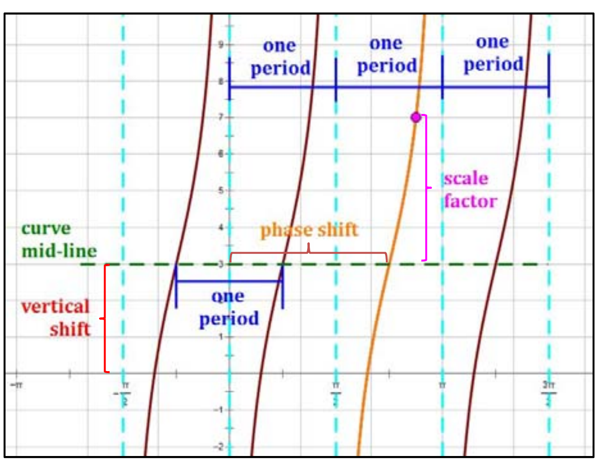
In this equation, we find several parameters of the function which will help us graph it. In particular:

- Scale factor: $|A|$. The tangent function does not have amplitude. $|A|$ is the magnitude of the stretch or compression of the function from its parent function: $y = \tan x$.
- Period: $P = \frac{\pi}{B}$. The period of a trigonometric function is the horizontal distance over which the curve travels before it begins to repeat itself (i.e., begins a new cycle). For a tangent or cotangent function, this is the horizontal distance between consecutive asymptotes (it is also the distance between x-intercepts). Note that π is the period of $y = \tan x$.
- Phase Shift: $PS = \frac{C}{B}$. The phase shift is the distance of the horizontal translation of the function. Note that the value of C in the general form has a minus sign in front of it, just like h does in the vertex form of a quadratic equation: $y = (x - h)^2 + k$. So,
  - o A minus sign in front of the C implies a translation to the right, and
  - o A plus sign in front of the C implies a implies a translation to the left.
- Vertical Shift: $VS = D$. This is the distance of the vertical translation of the function. This is equivalent to k in the vertex form of a quadratic equation: $y = (x - h)^2 + k$.

Example 2.3: $y = 4\tan\left(2x - \frac{3}{2}\pi\right) + 3$

The midline has the equation $y = D$. In this example, the midline is: $y = 3$. One cycle, shifted to the right, is shown in orange below.

Note that, for the tangent curve, we typically graph half of the principal cycle at the point of the phase shift, and then fill in the other half of the cycle to the left (see next page).



For this example:

$A = 4$; $B = 2$; $C = \frac{3}{2}\pi$; $D = 3$

Scale Factor: $|A| = |4| = 4$

Period: $P = \frac{\pi}{B} = \frac{\pi}{2} = \frac{1}{2}\pi$

Phase Shift: $PS = \frac{C}{B} = \frac{\frac{3}{2}\pi}{2} = \frac{3}{4}\pi$

Vertical Shift: $VS = D = 3$

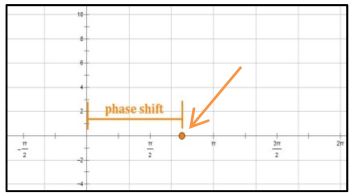
Graphing a Tangent Function with No Vertical Shift: $y = A\tan(Bx - C)$

A cycle of **the tangent function** has two asymptotes and a zero point halfway in-between. It flows upward to the right if $A > 0$ and downward to the right if $A < 0$.

Example:

$y = 4\tan\left(2x - \frac{3}{2}\pi\right)$.

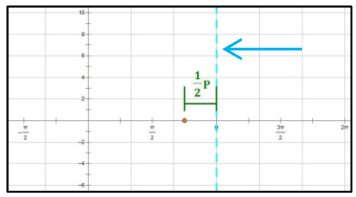
Step 1: Phase Shift: $PS = \frac{C}{B}$.
The first cycle begins at the "zero" point **PS units to the right of the Origin.**



$PS = \frac{C}{B} = \frac{\frac{3}{2}\pi}{2} = \frac{3}{4}\pi$.

The point is: $\left(\frac{3}{4}\pi, 0\right)$

Step 2: Period: $P = \frac{\pi}{B}$.
Place a vertical asymptote $\frac{1}{2}P$ units to the right of the beginning of the cycle.

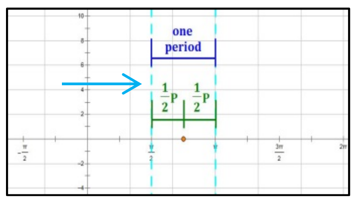


$P = \frac{\pi}{B} = \frac{\pi}{2} = \frac{1}{2}\pi$. $\frac{1}{2}P = \frac{1}{4}\pi$.

The right asymptote is at:

$x = \frac{3}{4}\pi + \frac{1}{4}\pi = \pi$

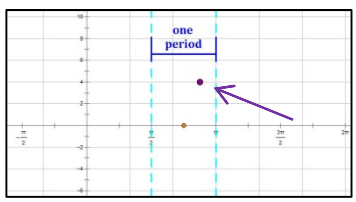
Step 3: Place a vertical asymptote $\frac{1}{2}P$ units to the left of the beginning of the cycle.



The left asymptote is at:

$x = \frac{3}{4}\pi - \frac{1}{4}\pi = \frac{1}{2}\pi$

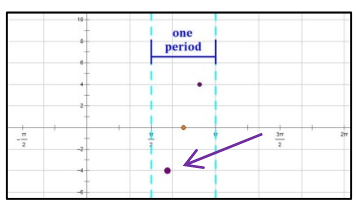
Step 4: The y-value of the point halfway between the zero point and the right asymptote is "A".



The point is:

$\left(\frac{\frac{3}{4}\pi + \pi}{2}, 4\right) = \left(\frac{7}{8}\pi, 4\right)$

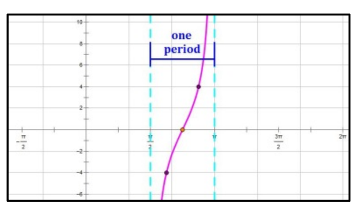
Step 5: The y-value of the point halfway between the left asymptote and the zero point is "$-A$".



The point is:

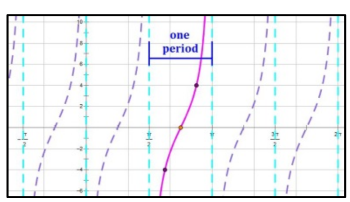
$\left(\frac{\frac{1}{2}\pi + \frac{3}{4}\pi}{2}, -4\right) = \left(\frac{5}{8}\pi, -4\right)$

Step 6: Draw a smooth curve through the three key points, approaching the asymptotes on each side.



This will produce the graph of one cycle of the function.

Step 7: Duplicate the cycle to the left and right as desired.



Note: If $D \neq 0$, all points on the curve are shifted vertically by D units.

Graph of a General Cotangent Function

General Form

The general form of a cotangent function is: $\boldsymbol{y = A\ \cot(Bx - C) + D}$.

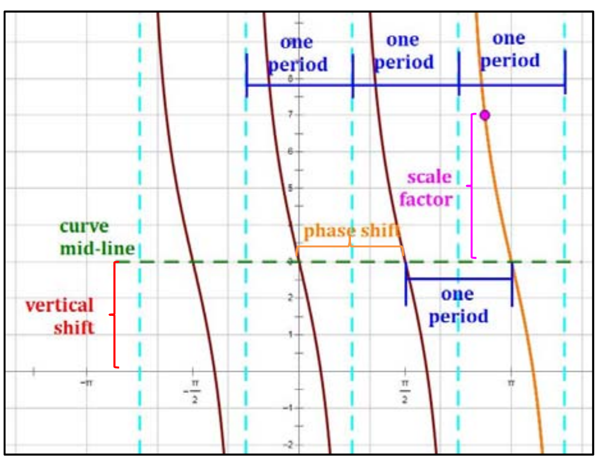
In this equation, we find several parameters of the function which will help us graph it. In particular:

- Scale factor: $|\boldsymbol{A}|$. The cotangent function does not have amplitude. $|A|$ is the magnitude of the stretch or compression of the function from its parent function: $y = \cot x$.
- Period: $\boldsymbol{P = \frac{\pi}{B}}$. The period of a trigonometric function is the horizontal distance over which the curve travels before it begins to repeat itself (i.e., begins a new cycle). For a tangent or cotangent function, this is the horizontal distance between consecutive asymptotes (it is also the distance between x-intercepts). Note that π is the period of $y = \cot x$.
- Phase Shift: $\boldsymbol{PS = \frac{C}{B}}$. The phase shift is the distance of the horizontal translation of the function. Note that the value of $\boldsymbol{C}$ in the general form has a minus sign in front of it, just like $\boldsymbol{h}$ does in the vertex form of a quadratic equation: $\boldsymbol{y = (x - h)^2 + k}$. So,
  - A minus sign in front of the $\boldsymbol{C}$ implies a translation to the right, and
  - A plus sign in front of the $\boldsymbol{C}$ implies a implies a translation to the left.
- Vertical Shift: $\boldsymbol{VS = D}$. This is the distance of the vertical translation of the function. This is equivalent to $\boldsymbol{k}$ in the vertex form of a quadratic equation: $\boldsymbol{y = (x - h)^2 + k}$.

Example 2.4: $\boldsymbol{y = 4\cot\left(2x - \frac{3}{2}\pi\right) + 3}$

The midline has the equation $\mathrm{y = D}$. In this example, the midline is: $\mathrm{y = 3}$. One cycle, shifted to the right, is shown in orange below.

Note that, for the cotangent curve, we typically graph the asymptotes first, and then graph the curve between them (see next page).



For this example:

$\boldsymbol{A = 4;\ \ B = 2;\ \ C = \frac{3}{2}\pi;\ \ D = 3}$

Scale Factor: $|\boldsymbol{A}| = |\boldsymbol{4}| = \boldsymbol{4}$

Period: $\boldsymbol{P = \frac{\pi}{B} = \frac{\pi}{2} = \frac{1}{2}\pi}$

Phase Shift: $\boldsymbol{PS = \frac{C}{B} = \frac{\frac{3}{2}\pi}{2} = \frac{3}{4}\pi}$

Vertical Shift: $\boldsymbol{VS = D = 3}$

Graphing a Cotangent Function with No Vertical Shift: $y = A\cot(Bx - C)$

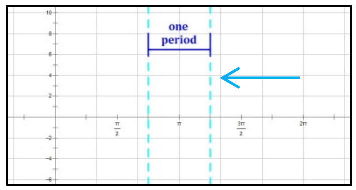
A cycle of **the cotangent function** has two asymptotes and a zero point halfway in-between. It flows downward to the right if $A > 0$ and upward to the right if $A < 0$.

Example: $y = 4\cot\left(2x - \frac{3}{2}\pi\right)$.

Step 1: Phase Shift: $PS = \frac{C}{B}$. Place a vertical asymptote ***PS* units to the right of the *y*-axis.**

$PS = \frac{C}{B} = \frac{\frac{3}{2}\pi}{2} = \frac{3}{4}\pi$. The left asymptote is at: $x = \frac{3}{4}\pi$

Step 2: Period: $P = \frac{\pi}{B}$. Place another vertical asymptote P units to the right of the first one.

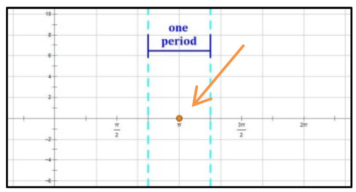


$P = \frac{\pi}{B} = \frac{\pi}{2} = \frac{1}{2}\pi$.

The right asymptote is at:

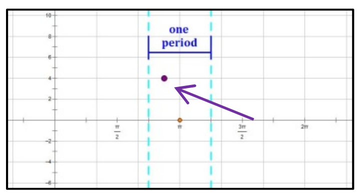
$x = \frac{3}{4}\pi + \frac{1}{2}\pi = \frac{5}{4}\pi$

Step 3: A zero point exists halfway between the two asymptotes.



The point is: $\left(\frac{\frac{3}{4}\pi + \frac{5}{4}\pi}{2}, 0\right) = (\pi, 0)$

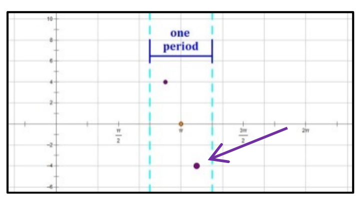
Step 4: The y-value of the point halfway between the left asymptote and the zero point is "A".



The point is:

$\left(\frac{\frac{3}{4}\pi + \pi}{2}, 4\right) = \left(\frac{7}{8}\pi, 4\right)$

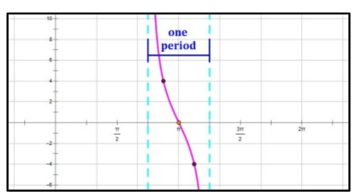
Step 5: The y-value of the point halfway between the zero point and the right asymptote is "$-A$".



The point is:

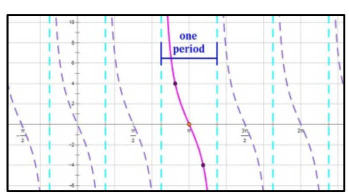
$\left(\frac{\pi + \frac{5}{4}\pi}{2}, -4\right) = \left(\frac{9}{8}\pi, -4\right)$

Step 6: Draw a smooth curve through the three key points, approaching the asymptotes on each side.



This will produce the graph of one cycle of the function.

Step 7: Duplicate the cycle to the left and right as desired.



Note: If $D \neq 0$, all points on the curve are shifted vertically by D units.

Graph of a General Secant Function

General Form

The general form of a secant function is: $\boldsymbol{y = A\ \sec(Bx - C) + D}$.

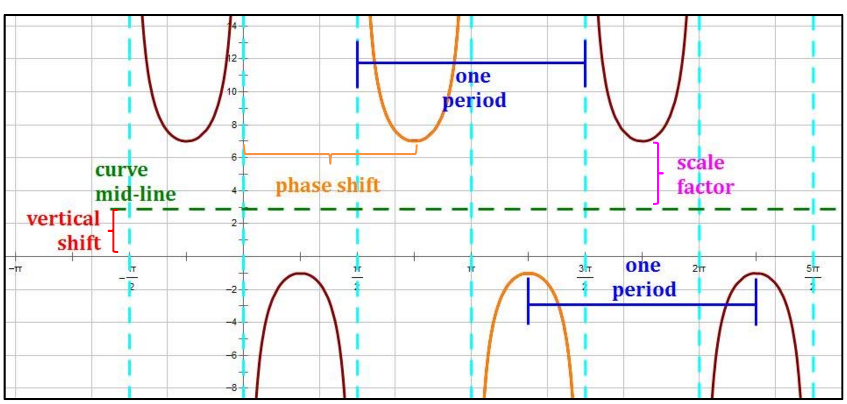
In this equation, we find several parameters of the function which will help us graph it. In particular:

- Scale factor: $|\boldsymbol{A}|$. The secant function does not have amplitude. $|A|$ is the magnitude of the stretch or compression of the function from its parent function: $y = \sec x$.
- Period: $\boldsymbol{P = \frac{2\pi}{B}}$. The period of a trigonometric function is the horizontal distance over which the curve travels before it begins to repeat itself (i.e., begins a new cycle). For a secant or cosecant function, this is the horizontal distance between consecutive maxima or minima (it is also the distance between every second asymptote). Note that 2π is the period of $y = \sec x$.
- Phase Shift: $\boldsymbol{PS = \frac{C}{B}}$. The phase shift is the distance of the horizontal translation of the function. Note that the value of $\boldsymbol{C}$ in the general form has a minus sign in front of it, just like $\boldsymbol{h}$ does in the vertex form of a quadratic equation: $\boldsymbol{y = (x - h)^2 + k}$. So,
  - o A minus sign in front of the $\boldsymbol{C}$ implies a translation to the right, and
  - o A plus sign in front of the $\boldsymbol{C}$ implies a implies a translation to the left.
- Vertical Shift: $\boldsymbol{VS = D}$. This is the distance of the vertical translation of the function. This is equivalent to $\boldsymbol{k}$ in the vertex form of a quadratic equation: $\boldsymbol{y = (x - h)^2 + k}$.

Example 2.5: $\boldsymbol{y = 4\sec\left(2x - \frac{3}{2}\pi\right) + 3}$

The midline has the equation $\mathrm{y = D}$. In this example, the midline is: $\mathrm{y = 3}$. One cycle, shifted to the right, is shown in orange below.

One cycle of the secant curve contains two U-shaped curves, one opening up and one opening down.



For this example:

$\boldsymbol{A = 4;\ \ B = 2;\ \ C = \frac{3}{2}\pi;\ \ D = 3}$

Scale Factor: $|\boldsymbol{A}| = |\boldsymbol{4}| = \boldsymbol{4}$

Period: $\boldsymbol{P = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi}$

Phase Shift: $\boldsymbol{PS = \frac{C}{B} = \frac{\frac{3}{2}\pi}{2} = \frac{3}{4}\pi}$

Vertical Shift: $\boldsymbol{VS = D = 3}$

Graphing a Secant Function with No Vertical Shift: $y = A\sec(Bx - C)$

A cycle of **the secant function** can be developed by first plotting a cycle of the corresponding cosine function because $\sec x = \frac{1}{\cos x}$.

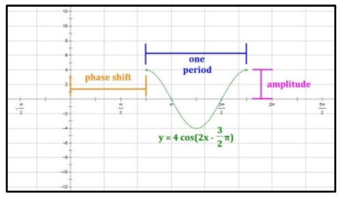
- The cosine function's zero points produce asymptotes for the secant function.
- Maxima for the cosine function produce minima for the secant function.
- Minima for the cosine function produce maxima for the secant function.
- Secant curves are U-shaped, alternately opening up and opening down.

Example:

$y = 4\sec\left(2x - \frac{3}{2}\pi\right)$.

Step 1: Graph one wave of the corresponding cosine function.

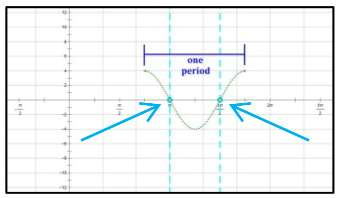
$y = A\cos(Bx - C)$



The equation of the corresponding cosine function for the example is:

$y = 4\cos\left(2x - \frac{3}{2}\pi\right)$

Step 2: Asymptotes for the secant function occur at the zero points of the cosine function.



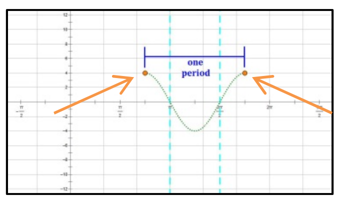
The zero points occur at:

$(\pi, 0)$ and $\left(\frac{3}{2}\pi, 0\right)$

Secant asymptotes are:

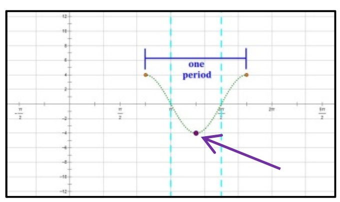
$x = \pi$ and $x = \frac{3}{2}\pi$

Step 3: Each maximum of the cosine function represents a minimum for the secant function.



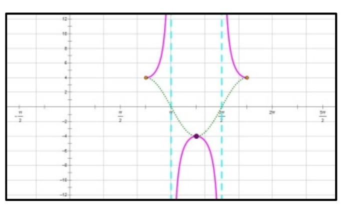
Cosine maxima and, therefore, secant minima are at: $\left(\frac{3}{4}\pi, 4\right)$ and $\left(\frac{7}{4}\pi, 4\right)$

Step 4: Each minimum of the cosine function represents a maximum for the secant function.



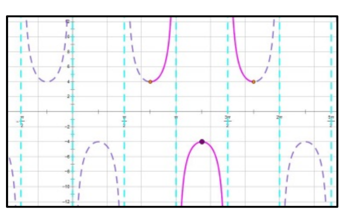
The cosine minimum and, therefore, the secant maximum is at: $\left(\frac{5}{4}\pi, -4\right)$

Step 5: Draw smooth U-shaped curves through each key point, approaching the asymptotes on each side.



This will produce the graph of one cycle of the function.

Step 6: Duplicate the cycle to the left and right as desired. Erase the cosine function if necessary.



Note: If $D \neq 0$, all points on the curve are shifted vertically by D units.

Graph of a General Cosecant Function

General Form

The general form of a cosecant function is: $\boldsymbol{y = A\ \csc(Bx - C) + D}$.

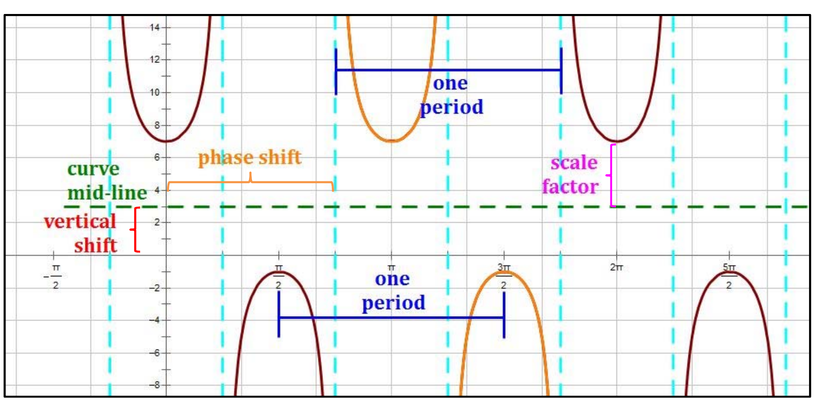
In this equation, we find several parameters of the function which will help us graph it. In particular:

- Scale factor: $\boldsymbol{|A|}$. The cosecant function does not have amplitude. $|A|$ is the magnitude of the stretch or compression of the function from its parent function: $y = \csc x$.
- Period: $\boldsymbol{P = \frac{2\pi}{B}}$. The period of a trigonometric function is the horizontal distance over which the curve travels before it begins to repeat itself (i.e., begins a new cycle). For a secant or cosecant function, this is the horizontal distance between consecutive maxima or minima (it is also the distance between every second asymptote). Note that 2π is the period of $y = \csc x$.
- Phase Shift: $\boldsymbol{PS = \frac{C}{B}}$. The phase shift is the distance of the horizontal translation of the function. Note that the value of $\boldsymbol{C}$ in the general form has a minus sign in front of it, just like $\boldsymbol{h}$ does in the vertex form of a quadratic equation: $\boldsymbol{y = (x - h)^2 + k}$. So,
  - A minus sign in front of the $\boldsymbol{C}$ implies a translation to the right, and
  - A plus sign in front of the $\boldsymbol{C}$ implies a implies a translation to the left.
- Vertical Shift: $\boldsymbol{VS = D}$. This is the distance of the vertical translation of the function. This is equivalent to $\boldsymbol{k}$ in the vertex form of a quadratic equation: $\boldsymbol{y = (x - h)^2 + k}$.

Example 2.6: $\boldsymbol{y = 4\csc\left(2x - \frac{3}{2}\pi\right) + 3}$

The midline has the equation $\mathrm{y = D}$. In this example, the midline is: $\mathrm{y = 3}$. One cycle, shifted to the right, is shown in orange below.

One cycle of the cosecant curve contains two U-shaped curves, one opening up and one opening down.



For this example:

$\boldsymbol{A = 4;\ \ B = 2;\ \ C = \frac{3}{2}\pi;\ \ D = 3}$

Scale Factor: $\boldsymbol{|A| = |4| = 4}$

Period: $\boldsymbol{P = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi}$

Phase Shift: $\boldsymbol{PS = \frac{C}{B} = \frac{\frac{3}{2}\pi}{2} = \frac{3}{4}\pi}$

Vertical Shift: $\boldsymbol{VS = D = 3}$

Graphing a Cosecant Function with No Vertical Shift: $y = A\csc(Bx - C)$

A cycle of **the cosecant function** can be developed by first plotting a cycle of the corresponding sine function because $\csc x = \frac{1}{\sin x}$.

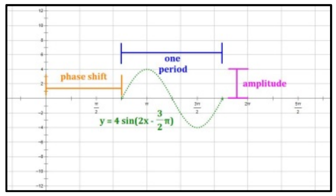
- The sine function's zero points produce asymptotes for the cosecant function.
- Maxima for the sine function produce minima for the cosecant function.
- Minima for the sine function produce maxima for the cosecant function.
- Cosecant curves are U-shaped, alternately opening up and opening down.

Example:

$y = 4\csc\left(2x - \frac{3}{2}\pi\right)$.

Step 1: Graph one wave of the corresponding sine function.

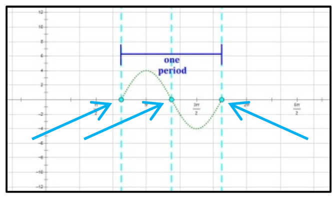
$y = A\sin(Bx - C)$



The equation of the corresponding sine function for the example is:

$y = 4\sin\left(2x - \frac{3}{2}\pi\right)$

Step 2: Asymptotes for the cosecant function occur at the zero points of the sine function.



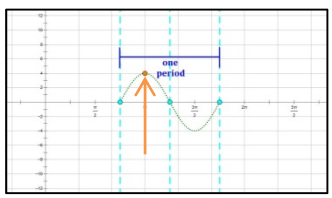
The zero points occur at:

$\left(\frac{3}{4}\pi, 0\right), \left(\frac{5}{4}\pi, 0\right), \left(\frac{7}{4}\pi, 0\right)$

Cosecant asymptotes are:

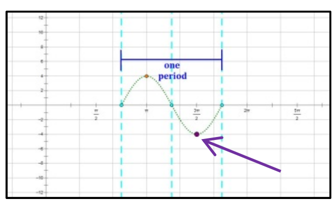
$x = \frac{3}{4}\pi,\ x = \frac{5}{4}\pi,\ x = \frac{7}{4}\pi$

Step 3: Each maximum of the sine function represents a minimum for the cosecant function.



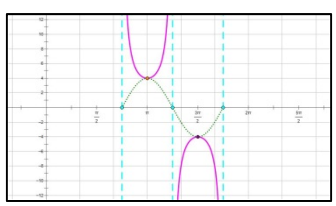
The sine maximum and, therefore, the cosecant minimum is at: $(\pi, 4)$

Step 4: Each minimum of the sine function represents a maximum for the cosecant function.



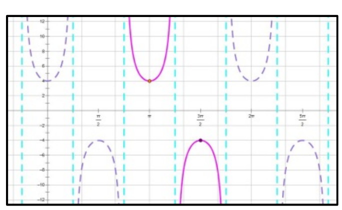
The sine minimum and, therefore, the cosecant maximum is at: $\left(\frac{3}{2}\pi, -4\right)$

Step 5: Draw smooth U-shaped curves through each key point, approaching the asymptotes on each side.



This will produce the graph of one cycle of the function.

Step 6: Duplicate the cycle to the left and right as desired. Erase the sine function if necessary.



Note: If $D \neq 0$, all points on the curve are shifted vertically by D units.

Simple Harmonic Motion

In Physics, Simple Harmonic Motion is an oscillating motion (think: repeating up and down motion) where the force applied to an object is proportional to and in the opposite direction of its displacement. A common example is the action of a coiled spring, which oscillates up and down when released. Such motion can be modeled by the sine and cosine functions, using the following equations (note: ω is the lower case Greek letter "omega," not the English letter w):

Harmonic motion equations: $d = a\cos\omega t$ or $d = a\sin\omega t$

Period: $\frac{2\pi}{\omega}$

Frequency: $f = \frac{1}{\text{period}} = \frac{\omega}{2\pi}$ or $\omega = 2\pi f$ with $\omega > 0$

Situations in which an object starts at rest at the center of its oscillation, or at rest, use the sine function (because $\sin 0 = 0$); situations in which an object starts in an up or down position prior to its release use the cosine function (because $\cos 0 = 1$).

Example 2.7: An object is attached to a coiled spring**.** The object is pulled up and then released. If the amplitude is 5 cm and the period is 7 seconds, write an equation for the distance of the object from its starting position after t seconds.

The spring will start at a y-value of $+5$ (since it is pulled up), and oscillate between $+5$ and -5 (absent any other force) over time. A good representation of this would be a cosine curve with lead coefficient $a = +5$.

The period of the function is 7 seconds. So, we get:

$$f = \frac{1}{\text{period}} = \frac{1}{7} \quad \text{and} \quad \omega = 2\pi f = 2\pi \cdot \frac{1}{7} = \frac{2\pi}{7}$$

The resulting equation, then, is: $d = 5\cos\left(\frac{2\pi}{7}t\right)$

Example 2.8: An object in simple harmonic motion has a frequency of 1.5 oscillations per second and an amplitude of 13 cm. Write an equation for the distance of the object from its rest position after t seconds.

Assuming that distance $= 0$ at time $t = 0$, it makes sense to use a sine function for this problem. Since the amplitude is 13 cm, a good representation of this would be a sine curve with lead coefficient $a = 13$. Note that a lead coefficient $a = -13$ would work as well.

Recalling that $\omega = 2\pi f$, with $f = 1.5$ we get: $\omega = 2\pi \cdot 1.5 = 3\pi$.

The resulting equations, then, are: $d = 13\sin(3\pi t)$ or $d = -13\sin(3\pi t)$

Inverse Trigonometric Functions

Inverse Trigonometric Functions

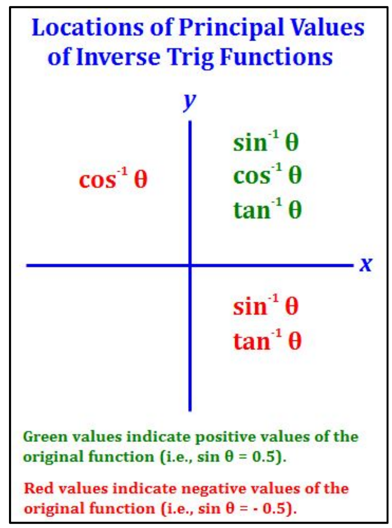
Inverse trigonometric functions are shown with a $"-1"$ exponent or an "arc" prefix. So, the inverse sine of x may be shown as $\sin^{-1}(x)$ or $\arcsin(x)$. Inverse trigonometric functions ask the question: which angle θ has a function value of x? For example:

$\theta = \sin^{-1}(0.5)$ asks which angle has a sine value of 0.5. It is equivalent to: $\sin\theta = 0.5$.

$\theta = \arctan(1)$ asks which angle has a tangent value of 1. It is equivalent to: $\tan\theta = 1$.

Principal Values of Inverse Trigonometric Functions

There are an infinite number of angles that answer the above questions, so the inverse trigonometric functions are referred to as multi-valued functions. Because of this, mathematicians have defined a principal solution for problems involving inverse trigonometric functions. The angle which is the principal solution (or principal value) is defined to be the solution that lies in the quadrants identified in the figure at right. For example:



The solutions to the equation $\theta = \sin^{-1} 0.5$ are all x-values in the intervals $\left\{\left(\frac{\pi}{6} + 2n\pi\right) \cup \left(\frac{5\pi}{6} + 2n\pi\right)\right\}$. That is, the set of all solutions to this equation contains the two solutions in the interval $[0, 2\pi)$, as well as all angles that are integer multiples of 2π less than or greater than those two angles. Given the confusion this can create, mathematicians have defined a principal value for the solution to these kinds of equations.

The principal value of θ for which $\theta = \sin^{-1} 0.5$ lies in Q1 because 0.5 is positive, and is $\theta = \frac{\pi}{6}$.

Ranges of Inverse Trigonometric Functions

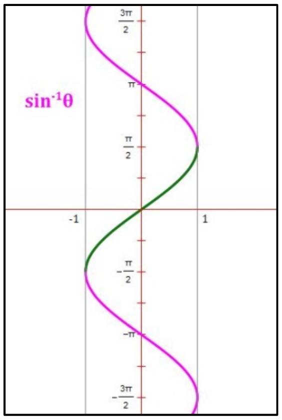
The ranges of inverse trigonometric functions are generally defined to be the ranges of the principal values of those functions. A table summarizing these is provided at right.

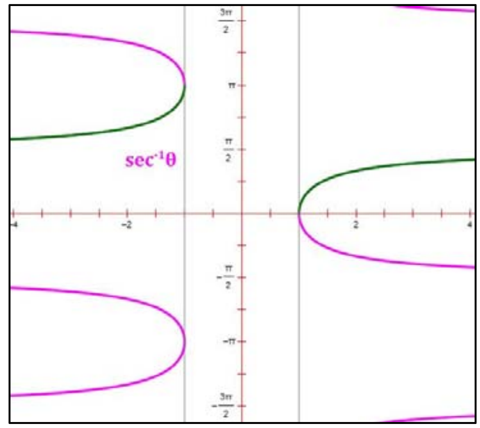
Angles in Q4 are expressed as negative angles.

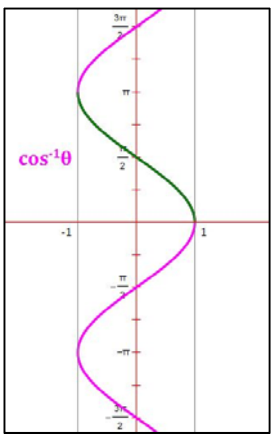
Ranges of Inverse Trigonometric Functions	
Function	**Range**
$\sin^{-1}\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\cos^{-1}\theta$	$0 \le \theta \le \pi$
$\tan^{-1}\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

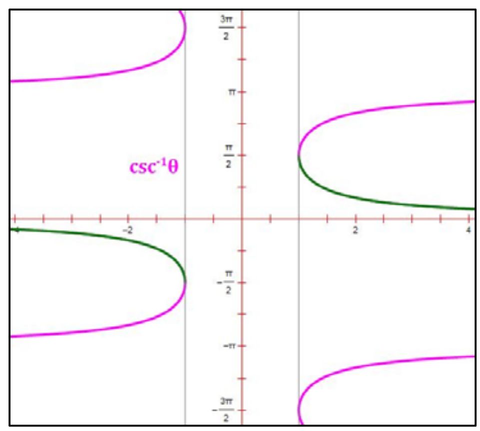
Graphs of Inverse Trigonometric Functions

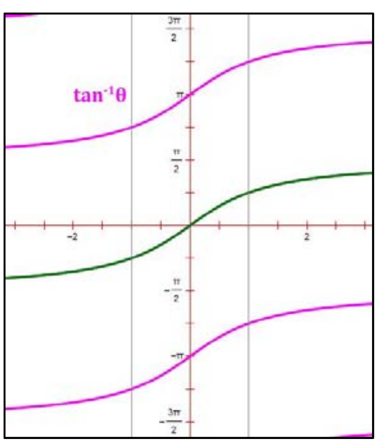
Principal values are shown in **green**.

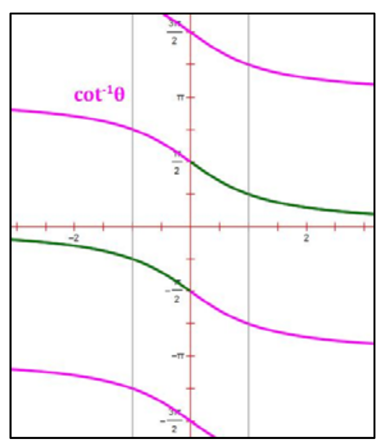






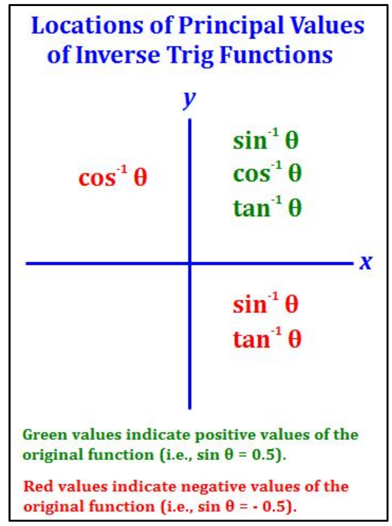






Problems Involving Inverse Trigonometric Functions

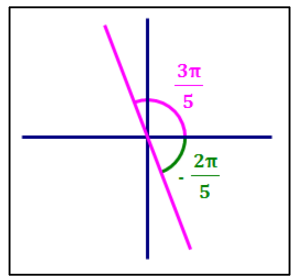
It is tempting to believe, for example, that $\sin^{-1}(\sin x) = x$ or $\tan^{-1}(\tan x) = x$. The two functions are, after all inverses. However, this is not always the case because the inverse function value desired is typically its principal value, which the student will recall is defined only in certain quadrants (see the table at right).



Let's look at a couple of problems to see how they are solved.

Example 3.1: Calculate the principal value of $\tan^{-1}\left(\tan\frac{3\pi}{5}\right)$.

Begin by noticing that $\tan^{-1}$ and tan are inverse functions, so the solution to this problem is related to the angle given: $\frac{3\pi}{5}$. This angle is in Q2, but the inverse tangent function is defined only in Q1 and Q4, on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



We seek the angle in Q1 or Q4 that has the same tangent value as $\frac{3\pi}{5}$. Since the tangent function has period π, we can calculate:

$$\tan^{-1}\left(\tan\frac{3\pi}{5}\right) = \frac{3\pi}{5} - \pi = -\frac{2\pi}{5}$$ (in Q4) as our solution.

Example 3.2: Calculate the principal value of $\sin^{-1}\left(\cos\frac{5\pi}{4}\right)$.

We are looking for the angle whose sine value is $\cos\frac{5\pi}{4}$ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Method 1: $\sin^{-1}\left(\cos\frac{5\pi}{4}\right) = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$ since sine values are negative in Q4.

Method 2: Recall: $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$. Then, $\cos\frac{5\pi}{4} = \sin\left(\frac{5\pi}{4} + \frac{\pi}{2}\right) = \sin\frac{7\pi}{4}$.

Then, $\sin^{-1}\left(\cos\frac{5\pi}{4}\right)$

$= \sin^{-1}\left(\sin\frac{7\pi}{4}\right)$ because $\cos\frac{5\pi}{4} = \sin\left(\frac{5\pi}{4} + \frac{\pi}{2}\right) = \sin\frac{7\pi}{4}$

$= \sin^{-1}\left(\sin\frac{-\pi}{4}\right)$ because $\frac{7\pi}{4} \equiv -\frac{\pi}{4}$ and $-\frac{\pi}{4}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

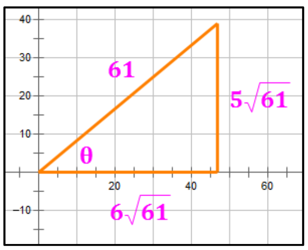
$= -\frac{\pi}{4}$ because inverse functions work nicely in quadrants in which the principal values of the inverse functions are defined.

Problems Involving Inverse Trigonometric Functions

When the inverse trigonometric function is the inner function in a composition of functions, it will usually be necessary to draw a triangle to solve the problem. In these cases, draw the triangle defined by the inner (inverse trig) function. Then derive the value of the outer (trig) function.

Example 3.3: Calculate the value of $\cot\left(\sin^{-1}\left[\frac{5\sqrt{61}}{61}\right]\right)$.

Recall that the argument of the $\sin^{-1}$ function, $\frac{5\sqrt{61}}{61}=\frac{y}{r}$. Draw the triangle based on this. Next, calculate the value of the triangle's horizontal leg:



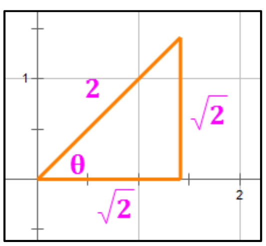
$$x=\sqrt{61^2-\left(5\sqrt{61}\right)^2}=6\sqrt{61}.$$

Based on the diagram, then,

$$\cot\left(\sin^{-1}\left[\frac{5\sqrt{61}}{61}\right]\right)=\frac{x}{y}=\frac{6\sqrt{61}}{5\sqrt{61}}=\frac{6}{5}$$

Example 3.4: Calculate the value of $\tan\left(\cos^{-1}\frac{\sqrt{2}}{2}\right)$.

Recall that the argument of the $\cos^{-1}$ function, $\frac{\sqrt{2}}{2}=\frac{x}{r}$. Draw the triangle based on this. Next, calculate the value of the triangle's vertical leg:



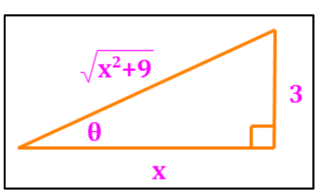
$$y=\sqrt{2^2-\left(\sqrt{2}\right)^2}=\sqrt{2}.$$

Based on the diagram, then,

$$\tan\left(\cos^{-1}\left[\frac{\sqrt{2}}{2}\right]\right)=\frac{y}{x}=\frac{\sqrt{2}}{\sqrt{2}}=1$$

Example 3.5: Calculate an algebraic expression for $\sin\left(\sec^{-1}\left[\frac{\sqrt{x^2+9}}{x}\right]\right)$.

Recall that the argument of the $\sec^{-1}$ function, $\frac{\sqrt{x^2+9}}{x}=\frac{r}{x}$. Draw the triangle based on this. Next, calculate the value of the triangle's vertical leg:



$$y=\sqrt{\left(\sqrt{x^2+9}\right)^2-x^2}=3$$

Based on the diagram, then,

$$\sin\left(\sec^{-1}\left[\frac{\sqrt{x^2+9}}{x}\right]\right)=\frac{y}{r}=\frac{3}{\sqrt{x^2+9}}=\boldsymbol{\frac{3\sqrt{x^2+9}}{x^2+9}}$$

Key Angle Formulas

Angle Addition Formulas

$$\sin\ (\alpha+\beta)=\sin\alpha\cos\beta\ +\ \cos\alpha\sin\beta$$
$$\sin\ (\alpha-\beta)=\sin\alpha\cos\beta\ -\ \cos\alpha\sin\beta$$

$$\cos\ (\alpha+\beta)=\cos\alpha\cos\beta\ -\ \sin\alpha\sin\beta$$
$$\cos\ (\alpha-\beta)=\cos\alpha\cos\beta\ +\ \sin\alpha\sin\beta$$

$$\tan\ (\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\ \tan\beta}$$

$$\tan\ (\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\ \tan\beta}$$

Double Angle Formulas

$$\sin\ 2\theta\ =\ 2\sin\theta\cos\theta$$

$$\cos\ 2\theta\ =\ \cos^2\theta-\sin^2\theta$$
$$=\ 1-2\sin^2\theta$$
$$=\ 2\cos^2\theta-1$$

$$\tan\ 2\theta\ =\ \frac{2\ \tan\theta}{1-\tan^2\theta}$$

Half Angle Formulas

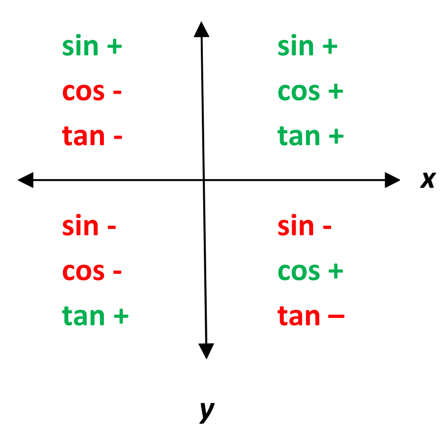
$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$
$$=\frac{1-\cos\theta}{\sin\theta}$$
$$=\frac{\sin\theta}{1+\cos\theta}$$

The use of a "+" or "-" sign in the half angle formulas depends on the quadrant in which the angle $\frac{\theta}{2}$ resides. See chart below.

Signs of Trig Functions By Quadrant



Key Angle Formulas – Examples

Example 4.1: Find the exact value of: $\cos(175^\circ)\cos(55^\circ)+\sin(175^\circ)\sin(55^\circ)$.

Recall: $\cos\,(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$\cos(175^\circ)\cos(55^\circ)+\sin(175^\circ)\sin(55^\circ)=\cos(175^\circ-55^\circ)$

$=\cos(120^\circ)$

$=-\cos(60^\circ)$ Converting to an angle in Q1

$=-\frac{1}{2}$

Example 4.2: Find the exact value of: $\tan 255°$ Recall: $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}$

$\tan 255° = \tan(315°-60°)$

$=\dfrac{\tan 315°-\tan 60°}{1+\tan 315°\cdot\tan 60°}$ Angles in Q4 and Q1

$=\dfrac{-\tan 45°-\tan 60°}{1+(-\tan 45°)\cdot\tan 60°}$ Converting to Q1 angles

$=\dfrac{-1-\sqrt{3}}{1+(-1)\cdot\sqrt{3}}=\dfrac{-(1+\sqrt{3})}{1-\sqrt{3}}$

$=\dfrac{-(1+\sqrt{3})}{1-\sqrt{3}}\cdot\dfrac{1+\sqrt{3}}{1+\sqrt{3}}=\dfrac{-(4+2\sqrt{3})}{-2}=2+\sqrt{3}$

Example 4.3: Find the exact value of: $\sin(105°)$. Recall: $\sin\,(\alpha+\beta)=\sin\alpha\cos\beta+\sin\beta\cos\alpha$

$\sin(105°)=\sin(60°+45°)$ Note: both angles are in Q1, which makes things easier.

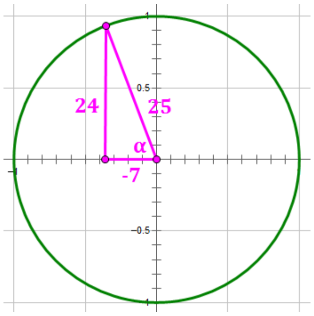
$=(\sin 60°\cdot\cos 45°)+(\sin 45°\cdot\cos 60°)$

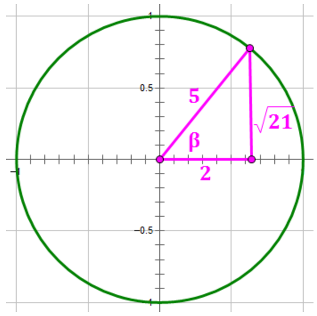
$=\left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}\right)+\left(\frac{\sqrt{2}}{2}\cdot\frac{1}{2}\right)$

$=\dfrac{\sqrt{2}\cdot(\sqrt{3}+1)}{4}$ or $\dfrac{\sqrt{6}+\sqrt{2}}{4}$

Example 4.4: $\sin\alpha = \frac{24}{25}$, α lies in quadrant II, and $\cos\beta = \frac{2}{5}$, β lies in quadrant I. Find $\cos(\alpha - \beta)$.

Construct triangles for the two angles, being careful to consider the signs of the values in each quadrant:

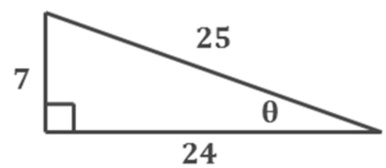




Then, $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

$$= \left(\frac{-7}{25}\cdot\frac{2}{5}\right) + \left(\frac{24}{25}\cdot\frac{\sqrt{21}}{5}\right) = \frac{-14 + 24\sqrt{21}}{125}$$

Example 4.5: Given the diagram at right, find: $\tan 2\theta$

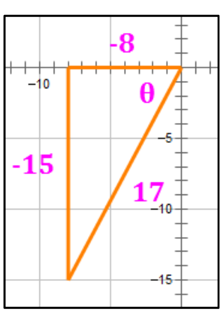


$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$= \frac{2\cdot\frac{7}{24}}{1 - \left(\frac{7}{24}\right)^2} = \frac{\frac{7}{12}}{\frac{527}{576}} = \frac{7}{12}\cdot\frac{576}{527} = \frac{336}{527}$$

Example 4.6: $\tan\theta = \frac{15}{8}$, and θ lies in quadrant III. Find $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$.

Draw the triangle below, then apply the appropriate formulas.



$$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\left(-\frac{15}{17}\right)\cdot\left(-\frac{8}{17}\right) = \frac{240}{289}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(-\frac{8}{17}\right)^2 - \left(-\frac{15}{17}\right)^2 = -\frac{161}{289}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = -\frac{240}{161}$$

Example 4.7: Find the exact value of: $\cos\frac{5\pi}{12}$ Recall: $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$

Note that $\frac{5\pi}{12}$ is in Q1, so the value of $\cos\frac{5\pi}{12}$ is positive.

$\cos\frac{5\pi}{12} = \cos\left(\frac{\frac{5\pi}{6}}{2}\right)$

$= +\sqrt{\frac{1+\cos\frac{5\pi}{6}}{2}}$ Using the half-angle formula above

$= \sqrt{\frac{1-\cos\frac{\pi}{6}}{2}}$ Converting to an angle in Q1

$= \sqrt{\frac{\frac{2}{2}-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$

Example 4.8: $\csc\theta = -4$, θ lies in quadrant IV. Find $\sin\frac{\theta}{2}$. Recall: $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$

Note that if θ is in Q4, then $\frac{\theta}{2}$ is in Q2, so the value of $\sin\frac{\theta}{2}$ is positive.

$\sin\theta = \frac{1}{\csc\theta}$ so, $\sin\theta = -\frac{1}{4}$

$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(-\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$ Note: cosine is positive in Q4

$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$ Using the half-angle formula above

$= +\sqrt{\frac{1-\frac{\sqrt{15}}{4}}{2}} = \sqrt{\frac{\frac{4-\sqrt{15}}{4}}{2}}$

$= \sqrt{\frac{4-\sqrt{15}}{8}} \cdot \sqrt{\frac{2}{2}} = \sqrt{\frac{8-2\sqrt{15}}{16}} = \frac{\sqrt{8-2\sqrt{15}}}{4}$

Key Angle Formulas

Power Reducing Formulas

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

Product-to-Sum Formulas

$$\sin\alpha\cdot\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\cos\alpha\cdot\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

$$\sin\alpha\cdot\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$\cos\alpha\cdot\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

Sum-to-Product Formulas

$$\sin\alpha + \sin\beta = 2\cdot\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cdot\sin\left(\frac{\alpha-\beta}{2}\right)\cdot\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cdot\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\cdot\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{\alpha-\beta}{2}\right)$$

Key Angle Formulas – Examples

Example 4.9: Convert to a sum formula: $\sin 8x \cdot \cos 5x$

Use: $\sin\boldsymbol{\alpha}\cdot\cos\boldsymbol{\beta} = \frac{1}{2}\left[\sin(\boldsymbol{\alpha}+\boldsymbol{\beta}) + \sin(\boldsymbol{\alpha}-\boldsymbol{\beta})\right]$

$$\sin 8x \cdot \cos 5x = \frac{1}{2}\left[\sin(8x+5x) + \sin(8x-5x)\right]$$

$$= \frac{1}{2}\left[\sin(13x) + \sin(3x)\right]$$

Example 4.10: Convert to a sum formula: $\cos\frac{7x}{2} \cdot \cos\frac{x}{2}$

Use: $\cos\boldsymbol{\alpha}\cdot\cos\boldsymbol{\beta} = \frac{1}{2}\left[\cos(\boldsymbol{\alpha}-\boldsymbol{\beta}) + \cos(\boldsymbol{\alpha}+\boldsymbol{\beta})\right]$

$$\cos\frac{7x}{2} \cdot \cos\frac{x}{2} = \frac{1}{2}\left[\cos\left(\frac{7x}{2}-\frac{x}{2}\right) + \cos\left(\frac{7x}{2}+\frac{x}{2}\right)\right]$$

$$= \frac{1}{2}\left[\cos(3x) + \cos(4x)\right]$$

Example 4.11: Convert to a product formula: $\sin 8x + \sin 2x$

Use: $\sin\boldsymbol{\alpha} + \sin\boldsymbol{\beta} = 2\cdot\sin\left(\frac{\boldsymbol{\alpha}+\boldsymbol{\beta}}{2}\right)\cdot\cos\left(\frac{\boldsymbol{\alpha}-\boldsymbol{\beta}}{2}\right)$

$$\sin 8x + \sin 2x = 2\cdot\sin\left(\frac{8x+2x}{2}\right)\cdot\cos\left(\frac{8x-2x}{2}\right)$$

$$= 2\cdot\sin(5x)\cdot\cos(3x)$$

Example 4.12: Convert to a product formula: $\cos 8x - \cos 2x$

Use: $\cos\boldsymbol{\alpha} - \cos\boldsymbol{\beta} = -2\cdot\sin\left(\frac{\boldsymbol{\alpha}+\boldsymbol{\beta}}{2}\right)\cdot\sin\left(\frac{\boldsymbol{\alpha}-\boldsymbol{\beta}}{2}\right)$

$$\cos 8x - \cos 2x = -2\cdot\sin\left(\frac{8x+2x}{2}\right)\cdot\sin\left(\frac{8x-2x}{2}\right)$$

$$= -2\cdot\sin(5x)\cdot\sin(3x)$$

Verifying Identities

A significant portion of any trigonometry course deals with verifying Trigonometric Identities, i.e., statements that are always true (assuming the trigonometric values involved exist). This section deals with how the student may approach verification of identities such as:

$$(1+\tan^2\theta)\cdot(1-\sin^2\theta)=1$$

In verifying a Trigonometric Identity, the student is asked to work with only one side of the identity and, using the standard rules of mathematical manipulation, derive the other side. The student may work with either side of the identity, so generally it is best to work on the side that is most complex. The steps below present a strategy that may be useful in verifying identities.

Verification Steps

1. **Identify which side you want to work on.** Let's call this **Side A.** Let's call the side you are not working on **Side B.** So, you will be working on Side A to make it look like Side B.
    a. If one side has a multiple of an angle (e.g., $\tan 3x$) and the other side does not (e.g., $\cos x$), work with the side that has the multiple of an angle.
    b. If one side has only sines and cosines and the other does not, work with the side that does not have only sines and cosines.
    c. If you get part way through the exercise and realize you should have started with the other side, start over and work with the other side.
2. If necessary, **investigate Side B** by working on it a little. This is not a violation of the rules as long as, in your verification, you completely manipulate Side A to look like Side B. If you choose to investigate Side B, move your work off a little to the side so it is clear you are "investigating" and not actually "working" side B.
3. **Simplify** Side A as much as possible, but remember to look at the other side to make sure you are moving in that direction. Do this also at each step along the way, as long as it makes Side A look more like Side B.
    a. Use the Pythagorean Identities to simplify, e.g., if one side contains $(1-\sin^2 x)$ and the other side contains cosines but not sines, replace $(1-\sin^2 x)$ with $\cos^2 x$.
    b. Change any multiples of angles, half angles, etc. to expressions with single angles (e.g., replace $\sin 2x$ with $2\sin x\cos x$).
    c. Look for 1's. Often changing a 1 into $\sin^2\theta+\cos^2\theta$ (or vice versa) will be helpful.
4. **Rewrite Side A in terms of sines and cosines.**
5. **Factor** where possible.
6. **Separate or combine fractions** to make Side A look more like Side B.

The following pages illustrate a number of techniques that can be used to verify identities.

Verifying Identities – Techniques

Technique: Investigate One or Both Sides

Often, when looking at an identity, it is not immediately obvious how to proceed. In many cases, investigating both sides will provide the necessary hints to proceed.

Example 5.1:

$$\frac{\frac{1}{\sin x} - \frac{1}{\cos x}}{\frac{1}{\sin x} + \frac{1}{\cos x}} \qquad = \frac{\cot x - 1}{\cot x + 1}$$

Yuk! This identity looks difficult to deal with – there are lots of fractions. Let's investigate it by converting the right side to sines and cosines. Note that on the right, we move the new fraction off to the side to indicate we are investigating only. We do this because we must verify an identity by working on only one side until we get the other side.

$$\frac{\frac{1}{\sin x} - \frac{1}{\cos x}}{\frac{1}{\sin x} + \frac{1}{\cos x}} \qquad = \frac{\cot x - 1}{\cot x + 1} \qquad = \frac{\frac{\cos x}{\sin x} - \frac{\cos x}{\cos x}}{\frac{\cos x}{\sin x} + \frac{\cos x}{\cos x}}$$

In manipulating the right side, we changed each 1 in the green expression to $\frac{\cos x}{\cos x}$ because we want something that looks more like the expression on the left.

Notice that the orange expression looks a lot like the expression on the left, except that every place we have a 1 in the expression on the left we have $\cos x$ in the orange expression.

What is our next step? We need to change all the 1's in the expression on the left to $\cos x$. We can do this by multiplying the expression on the left by $\frac{\cos x}{\cos x}$, as follows:

$$\frac{\cos x}{\cos x} \cdot \frac{\frac{1}{\sin x} - \frac{1}{\cos x}}{\frac{1}{\sin x} + \frac{1}{\cos x}}$$

$$\frac{\frac{\cos x}{\sin x} - \frac{\cos x}{\cos x}}{\frac{\cos x}{\sin x} + \frac{\cos x}{\cos x}}$$ Notice that this matches the orange expression above.

$$\frac{\cot x - 1}{\cot x + 1} \qquad = \frac{\cot x - 1}{\cot x + 1}$$

Verifying Identities – Techniques (cont'd)

Technique: Break a Fraction into Pieces

When a fraction contains multiple terms in the numerator, it is sometimes useful to break it into separate terms. This works especially well when the resulting numerator has the same number of terms as exist on the other side of the equal sign.

Example 5.2:

$$\frac{\cos(\alpha-\beta)}{\cos\alpha\cos\beta} \qquad = 1-\tan\alpha\tan\beta$$

First, it's a good idea to replace $\cos(\alpha-\beta)$ with $\cos\alpha\cos\beta-\sin\alpha\sin\beta$:

$$\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}$$

Next, break the fraction into two pieces:

$$\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}$$

Finally, simplify the expression:

$$1-\left(\frac{\sin\alpha}{\cos\alpha}\right)\cdot\left(\frac{\sin\beta}{\cos\beta}\right)$$

$$1-\tan\alpha\tan\beta \qquad = 1-\tan\alpha\tan\beta$$

Verifying Identities – Techniques (cont'd)

Technique: Get a Common Denominator on One Side

If it looks like you would benefit from getting a common denominator for the two sides of an identity, try converting one side so that it has that denominator. In many cases, this will result in an expression that will simplify into a more useful form.

Example 5.3:

$$\frac{\cos x}{1-\sin x} \qquad = \frac{1+\sin x}{\cos x}$$

If we were to solve this like an equation, we might create a common denominator. Remember, however, that we can only work on one side, so we will obtain the common denominator on only one side. In this example, the common denominator would be: $\cos x\,(1-\sin x)$.

$$\frac{\cos x}{\cos x} \cdot \frac{\cos x}{1-\sin x}$$

$$\frac{\cos^2 x}{\cos x\,(1-\sin x)}$$

Once we have manipulated one side of the identity to have the common denominator, the rest of the expression should simplify. To keep the $\cos x$ in the denominator of the expression on the left, we need to work with the numerator. A common substitution is to convert between $\sin^2 x$ and $\cos^2 x$ using the Pythagorean identity $\sin^2 x + \cos^2 x = 1$.

$$\frac{1-\sin^2 x}{\cos x\,(1-\sin x)}$$

Notice that the numerator is a difference of squares. Let's factor it.

$$\frac{(1+\sin x)(1-\sin x)}{\cos x\,(1-\sin x)}$$

Finally, we simplify by eliminating the common factor in the numerator and denominator.

$$\frac{1+\sin x}{\cos x} \qquad = \frac{1+\sin x}{\cos x}$$

Solving Trigonometric Equations

Solving trigonometric equations involves many of the same skills as solving equations in general. Some specific things to watch for in solving trigonometric equations are the following:

- **Arrangement.** It is often a good idea to get arrange the equation so that all terms are on one side of the equal sign, and zero is on the other. For example, $\tan^2 x \sin x = \tan^2 x$ can be rearranged to become $\tan^2 x \sin x - \tan^2 x = 0$.
- **Quadratics.** Look for quadratic equations. Any time an equation contains a single Trig function with multiple exponents, there may be a way to factor it like a quadratic equation. For example, $\cos^2 x + 2\cos x + 1 = (\cos x + 1)^2$.
- **Factoring.** Look for ways to factor the equation and solve the individual terms separately. For example, $\tan^2 x \sin x - \tan^2 x = \tan^2 x\,(\sin x - 1)$.
- **Terms with No Solution**. After factoring, some terms will have no solution and can be discarded. For example, $\sin x - 2 = 0$ requires $\sin x = 2$, which has no solution since the sine function never takes on a value of 2.
- **Replacement.** Having terms with different Trig functions in the same equation is not a problem if you are able to factor the equation so that the different Trig functions are in different factors. When this is not possible, look for ways to replace one or more Trig functions with others that are also in the equation. The Pythagorean Identities are particularly useful for this purpose. For example, in the equation $\cos^2 x - \sin x - 1 = 0$, $\cos^2 x$ can be replaced by $1 - \sin^2 x$, resulting in an equation containing only one Trig function.
- **Extraneous Solutions.** Check each solution to make sure it works in the original equation. A solution of one factor of an equation may fail as a solution overall because the original function does not exist at that value. See Example 5.6 below.
- **Infinite Number of Solutions.** Trigonometric equations often have an infinite number of solutions because of their periodic nature. In such cases, we append "$+2n\pi$" or another term to the solutions to indicate this. See Example 5.9 below.
- **Solutions in an Interval.** Be careful when solutions are sought in a specific interval. For the interval $[0, 2\pi)$, there are typically two solutions for each factor containing a Trig function as long as the variable in the function has lead coefficient of 1 (e.g., x or θ). If the lead coefficient is other than 1 (e.g., $5x$ or 5θ), the number of solutions will typically be two multiplied by the lead coefficient (e.g., 10 solutions in the interval $[0, 2\pi)$ for a term involving $5x$). See Example 5.5 below, which has 8 solutions on the interval $[0, 2\pi)$.

A number of these techniques are illustrated in the examples that follow.

Solving Trigonometric Equations – Examples

Example 5.4: Solve for x on the interval $[0, 2\pi)$: $\cos^2 x + 2\cos x + 1 = 0$

The trick on this problem is to recognize the expression as a quadratic equation. Replace the trigonometric function, in this case, $\cos x$, with a variable, like u, that will make it easier to see how to factor the expression. If you can see how to factor the expression without the trick, by all means proceed without it.

Let $u = \cos x$, and our equation becomes: $u^2 + 2u + 1 = 0$.

This equation factors to get: $(u + 1)^2 = 0$

Substituting $\cos x$ back in for u gives: $(\cos x + 1)^2 = 0$

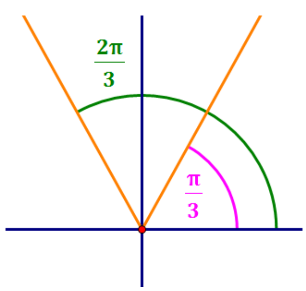
And finally: $\cos x + 1 = 0 \quad \Rightarrow \quad \cos x = -1$

The only solution for this on the interval $[0, 2\pi)$ is: $\boldsymbol{x = \pi}$

Example 5.5: Solve for x on the interval $[0, 2\pi)$: $\sin 4x = \frac{\sqrt{3}}{2}$

When working with a problem in the interval $[0, 2\pi)$ that involves a function of kx, it is useful to expand the interval to $[0, 2k\pi)$ for the first steps of the solution.

In this problem, $k = 4$, so we want all solutions to $\sin u = \frac{\sqrt{3}}{2}$ where $u = 4x$ is an angle in the interval $[0, 8\pi)$. Note that, beyond the two solutions suggested by the diagram, additional solutions are obtained by adding multiples of 2π to those two solutions.



Note that there are 8 solutions because the usual number of solutions (i.e., 2) is increased by a factor of $k = 4$.

Using the diagram at left, we get the following solutions:

$$u = 4x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}, \frac{13\pi}{3}, \frac{14\pi}{3}, \frac{19\pi}{3}, \frac{20\pi}{3}$$

Then, dividing by 4, we get:

$$x = \frac{\pi}{12}, \frac{2\pi}{12}, \frac{7\pi}{12}, \frac{8\pi}{12}, \frac{13\pi}{12}, \frac{14\pi}{12}, \frac{19\pi}{12}, \frac{20\pi}{12}$$

And simplifying, we get:

$$\boldsymbol{x = \frac{\pi}{12}, \frac{\pi}{6}, \frac{7\pi}{12}, \frac{2\pi}{3}, \frac{13\pi}{12}, \frac{7\pi}{6}, \frac{19\pi}{12}, \frac{5\pi}{3}}$$

Solving Trigonometric Equations – Examples

Example 5.6: Solve for x on the interval $[0, 2\pi)$: $\tan^2 x \sin x = \tan^2 x$

$\tan^2 x \sin x - \tan^2 x = 0$

$\tan^2 x\,(\sin x - 1) = 0$

$\tan x = 0$ or $(\sin x - 1) = 0$

$x = 0, \pi$

$\sin x = 1$

$x = \frac{\pi}{2}$

While $x = \frac{\pi}{2}$ is a solution to the equation $\sin x = 1$, $\tan x$ is undefined at $x = \frac{\pi}{2}$, so $\frac{\pi}{2}$ is not a solution to this equation.

$\boldsymbol{x = 0, \pi}$

Example 5.7: Solve for x on the interval $[0, 2\pi)$: $\cos x + 2\cos x \sin x = 0$

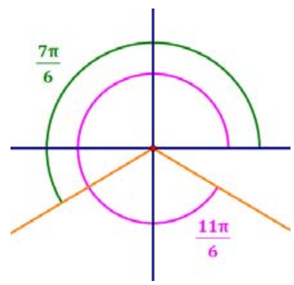
$\cos x\ (1 + 2\sin x) = 0$

$\cos x = 0$ or $(1 + 2\sin x) = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\sin x = -\frac{1}{2}$

$x = \frac{7\pi}{6}, \frac{11\pi}{6}$



$\boldsymbol{x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}}$

Example 5.8: Solve for x on the interval $[0, 2\pi)$: $\cos\left(x + \frac{\pi}{3}\right) + \cos\left(x - \frac{\pi}{3}\right) = 1$

Use: $\cos\ (\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ $\quad\cos\ (\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\cos\left(x + \frac{\pi}{3}\right) + \cos\left(x - \frac{\pi}{3}\right) = 1$

$\cos x \cos\frac{\pi}{3} - \sin x \sin\frac{\pi}{3} + \cos x \cos\frac{\pi}{3} + \sin x \sin\frac{\pi}{3} = 1$

$2\cos x \cos\frac{\pi}{3} = 1$

$2\cos x \cdot \frac{1}{2} = 1$

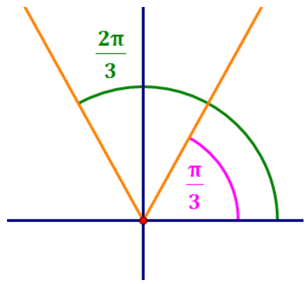
$\cos x = 1 \quad\Rightarrow\quad \boldsymbol{x = 0}$

Solving Trigonometric Equations – Examples

Example 5.9: Solve for all solutions of x: $2\sin x - \sqrt{3} = 0$

$2\sin x = \sqrt{3}$

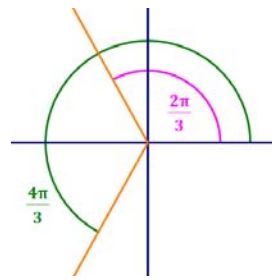
$\sin x = \dfrac{\sqrt{3}}{2}$



The drawing at left illustrates the two angles in $[0, 2\pi)$ for which $\sin x = \frac{\sqrt{3}}{2}$. To get all solutions, we need to add all integer multiples of 2π to these solutions. So,

$$\boldsymbol{x \in \left\{\frac{\pi}{3} + 2n\pi\right\} \cup \left\{\frac{2\pi}{3} + 2n\pi\right\}}$$

Example 5.10: Solve for all solutions of x: $\tan x \sec x = -2\tan x$



$\tan x \sec x + 2\tan x = 0$

$\tan x\,(\sec x + 2) = 0$

$\tan x = 0 \quad \text{or} \quad (\sec x + 2) = 0$

$x = 0 + n\pi = n\pi$

$(\sec x + 2) = 0$

$\sec x = -2$

$\cos x = -\frac{1}{2}$

$x = \frac{2\pi}{3} + 2n\pi$ or $x = \frac{4\pi}{3} + 2n\pi$

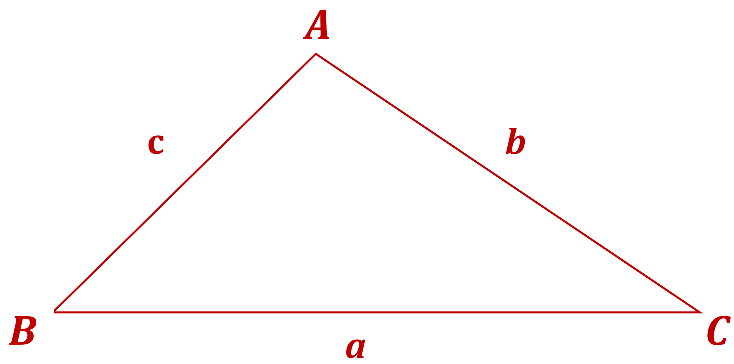
Collecting the various solutions, $\boldsymbol{x \in \{n\pi\} \cup \left\{\frac{2\pi}{3} + 2n\pi\right\} \cup \left\{\frac{4\pi}{3} + 2n\pi\right\}}$

Note: the solution involving the tangent function has two answers in the interval $[0, 2\pi)$. However, they are π radians apart, as most solutions involving the tangent function are. Therefore, we can simplify the answers by showing only one base answer and adding $n\pi$, instead of showing two base answers that are π apart, and adding $2n\pi$ to each.

For example, the following two solutions for $\tan x = 0$ are telescoped into the single solution given above:

$$\left.\begin{aligned} x &= 0 + 2n\pi = \{\dots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \dots\} \\ x &= \pi + 2n\pi = \{\dots, -3\pi, -\pi, \pi, 3\pi, 5\pi \dots\} \end{aligned}\right\} \quad x = 0 + n\pi = \{\dots, -2\pi, -\pi, 0, \pi, 2\pi, \dots\}$$

Laws of Sines and Cosines



The triangle above can be oriented in any manner. It does not matter which angle is A, B or C. However,

- **Side a** is always opposite (across from) $\angle A$.
- **Side b** is always opposite (across from) $\angle B$.
- **Side c** is always opposite (across from) $\angle C$.

Law of Sines (see above illustration)

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of Cosines (see above illustration)

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

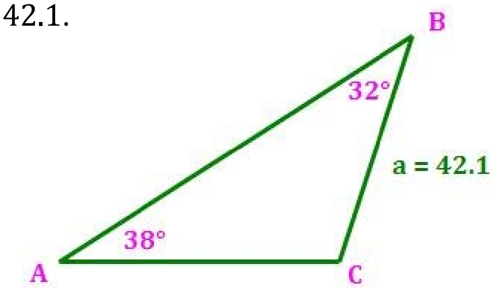
$$c^2 = a^2 + b^2 - 2ab \cos C$$

The law of cosines can be described in words as follow: The square of any side is the sum of the squares of the other two sides minus twice the product of those two sides and the cosine of the angle between them.

It looks a lot like the Pythagorean Theorem, with the minus term appended.

Laws of Sines and Cosines – Examples

Example 6.1: Solve the triangle, given: $\mathrm{A} = 38°,\ \mathrm{B} = 32°,\ \mathrm{a} = 42.1$.



To solve: find the third angle, and then use the Law of Sines.

$$m\angle C = 180° - 38° - 32° = 110°$$

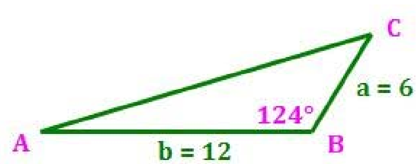
Then use the Law of Sines to find the lengths of the two remaining sides.

$$\frac{42.1}{\sin 38°} = \frac{b}{\sin 32°} \quad \Rightarrow \quad b = \frac{42.1 \cdot \sin 32°}{\sin 38°} = 36.2$$

$$\frac{42.1}{\sin 38°} = \frac{c}{\sin 110°} \quad \Rightarrow \quad c = \frac{42.1 \cdot \sin 110°}{\sin 38°} = 64.3$$

Example 6.2: Solve the triangle, given: $\mathrm{a} = 6,\ \mathrm{c} = 12,\ \mathrm{B} = 124°$.

First, draw the triangle from the information you are given. This will help you get an idea of whether the values you calculate in this problem are reasonable.



Next, find the length of the 3rd side of the triangle using the Law of Cosines: $b^2 = a^2 + c^2 - 2ac\ \cos B$

$$b^2 = 6^2 + 12^2 - 2(6)(12)(\cos 124°) = 236.52378$$

$$b = \sqrt{236.52378} = 16.14075 \sim 16.1$$

Use the Law of Sines to find the measure of one of the remaining angles.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \Rightarrow \quad \frac{6}{\sin A} = \frac{16.14075}{\sin 124°} \quad \Rightarrow \quad \sin A = 0.3082$$

$$m\angle A = \sin^{-1} 0.3082 = 18°$$

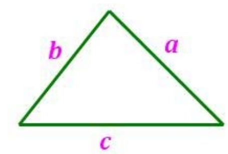
The measure of the remaining angle can be calculated either from the Law of Sines or from knowledge that the sum of the three angles inside a triangle is 180°.

$$m\angle C = 180° - 124° - 18° = 38°$$

Solving an Oblique Triangle

Several methods exist to solve an oblique triangle, i.e., a triangle with no right angle. The appropriate method depends on the information available for the triangle. All methods require that the length of at least one side be provided. In addition, one or two angle measures may be provided. Note that if two angle measures are provided, the measure of the third is determined (because the sum of all three angle measures must be 180°). The methods used for each situation are summarized below.

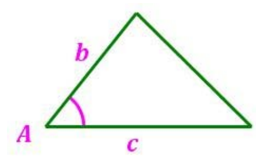
Given Three Sides and no Angles (SSS)



Given three segment lengths and no angle measures, do the following:

- Use the Law of Cosines to determine the measure of one angle.
- Use the Law of Sines to determine the measure of one of the two remaining angles.
- Subtract the sum of the measures of the two known angles from 180° to obtain the measure of the remaining angle.

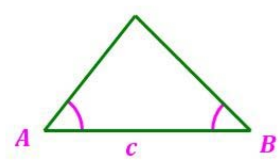
Given Two Sides and the Angle between Them (SAS)



Given two segment lengths and the measure of the angle that is between them, do the following:

- Use the Law of Cosines to determine the length of the remaining leg.
- Use the Law of Sines to determine the measure of one of the two remaining angles.
- Subtract the sum of the measures of the two known angles from 180° to obtain the measure of the remaining angle.

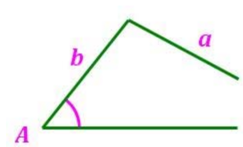
Given One Side and Two Angles (ASA or AAS)



Given one segment length and the measures of two angles, do the following:

- Subtract the sum of the measures of the two known angles from 180° to obtain the measure of the remaining angle.
- Use the Law of Sines to determine the lengths of the two remaining legs.

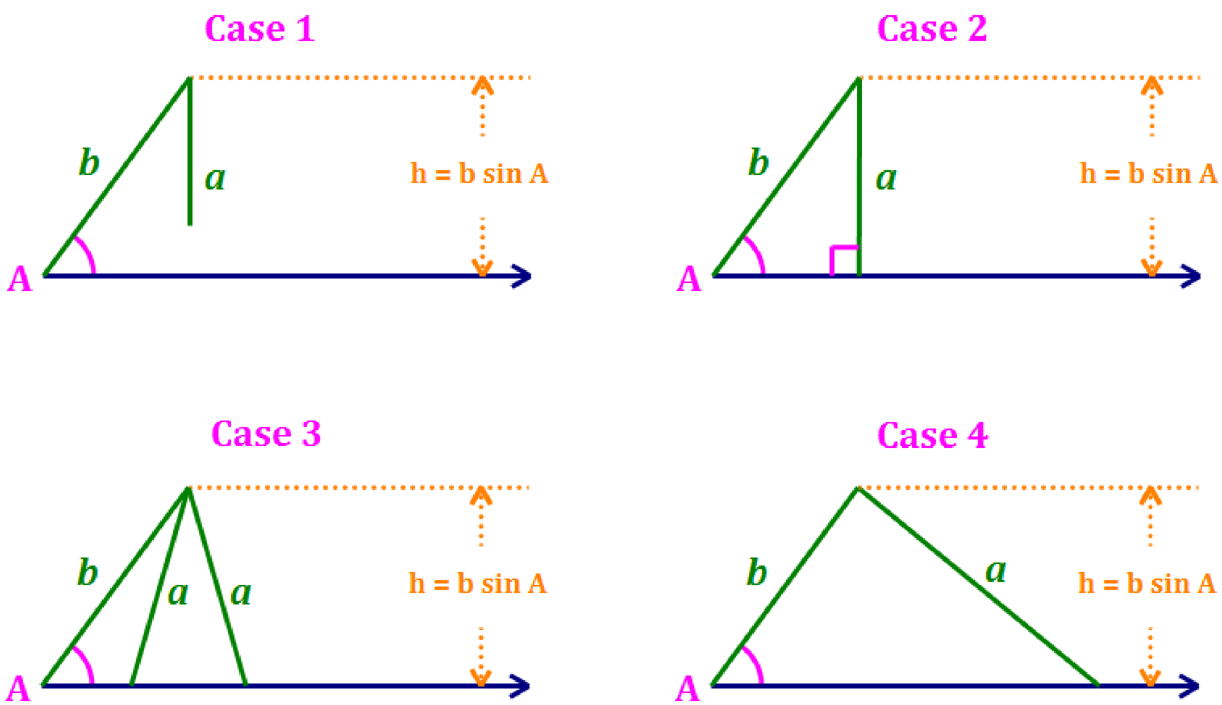
Given Two Sides and an Angle not between Them (SSA)



This is the Ambiguous Case (see below). Several possibilities exist, depending on the lengths of the sides and the measure of the angle. The possibilities are considered on the following pages.

The Ambiguous Case (SSA)

Given two segment lengths and an angle that is not between them, it is not clear whether a triangle is defined. It is possible that the given information will define a single triangle, two triangles, or even no triangle. Because there are multiple possibilities in this situation, it is called the ambiguous case. Here are the possibilities:



There are three cases in which $a < b$.

Case 1: $a < b \sin A$ Produces no triangle because a is not long enough to reach the base.

Case 2: $a = b \sin A$ Produces one (right) triangle because a is exactly long enough to reach the base. a forms a right angle with the base, and is the height of the triangle.

Case 3: $a > b \sin A$ Produces two triangles because a is the right size to reach the base in two places. The angle from which a swings from its apex to meet the base can take two values.

There is one case in which $a \geq b$.

Case 4: $a \geq b$ Produces one triangle because a is too long to reach the base in more than one location.

The Ambiguous Case (SSA) – Method Comparison

Three methods to solve the ambiguous case are provided in this handbook – the Height Comparison Method, the Butterfly Method, and the Sine Validity Method. The Height Comparison Method and the Butterfly Method are explored further in this chapter. The Sine Validity Method is explored further in Appendix B.

Each method is briefly described, along with its pros and cons, in the table below. Assume the information given in the problem is the measure of Angle A and the lengths of sides a and b. This is consistent with the diagrams presented on the previous page.

Method	**Height Comparison Method**	**Butterfly Method**	**Sine Validity Method**
Description	Calculate the height, h, of the ambiguous triangle. Compare a to h and b to determine the number of possible triangles. If there are one or more triangles, continue to solve the triangles, starting with Angle B, using the Law of Sines and the fact that the sum of the angles in a triangle is $180°$.	Calculate the length of the missing side, c, using the Law of Cosines. This will result in a quadratic equation with zero, one, two real solutions, indicating the number of possible triangles. The Law of Sines is used to determine the measure of Angle B in any triangles. Angle C is found using the fact that the sum of the angles in a triangle is $180°$.	Calculate the sine of Angle B using the Law of Sines. The number of triangles is based on whether $\sin B$ is valid (in the closed interval -1 to 1). If there are one or more triangles, continue to solve the triangles using the Law of Sines and the fact that the sum of the angles in a triangle is $180°$.
Pros	The Law of Sines is used multiple times. It is easy to remember and straightforward to use. This is the method preferred by most students.	Use of the Law of Cosines answers two questions with one calculation: 1) how many triangles exist, and 2) the length of the missing side, c, of any triangles that do exist. Length(s) of c may be more accurate.	Finding valid Angle $B's$ answers: 1) whether any triangles exist, and 2) the value of $\sin B$ for any triangles that do exist.
Cons	An additional length must be calculated – the height, h. Students may forget how the relationship among lengths $a, h,$ and b determines the number of triangles. Rounding the measures of the angles produces less accuracy throughout.	The Law of Cosines is more difficult to remember than the Law of Sines. Both the Law of Sines and the Law of Cosines must be used. A quadratic equation in the variable c must be developed and solved.	This method is more difficult for students to understand than the other two methods. If $\sin B$ is valid, the student must still determine whether one or two triangles exist by comparing the lengths of a and b.

The Ambiguous Case (SSA)

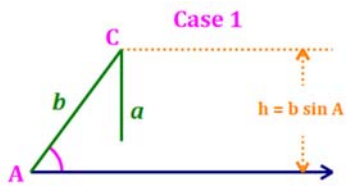
Solving the Ambiguous Case – Height Comparison Method

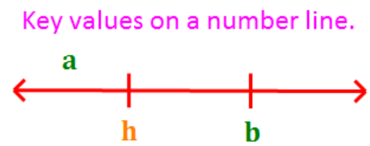
How do you solve a triangle (or two) in the ambiguous case? Assume the information given is the lengths of sides a and b, and the measure of Angle A. Use the following steps:

Step 1: Calculate the height of the triangle (in this development, $h = b \sin A$).

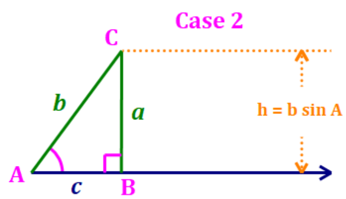
Step 2: Compare a to the height of the triangle, h:

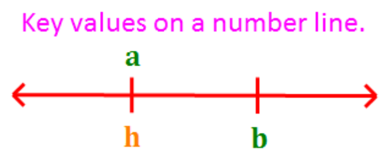
- If $a < h$, then we have Case 1 – there is no triangle. Stop here.





- If $a = h$, then $m\angle B = 90°$, and we have Case 2 – a right triangle. Proceed to Step 4.

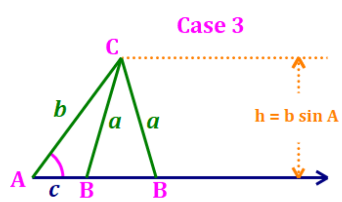


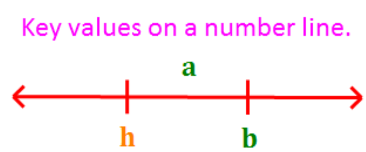


- If $a > h$, then we have Case 3 or Case 4. Proceed to the Step 3 to determine which.

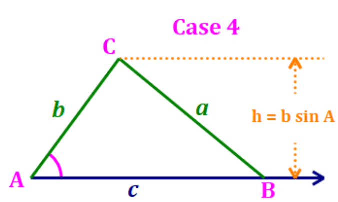
Step 3: Compare a to b.

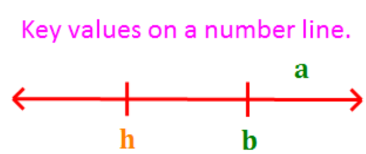
- If $a < b$, then we have Case 3 – two triangles. Calculate $\sin B$ using the Law of Sines. Find the two angles in the interval $(0°, 180°)$ with this sine value; each of these $\angle B$'s produces a separate triangle. Proceed to Step 4 and calculate the remaining values for each.





- If $a \geq b$, then we have Case 4 – one triangle. Find $m\angle B$ using the Law of Sines. Proceed to Step 4.





The Ambiguous Case (SSA)

Solving the Ambiguous Case – Height Comparison Method

Step 4: Calculate C. At this point, we have the lengths of sides a and b, and the measures of Angles A and B. If we are dealing with Case 3 – two triangles, we must perform Steps 4 and 5 for each triangle.

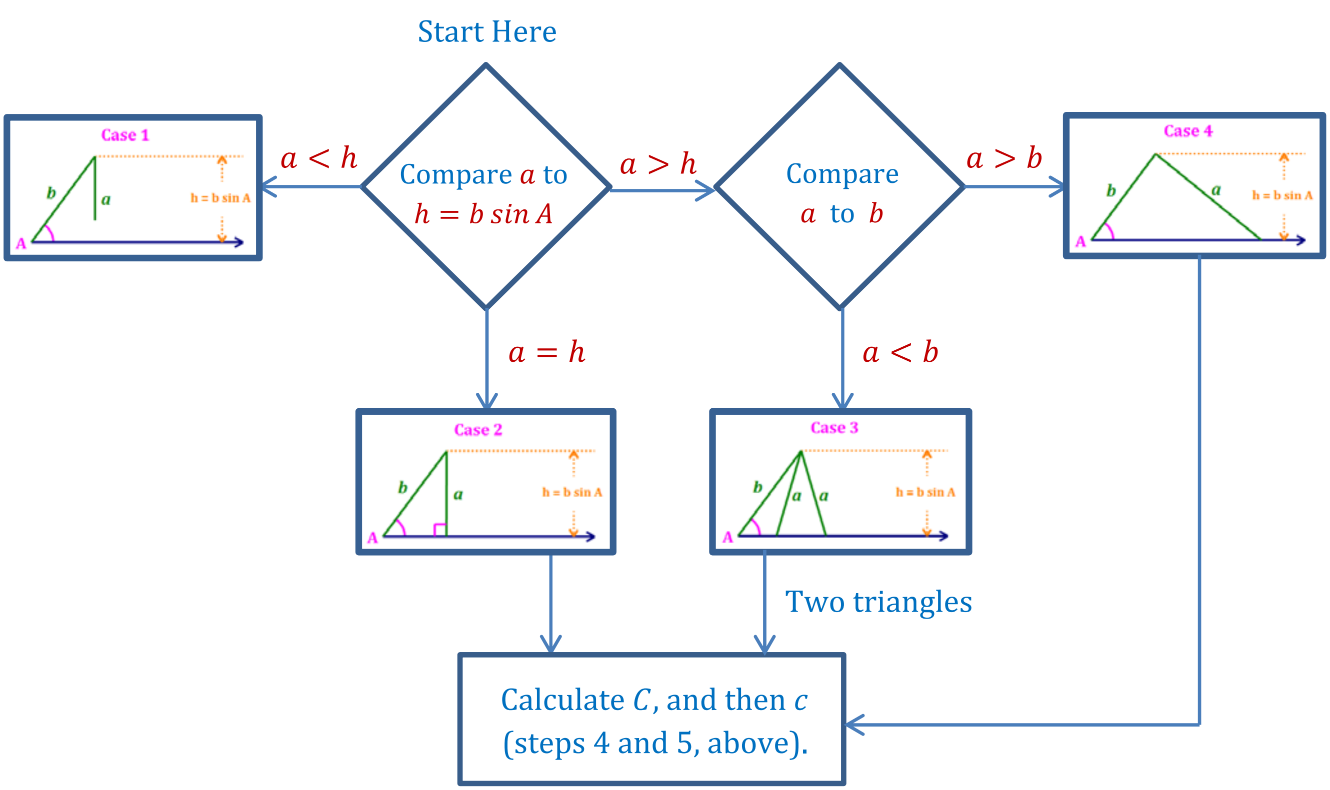
Step 4 is to calculate the measure of Angle C as follows: $m\angle C = 180° - m\angle A - m\angle B$

Step 5: Calculate c. Finally, we calculate the value of c using the Law of Sines.

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \Rightarrow \quad c = \frac{a \sin C}{\sin A} \qquad \text{or} \qquad \frac{b}{\sin B} = \frac{c}{\sin C} \quad \Rightarrow \quad c = \frac{b \sin C}{\sin B}$$

Note: using a and $\angle A$ may produce more accurate results since both of these values are given.

Ambiguous Case Flowchart – Height Comparison Method



Ambiguous Case – Height Comparison Method Examples

Example 6.3: Determine whether the following measurements produce one triangle, two triangles, or no triangle: $m\angle C = 35°$, $a = 18.7$, $c = 16.1$. Solve any triangles that result.

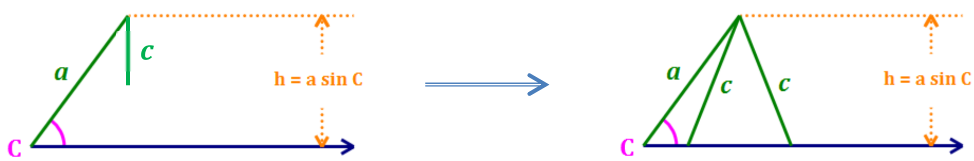
Since we are given two sides and an angle that is not between them, this is the ambiguous case.

We draw this situation with $\angle C$ on the left and c hanging down, as shown below.

Step 1: Calculate $h = a \sin C$. $h = 18.7 \cdot \sin 35° = 10.725$

Step 2: Compare c to h. $c = 16.1 > h = 10.725$.

Step 3: Compare c to a. $c = 16.1 < a = 18.7$, so we have Case 3 – two triangles.



Calculate $\sin A$ using the Law of Sines:

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \Rightarrow \quad \frac{18.7}{\sin A} = \frac{16.1}{\sin 35°} \quad \Rightarrow \quad \sin A = 0.6662$$

Two angles in the interval $(0°, 180°)$ have this sine value. Let's find them:

$$m\angle A_1 = \sin^{-1} 0.6662 = 42° \quad \text{or} \quad m\angle A_2 = 180° - 42° = 138°$$

Since we will have two triangles, we must solve each.

Triangle 1 – Start with:	**Triangle 2 – Start with:**
$a = 18.7,\ c = 16.1$ $m\angle C = 35°,\ m\angle A_1 = 42°$	$a = 18.7,\ c = 16.1$ $m\angle C = 35°,\ m\angle A_2 = 138°$
Step 4:	**Step 4:**
$m\angle B = 180° - 35° - 42° = 103°$	$m\angle B = 180° - 35° - 138° = 7°$
Step 5:	**Step 5:**
$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{b}{\sin 103°} = \frac{16.1}{\sin 35°}$	$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{b}{\sin 7°} = \frac{16.1}{\sin 35°}$
$b = 27.4$	$b = 3.4$

Ambiguous Case – Height Comparison Method Examples

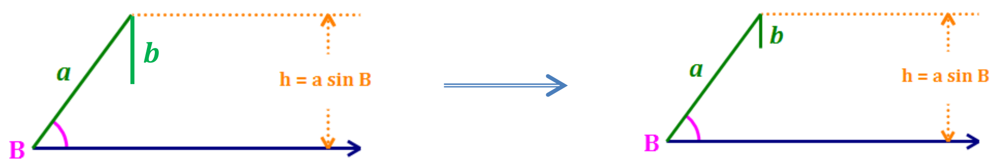
Example 6.4: Determine whether the following measurements produce one triangle, two triangles, or no triangle: $m\angle \text{B} = 88°$, $\text{b} = 2$, $\text{a} = 23$. Solve any triangles that result.

Since we are given two sides and an angle that is not between them, this is the ambiguous case.

We draw this situation with $\angle B$ on the left and b hanging down, as shown below.

Step 1: Calculate $h = a \sin B$. $h = 23 \cdot \sin 88° = 22.986$

Step 2: Compare b to h. $b = 2 < h = 22.986.$



Stop. Since $b < h$, we have Case 1 – no triangle.

The Ambiguous Case (SSA)

Solving the Ambiguous Case – Butterfly Method

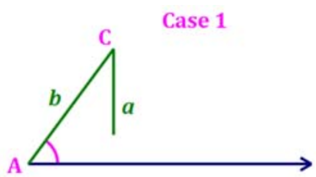
Want another method to solve the ambiguous case? Assume the information given is the lengths of sides a and b, and the measure of Angle A. Use the following steps involving the Law of Cosines:

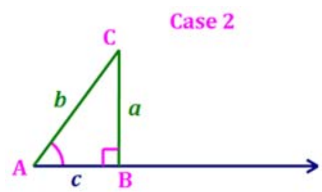
Step 1: Find the length of side c using the law of cosines that includes the given angle A:

- $a^2 = b^2 + c^2 - 2bc\cos A$. The values of $a, b, \cos A$ are known.
- This results in a quadratic function in c: $c^2 - (2b\cos A)c + (b^2 - a^2) = 0$
- Solve the quadratic for the two values of c.

Step 2: How many triangles are there? $0, 1$ or 2?

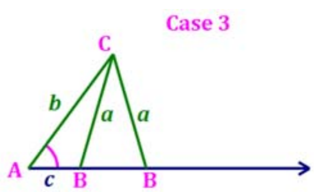
- If the values of c are imaginary, then we have Case 1 – there is no triangle. Stop here.
- If the values of c are the same, then we have Case 2 – a right triangle. $m\angle B = 90°$.

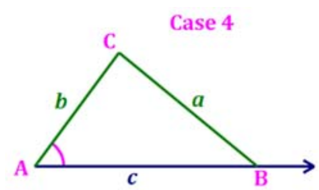




- If the values of c are distinct and both positive, then we have Case 3 – two triangles.
- If the values of c are distinct and one is negative, then we have Case 4 – one triangle.

In cases 3 and 4, proceed to the next step using only positive values of c.





Step 3: Calculate Angles B and C. At this point, we have the lengths of all three sides and the measure of Angle A. If we are dealing with Case 3 – two triangles, we must perform this step for each of the triangles. Using the Law of Sines,

$$\left.\begin{array}{l} \dfrac{a}{\sin A} = \dfrac{b}{\sin B} \quad \Rightarrow \quad \sin B = \dfrac{b\sin A}{a} \\[2ex] m\angle B = \sin^{-1}\dfrac{b\sin A}{a} \\[2ex] m\angle C = 180° - m\angle A - m\angle B \end{array}\right\} \text{ or } \left\{\begin{array}{l} \dfrac{a}{\sin A} = \dfrac{c}{\sin C} \quad \Rightarrow \quad \sin C = \dfrac{c\sin A}{a} \\[2ex] m\angle C = \sin^{-1}\dfrac{c\sin A}{a} \\[2ex] m\angle B = 180° - m\angle A - m\angle C \end{array}\right.$$

Keep extra decimals throughout the calculation, rounding to the desired accuracy only at the end of the process.

Ambiguous Case – Butterfly Method Examples

Example 6.5: Determine whether the following measurements produce one triangle, two triangles, or no triangle: $m\angle C = 35°$, $a = 18.7$, $c = 16.1$. Solve any triangles that result.

Since we are given two sides and an angle that is not between them, this is the ambiguous case.

Using the Butterfly Method, we do not yet need to draw the triangle(s).

Step 1: To find the length of side b, use the **Law of Cosines** that includes the given angle C:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

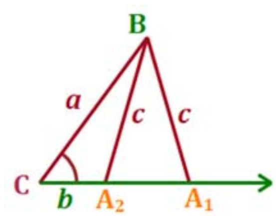
$$16.1^2 = 18.7^2 + b^2 - 2(18.7)b(\cos 35°)$$

$$b^2 - 30.6363\,b + 90.48 = 0 \quad \rightarrow \quad b = \frac{30.6363 \pm \sqrt{(-30.6363)^2 - 4(1)(90.48)}}{2(1)}$$

$$b = \{27.33, 3.31\}$$

Step 2: How many triangles are there? 0, 1 or 2?

Since the two solutions for b are both real and positive, they are both possible lengths of side b, so there are two triangles. Let's draw:



Step 3: Calculate the measures of angles A and B.

Calculate $\sin A$ using the Law of Sines:

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \Rightarrow \quad \frac{18.7}{\sin A} = \frac{16.1}{\sin 35°} \quad \Rightarrow \quad \sin A = 0.6662$$

Two angles in the interval $(0°, 180°)$ have this sine value. Let's find them:

$$m\angle A_1 = \sin^{-1} 0.6662 = 42° \quad \text{or} \quad m\angle A_2 = 180° - 42° = 138°$$

The diagram shows that the larger value of b corresponds to Angle A_1.

Triangle 1 – Start with:

$a = 18.7$, $b = 27.33$, $c = 16.1$
$m\angle C = 35°$, $m\angle A_1 = 42°$

Complete the triangle:

$m\angle B = 180° - 35° - 42° = 103°$

Triangle 2 – Start with:

$a = 18.7$, $b = 3.31$, $c = 16.1$
$m\angle C = 35°$, $m\angle A_2 = 138°$

Complete the triangle:

$m\angle B = 180° - 35° - 138° = 7°$

Note: the solutions using the Butterfly Method are more accurate than those in Example 6.3 because the Butterfly Method uses less rounding throughout.

Ambiguous Case – Butterfly Method Examples

Example 6.6: Determine whether the following measurements produce one triangle, two triangles, or no triangle: $m\angle \mathrm{B} = 88°$, $\mathrm{b} = 2$, $\mathrm{a} = 23$. Solve any triangles that result.

Since we are given two sides and an angle that is not between them, this is the ambiguous case.

Using the Butterfly Method, we do not yet need to draw the triangle(s).

Step 1: Find the length of side c using the law of cosines that includes the given angle B:

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$2^2 = 23^2 + c^2 - 2(23)c(\cos 88°)$$

$$c^2 - 1.6054\,b + 525 = 0 \quad \rightarrow \quad b = \frac{1.6054 \pm \sqrt{(-1.6054)^2 - 4(1)(525)}}{2(1)}$$

$$c = \{0.80 \pm 22.90i\}$$

Notice that the solutions have an imaginary component; they are not real.

Step 2: How many triangles are there? 0, 1 or 2?

Since the two solutions are not real, they are not possible lengths of the side of a triangle. Therefore, there are no triangles possible with the information given.

Sine Validity Method

A third method begins by calculating the measure of angle A using the Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \Rightarrow \quad \frac{23}{\sin A} = \frac{2}{\sin 88°} \quad \Rightarrow \quad \sin A = 11.493$$

$$m\angle A = \sin^{-1} 11.493$$

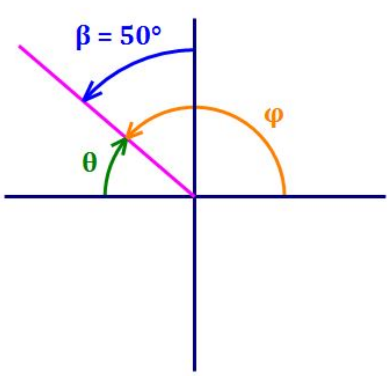
11.493 is not a valid sine value (recall that sine values range from -1 to 1). Therefore, the given values do not define a triangle.

This method for dealing with the ambiguous case is explored in Appendix B.

Bearings

Bearings are described differently from other angles in Trigonometry. A bearing is a clockwise or counterclockwise angle whose initial side is either due north or due south. The student will need to translate these into reference angles and/or polar angles to solve problems involving bearings.

Some bearings, along with the key associated angles are shown in the illustrations below. The bearing angle is shown as $\boldsymbol{\beta}$, the reference angle is shown as $\boldsymbol{\theta}$, and the polar angle is shown as $\boldsymbol{\varphi}$.

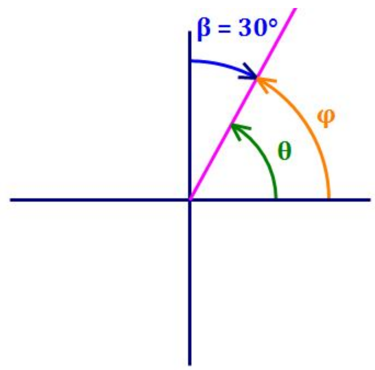


Bearing: N 50° W

Bearing Angle: $\beta = 50°$

Reference Angle: $\theta = 40°$

Polar Angle: $\varphi = 140°$

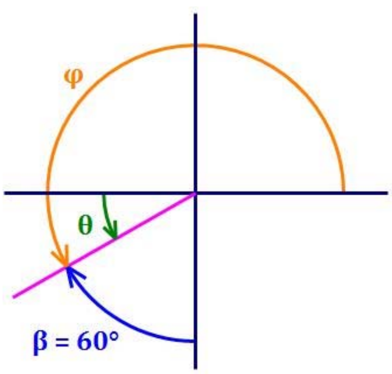


Bearing: N 30° E

Bearing Angle: $\beta = 30°$

Reference Angle: $\theta = 60°$

Polar Angle: $\varphi = 60°$

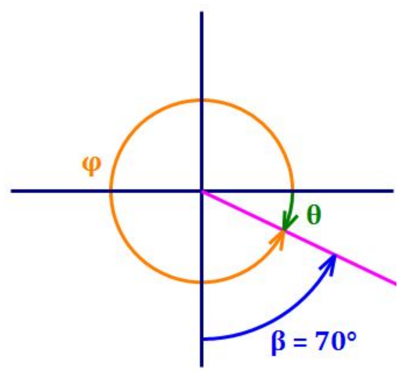


Bearing: S 60° W

Bearing Angle: $\beta = 60°$

Reference Angle: $\theta = 30°$

Polar Angle: $\varphi = 210°$



Bearing: S 70° E

Bearing Angle: $\beta = 70°$

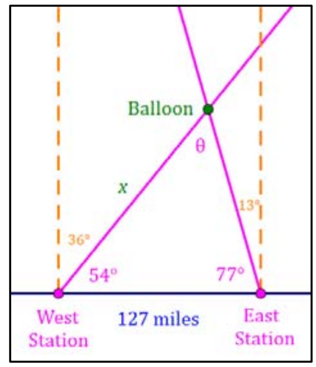
Reference Angle: $\theta = 20°$

Polar Angle: $\varphi = 340°$

Bearings – Examples

Example 6.7: Two tracking stations are on the equator 127 miles apart. A weather balloon is located on a bearing of N 36° E from the western station and on a bearing of N 13° W from the eastern station. How far is the balloon from the western station?

The bearing angles given are those shown in orange in the diagram at right. The first step is to calculate the reference angles shown in magenta in the diagram.



$$90° - 36° = 54°$$

$$90° - 13° = 77°$$

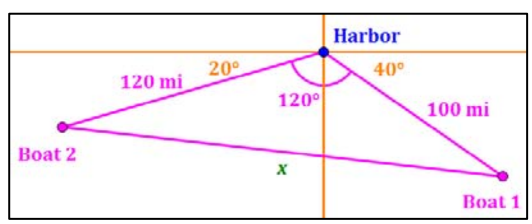
$$\theta = 180° - 54° - 77° = 49°$$

Then, use the Law of Sines, as follows:

$$\frac{127}{\sin 49°} = \frac{x}{\sin 77°} \quad \Rightarrow \quad x = 164.0 \text{ miles}$$

Example 6.8: Two sailboats leave a harbor in the Bahamas at the same time. The first sails at 25 mph in a direction S 50° E. The second sails at 30 mph in a direction S 70° W. Assuming that both boats maintain speed and heading, after 4 hours, how far apart are the boats?

Let's draw a diagram to illustrate this situation. The lengths of two sides of a triangle are based on the distances the boats travel in four hours. The bearing angles given are used to calculate the reference shown in orange in the diagram below.



Boat 1 travels: 25 mph $\cdot$ 4 hours $= 100$ miles at a heading of S 50° E. This gives a reference angle of $90° - 50° = 40°$ below the positive x-axis.

Boat 2 travels: 30 mph $\cdot$ 4 hours $= 120$ mi. at a heading of S 70° W. This gives a reference angle of $90° - 70° = 20°$ below the negative x-axis.

Using the Law of Cosines, we can calculate:

$$x^2 = 100^2 + 120^2 - 2(100)(120)(\cos 120°) = 36{,}400 \quad \Rightarrow \quad x = 190.8 \text{ miles}$$

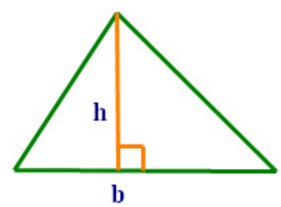
Area of a Triangle

Area of a Triangle

There are a number of formulas for the area of a triangle, depending on what information about the triangle is available.

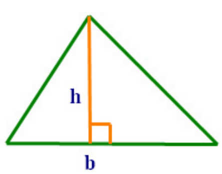
Geometry Formula: This formula, learned in Elementary Geometry, is probably most familiar to the student. It can be used when the base and height of a triangle are either known or can be determined.

$$A = \frac{1}{2}bh$$

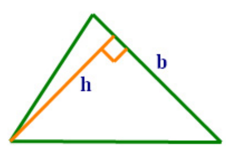


where, b is the length of the base of the triangle.
h is the height of the triangle.

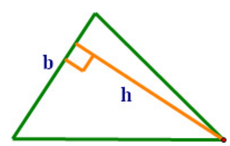
Note: The base can be any side of the triangle. The height is the length of the altitude of whichever side is selected as the base. So, you can use:



or

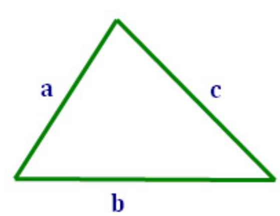


or



Heron's Formula: **Heron's formula** for the area of a triangle can be used when the lengths of all of the sides are known. Sometimes this formula, though less appealing, can be very useful.

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$



where, $s = \frac{1}{2}P = \frac{1}{2}(a+b+c)$.
a, b, c are the lengths of the sides of the triangle.

Note: s is called the semi-perimeter of the triangle because it is half of the triangle's perimeter.

Area of a Triangle (cont'd)

Trigonometric Formulas

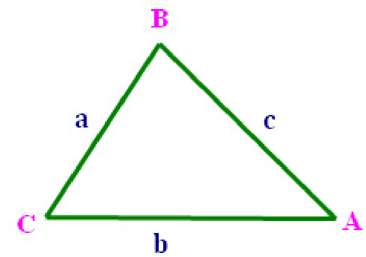
The following formulas for the area of a triangle can be derived from the Geometry formula, $A = \frac{1}{2}bh$, using Trigonometry. Which one to use depends on the information available:

Two angles and one side:

$$A = \frac{1}{2} \cdot \frac{a^2 \cdot \sin B \cdot \sin C}{\sin A} = \frac{1}{2} \cdot \frac{b^2 \cdot \sin A \cdot \sin C}{\sin B} = \frac{1}{2} \cdot \frac{c^2 \cdot \sin A \cdot \sin B}{\sin C}$$

Two sides and the angle between them:

$$A = \frac{1}{2}\, ab\, \sin C = \frac{1}{2}\, ac\, \sin B = \frac{1}{2}\, bc\, \sin A$$



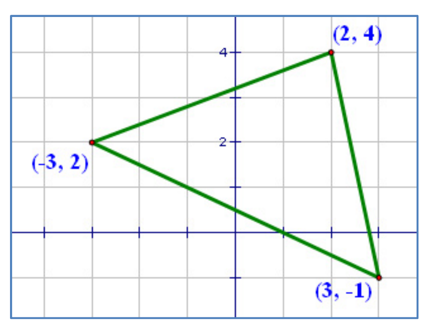
Coordinate Geometry Formula

If the three vertices of a triangle are displayed in a coordinate plane, the formula below, using a determinant, will give the area of a triangle.

Let vertices of a triangle in the coordinate plane be: (x_1, y_1), (x_2, y_2), (x_3, y_3). Then, the area of the triangle is:

$$A = \frac{1}{2} \cdot \left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$

Example 7.1: For the triangle in the figure at right, the area is:



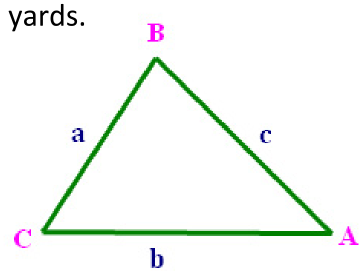
$$A = \frac{1}{2} \cdot \left| \begin{vmatrix} 2 & 4 & 1 \\ -3 & 2 & 1 \\ 3 & -1 & 1 \end{vmatrix} \right|$$

$$= \frac{1}{2} \cdot \left| \, 2\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} - 4\begin{vmatrix} -3 & 1 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} -3 & 2 \\ 3 & -1 \end{vmatrix} \, \right|$$

$$= \frac{1}{2} \cdot \left| \, (2(3) - 4(-6) + 1(-3)) \, \right| = \frac{1}{2} \cdot 27 = \frac{27}{2}$$

Area of a Triangle – Examples

Example 7.2: Find the area of the triangle if: C = 120°, a = 4 yards, b = 5 yards.



$$Area = \frac{1}{2}\, ab \sin C$$

$$= \frac{1}{2} \cdot 4 \cdot 5 \cdot \sin 120° = 10 \cdot \frac{\sqrt{3}}{2} = 8.66 \text{ yards}^2$$

Example 7.3: Find the area of the triangle if: $a = 10$ yards, $b = 11$ yards, $c = 15$ yards.

To solve this problem, we will use Heron's formula:

$$Area = \sqrt{s(s-a)(s-b)(s-c)} \qquad s = \frac{1}{2}(a+b+c)$$

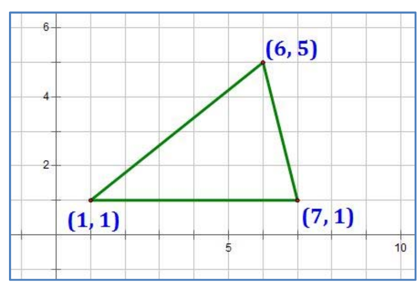
First calculate: $s = \frac{1}{2}(10 + 11 + 15) = 18$

Then, $Area = \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{18(18-10)(18-11)(18-15)}$$

$$= \sqrt{18 \cdot 8 \cdot 7 \cdot 3} = 12\sqrt{21} = 54.99 \text{ yards}^2$$

Example 7.4: Find the area of the triangle in the figure below using Coordinate Geometry:



$$A = \frac{1}{2} \cdot \left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right| = \frac{1}{2} \cdot \left| \begin{vmatrix} 1 & 1 & 1 \\ 7 & 1 & 1 \\ 6 & 5 & 1 \end{vmatrix} \right|$$

$$= \frac{1}{2} \cdot \left| 1 \begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix} - 1 \begin{vmatrix} 7 & 1 \\ 6 & 1 \end{vmatrix} + 1 \begin{vmatrix} 7 & 1 \\ 6 & 5 \end{vmatrix} \right|$$

$$= \frac{1}{2} \cdot \left| \left(1(-4) - 1(1) + 1(29)\right) \right| = \frac{1}{2} \cdot 24 = 12$$

Note: It is easy to see that this triangle has a base of length 6 and a height of 4, so from Elementary Geometry, the area of the triangle is: $A = \frac{1}{2} bh = \frac{1}{2} \cdot 6 \cdot 4 = 12$ (same answer). The student may wish to test the other methods for calculating area that are presented in this chapter to see if they produce the same result. (Hint: they do.)

Polar Coordinates

Polar coordinates are an alternative method of describing a point in a Cartesian plane based on the distance of the point from the origin and the polar angle whose terminal side contains the point.

Let's take a look at the relationship between a point's rectangular coordinates (x, y) and its polar coordinates (r, θ).

The magnitude, r, is the distance of the point from the origin: $r = \sqrt{x^2 + y^2}$

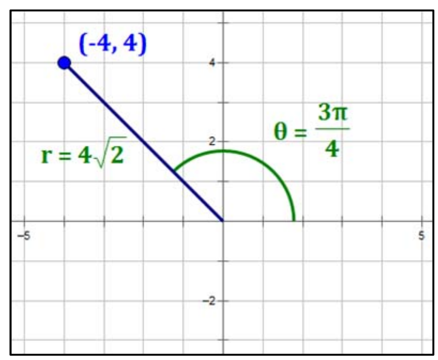
The angle, θ, is the polar angle whose terminal side contains the point. Generally, this angle is expressed in radians, not degrees:

$\tan\theta = \frac{y}{x}$ so $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, adjusted to be in the appropriate quadrant.

Conversion from polar coordinates to rectangular coordinates is straightforward:

$x = r\cos\theta$ and $y = r\sin\theta$

Example 8.1: Express the rectangular form $(-4, 4)$ in polar coordinates:



Given: $x = -4$ $y = 4$

$r = \sqrt{x^2 + y^2} = \sqrt{(-4)^2 + 4^2} = 4\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{4}{-4}\right) = \tan^{-1}(-1)$ in Quadrant II,

so $\theta = \frac{3\pi}{4}$

So, the coordinates of the point are as follows:

Rectangular coordinates: $(-4, 4)$ Polar Coordinates: $\left(4\sqrt{2}, \frac{3\pi}{4}\right)$

Example 8.2: Express the polar form $\left(4\sqrt{2}, \frac{3\pi}{4}\right)$ in rectangular coordinates:

Given: $r = 4\sqrt{2}$ $\theta = \frac{3\pi}{4}$

$x = r\cos\theta = 4\sqrt{2}\cdot\cos\frac{3\pi}{4} = 4\sqrt{2}\cdot\left(-\frac{\sqrt{2}}{2}\right) = -4$

$y = r\sin\theta = 4\sqrt{2}\cdot\sin\frac{3\pi}{4} = 4\sqrt{2}\cdot\left(\frac{\sqrt{2}}{2}\right) = 4$

So, the coordinates of the point are as follows:

Polar Coordinates: $\left(4\sqrt{2}, \frac{3\pi}{4}\right)$ Rectangular coordinates: $(-4, 4)$

Polar Form of Complex Numbers

Expressing Complex Numbers in Polar Form

A complex number can be represented as point in the Cartesian Plane, using the horizontal axis for the real component of the number and the vertical axis for the imaginary component of the number.

If we express a complex number in rectangular coordinates as $z = a + bi,$ we can also express it in polar coordinates as $z = r(\cos\theta + i\sin\theta)$, with $\theta \in [0, 2\pi)$. Then, the equivalences between the two forms for z are:

Convert Rectangular to Polar	Convert Polar to Rectangular
Magnitude: $\lvert z\rvert = r = \sqrt{a^2 + b^2}$	x-coordinate: $a = r\cos\theta$
Angle: $\theta = \tan^{-1}\left(\frac{b}{a}\right)$	y-coordinate: $b = r\sin\theta$

Since θ will generally have two values on $[0, 2\pi)$, we need to be careful to select the angle in the quadrant in which $z = a + bi$ resides.

Operations on Complex Numbers in Polar Form

Around 1740, Leonhard Euler proved that: $e^{i\theta} = \cos\theta + i\sin\theta$. As a result, we can express any complex number as an exponential form of e. That is:

$$z = a + bi = r(\cos\theta + i\sin\theta) = r \cdot e^{i\theta}$$

Thinking of each complex number as being in the form $z = r \cdot e^{i\theta}$, the following rules regarding operations on complex numbers can be easily derived based on the properties of exponents.

Let: $z_1 = a_1 + b_1 i = r_1(\cos\theta + i\sin\theta)$, $z_2 = a_2 + b_2 i = r_2(\cos\varphi + i\sin\varphi)$. Then,

Multiplication: $z_1 \cdot z_2 = r_1 r_2 [\cos(\theta + \varphi) + i\sin(\theta + \varphi)]$

So, to multiply complex numbers, you multiply their magnitudes and add their angles.

Division: $z_1 \div z_2 = \frac{r_1}{r_2} [\cos(\theta - \varphi) + i\sin(\theta - \varphi)]$

So, to divide complex numbers, you divide their magnitudes and subtract their angles.

Powers: ${z_1}^n = {r_1}^n(\cos n\theta + i\sin n\theta)$

This results directly from the multiplication rule.

Roots: $\sqrt[n]{z_1} = \sqrt[n]{r_1}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$ *also, see "DeMoivre's Theorem" below*

This results directly from the power rule if the exponent is a fraction.

Operations on Complex Numbers - Examples

Example 8.3: Find the product: $z_1 \cdot z_2$.

$z_1 = \sqrt{3}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) \longleftarrow$ shorthand is: $z_1 = \sqrt{3}\text{ cis}\left(\frac{7\pi}{4}\right) \quad = \sqrt{3}e^{i\cdot\frac{7\pi}{4}}$

$z_2 = \sqrt{6}\left(\cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}\right) \longleftarrow$ shorthand is: $z_2 = \sqrt{6}\text{ cis}\left(\frac{9\pi}{4}\right) \quad = \sqrt{6}e^{i\cdot\frac{9\pi}{4}}$

To multiply two numbers in polar form, multiply the r-values and add the angles.

$$z_1 \cdot z_2 = \sqrt{3} \cdot \sqrt{6} \cdot \text{cis}\left(\frac{7\pi}{4} + \frac{9\pi}{4}\right)$$

$$= 3\sqrt{2}\text{ cis}(4\pi) = 3\sqrt{2}\text{ cis } 0 = 3\sqrt{2} \quad \text{because cis } 0 = 1.$$

Note: multiplication may be easier to understand in exponential form, since exponents are added when values with the same base are multiplied:

$$\sqrt{3}e^{i\cdot\frac{7\pi}{4}} \cdot \sqrt{6}e^{i\cdot\frac{9\pi}{4}} = \sqrt{3} \cdot \sqrt{6} \cdot e^{i\left(\frac{7\pi}{4} + \frac{9\pi}{4}\right)} = 3\sqrt{2}\, e^{i(4\pi)} = 3\sqrt{2}\, e^{i(0)} = 3\sqrt{2}$$

Example 8.4: Find the quotient: $z_1 \div z_2$.

$z_1 = \sqrt{3}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) \longleftarrow$ shorthand is: $z_1 = \sqrt{3}\text{ cis}\left(\frac{7\pi}{4}\right) \quad = \sqrt{3}e^{i\cdot\frac{7\pi}{4}}$

$z_2 = \sqrt{6}\left(\cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}\right) \longleftarrow$ shorthand is: $z_2 = \sqrt{6}\text{ cis}\left(\frac{9\pi}{4}\right) \quad = \sqrt{6}e^{i\cdot\frac{9\pi}{4}}$

To divide two numbers in polar form, divide the r-values and subtract the angles.

$$z_1 \cdot z_2 = \frac{\sqrt{3}}{\sqrt{6}} \cdot \text{cis}\left(\frac{7\pi}{4} - \frac{9\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}\text{ cis}\left(-\frac{\pi}{2}\right) = \frac{\sqrt{2}}{2}\text{cis}\left(-\frac{\pi}{2} + 2\pi\right) = \frac{\sqrt{2}}{2}\text{cis}\left(\frac{3\pi}{2}\right) = -\frac{i\sqrt{2}}{2} \quad \text{because cis}\left(\frac{3\pi}{2}\right) = -\text{i}.$$

Note: division may be easier to understand in exponential form, since exponents are subtracted when values with the same base are divided:

$$\frac{\sqrt{3}\, e^{i\cdot\frac{7\pi}{4}}}{\sqrt{6}\, e^{i\cdot\frac{9\pi}{4}}} = \frac{1}{\sqrt{2}}\, e^{i\left(\frac{7\pi}{4} - \frac{9\pi}{4}\right)} = \frac{\sqrt{2}}{2}e^{i\left(-\frac{\pi}{2}\right)} = \frac{\sqrt{2}}{2}e^{i\left(-\frac{\pi}{2} + 2\pi\right)} = \frac{\sqrt{2}}{2}e^{i\left(\frac{3\pi}{2}\right)} = -\frac{i\sqrt{2}}{2}$$

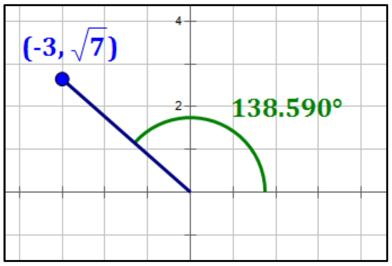
DeMoivre's Theorem

Abraham de Moivre (1667-1754) was a French mathematician who developed a very useful Theorem for dealing with operations on complex numbers.

If we let $z = r(\cos\theta + i\sin\theta)$, DeMoivre's Theorem gives us the power rule expressed on the prior page:

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Example 8.5: Find $\left(-3 + i\sqrt{7}\right)^6$



First, since $z = a + bi$, we have $a = -3$ and $b = \sqrt{7}$.

Then, $r = \sqrt{(-3)^2 + \left(\sqrt{7}\right)^2} = 4;\ r^6 = 4^6 = 4{,}096$

And, $\theta = \tan^{-1}\left(-\frac{\sqrt{7}}{3}\right) = 138.590°$ in Q2

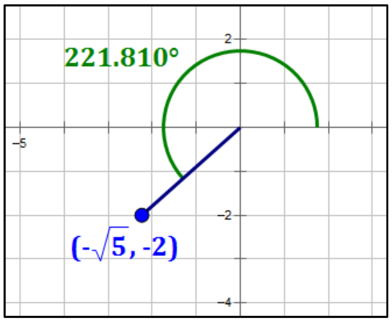
$6\theta = 831.542° \sim 111.542°$

So,

$$\left(-3 + i\sqrt{7}\right)^6 = 4{,}096\ \text{cis}(111.542°) = 4{,}096 \cdot [\cos(111.542°) + i\sin(111.542°)]$$

$$= -1{,}504.0 + 3{,}809.9i$$

Example 8.6: Find $\left(-\sqrt{5} - 2i\right)^5$



First, since $z = a + bi$, we have $a = -\sqrt{5}$ and $b = -2$.

Then, $r = \sqrt{\left(-\sqrt{5}\right)^2 + (-2)^2} = 3;\ r^5 = 3^5 = 243$

And, $\theta = \tan^{-1}\left(\frac{2}{\sqrt{5}}\right) = 221.810°$ in Q3

$5\theta = 1{,}109.052° \sim 29.052°$

So,

$$\left(-\sqrt{5} - 2i\right)^5 = 243\ \text{cis}(29.052°) = 243 \cdot [\cos(29.052°) + i\sin(29.052°)]$$

$$= 212.4 + 118.0i$$

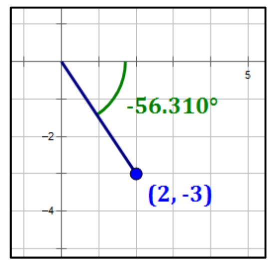
DeMoivre's Theorem for Roots

Let $z = r(\cos\theta + i\sin\theta)$. Then, z has n distinct complex n-th roots that occupy positions equidistant from each other on a circle of radius $\sqrt[n]{r}$. Let's call the roots: $z_0,\ z_1,\ z_2, \dots, z_{n-1}$. Then, these roots can be calculated as follows $(k = 0, 1, 2, \dots, n-1)$:

$$z_k = \sqrt[n]{r} \cdot \left[\cos\left(\frac{\theta + k(2\pi)}{n}\right) + i\sin\left(\frac{\theta + k(2\pi)}{n}\right)\right] = \sqrt[n]{r} \cdot \operatorname{cis}\left(\frac{\theta + k(2\pi)}{n}\right)$$

The formula could also be restated with 2π replaced by $360°$ if this helps in the calculation.

Example 8.7: Find the fifth roots of $\mathbf{2 - 3}\boldsymbol{i}$.



First, since $z = a + bi$, we have $a = 2$ and $b = -3$.

Then, $r = \sqrt{2^2 + (-3)^2} = \sqrt{13}$; $\sqrt[5]{r} = \sqrt[10]{13} \sim 1.2924$

And, $\theta = \tan^{-1}\left(-\frac{3}{2}\right) = -56.310°$; $\frac{\theta}{5} = -11.262°$ in Q4

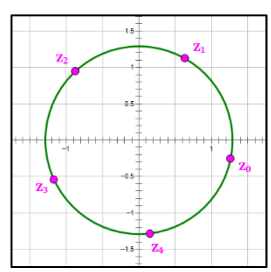
The incremental angle for successive roots is: $360° \div 5 \text{ roots} = 72°$.

Then create a chart like this:

Fifth roots of $(2-3i)$	$\sqrt[5]{r} = \sqrt[10]{13} \sim 1.2924$	$\frac{\theta}{5} = -11.262°$		
k	Angle (θ_k)	$z_k =$	$\sqrt[n]{r} \cdot \cos\theta_k$	$+\ \sqrt[n]{r} \cdot \sin\theta_k \cdot i$
0	$-11.262°$	$z_0 =$	1.2675	$-\ 0.2524\,i$
1	$-11.262° + 72° = 60.738°$	$z_1 =$	0.6317	$+\ 1.1275\,i$
2	$60.738° + 72° = 132.738°$	$z_2 =$	-0.8771	$+\ 0.9492\,i$
3	$132.738° + 72° = 204.738°$	$z_3 =$	-1.1738	$-\ 0.5408\,i$
4	$204.738° + 72° = 276.738°$	$z_4 =$	0.1516	$-\ 1.2835\,i$

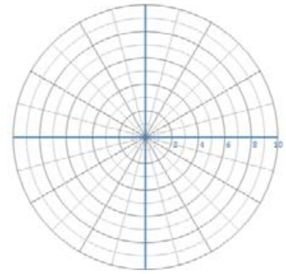
Notice that if we add another $72°$, we get $348.738°$, which is equivalent to our first angle, $-11.262°$ because $348.738° - 360° = -11.262°$. This is a good thing to check. The "next angle" will always be equivalent to the first angle! If it isn't, go back and check your work.

Roots fit on a circle: Notice that, since all of the roots of $\mathbf{2 - 3}\boldsymbol{i}$ have the same magnitude, and their angles are $72°$ apart from each other, they occupy equidistant positions on a circle with center $(0, 0)$ and radius $\sqrt[5]{r} = \sqrt[10]{13} \sim 1.2924$.



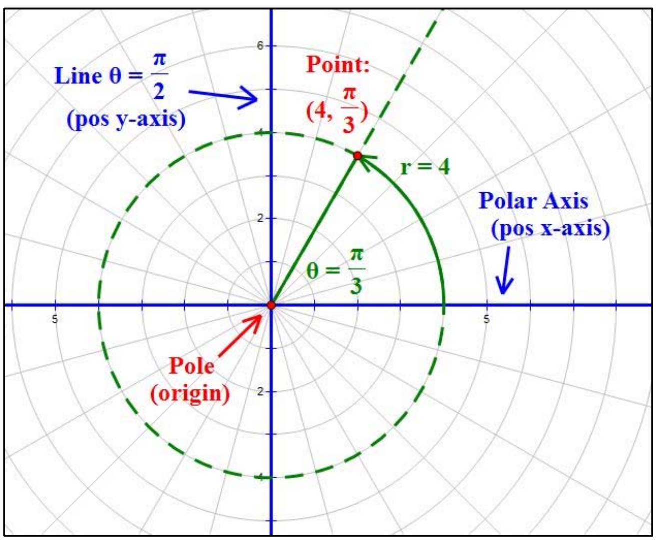
Polar Graphs

Typically, Polar Graphs will be plotted on polar graph paper such as that illustrated at right. On this graph, a point (r, θ) can be considered to be the intersection of the circle of radius r and the terminal side of the angle θ (see the illustration below). Note: a free PC app that can be used to design and print your own polar graph paper is available at www.mathguy.us.



Parts of the Polar Graph

The illustration below shows the key parts of a polar graph, along with a point, $\left(4, \frac{\pi}{3}\right)$.



The Pole is the point $(0, 0)$ (i.e., the origin).

The Polar Axis is the positive x-axis.

The Line: $\theta = \frac{\pi}{2}$ is the positive y-axis.

Many equations that contain the cosine function are symmetric about the x-axis.

Many equations that contain the sine function are symmetric about the y-axis.

Polar Equations – Symmetry

Following are the three main types of symmetry exhibited in many polar equation graphs:

Symmetry about:	Quadrants Containing Symmetry	Symmetry Test[1]
Pole	Opposite (I and III or II and IV)	Replace r with $-r$ in the equation
x-axis	Left hemisphere (II and III) or right hemisphere (I and IV)	Replace θ with $-\theta$ in the equation
y-axis	Upper hemisphere (I and II) or lower hemisphere (III and IV)	Replace (r, θ) with $(-r, -\theta)$ in the equation

(1) If performing the indicated replacement results in an equivalent equation, the equation passes the symmetry test and the indicated symmetry exists. If the equation fails the symmetry test, symmetry may or may not exist.

Graphs of Polar Equations

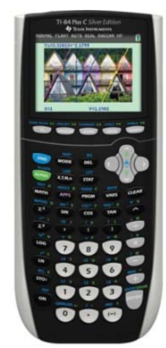
Graphing Methods

Method 1: Point plotting

- Create a two-column chart that calculates values of r for selected values of θ. This is akin to a two-column chart that calculates values of y for selected values of x that can be used to plot a rectangular coordinates equation (e.g., $y = x^2 - 4x + 3$).
- The θ-values you select for purposes of point plotting should vary depending on the equation you are working with (in particular, the coefficient of θ in the equation). However, a safe bet is to start with multiples of $\pi/6$ (including $\theta = 0$). Plot each point on the polar graph and see what shape emerges. If you need more or fewer points to see what curve is emerging, adjust as you go.
- If you know anything about the curve (typical shape, symmetry, etc.), use it to facilitate plotting points.
- Connect the points with a smooth curve. Admire the result; many of these curves are aesthetically pleasing.

Method 2: Calculator

Using a TI-84 Plus Calculator or its equivalent, do the following:

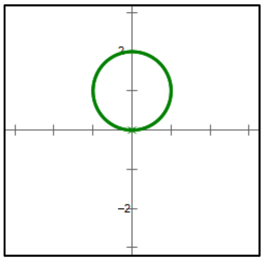


- Make sure your calculator is set to radians and polar functions. Hit the **MODE** key; select **RADIANS** in row 4 and **POLAR** in row 5. After you do this, hitting **CLEAR** will get you back to the main screen.
- Hit **Y=** and enter the equation in the form $r = f(\theta)$. Use the **X,T,θ,n** key to enter θ into the equation. If your equation is of the form $r^2 = f(\theta)$, you may need to enter two functions, $r = \sqrt{f(\theta)}$ and $r = -\sqrt{f(\theta)}$, and plot both.
- Hit **GRAPH** to plot the function or functions you entered in the previous step.
- If necessary, hit **WINDOW** to adjust the parameters of the plot.
  - If you cannot see the whole function, adjust the **X-** and **Y-** variables (or use **ZOOM**).
  - If the curve is not smooth, reduce the value of the **θstep** variable. This will plot more points on the screen. Note that smaller values of **θstep** require more time to plot the curve, so choose a value that plots the curve well in a reasonable amount of time.
  - If the entire curve is not plotted, adjust the values of the **θmin** and **θmax** variables until you see what appears to be the entire plot.

Note: You can view the table of points used to graph the polar function by hitting **2ND – TABLE**.

Graph of Polar Equations

Circle



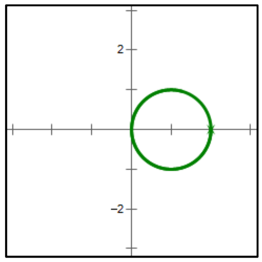
Equation: $r = a\sin\theta$

Location:

above x-axis if $a > 0$
below x-axis if $a < 0$

Radius: $a/2$

Symmetry: y-axis



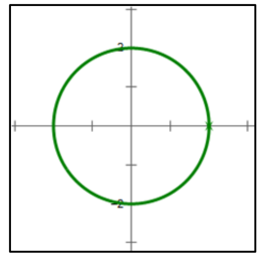
Equation: $r = a\cos\theta$

Location:

right of y-axis if $a > 0$
left of y-axis if $a < 0$

Radius: $a/2$

Symmetry: x-axis



Equation: $r = a$

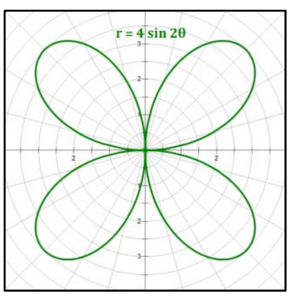
Location:

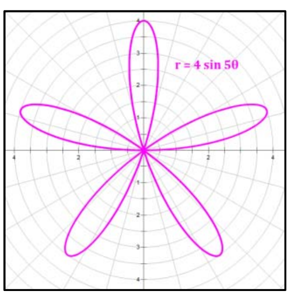
Centered on the Pole

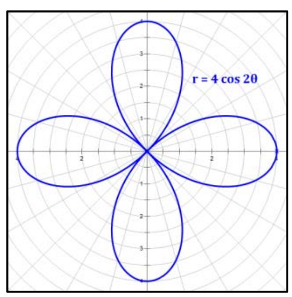
Radius: a

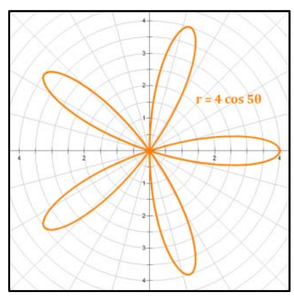
Symmetry: Pole, x-axis, y-axis

Rose







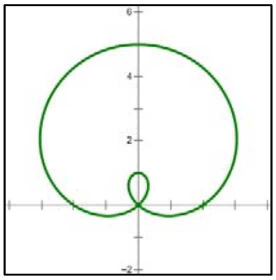


Characteristics of roses:

- Equation: $r = a\sin n\theta$
  - Symmetric about the y-axis
- Equation: $r = a\cos n\theta$
  - Symmetric about the x-axis
- Contained within a circle of radius $r = a$
- If n is odd, the rose has n petals.
- If n is even the rose has $2n$ petals.
- Note that a circle is a rose with one petal (i.e, $n = 1$).

Graphs of Polar Equations

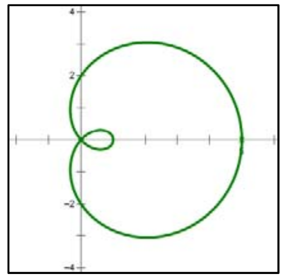
Limaçon of Pascal



Equation: $r = a + b \sin \theta$

Location: bulb above x-axis if $b > 0$
bulb below x-axis if $b < 0$

Symmetry: y-axis

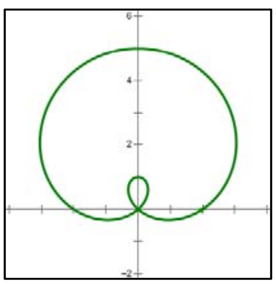


Equation: $r = a + b \cos \theta$

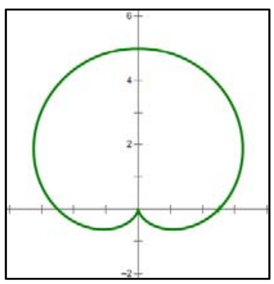
Location: bulb right of y-axis if $b > 0$
bulb left of y-axis if $b < 0$

Symmetry: x-axis

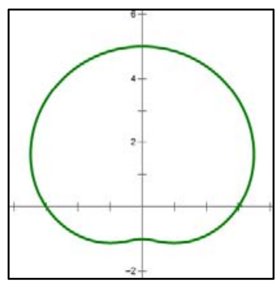
Four Limaçon Shapes



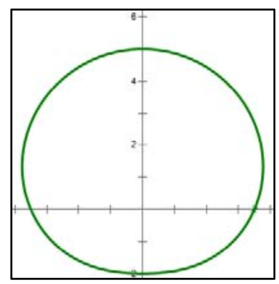
$a < b$
Inner loop



$a = b$
"Cardioid"

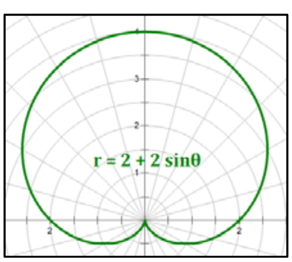


$b < a < 2b$
Dimple

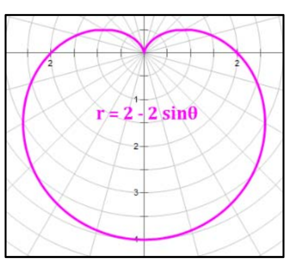


$a \geq 2b$
No dimple

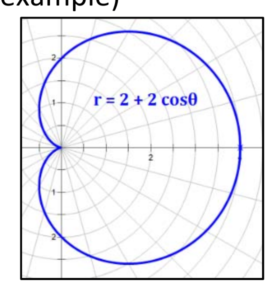
Four Limaçon Orientations (using the Cardioid as an example)



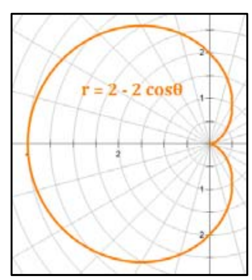
sine function
$b > 0$



sine function
$b < 0$



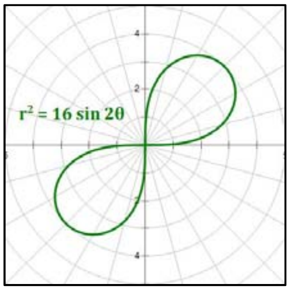
cosine function
$b > 0$

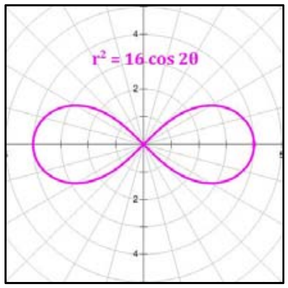


cosine function
$b < 0$

Graph of Polar Equations

Lemniscate of Bernoulli



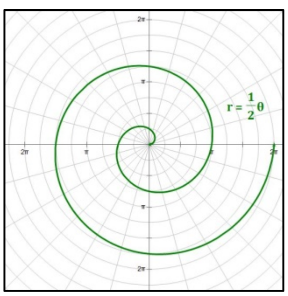


The lemniscate is the set of all points for which the product of the distances from two points (i.e., foci) which are "2c" apart is c^2.

Characteristics of lemniscates:

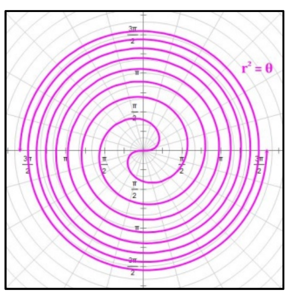
- Equation: $r^2 = a^2 \sin 2\theta$
  - Symmetric about the line $y = x$
- Equation: $r^2 = a^2 \cos 2\theta$
  - Symmetric about the x-axis
- Contained within a circle of radius $r = a$

Spirals



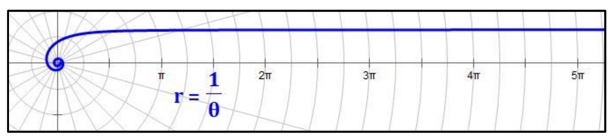
Archimedes' Spiral

$r = a\theta$

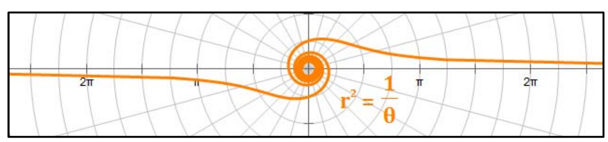


Fermat's Spiral

$r^2 = a^2\theta$



Hyperbolic Spiral $r = \dfrac{a}{\theta}$



Lituus $r^2 = \dfrac{a^2}{\theta}$

Characteristics of spirals:

- Equation: $r^b = a^b\theta,\ b > 0$
  - Distance from the Pole increases with θ
- Equation: $r^b = \dfrac{a^b}{\theta},\ b > 0$
  - Hyperbolic Spiral ($b = 1$): asymptotic to the line a units from the x-axis
  - Lituus ($b = 2$): asymptotic to the x-axis
- Not contained within any circle

Graphing Polar Equations – The Rose

Example 9.1: $\mathbf{r = 4\sin 2\theta}$

This function is a rose. Consider the forms $r = a\sin b\theta$ and $r = a\cos b\theta$.

The number of petals on the rose depends on the value of b.

- If b is an even integer, the rose will have $2b$ petals.
- If b is an odd integer, it will have b petals.

Let's create a table of values and graph the equation:

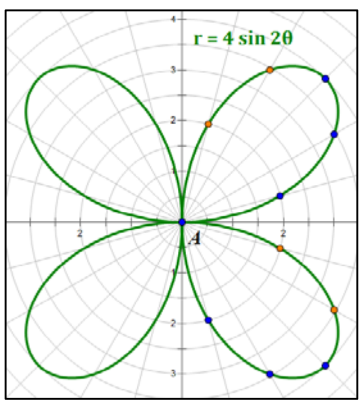
$\mathbf{r = 4\sin 2\theta}$			
$\boldsymbol{\theta}$	$\boldsymbol{r}$	$\boldsymbol{\theta}$	$\boldsymbol{r}$
0	0		
$\pi/12$	2	$7\pi/12$	-2
$\pi/6$	3.464	$2\pi/3$	-3.464
$\pi/4$	4	$3\pi/4$	-4
$\pi/3$	3.464	$5\pi/6$	-3.464
$5\pi/12$	2	$11\pi/12$	-2
$\pi/2$	0	π	0

Because this function involves an argument of 2θ, we want to start by looking at values of θ in $[0, 2\pi] \div 2 = [0, \pi]$. You could plot more points, but this interval is sufficient to establish the nature of the curve; so you can graph the rest easily.

Once symmetry is established, these values are easily determined.

The values in the table generate the points in the two petals right of the y-axis.

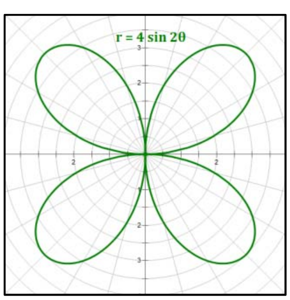
Knowing that the curve is a rose allows us to graph the other two petals without calculating more points.

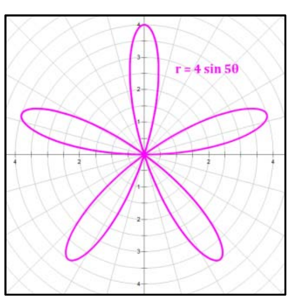


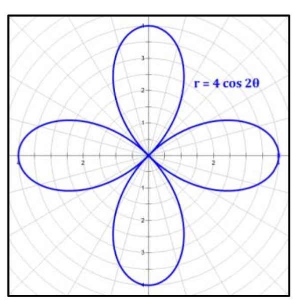
Blue points on the graph correspond to blue values in the table.

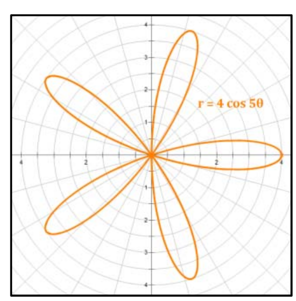
Orange points on the graph correspond to orange values in the table.

The four Rose forms:









Graphing Polar Equations – The Cardioid
Example 9.2: $r = 2 + 2\sin\theta$

This cardioid is also a limaçon of form $r = a + b\sin\theta$ with $a = b$. The use of the sine function indicates that the large loop will be symmetric about the y-axis. The $+$ sign indicates that the large loop will be above the x-axis. Let's create a table of values and graph the equation:

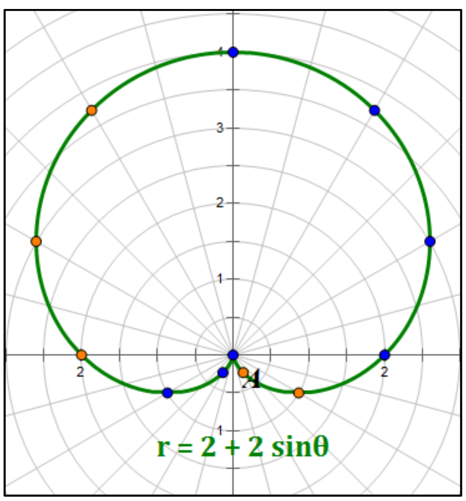
$r = 2 + 2\sin\theta$			
θ	r	θ	r
0	2		
$\pi/6$	3	$7\pi/6$	1
$\pi/3$	3.732	$4\pi/3$	0.268
$\pi/2$	4	$3\pi/2$	0
$2\pi/3$	3.732	$5\pi/3$	0.268
$5\pi/6$	3	$11\pi/6$	1
π	2	2π	2

Generally, you want to look at values of θ in $[0, 2\pi]$. However, some functions require larger intervals. The size of the interval depends largely on the nature of the function and the coefficient of θ.

Once symmetry is established, these values are easily determined.

The portion of the graph above the x-axis results from θ in Q1 and Q2, where the sine function is positive.

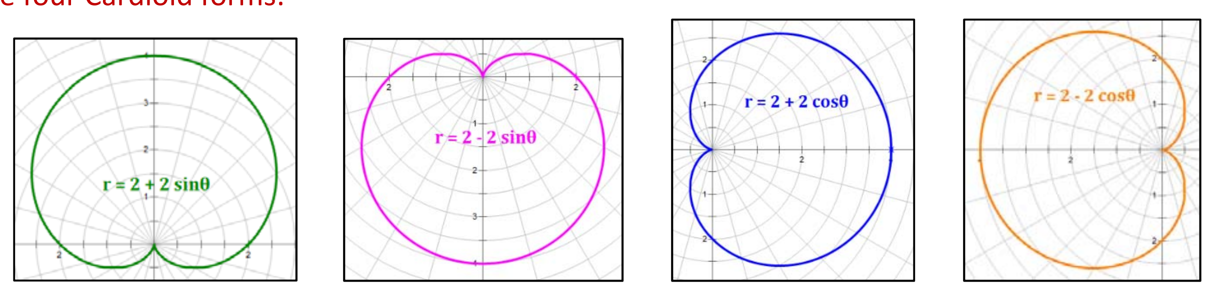
Similarly, the portion of the graph below the x-axis results from θ in Q3 and Q4, where the sine function is negative.



Blue points on the graph correspond to blue values in the table.

Orange points on the graph correspond to orange values in the table.

The four Cardioid forms:



Converting Between Polar and Rectangular Forms of Equations

Rectangular to Polar

To convert an equation from Rectangular Form to Polar Form, use the following equivalences:

$x = r\cos\theta$	Substitute $r\cos\theta$ for x
$y = r\sin\theta$	Substitute $r\sin\theta$ for y
$x^2 + y^2 = r^2$	Substitute r^2 for $x^2 + y^2$

Example 9.3: Convert $8x - 3y + 10 = 0$ to a polar equation of the form $r = f(\theta)$.

Starting Equation:	$8x - 3y + 10 = 0$
Substitute $x = r\cos\theta$ and $y = r\sin\theta$:	$8 \cdot r\cos\theta - 3 \cdot r\sin\theta + 10 = 0$
Factor out r:	$r\,(8\cos\theta - 3\sin\theta) = -10$
Divide by $(8\cos\theta - 3\sin\theta)$:	$\boldsymbol{r = \dfrac{-10}{8\cos\theta - 3\sin\theta}}$

Polar to Rectangular

To convert an equation from Polar Form to Rectangular Form, use the following equivalences:

$\cos\theta = \dfrac{x}{r}$	Substitute $\dfrac{x}{r}$ for $\cos\theta$
$\sin\theta = \dfrac{y}{r}$	Substitute $\dfrac{y}{r}$ for $\sin\theta$
$r^2 = x^2 + y^2$	Substitute $x^2 + y^2$ for r^2

Example 9.4: Convert $r = 8\cos\theta + 9\sin\theta$ to a rectangular equation.

Starting Equation:	$r = 8\cos\theta + 9\sin\theta$
Substitute $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$:	$r = 8\left(\frac{x}{r}\right) + 9\left(\frac{y}{r}\right)$
Multiply by r:	$r^2 = 8x + 9y$
Substitute $r^2 = x^2 + y^2$:	$x^2 + y^2 = 8x + 9y$
Subtract $8x + 9y$:	$x^2 - 8x \quad + y^2 - 9y \quad = 0$
Complete the square:	$(x^2 - 8x + 16) + \left(y^2 - 9y + \frac{81}{4}\right) = 16 + \frac{81}{4}$
Simplify to standard form for a circle:	$\boldsymbol{(x-4)^2 + \left(y - \frac{9}{2}\right)^2 = \frac{145}{4}}$

Parametric Equations

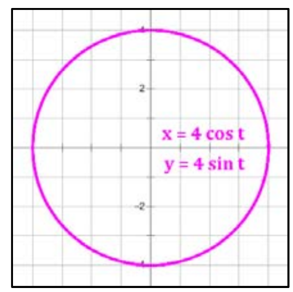
One way to define a curve is by making x and y (or r and θ) functions of a third variable, often t (for time). The third variable is called the Parameter, and functions defined in this manner are said to be in Parametric Form. The equations that define the desired function are called Parametric Equations.

In Parametric Equations, the parameter is the independent variable. Each of the other two (or more) variables is dependent on the value of the parameter. As the parameter changes, the other variables change, generating the points of the function.

Example 9.5: A relatively simple example is a circle, which we can define as follows:

Circle: $x = r\cos t \quad y = r\sin t$

As the variable t progresses from 0 to 2π, a circle of radius r is born.



The circle in the illustration at right can be defined in several ways:

Cartesian form: $x^2 + y^2 = 16$

Polar form: $r = 4$

Parametric form: $x = 4\cos t \qquad y = 4\sin t$

Familiar Curves

Many curves with which the student may be familiar have parametric forms. Among those are the following:

Curve	Cartesian Form	Polar Form	Parametric Form
Parabola with horizontal directrix	$y = a(x-h)^2 + k$	$r = \dfrac{p}{1 \pm \sin\theta}$	$x = 2pt$ $y = pt^2$
Ellipse with horizontal major axis	$\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$	$r = \dfrac{ep}{1 \pm e\cdot\cos\theta}$ $(0 < e < 1)$	$x = a\cos t$ $y = b\sin t$
Hyperbola with horizontal transverse axis	$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$	$r = \dfrac{ep}{1 \pm e\cdot\cos\theta}$ $(e > 1)$	$x = a\sec t$ $y = b\tan t$

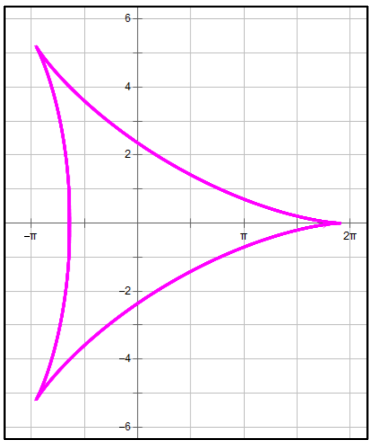
As can be seen from this chart, sometimes the parametric form of a function is its simplest. In fact, parametric equations often allow us to graph curves that would be very difficult to graph in either Polar form or Cartesian form. Some of these are illustrated on the next page.

Some Functions Defined by Parametric Equations

(Star Wars fans: are these the "oids" you are looking for?)

The graphs below are examples of functions defined by parametric equations. The equations and a brief description of the curve are provided for each function.

Deltoid

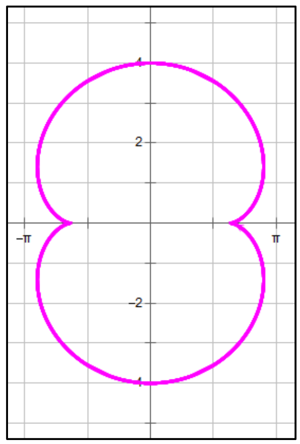


Parametric equations:

$$x = 2a\cos t + a\cos 2t$$
$$y = 2a\sin t - a\sin 2t$$

The deltoid is the path of a point on the circumference of a circle as it makes three complete revolutions on the inside of a larger circle.

Nephroid

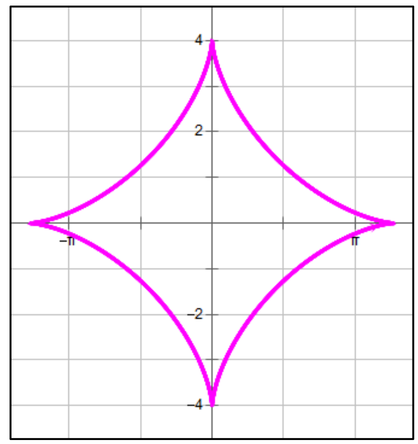


Parametric equations:

$$x = a(3\cos t - \cos 3t)$$
$$y = a(3\sin t - \sin 3t)$$

The nephroid is the path of a point on the circumference of a circle as it makes two complete revolutions on the outside of a larger circle.

Astroid

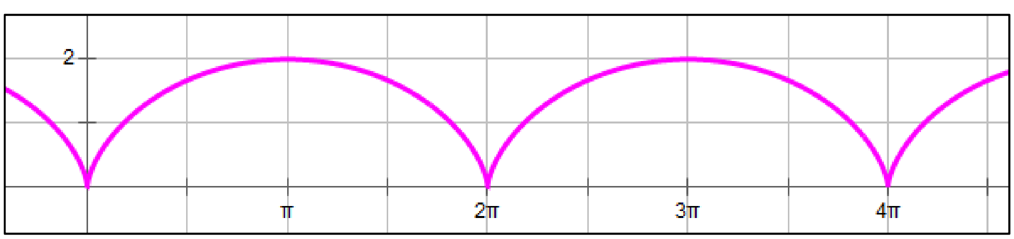


Parametric equations:

$$x = a\cos^3 t$$
$$y = a\sin^3 t$$

The astroid is the path of a point on the circumference of a circle as it makes four complete revolutions on the inside of a larger circle.

Cycloid



Parametric equations:

$$x = a(t - \sin t)$$
$$y = a\,(1 - \cos t)$$

The cycloid is the path of a point on the circumference of a circle as the circle rolls along a flat surface (think: the path of a point on the outside of a bicycle tire as you ride on the sidewalk). The cycloid is both a *brachistochrone* and a *tautochrone* (look these up if you are interested).

Vectors

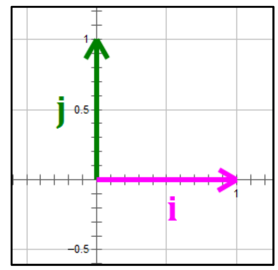
A vector is a quantity that has both magnitude and direction. An example would be wind blowing toward the east at 30 miles per hour. Another example would be the force of a 10 kg weight being pulled toward the earth (a force you can feel if you are holding the weight).

Special Unit Vectors

We define unit vectors to be vectors of length 1. Unit vectors having the direction of the positive axes are very useful. They are described in the chart and graphic below.

Unit Vector	Direction
$\mathbf{i}$	positive x-axis
$\mathbf{j}$	positive y-axis
$\mathbf{k}$	positive z-axis

Graphical representation of unit vectors $\mathbf{i}$ and $\mathbf{j}$ in two dimensions. →



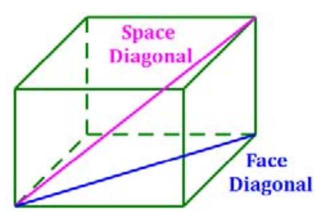
Vector Components

The length of a vector, $\mathbf{v}$, is called its magnitude and is represented by the symbol $\|\mathbf{v}\|$. If a vector's initial point (starting position) is (x_1, y_1, z_1), and its terminal point (ending position) is (x_2, y_2, z_2), then the vector displaces $\boldsymbol{a = x_2 - x_1}$ in the x-direction, $\boldsymbol{b = y_2 - y_1}$ in the y-direction, and $\boldsymbol{c = z_2 - z_1}$ in the z-direction. We can, then, represent the vector as follows:

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

The magnitude of the vector, $\mathbf{v}$, is calculated as:

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2}$$



If this looks familiar, it should. The magnitude of a vector in three dimesnsions is determined as the length of the space diagonal of a rectangular prism with sides a, b and c.

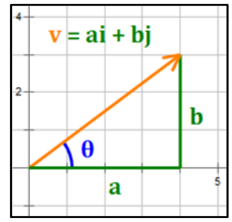
In two dimensions, these concepts contract to the following:

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} \qquad \|\mathbf{v}\| = \sqrt{a^2 + b^2}$$

In two dimensions, the magnitude of the vector is the length of the hypotenuse of a right triangle with sides a and b.

Vector Properties

Vectors have a number of nice properties that make working with them both useful and relatively simple. Let m and n be scalars, and let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Then,



- If $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$, then $a = \|\mathbf{v}\| \cos\theta$ and $b = \|\mathbf{v}\| \sin\theta$
- Then, $\mathbf{v} = \|\mathbf{v}\| \cos\theta \ \mathbf{i} + \|\mathbf{v}\| \sin\theta \ \mathbf{j}$ (note: this formula is often used in Force calculations)
- If $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j}$, then $\mathbf{u} + \mathbf{v} = (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j}$
- If $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$, then $m\mathbf{v} = (ma)\mathbf{i} + (mb)\mathbf{j}$
- Define $\mathbf{0}$ to be the zero vector (i.e., it has zero length, so that $a = b = 0$). Note: the zero vector is also called the null vector.

Note: $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ can also be shown with the following notation: $\mathbf{v} = \langle a, b\rangle$. This notation is useful in calculating dot products and performing operations with vectors.

Properties of Vectors

- $\mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v}$ Additive Identity
- $\mathbf{v} + (-\mathbf{v}) = (-\mathbf{v}) + \mathbf{v} = \mathbf{0}$ Additive Inverse
- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ Commutative Property
- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ Associative Property
- $m(n\boldsymbol{u}) = (mn)\boldsymbol{u}$ Associative Property
- $m(\boldsymbol{u} + \boldsymbol{v}) = m\boldsymbol{u} + m\boldsymbol{v}$ Distributive Property
- $(m + n)\boldsymbol{u} = m\boldsymbol{u} + n\boldsymbol{u}$ Distributive Property
- $1(\mathbf{v}) = \mathbf{v}$ Multiplicative Identity

Also, note that:

- $\|m\mathbf{v}\| = |m| \, \|\mathbf{v}\|$ Magnitude Property
- $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ Unit vector in the direction of $\mathbf{v}$

Vector Properties – Examples

Example 10.1: **u** = -3**i** - 6**j**, **v** = 6**i** + 8**j**; Find **u** + **v**.

An alternative notation for a vector in the form $a\mathbf{i} + b\mathbf{j}$ is $\langle a, b\rangle$. Using this alternative notation makes many vector operations much easier to work with.

To add vectors, simply line them up vertically and add:

$$\mathbf{u} = \langle -3, -6\rangle$$
$$\mathbf{v} = \langle\ \ 6,\ \ \ 8\rangle$$
$$\mathbf{u} + \mathbf{v} = \langle -3 + 6, -6 + 8\rangle$$

$$\mathbf{u} + \mathbf{v} = \langle 3, 2\rangle = 3\mathbf{i} + 2\mathbf{j}$$

Example 10.2: **u** = -2**i** - 7**j** and **v** = -4**i** - 21**j**; Find $\|\mathbf{v} - \mathbf{u}\|$.

$$\mathbf{v} = \langle -4, -21\rangle$$
$$+\ \ -\mathbf{u} = \langle\ \ 2,\ \ \ 7\rangle$$
$$\mathbf{v} - \mathbf{u} = \langle -2, -14\rangle$$

Subtracting $\mathbf{u}$ is the same as adding $-\mathbf{u}$. To get $-\mathbf{u}$, simply change the sign of each element of $\mathbf{u}$. If you find it easier to add than to subtract, you may want to adopt this approach to subtracting vectors.

$$\|\mathbf{v} - \mathbf{u}\| = \sqrt{(-2)^2 + (-14)^2}$$
$$= \sqrt{200} = \sqrt{100} \cdot \sqrt{2} = 10\sqrt{2}$$

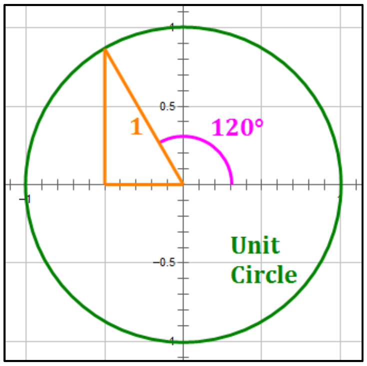
Example 10.3: Find the unit vector that has the same direction as the vector **v** = 5**i** - 12**j.**

A unit vector has **magnitude 1**. To get a unit vector in the same direction as the original vector, divide the vector by its magnitude.

The unit vector is: $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{5\mathbf{i} - 12\mathbf{j}}{\sqrt{5^2 + 12^2}} = \dfrac{5\mathbf{i} - 12\mathbf{j}}{13} = \dfrac{5}{13}\mathbf{i} - \dfrac{12}{13}\mathbf{j}$

Vector Properties – Examples

Example 10.4: Write the vector **v** in terms of **i** and **j** if $\|\mathbf{v}\| = 10$ and direction angle $\theta = 120°$.



It helps to graph the vector identified in the problem.

The unit vector in the direction $\theta = 120°$ is:

$$\langle \cos 120°, \sin 120° \rangle = \langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \rangle = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

Multiply this by $\|\mathbf{v}\| = \mathbf{10}$ to get $\mathbf{v}$:

$$\mathbf{v} = 10\left(-\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right) = -5\mathbf{i} + 5\sqrt{3}\,\mathbf{j}$$

Vector Dot Product

The Dot Product of two vectors, $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$, is defined as follows:

$$\mathbf{u} \circ \mathbf{v} = (a_1 \cdot a_2) + (b_1 \cdot b_2) + (c_1 \cdot c_2)$$

It is important to note that the dot product is a scalar (i.e., a number), not a vector. It describes something about the relationship between two vectors, but is not a vector itself. A useful approach to calculating the dot product of two vectors is illustrated here:

$$\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j} + \mathbf{c_1k} = \langle a_1, b_1, c_1 \rangle$$
$$\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j} + \mathbf{c_2k} = \langle a_2, b_2, c_2 \rangle$$

alternative vector notation

In the example at right the vectors are lined up vertically. The numbers in the each column are multiplied and the results are added to get the dot product. In the example, $\langle 4, -3, 2 \rangle \circ \langle 2, -2, 5 \rangle = 8 + 6 + 10 = 24$.

General	Example
$\langle a_1, b_1, c_1 \rangle$	$\langle 4, -3, 2 \rangle$
$\circ \langle a_2, b_2, c_2 \rangle$	$\circ \langle 2, -2, 5 \rangle$
$a_1a_2 + b_1b_2 + c_1c_2$	$8 + 6 + 10$ $= 24$

Properties of the Dot Product

Let m be a scalar, and let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Then,

- $\mathbf{0} \circ \mathbf{u} = \mathbf{u} \circ \mathbf{0} = 0$ Zero Property
- $\mathbf{i} \circ \mathbf{j} = \mathbf{j} \circ \mathbf{k} = \mathbf{k} \circ \mathbf{i} = 0$ $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are orthogonal to each other.
- $\mathbf{u} \circ \mathbf{v} = \mathbf{v} \circ \mathbf{u}$ Commutative Property
- $\mathbf{u} \circ \mathbf{u} = \|\mathbf{u}\|^2$ Magnitude Square Property
- $\mathbf{u} \circ (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \circ \mathbf{v}) + (\mathbf{u} \circ \mathbf{w})$ Distributive Property
- $m(\mathbf{u} \circ \mathbf{v}) = (m\boldsymbol{u}) \circ \mathbf{v} = \boldsymbol{u} \circ (m\mathbf{v})$ Multiplication by a Scalar Property

More properties:

- If $\mathbf{u} \circ \mathbf{v} = 0$ and $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal (perpendicular).
- If there is a scalar m such that $m\boldsymbol{u} = \mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel.
- If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then $\cos\theta = \dfrac{\mathbf{u} \circ \mathbf{v}}{\|\mathbf{u}\|\ \|\mathbf{v}\|}$.

Vector Dot Product – Examples

Example 10.5: **u** = -5**i** + 3**j**, **v** = 5**i** - 6**j**, **w** = -3**i** + 12**j**; Find **u** · **w** + **v** · **w**.

The alternate notation for vectors comes in especially handy in doing these types of problems. Also, note that: (**u** · **w**) + (**v** · **w**) = (**u** + **v**) ∘ **w**. Let's calculate (**u** + **v**) ∘ **w**.

$$\mathbf{u} = \langle -5, \ \ 3\rangle$$
$$+ \ \ \mathbf{v} = \langle \ \ 5, -6\rangle$$
$$\mathbf{u}+\mathbf{v} = \langle \ \ 0, -3\rangle$$
$$\circ \ \mathbf{w} = \langle -3, \ 12\rangle$$
$$(\mathrm{u}+\mathrm{v}) \circ \mathrm{w} = (0 \cdot [-3]) + (-3 \cdot 12) = 0 - 36 = -36$$

Using the distributive property for dot products results in an easier problem with fewer calculations.

Example 10.6: Find the angle between the given vectors: **u** = **i** - **j**, **v** = 4**i** + 5**j.**

$$\cos\theta = \frac{\mathbf{u} \circ \mathbf{v}}{\|\mathbf{u}\| \ \|\mathbf{v}\|}$$

$$0° \le \theta \le 180°$$

$$\mathbf{u} = \langle 1, -1\rangle$$
$$\circ \, \mathbf{v} = \langle 4, \ \ 5\rangle$$
$$\mathbf{u} \circ \mathbf{v} = (1 \cdot 4) + ([-1] \cdot 5) = -1$$
$$\|\mathbf{u}\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$
$$\|\mathbf{v}\| = \sqrt{4^2 + 5^2} = \sqrt{41}$$
$$\cos\theta = \frac{\mathbf{u} \circ \mathbf{v}}{\|\mathbf{u}\| \ \|\mathbf{v}\|} = \frac{-1}{\sqrt{2} \cdot \sqrt{41}} = \frac{-1}{\sqrt{82}}$$
$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{82}}\right) = 96.3°$$

Example 10.7: Are the following vectors parallel, orthogonal, or neither? **v** = 4**i** + 3**j**, **w** = 3**i** - 4**j**

If vectors are parallel, one is a multiple of the other; also $\mathbf{v} \circ \mathbf{w} = \|\mathbf{v}\| \ \|\mathbf{w}\|$.

If vectors are perpendicular, their dot product is zero.

Calculate the dot product.

$$\mathbf{v} = \langle 4, \ \ 3\rangle$$
$$\circ \ \mathbf{w} = \langle 3, -4\rangle$$
$$\mathbf{v} \circ \mathbf{w} = (4 \cdot 3) + (3 \cdot [-4]) = 12 + (-12) = 0$$

So, the vectors are orthogonal.

Vector Dot Product – Examples

Example 10.8: Are the vectors are parallel, orthogonal, or neither. **v** = 3**i** + 4**j**, **w** = 6**i** + 8**j**

Vector Multiple Approach

$\mathbf{v} = \langle 3, 4\rangle$

$\mathbf{w} = \langle 6, 8\rangle$

Clearly, $\mathbf{w} = 2\mathbf{v}$

The vectors are parallel.

It is clearly easier to check whether one vector is a multiple of the other than to use the dot product method. The student may use either, unless instructed to use a particular method.

Dot Product Approach

To determine if two vectors are parallel using the dot product, we check to see if:

$$\mathbf{v} \circ \mathbf{w} = \|\mathbf{v}\|\ \|\mathbf{w}\|$$

$$\begin{array}{r} \mathbf{v} = \langle 3, 4\rangle \\ \circ\ \mathbf{w} = \langle 6, 8\rangle \\ \hline \mathbf{v} \circ \mathbf{w} = 18 + 32 = 50 \end{array}$$

$$\|\mathbf{v}\| = \sqrt{(3)^2 + (4)^2} = 5$$

$$\|\mathbf{w}\| = \sqrt{(6)^2 + (8)^2} = 10$$

$$\|\mathbf{v}\|\ \|\mathbf{w}\| = 5 \cdot 10 = 50 = \mathbf{v} \circ \mathbf{w}$$

The vectors are parallel.

Cross Product Approach (see Cross Product below)

To determine if two vectors are parallel using the cross product, we check to see if:

$$\mathbf{v} \times \mathbf{w} = 0$$

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = (v_1 w_2 - v_2 w_1)$$

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix} = (3 \cdot 8 - 4 \cdot 6) = 0$$

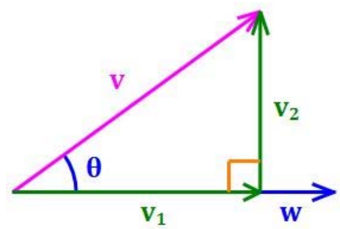
The vectors are parallel.

Applications of the Vector Dot Product

Vector Projection

The projection of a vector, $\mathbf{v}$, onto another vector $\mathbf{w}$, is obtained using the dot product. The formula used to determine the projection vector is:

$$\text{proj}_{\mathbf{w}}\mathbf{v} = \left(\frac{\mathbf{v} \circ \mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w} = \left(\frac{\mathbf{v} \circ \mathbf{w}}{\mathbf{w} \circ \mathbf{w}}\right)\mathbf{w}$$



Notice that $\left(\frac{\mathbf{v} \circ \mathbf{w}}{\|\mathbf{w}\|^2}\right)$ is a scalar, and that $\text{proj}_{\mathbf{w}}\mathbf{v}$ is a vector.

In the diagram at right, $\mathbf{v}_1 = \text{proj}_{\mathbf{w}}\mathbf{v}$.

Orthogonal Components of a Vector (Decomposition)

A vector, $\mathbf{v}$, can be expressed as the sum of two orthogonal vectors $\mathbf{v_1}$ and $\mathbf{v_2}$, as shown in the above diagram. The resulting vectors are:

$$\mathbf{v_1} = \text{proj}_{\mathbf{w}}\mathbf{v} = \left(\frac{\mathbf{v} \circ \mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w} \qquad \text{and} \qquad \mathbf{v_2} = \mathbf{v} - \mathbf{v_1}$$

$\mathbf{v_1}$ is parallel to $\mathbf{w}$ $\qquad$ $\mathbf{v_2}$ is orthogonal to $\mathbf{w}$

Work

Work is a scalar quantity in physics that measures the force exerted on an object over a particular distance. It is defined using vectors, as shown below. Let:

- $\mathbf{F}$ be the force vector acting on an object, moving it from point A to point B.
- $\overrightarrow{AB}$ be the vector from A to B.
- θ be the angle between $\mathbf{F}$ and $\overrightarrow{AB}$.

Then, we define work as:

$$W = \mathbf{F} \circ \overrightarrow{AB}$$

$$W = \|\mathbf{F}\| \, \|\overrightarrow{AB}\| \, \cos\theta$$

Both of these formulas are useful. Which one to use in a particular situation depends on what information is available.

Magnitude of Force ($\|\mathbf{F}\|$); Distance Traveled ($\|\overrightarrow{AB}\|$); Angle between Vectors (θ)

Applications of Vectors – Examples

Example 10.9: The magnitude and direction of two forces acting on an object are 60 pounds, N 40° E, and 70 pounds, N 40° W, respectively. Find the magnitude and the direction angle of the resultant force.

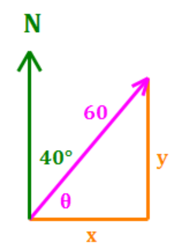
This problem requires the addition of two vectors. The approach used here is:

1) Convert each vector into its $\mathbf{i}$ and $\mathbf{j}$ components, call them x and y,
2) Add the resulting x and y values for the two vectors, and
3) Convert the sum to its polar form.

Keep additional accuracy throughout and round at the end. This will prevent error compounding and will preserve the required accuracy of your final solutions.

Step 1: Convert each vector into its $\mathbf{i}$ and $\mathbf{j}$ components

Let $\mathbf{F_1}$ be a force of 60 lbs. at bearing: N 40° E



From the diagram at right,

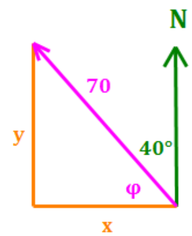
$$\theta = 90° - 40° = 50°$$

$$x = 60\cos 50° = 38.5673$$

$$y = 60\sin 50° = 45.9627$$

Let $\mathbf{F_2}$ be a force of 70 lbs. at bearing: N 40° W

From the diagram at right,



$$\varphi = 90° - 40° = 50°$$

$$x = -70\cos(50°) = -44.9951$$

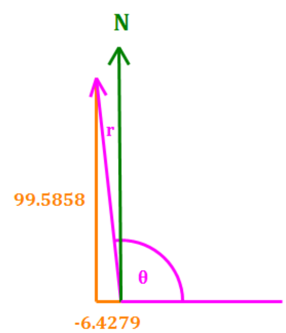
$$y = 70\sin(50°) = 53.6231$$

Step 2: Add the results for the two vectors

$$\mathbf{F_1} = \langle\ \ 38.5673, 45.9627\rangle$$

$$\mathbf{F_2} = \langle -44.9951, 53.6231\rangle$$

$$\mathbf{F_1} + \mathbf{F_2} = \langle\ \ -6.4278, 99.5858\rangle$$



Step 3: Convert the sum to its polar form

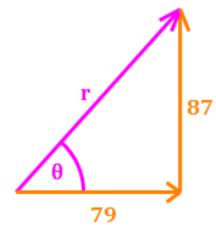
Direction Angle $= \theta = \tan^{-1}\left(\frac{99.5858}{-6.4278}\right) = 93.7°$

Magnitude $= r = \sqrt{(-6.4278)^2 + 99.5858^2} = 99.79$ lbs.

Applications of Vectors – Examples

Example 10.10: One rope pulls a barge directly east with a force of 79 newtons, and another rope pulls the barge directly north with a force of 87 newtons. Find the magnitude and direction angle of the resulting force acting on the barge.

The process of adding two vectors whose headings are north, east, west or south (NEWS) is very similar to converting a set of rectangular coordinates to polar coordinates. So, if this process seems familiar, that's because it is.



Magnitude $= r = \sqrt{(79)^2 + (87)^2} = 117.52 \text{ newtons}$

Direction Angle $= \theta = \tan^{-1}\left(\frac{87}{79}\right) = 47.8°$

Example 10.11: A force is given by the vector **F** = 5**i** + 2**j**. The force moves an object along a straight line from the point (5, 7) to the point (18, 13). Find the work done if the distance is measured
in feet and the force is measured in pounds.

For this problem it is sufficient to use the work formula, $W = \mathbf{F} \cdot \overrightarrow{AB}$

We are given $\mathbf{F} = \langle 5, 2 \rangle$.

We can calculate $\overrightarrow{AB}$ as the difference between the two given points.

$$\begin{array}{r} (18, 13) \\ -(\ 5,\ \ 7) \\ \hline \overrightarrow{AB} = \langle 13,\ \ 6 \rangle \end{array}$$

Note that the difference between two points is a vector.

Then, calculate $W = \mathbf{F} \circ \overrightarrow{AB}$

$$\begin{array}{r} \mathbf{F} = \langle\ 5, 2 \rangle \\ \circ\ \overrightarrow{AB} = \langle 13, 6 \rangle \\ \hline W = \mathbf{F} \circ \overrightarrow{AB} = (5 \cdot 13) + (2 \cdot 6) = 77 \text{ foot pounds} \end{array}$$

Applications of Vectors – Examples

Example 10.12: Decompose $\mathbf{v}$ into two vectors $\mathbf{v_1}$ and $\mathbf{v_2}$, where $\mathbf{v_1}$ is parallel to $\mathbf{w}$ and $\mathbf{v_2}$ is orthogonal to w. **v** = **i** - 4**j**, **w** = 2**i** + **j**

The formulas for this are:

$$\mathbf{v_1} = \text{proj}_{\mathbf{w}}\mathbf{v} = \left(\frac{\mathbf{v} \circ \mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w}$$

$$\mathbf{v_2} = \mathbf{v} - \mathbf{v_1}$$

Let's do the calculations.

$$\mathbf{v} = \langle 1, -4 \rangle$$

$$\circ\ \mathbf{w} = \langle 2,\ \ 1 \rangle$$

$$\mathbf{v} \circ \mathbf{w} = (1 \cdot 2) + (-4 \cdot 1) = -2$$

$$\|\mathbf{w}\|^2 = 2^2 + 1^2 = 5$$

Then,

$$\mathbf{v_1} = \text{proj}_{\mathbf{w}}\mathbf{v} = \left(\frac{\mathbf{v} \circ \mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w} = \left(\frac{-2}{5}\right)\langle 2,\ \ 1 \rangle = \left\langle -\frac{4}{5}, -\frac{2}{5} \right\rangle$$

$$\mathbf{v_1} = -\frac{4}{5}\mathbf{i} - \frac{2}{5}\mathbf{j}$$

And,

$$\mathbf{v} = \langle 1, -4 \rangle$$

$$+\ \ -\mathbf{v_1} = \langle \tfrac{4}{5},\ \ \tfrac{2}{5} \rangle$$

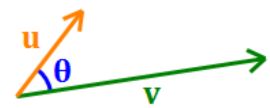
$$\mathbf{v_2} = \mathbf{v} - \mathbf{v_1} = \langle \frac{9}{5}, \frac{-18}{5} \rangle$$

$$\mathbf{v_2} = \frac{9}{5}\mathbf{i} - \frac{18}{5}\mathbf{j}$$

Vector Cross Product

Cross Product

In three dimensions,



Let: $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

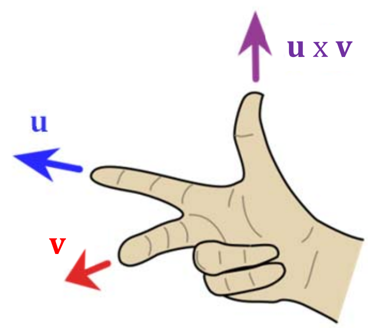
Then, the Cross Product is given by:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\,\mathbf{i} + (u_3v_1 - u_1v_3)\,\mathbf{j} + (u_1v_2 - u_2v_1)\,\mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sin\theta\;\mathbf{n}$$

Explanation: The cross product of two nonzero vectors in three dimensions produces a third vector that is orthogonal to each of the first two. This resulting vector $\mathbf{u} \times \mathbf{v}$ is, therefore, normal to the plane containing the first two vectors (assuming $\mathbf{u}$ and $\mathbf{v}$ are not parallel). In the second formula above, $\mathbf{n}$ is the unit vector normal to the plane containing the first two vectors. Its orientation (direction) is determined using the right hand rule.

Right Hand Rule



Using your right hand:

- Point your forefinger in the direction of $\mathbf{u}$, and
- Point your middle finger in the direction of $\mathbf{v}$.

Then:

- Your thumb will point in the direction of $\mathbf{u} \times \mathbf{v}$.

In two dimensions,

Let: $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$

Then, $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = (u_1v_2 - u_2v_1)$ which is a scalar (in two dimensions).

The cross product of two nonzero vectors in two dimensions is zero if the vectors are parallel. That is, vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if $\mathbf{u} \times \mathbf{v} = 0$.

The area of a parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides and angle θ between them:

$$Area = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sin\theta.$$

Vector Cross Product

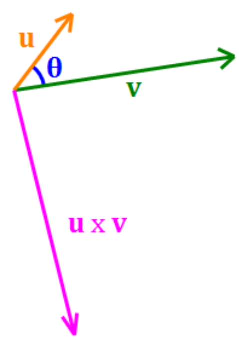
Properties of the Cross Product

Let m be a scalar, and let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Then,

- $\mathbf{0} \times \mathbf{u} = \mathbf{u} \times \mathbf{0} = \mathbf{0}$ — Zero Property
- $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ — $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are orthogonal to each other
- $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$, $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$ — Reverse orientation orthogonality
- $\mathbf{u} \times \mathbf{u} = \mathbf{0}$ — Every non-zero vector is parallel to itself
- $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$ — Anti-commutative Property
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ — Distributive Property
- $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$ — Distributive Property
- $(m\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (m\mathbf{v}) = m(\mathbf{u} \times \mathbf{v})$ — Scalar Multiplication

More properties:

- If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel.
- If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then $\sin\theta = \dfrac{\|\mathbf{u} \times \mathbf{v}\|}{\|\mathbf{u}\|\ \|\mathbf{v}\|}$.



Angle Between Two Vectors

Notice the similarities in the formulas for the angle between two vectors using the dot product and the cross product:

$$\cos\theta = \frac{\mathbf{u} \circ \mathbf{v}}{\|\mathbf{u}\|\ \|\mathbf{v}\|} \qquad \sin\theta = \frac{\|\mathbf{u} \times \mathbf{v}\|}{\|\mathbf{u}\|\ \|\mathbf{v}\|}$$

Vector Triple Products

Scalar Triple Product

Let: $\mathbf{u} = \mathrm{u}_1\mathbf{i} + \mathrm{u}_2\mathbf{j} + \mathrm{u}_3\mathbf{k}$, $\mathbf{v} = \mathrm{v}_1\mathbf{i} + \mathrm{v}_2\mathbf{j} + \mathrm{v}_3\mathbf{k}$, $\mathbf{w} = \mathrm{w}_1\mathbf{i} + \mathrm{w}_2\mathbf{j} + \mathrm{w}_3\mathbf{k}$.

Then the triple product $\mathbf{u} \circ (\mathbf{v} \times \mathbf{w})$ gives a scalar representing the volume of a parallelepiped (a 3D parallelogram) with $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as edges:

$$\mathbf{u} \circ (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathrm{u}_1 & \mathrm{u}_2 & \mathrm{u}_3 \\ \mathrm{v}_1 & \mathrm{v}_2 & \mathrm{v}_3 \\ \mathrm{w}_1 & \mathrm{w}_2 & \mathrm{w}_3 \end{vmatrix}$$

$$\mathbf{u} \circ (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \circ \mathbf{w}$$

Note: vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are coplanar if and only if $\mathbf{u} \circ (\mathbf{v} \times \mathbf{w}) = 0$.

Other Triple Products

$\mathbf{u} \circ (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \circ (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$ Duplicating a vector results in a product of $\mathbf{0}$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \circ \mathbf{w})\,\mathbf{v} - (\mathbf{u} \circ \mathbf{v})\,\mathbf{w}$$

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \circ \mathbf{w})\,\mathbf{v} - (\mathbf{v} \circ \mathbf{w})\,\mathbf{u}$$

$$\mathbf{u} \circ (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \circ (\mathbf{w} \times \mathbf{x}) = \mathbf{w} \circ (\mathbf{x} \times \mathbf{v})$$

No Associative Property

The associative property of real numbers does not translate to triple products. In particular,

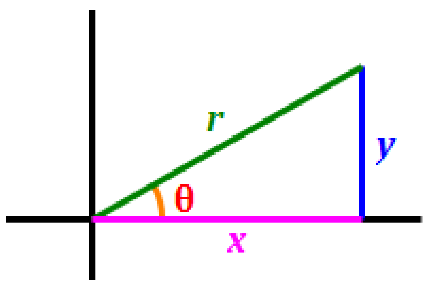
$(\mathbf{u} \circ \mathbf{v}) \cdot \mathbf{w} \neq \mathbf{u} \cdot (\mathbf{v} \circ \mathbf{w})$ No associative property of dot products/multiplication

$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ No associative property of cross products

Appendix A

Summary of Trigonometric Formulas

Trigonometric Functions (x- and y- axes)



$\sin\theta = \frac{y}{r}$ $\quad \sin\theta = \frac{1}{\csc\theta}$

$\cos\theta = \frac{x}{r}$ $\quad \cos\theta = \frac{1}{\sec\theta}$

$\tan\theta = \frac{y}{x}$ $\quad \tan\theta = \frac{1}{\cot\theta}$ $\quad \tan\theta = \frac{\sin\theta}{\cos\theta}$

$\cot\theta = \frac{x}{y}$ $\quad \cot\theta = \frac{1}{\tan\theta}$ $\quad \cot\theta = \frac{\cos\theta}{\sin\theta}$

$\sec\theta = \frac{r}{x}$ $\quad \sec\theta = \frac{1}{\cos\theta}$

$\csc\theta = \frac{r}{y}$ $\quad \csc\theta = \frac{1}{\sin\theta}$

Pythagorean Identities

(for any angle θ)

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sec^2\theta = 1 + \tan^2\theta$$

$$\csc^2\theta = 1 + \cot^2\theta$$

Sine-Cosine Relationship

$$\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$$

$$\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$$

Key Angles

($180° = \pi$ radians)

$$0° = 0 \text{ radians}$$

$$30° = \frac{\pi}{6} \text{ radians}$$

$$45° = \frac{\pi}{4} \text{ radians}$$

$$60° = \frac{\pi}{3} \text{ radians}$$

$$90° = \frac{\pi}{2} \text{ radians}$$

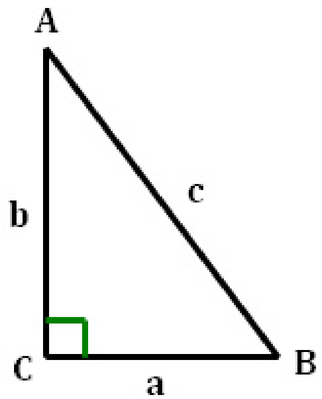
Cofunctions (in Quadrant I)

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right) \quad \Leftrightarrow \quad \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right) \quad \Leftrightarrow \quad \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$

$$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right) \quad \Leftrightarrow \quad \csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$$

Trigonometric Functions (Right Triangle)



SOH-CAH-TOA

$$\sin = \frac{opposite}{hypoteneuse} \qquad \sin A = \frac{a}{c} \qquad \sin B = \frac{b}{c}$$

$$\cos = \frac{adjacent}{hypoteneuse} \qquad \cos A = \frac{b}{c} \qquad \cos B = \frac{a}{c}$$

$$\tan = \frac{opposite}{adjacent} \qquad \tan A = \frac{a}{b} \qquad \tan B = \frac{b}{a}$$

Laws of Sines and Cosines (Oblique Triangle)

Law of Sines (see illustration below)

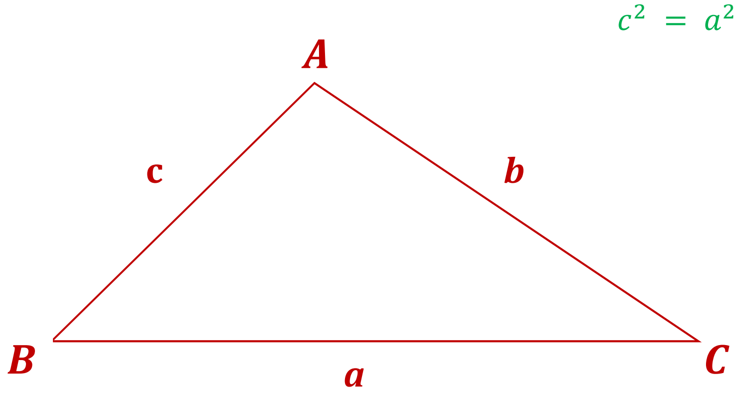
$$\frac{a}{\sin\ A} = \frac{b}{\sin\ B} = \frac{c}{\sin\ C}$$

Law of Cosines (see illustration below)

$$a^2 = b^2 + c^2 - 2bc\ \cos A$$

$$b^2 = a^2 + c^2 - 2ac\ \cos B$$

$$c^2 = a^2 + b^2 - 2ab\ \cos C$$



Angle Addition Formulas

$$\sin\ (\alpha+\beta) = \sin\alpha\cos\beta\ +\ \cos\alpha\sin\beta$$
$$\sin\ (\alpha-\beta) = \sin\alpha\cos\beta\ -\ \cos\alpha\sin\beta$$

$$\cos\ (\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos\ (\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan\ (\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\ \tan\beta}$$

$$\tan\ (\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\ \tan\beta}$$

Double Angle Formulas

$$\sin\ 2\theta\ =\ 2\sin\theta\cos\theta$$

$$\cos\ 2\theta\ =\ \cos^2\theta-\sin^2\theta$$
$$=\ 1-2\sin^2\theta$$
$$=\ 2\cos^2\theta-1$$

$$\tan\ 2\theta\ =\ \frac{2\ \tan\theta}{1-\tan^2\theta}$$

Half Angle Formulas

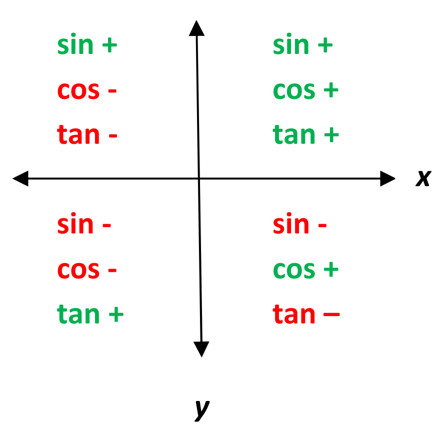
$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$
$$= \frac{1-\cos\theta}{\sin\theta}$$
$$= \frac{\sin\theta}{1+\cos\theta}$$

The use of a "+" or "-" sign in the half angle formulas depends on the quadrant in which the angle $\frac{\theta}{2}$ resides. See chart below.

Signs of Trig Functions By Quadrant



Power Reducing Formulas

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

Product-to-Sum Formulas

$$\sin\alpha\cdot\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\cos\alpha\cdot\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

$$\sin\alpha\cdot\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$\cos\alpha\cdot\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

Sum-to-Product Formulas

$$\sin\alpha + \sin\beta = 2\cdot\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cdot\sin\left(\frac{\alpha-\beta}{2}\right)\cdot\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cdot\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\cdot\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{\alpha-\beta}{2}\right)$$

Triangle Area Formulas

Geometry

$$A = \frac{1}{2} bh$$

where, b is the length of the base of the triangle.
h is the height of the triangle.

Heron's Formula

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where, $s = \frac{1}{2}P = \frac{1}{2}(a+b+c)$.

a, b, c are the lengths of the sides of the triangle.

Using Both Lengths and Angles

$$A = \frac{1}{2} \cdot \frac{a^2 \cdot \sin B \cdot \sin C}{\sin A} = \frac{1}{2} \cdot \frac{b^2 \cdot \sin A \cdot \sin C}{\sin B} = \frac{1}{2} \cdot \frac{c^2 \cdot \sin A \cdot \sin B}{\sin C}$$

$$A = \frac{1}{2}\, ab\, \sin C = \frac{1}{2}\, ac\, \sin B = \frac{1}{2}\, bc\, \sin A$$

Coordinate Geometry

Let three vertices of a triangle in the coordinate plane be: (x_1, y_1), (x_2, y_2), (x_3, y_3).

$$A = \frac{1}{2} \cdot \left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$

Complex Numbers in Polar Form

$e^{i\theta} = \cos\theta + i\sin\theta = \text{cis}\,\theta$

$z = a + bi = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta = r \cdot e^{i\theta}$

Operations

Let: $z_1 = a_1 + b_1 i = r_1(\cos\theta + i\sin\theta)$

$z_2 = a_2 + b_2 i = r_2(\cos\varphi + i\sin\varphi)$

Multiplication: $z_1 \cdot z_2 = r_1\,r_2\,[\cos\,(\theta + \varphi)\; + \; i\sin(\theta + \varphi)]$

Division: $z_1 \div z_2 = \dfrac{r_1}{r_2}\;[\cos\,(\theta - \varphi)\; + \; i\sin(\theta - \varphi)]$

Powers: $z^n = r^n(\cos n\theta + i\sin n\theta)$

Roots: $\sqrt[n]{z} = z_k = \sqrt[n]{r} \cdot \left[\,\cos\left(\frac{\theta + k(2\pi)}{n}\right) + \; i\sin\left(\frac{\theta + k(2\pi)}{n}\right)\right]$,

k varies from 0 to $n - 1$

Note: z has n distinct complex n-th roots: $z_0,\ z_1,\ z_2, \ldots,\ z_{n-1}$

Vectors

$\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors in the x, y, z directions respectively.

2 dimensions	3 dimensions
$\mathbf{v} = a\mathbf{i} + b\mathbf{j}$	$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$
$\|\mathbf{v}\| = \sqrt{a^2 + b^2}$	$\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2}$

Properties

$\mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v}$	Additive Identity
$\mathbf{v} + (-\mathbf{v}) = (-\mathbf{v}) + \mathbf{v} = \mathbf{0}$	Additive Inverse
$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative Property
$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	Associative Property
$m(n\boldsymbol{u}) = (mn)\boldsymbol{u}$	Associative Property
$m(\boldsymbol{u} + \boldsymbol{v}) = m\boldsymbol{u} + m\boldsymbol{v}$	Distributive Property
$(m + n)\boldsymbol{u} = m\boldsymbol{u} + n\mathbf{u}$	Distributive Property
$1(\mathbf{v}) = \mathbf{v}$	Multiplicative Identity
$\|m\mathbf{v}\| = \|m\| \, \|\mathbf{v}\|$	Magnitude Property
$\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$	Unit vector in the direction of $\mathbf{v}$

Vector Dot Product

Let: $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$

$\mathbf{u} \circ \mathbf{v} = (a_1 \cdot a_2) + (b_1 \cdot b_2) + (c_1 \cdot c_2)$

Properties

$\mathbf{0} \circ \mathbf{u} = \mathbf{u} \circ \mathbf{0} = 0$ Zero Property

$\mathbf{i} \circ \mathbf{j} = \mathbf{j} \circ \mathbf{k} = \mathbf{k} \circ \mathbf{i} = 0$ $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are orthogonal to each other.

$\mathbf{u} \circ \mathbf{v} = \mathbf{v} \circ \mathbf{u}$ Commutative Property

$\mathbf{u} \circ \mathbf{u} = \|\mathbf{u}\|^2$ Magnitude Square Property

$\mathbf{u} \circ (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \circ \mathbf{v}) + (\mathbf{u} \circ \mathbf{w})$ Distributive Property

$m(\mathbf{u} \circ \mathbf{v}) = (m\boldsymbol{u}) \circ \mathbf{v} = \boldsymbol{u} \circ (m\mathbf{v})$ Multiplication by a Scalar Property

$\cos\theta = \dfrac{\mathbf{u} \circ \mathbf{v}}{\|\mathbf{u}\|\ \|\mathbf{v}\|}$ θ is the angle between $\mathbf{u}$ and $\mathbf{v}$

Vector Projection

$$\text{proj}_{\mathbf{w}}\mathbf{v} = \left(\frac{\mathbf{v} \circ \mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w} = \left(\frac{\mathbf{v} \circ \mathbf{w}}{\mathbf{w} \circ \mathbf{w}}\right)\mathbf{w}$$

Orthogonal Components of a Vector

$\mathbf{v_1} = \text{proj}_{\mathbf{w}}\mathbf{v} = \left(\dfrac{\mathbf{v} \circ \mathbf{w}}{\|\mathbf{w}\|^2}\right)\mathbf{w}$ and $\mathbf{v_2} = \mathbf{v} - \mathbf{v_1}$

Work

$\mathbf{F}$ is the force vector acting on an object, moving it from point A to point B.

$W = \mathbf{F} \circ \overrightarrow{AB}$

$W = \|\mathbf{F}\|\ \|\overrightarrow{AB}\|\ \cos\theta$ θ is angle between $\mathbf{F}$ and $\overrightarrow{AB}$.

Vector Cross Product

2 Dimensions

Let: $\mathbf{u} = \mathrm{u}_1\mathbf{i} + \mathrm{u}_2\mathbf{j}$ and $\mathbf{v} = \mathrm{v}_1\mathbf{i} + \mathrm{v}_2\mathbf{j}$

Then, $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathrm{u}_1 & \mathrm{u}_2 \\ \mathrm{v}_1 & \mathrm{v}_2 \end{vmatrix} = (\mathrm{u}_1\mathrm{v}_2 - \mathrm{u}_2\mathrm{v}_1)$

Area of a parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides and angle θ between them:

$Area = \|\mathbf{u}\|\ \|\mathbf{v}\|\ \sin\theta$

3 Dimensions

Let: $\mathbf{u} = \mathrm{u}_1\mathbf{i} + \mathrm{u}_2\mathbf{j} + \mathrm{u}_3\mathbf{k}$ and $\mathbf{v} = \mathrm{v}_1\mathbf{i} + \mathrm{v}_2\mathbf{j} + \mathrm{v}_3\mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathrm{u}_1 & \mathrm{u}_2 & \mathrm{u}_3 \\ \mathrm{v}_1 & \mathrm{v}_2 & \mathrm{v}_3 \end{vmatrix} = (\mathrm{u}_2\mathrm{v}_3 - \mathrm{u}_3\mathrm{v}_2)\,\mathbf{i} + (\mathrm{u}_3\mathrm{v}_1 - \mathrm{u}_1\mathrm{v}_3)\,\mathbf{j} + (\mathrm{u}_1\mathrm{v}_2 - \mathrm{u}_2\mathrm{v}_1)\,\mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\|\ \|\mathbf{v}\|\ \sin\theta\ \mathbf{n}$$

$\mathbf{n}$ is the unit vector normal to the plane containing the first two vectors with orientation determined using the right hand rule.

Properties

$\mathbf{0} \times \mathbf{u} = \mathbf{u} \times \mathbf{0} = \mathbf{0}$	Zero Property
$\mathbf{i} \times \mathbf{j} = \mathbf{k},\ \ \mathbf{j} \times \mathbf{k} = \mathbf{i},\ \ \mathbf{k} \times \mathbf{i} = \mathbf{j}$	$\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are orthogonal to each other
$\mathbf{j} \times \mathbf{i} = -\mathbf{k},\ \ \mathbf{k} \times \mathbf{j} = -\mathbf{i},\ \ \mathbf{i} \times \mathbf{k} = -\mathbf{j}$	Reverse orientation orthogonality
$\mathbf{u} \times \mathbf{u} = \mathbf{0}$	Every non-zero vector is parallel to itself
$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$	Anti-commutative Property
$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$	Distributive Property
$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$	Distributive Property
$(\mathrm{m}\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (\mathrm{m}\mathbf{v}) = \mathrm{m}(\mathbf{u} \times \mathbf{v})$	Scalar Multiplication
$\sin\theta = \dfrac{\|\mathbf{u} \times \mathbf{v}\|}{\|\mathbf{u}\|\ \|\mathbf{v}\|}$	θ is the angle between $\mathbf{u}$ and $\mathbf{v}$

Vector Triple Products

Let: $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$.

Scalar Triple Product

$$\mathbf{u} \circ (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$\mathbf{u} \circ (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \circ \mathbf{w}$$

Other Triple Products

$$\mathbf{u} \circ (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \circ (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \circ \mathbf{w})\,\mathbf{v} - (\mathbf{u} \circ \mathbf{v})\,\mathbf{w}$$

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \circ \mathbf{w})\,\mathbf{v} - (\mathbf{v} \circ \mathbf{w})\,\mathbf{u}$$

$$\mathbf{u} \circ (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \circ (\mathbf{w} \times \mathbf{x}) = \mathbf{w} \circ (\mathbf{x} \times \mathbf{v})$$

No Associative Property

$$(\mathbf{u} \circ \mathbf{v}) \cdot \mathbf{w} \neq \mathbf{u} \cdot (\mathbf{v} \circ \mathbf{w})$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$$

Appendix B

Solving the Ambiguous Case – Sine Validity Method

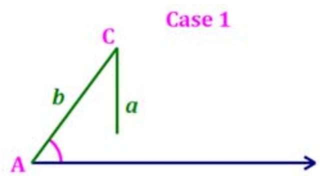
How do you solve a triangle (or two) in the ambiguous case? Assume the information given is the lengths of sides a and b, and the measure of Angle A. Use the following steps:

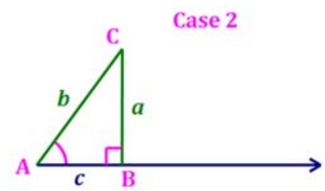
Step 1: Calculate the sine of the missing angle (in this development, angle B).

Step 1: Use $\dfrac{a}{\sin\ A} = \dfrac{b}{\sin\ B}$

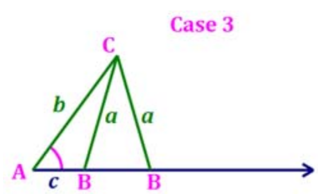
Step 2: Consider the value of $sin\ B$:

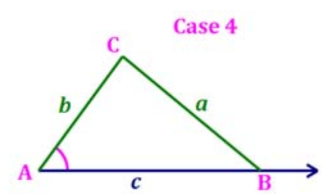
- If $sin\ B > 1$, then we have Case 1 – there is no triangle. Stop here.
- If $sin\ B = 1$, then $B = 90°$, and we have Case 2 – a right triangle. Proceed to Step 4.





- If $sin\ B < 1$, then we have Case 3 or Case 4. Proceed to the next step to determine which.





Step 3: Calculate the two values of $m\angle B$ from the sine value calculated in Step 2.

- For each angle B, if $m\angle A + m\angle B < 180°$, there is a valid triangle.
  - If this inequality is true for both values of angle B, then we have Case 3 – two triangles.
  - If $m\angle A + m\angle B \geq 180°$, then we have case 4 – one triangle.
- In either case, proceed to Step 4 for any valid triangles.

Step 4: Calculate C. At this point, we have the lengths of sides a and b, and the measures of Angles A and B. If we are dealing with Case 3 – two triangles, we must perform Steps 4 and 5 for each angle.

Step 4 is to calculate the measure of Angle C as follows: $m\angle C = 180° - m\angle A - m\angle B$

Step 5: Calculate c. Finally, we calculate the value of c using the Law of Sines.

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \Rightarrow \quad c = \frac{a \sin C}{\sin A} \qquad \text{or} \qquad \frac{b}{\sin B} = \frac{c}{\sin C} \quad \Rightarrow \quad c = \frac{b \sin C}{\sin B}$$

Note: using a and $\angle A$ in this calculation may produce more accurate results since both of these values are given.

Appendix C

Summary of Rectangular and Polar Forms

		Rectangular Form	Polar Form
Coordinates	Form	(x, y)	(r, θ)
	Conversion	$x = r\cos\theta$ $y = r\sin\theta$	$r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
Complex Numbers	Form	$a + bi$	$r(\cos\theta + i\sin\theta)$ or $r\ cis\theta$
	Conversion	$a = r\cos\theta$ $b = r\sin\theta$	$r = \sqrt{a^2 + b^2}$ $\theta = \tan^{-1}\left(\frac{b}{a}\right)$
Vectors	Form	$a\mathbf{i} + b\mathbf{j}$	$\lVert v \rVert\ \angle\theta$ $\lVert v \rVert$ = magnitude θ = direction angle
	Conversion	$a = \lVert v \rVert\cos\theta$ $b = \lVert v \rVert\sin\theta$	$\lVert v \rVert = \sqrt{a^2 + b^2}$ $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

Trigonometry Reference

Function Relationships

$\sin\theta = \dfrac{1}{\csc\theta}$ $\quad \csc\theta = \dfrac{1}{\sin\theta}$

$\cos\theta = \dfrac{1}{\sec\theta}$ $\quad \sec\theta = \dfrac{1}{\cos\theta}$

$\tan\theta = \dfrac{1}{\cot\theta}$ $\quad \cot\theta = \dfrac{1}{\tan\theta}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\quad \cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Opposite Angle Formulas

$\sin(-\theta) = -\sin(\theta)$

$\cos(-\theta) = \cos(\theta)$

$\tan(-\theta) = -\tan(\theta)$

$\cot(-\theta) = -\cot(\theta)$

$\sec(-\theta) = \sec(\theta)$

$\csc(-\theta) = -\csc(\theta)$

Cofunction Formulas (in Quadrant I)

$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ $\quad \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$

$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$ $\quad \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$

$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$ $\quad \csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$

Angle Addition Formulas

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$\cot^2\theta + 1 = \csc^2\theta$

Half Angle Formulas

$\sin\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$

$\cos\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$

$\tan\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$

$= \dfrac{1-\cos\theta}{\sin\theta}$

$= \dfrac{\sin\theta}{1+\cos\theta}$

Double Angle Formulas

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 1 - 2\sin^2\theta$

$= 2\cos^2\theta - 1$

$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Product-to-Sum Formulas

$\sin A\cdot\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$\cos A\cdot\cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

$\sin A\cdot\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$\cos A\cdot\sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

Triple Angle Formulas

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

Power Reducing Formulas

$\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$\tan^2\theta = \dfrac{1-\cos 2\theta}{1+\cos 2\theta}$

Sum-to-Product Formulas

$\sin A + \sin B = 2\cdot\sin\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right)$

$\sin A - \sin B = 2\cdot\sin\left(\frac{A-B}{2}\right)\cdot\cos\left(\frac{A+B}{2}\right)$

$\cos A + \cos B = 2\cdot\cos\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right)$

$\cos A - \cos B = -2\cdot\sin\left(\frac{A+B}{2}\right)\cdot\sin\left(\frac{A-B}{2}\right)$

Arc Length

$S = r\theta$

Law of Sines

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Law of Cosines

$a^2 = b^2 + c^2 - 2bc\cos A$

$b^2 = a^2 + c^2 - 2ac\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$

Law of Tangents

$\dfrac{a-b}{a+b} = \dfrac{\tan\left[\frac{1}{2}(A-B)\right]}{\tan\left[\frac{1}{2}(A+B)\right]}$

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Mollweide's Formulas

$\dfrac{a+b}{c} = \dfrac{\cos\left[\frac{1}{2}(A-B)\right]}{\sin\left(\frac{1}{2}C\right)}$

$\dfrac{a-b}{c} = \dfrac{\sin\left[\frac{1}{2}(A-B)\right]}{\cos\left(\frac{1}{2}C\right)}$

Euler's Formula

$e^{i\theta} = \cos\theta + i\sin\theta = \text{cis}\,\theta$

DeMoivre's Formula

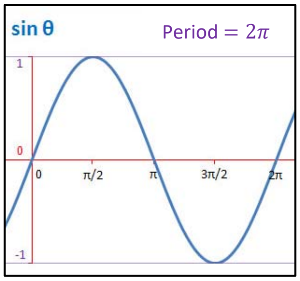
$(r\,\text{cis}\,\theta)^n = r^n\,\text{cis}\,(n\theta)$

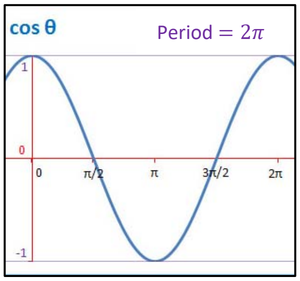
Polar Multiplication and Division

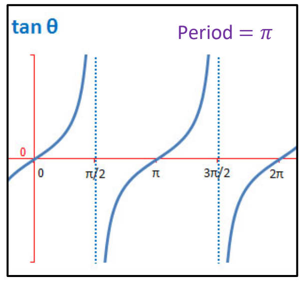
Let: $a = r_1\,\text{cis}\,\theta$ $\quad b = r_2\,\text{cis}\,\varphi$

$a\cdot b = r_1 r_2\,\text{cis}\,(\theta+\varphi)$ $\quad \dfrac{a}{b} = \dfrac{r_1}{r_2}\,\text{cis}\,(\theta-\varphi)$

Trigonometry Reference







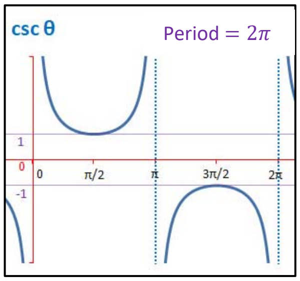
$y = A \cdot f(Bx - C) + D$

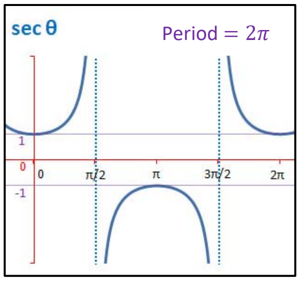
Amplitude: $|A|$

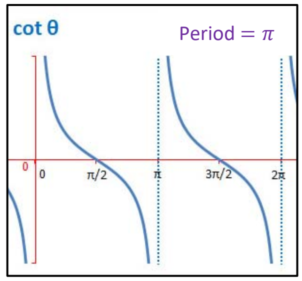
Period: $\frac{parent\ "f"\ period}{B}$

Phase Shift: $\frac{C}{B} \longrightarrow$

Vertical Shift: D







Harmonic Motion

$d = a \cos \omega t$ or

$d = a \sin \omega t$

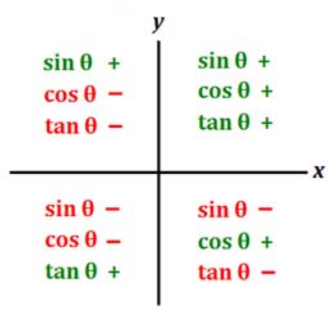
$f = \frac{1}{\text{period}} = \frac{\omega}{2\pi}$

$\omega = 2\pi f, \quad \omega > 0$

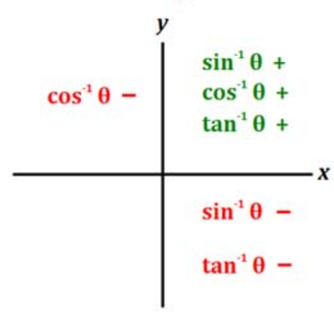
Trig Functions of Special Angles (Unit Circle)

θ Rad	$\theta°$	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0°	0	1	0
$\pi/6$	30°	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$\pi/4$	45°	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\pi/3$	60°	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	90°	1	0	*undefined*

Signs of Trig Functions by Quadrant



Locations of Principal Values of Inverse Trig Functions



Rectangular/Polar Conversion

Rectangular	Polar
(x, y)	(r, θ)
$x = r\cos\theta$ $y = r\sin\theta$	$r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
$a + bi$	$r\,(\cos\theta + i\sin\theta)$ or $r\ cis\theta$
$a = r\cos\theta$ $b = r\sin\theta$	$r = \sqrt{a^2 + b^2}$ $\theta = \tan^{-1}\left(\frac{b}{a}\right)$
$a\mathbf{i} + b\mathbf{j}$	$\lVert\mathbf{v}\rVert \angle\theta$
$a = \lVert\mathbf{v}\rVert\cos\theta$ $b = \lVert\mathbf{v}\rVert\sin\theta$	$\lVert\mathbf{v}\rVert = \sqrt{a^2 + b^2}$ $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

Triangle Area

$A = \frac{1}{2}bh$

$A = \sqrt{s(s-a)(s-b)(s-c)}$

$s = \frac{1}{2}P = \frac{1}{2}(a + b + c)$

$A = \frac{1}{2}\left(\frac{a^2 \sin B \sin C}{\sin A}\right)$

$A = \frac{1}{2}\ ab\sin C$

$A = \frac{1}{2}\left\lVert \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right\rVert$

$A = \frac{1}{2}\ \lVert\mathbf{u}\rVert\ \lVert\mathbf{v}\rVert\ \sin\theta$

Vector Properties

$0 + \mathbf{u} = \mathbf{u} + 0 = \mathbf{u}$

$\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = 0$

$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

$m(n\mathbf{u}) = (mn)\mathbf{u}$

$m(\mathbf{u} + \mathbf{v}) = m\mathbf{u} + m\mathbf{v}$

$(m + n)\mathbf{u} = m\mathbf{u} + n\mathbf{u}$

$1(\mathbf{v}) = \mathbf{v}$

$\lVert m\mathbf{v}\rVert = |m|\ \lVert\mathbf{v}\rVert$

Unit Vector: $\frac{\mathbf{v}}{\lVert\mathbf{v}\rVert}$

Vector Dot Product

$\mathbf{u} \circ \mathbf{v} = (u_1 \cdot v_1) + (u_2 \cdot v_2)$

$\mathbf{u} \circ (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \circ \mathbf{v}) + (\mathbf{u} \circ \mathbf{w})$

Vector Cross Product

$\mathbf{u}\ \text{x}\ \mathbf{v} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1$

$\mathbf{u}\ \text{x}\ (\mathbf{v} + \mathbf{w}) = (\mathbf{u}\ \text{x}\ \mathbf{v}) + (\mathbf{u}\ \text{x}\ \mathbf{w})$

Angle between Vectors

$\cos\theta = \frac{\mathbf{u} \circ \mathbf{v}}{\lVert\mathbf{u}\rVert\ \lVert\mathbf{v}\rVert}$ $\quad \sin\theta = \frac{\lVert\mathbf{u}\ \text{x}\ \mathbf{v}\rVert}{\lVert\mathbf{u}\rVert\ \lVert\mathbf{v}\rVert}$

$\perp$ iff $\mathbf{u} \circ \mathbf{v} = 0$ $\quad \parallel$ iff $\mathbf{u}\ \text{x}\ \mathbf{v} = 0$

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