HMMT November 2022

November 12, 2022

General Round

1. Emily's broken clock runs backwards at five times the speed of a regular clock. Right now, it is displaying the wrong time. How many times will it display the correct time in the next 24 hours? It is an analog clock (i.e. a clock with hands), so it only displays the numerical time, not AM or PM. Emily's clock also does not tick, but rather updates continuously.

   Proposed by: Brian Liu, Luke Robitaille, Selena Liu

   Answer: $\boxed{12}$

   Solution: When comparing Emily's clock with a normal clock, the difference between the two times decreases by 6 seconds for every 1 second that passes. Since this difference is treated as 0 whenever it is a multiple of 12 hours, the two clocks must agree once every $\frac{12}{6} = 2$ hours. Thus, in a 24 hour period it will agree 12 times.

2. How many ways are there to arrange the numbers $1, 2, 3, 4, 5, 6$ on the vertices of a regular hexagon such that exactly 3 of the numbers are larger than both of their neighbors? Rotations and reflections are considered the same.

   Proposed by: Ankit Bisain

   Answer: $\boxed{8}$

   Solution: Label the vertices of the hexagon $abcdef$.

   The numbers that are larger than both of their neighbors can't be adjacent, so assume (by rotation) that these numbers take up slots ace. We also have that 6 and 5 cannot be smaller than both of their neighbors, so assume (by rotation and reflection) that $a = 6$ and $c = 5$.

   Now, we need to insert $1, 2, 3, 4$ into b, d, e, f such that e is the largest among d, e, f. There are 4 ways to choose b, which uniquely determines e, and 2 ways to choose the ordering of d, f, giving $4 \cdot 2 = 8$ total ways.

3. Let $ABCD$ be a rectangle with $AB = 8$ and $AD = 20$. Two circles of radius 5 are drawn with centers in the interior of the rectangle - one tangent to AB and AD, and the other passing through both C and D. What is the area inside the rectangle and outside of both circles?

   Proposed by: Ankit Bisain

   Answer: $\boxed{112 - 25\pi}$

   Solution:

   Let O_1 and O_2 be the centers of the circles, and let M be the midpoint of $\overline{CD}$. We can see that $\triangle O_2MC$ and $\triangle O_2MD$ are both 3-4-5 right triangles. Now let C' be the intersection of circle O_2 and $\overline{BC}$ (that isn't C), and let D' be the intersection of circle O_2 and $\overline{AD}$ (that isn't D). We know that $AD' = BC' = 14$ because $BC' = 2O_2M = 6$.

   All of the area of $ABCD$ that lies outside circle O_2 must lie within rectangle $ABC'D'$ because $C'CDD'$ is completely covered by circle O_2. Now, notice that the area of circle O_2 that lies inside $ABC'D'$ is the same as the area of circle O_1 that lies outside $ABC'D'$. Thus, the total area of $ABC'D'$ that is covered by either of the two circles is exactly the area of one of the circles, 25π. The remaining area is $8 \cdot 14 - 25\pi$, which is our answer.

4. Let $x < 0.1$ be a positive real number. Let the *foury series* be $4 + 4x + 4x^2 + 4x^3 + \ldots$, and let the *fourier series* be $4 + 44x + 444x^2 + 4444x^3 + \ldots$. Suppose that the sum of the fourier series is four times the sum of the foury series. Compute x.

   Proposed by: Carl Schildkraut, Luke Robitaille, Priya Ganesh

   Answer: $\boxed{\frac{3}{40}}$

   Solution 1: The sum of the foury series can be expressed as $\frac{4}{1-x}$ by geometric series. The fourier series can be expressed as

$$\begin{aligned} &\frac{4}{9}\Big((10-1)+(100-1)x+(1000-1)x^2+\ldots\Big) \\ &= \frac{4}{9}\Big((10+100x+1000x^2+\ldots)-(1+x+x^2+\ldots)\Big) \\ &= \frac{4}{9}\left(\frac{10}{1-10x}-\frac{1}{1-x}\right). \end{aligned}$$

   Now we solve for x in the equation

$$\frac{4}{9}\left(\frac{10}{1-10x}-\frac{1}{1-x}\right) = 4\cdot\frac{4}{1-x}$$

   by multiplying both sides by $(1-10x)(1-x)$. We get $x = \frac{3}{40}$.

   Solution 2: Let R be the sum of the fourier series. Then the sum of the foury series is $(1-10x)R$. Thus, $1 - 10x = 1/4 \implies x = 3/40$.

5. An apartment building consists of 20 rooms numbered $1, 2, \ldots, 20$ arranged clockwise in a circle. To move from one room to another, one can either walk to the next room clockwise (i.e. from room i to room $(i+1) \pmod{20}$) or walk across the center to the opposite room (i.e. from room i to room $(i+10) \pmod{20}$). Find the number of ways to move from room 10 to room 20 without visiting the same room twice.

   Proposed by: Papon Lapate

   Answer: $\boxed{257}$

   Solution: One way is to walk directly from room 10 to 20. Else, divide the rooms into 10 pairs $A_0 = (10, 20), A_1 = (1, 11), A_2 = (2, 12), ..., A_9 = (9, 19)$. Notice that

   - each move is either between rooms in A_i and $A_{(i+1) \pmod{10}}$ for some $i \in \{0, 1, ..., 9\}$, or between rooms in the same pair, meaning that our path must pass through $A_0, A_1, ..., A_9$ in that order before coming back to room 20 in A_0,
   - in each of the pairs $A_1, A_2, ..., A_8$, we can choose to walk between rooms in that pair 0 or 1 times, and
   - we have to walk between rooms 9 and 19 if and only if we first reach A_9 at room 9 (so the choice of walking between A_9 is completely determined by previous choices).

   Thus, the number of ways to walk from room 10 to 20 is $1 + 2^8 = 257$.

6. In a plane, equilateral triangle ABC, square $BCDE$, and regular dodecagon $DEFGHIJKLMNO$ each have side length 1 and do not overlap. Find the area of the circumcircle of $\triangle AFN$.

   Proposed by: Kevin Zhao

   Answer: $\boxed{(2+\sqrt{3})\pi}$

Solution: Note that $\angle ACD = \angle ACB + \angle BCD = 60^\circ + 90^\circ = 150^\circ$. In a dodecagon, each interior angle is $180^\circ \cdot \frac{12-2}{12} = 150^\circ$ meaning that $\angle FED = \angle DON = 150^\circ$. since $EF = FD = 1$ and $DO = ON = 1$ (just like how $AC = CD = 1$), then we have that $\triangle ACD \cong \triangle DON \cong \triangle FED$ and because the triangles are isosceles, then $AD = DF = FN$ so D is the circumcenter of $\triangle AFN$. Now, applying the Law of Cosines gets that $AD^2 = 2 + \sqrt{3}$ so $AD^2\pi = (2 + \sqrt{3})\pi$.

7. In circle ω, two perpendicular chords intersect at a point P. The two chords have midpoints M_1 and M_2 respectively, such that $PM_1 = 15$ and $PM_2 = 20$. Line M_1M_2 intersects ω at points A and B, with M_1 between A and M_2. Compute the largest possible value of $BM_2 - AM_1$.

   Proposed by: Vidur Jasuja

   Answer: $\boxed{7}$

   Solution:

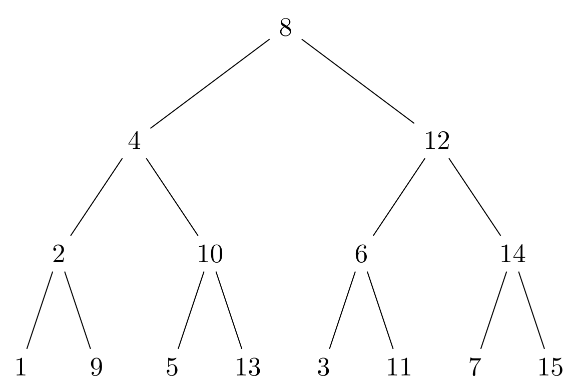
   Let O be the center of ω and let M be the midpoint of AB (so M is the foot of O to M_1M_2). Since OM_1PM_2 is a rectangle, we easily get that $MM_1 = 16$ and $MM_2 = 9$. Thus, $BM_2 - AM_1 = MM_1 - MM_2 = 7$.

8. Compute the number of sets S such that every element of S is a nonnegative integer less than 16, and if $x \in S$ then $(2x \bmod 16) \in S$.

   Proposed by: Vidur Jasuja

   Answer: $\boxed{678}$

   Solution:



   For any nonempty S we must have $0 \in S$. Now if we draw a directed graph of dependencies among the non-zero elements, it creates a balanced binary tree where every leaf has depth 3. In the diagram, if a is a parent of b it means that if $b \in S$, then a must also be in S.

   We wish to find the number of subsets of nodes such that every node in the set also has its parent in the set. We do this with recursion. Let $f(n)$ denote the number of such sets on a balanced binary tree of depth n. If the root vertex is not in the set, then the set must be empty. Otherwise, we can consider each subtree separately. This gives the recurrence $f(n) = f(n-1)^2 + 1$. We know $f(0) = 2$, so we can calculate $f(1) = 5, f(2) = 26, f(3) = 677$. We add 1 at the end for the empty set. Hence our answer is $f(3) + 1 = 678$.

9. Call a positive integer n *quixotic* if the value of

$$\operatorname{lcm}(1,2,3,\ldots,n)\cdot\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)$$

is divisible by 45. Compute the tenth smallest quixotic integer.

Proposed by: Vidur Jasuja

Answer: $\boxed{573}$

Solution: Let $L=\operatorname{lcm}(1,2,3,\ldots,n)$, and let $E=L\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$ denote the expression.

In order for n to be quixotic, we need $E\equiv 0 \pmod 5$ and $E\equiv 0 \pmod 9$. We consider these two conditions separately.

Claim: $E\equiv 0 \pmod 5$ if and only if $n\in[4\cdot 5^k, 5^{k+1})$ for some nonnegative integer k.

Proof. Let $k=\lfloor\log_5 n\rfloor$, which is equal to $\nu_5(L)$. In order for E to be divisible by 5, all terms in $\frac{L}{1},\frac{L}{2},\ldots,\frac{L}{n}$ that aren't multiples of 5 must sum to a multiple of 5. The potential terms that are not going to be multiples of 5 are $L/5^k, L/(2\cdot 5^k), L/(3\cdot 5^k)$, and $L/(4\cdot 5^k)$, depending on the value of n.

- If $n\in[5^k, 2\cdot 5^k)$, then only $L/5^k$ appears. Thus, the sum is $L/5^k$, which is not a multiple of 5.
- If $n\in[2\cdot 5^k, 3\cdot 5^k)$, then only $L/5^k$ and $L/(2\cdot 5^k)$ appear. The sum is $3L/(2\cdot 5^k)$, which is not a multiple of 5.
- If $n\in[3\cdot 5^k, 4\cdot 5^k)$, then only $L/5^k$, $L/(2\cdot 5^k)$, and $L/(3\cdot 5^k)$ appear. The sum is $11L/(6\cdot 5^k)$, which is not a multiple of 5.
- If $n\in[4\cdot 5^k, 5^{k+1})$, then $L/5^k$, $L/(2\cdot 5^k)$, $L/(3\cdot 5^k)$, and $L/(4\cdot 5^k)$ all appear. The sum is $25L/(12\cdot 5^k)$, which is a multiple of 5. Thus, this case works.

Only the last case works, implying the claim. $\square$

Claim: $E\equiv 0 \pmod 9$ if and only if $n\in[7\cdot 3^{k-1}, 8\cdot 3^{k-1})$ for some positive integer k.

Proof. This is a repeat of the previous proof, so we will only sketch it. Let $k=\lfloor\log_3 n\rfloor$, which is equal to $\nu_3(L)$. This time, the terms we need to consider are those that are not multiples of 9, which are

$$\frac{L}{3^{k-1}},\frac{L}{2\cdot 3^{k-1}},\ldots,\frac{L}{8\cdot 3^{k-1}}.$$

Similar to the above, we need to check that the sum of the first j terms is divisible by 9 if and only if $j=7$. There are 8 cases, but we could reduce workload by showing first that it is divisible by 3 if and only if $j\in\{6,7,8\}$ (there are only $L/3^k$ and $L/(2\cdot 3^k)$ to consider), then eliminate 6 and 8 by using $\pmod 9$. $\square$

Doing a little bit of arithmetic, we'll get the first 10 quixotic numbers: 21, 22, 23, 567, 568, 569, 570, 571, 572, 573.

10. Compute the number of distinct pairs of the form

$$(\text{first three digits of } x,\ \text{first three digits of } x^4)$$

over all integers $x>10^{10}$.

For example, one such pair is $(100,100)$ when $x=10^{10^{10}}$.

Proposed by: Albert Wang

Answer: $\boxed{4495}$

Solution: Graph these points on an x, y-plane. We claim that there are integers $100 = a_0 < a_1 < a_2 < a_3 < a_4 = 999$, for which the locus of these points is entirely contained in four taxicab (up/right movement by 1 unit) paths from $(a_i, 100)$ to $(a_{i+1}, 999)$, $i = 0, 1, 2, 3$.

As we increment x very slowly over all reals in $[100, 1000)$, which would produce the same set of tuples as we want (some small details missing here, but for large enough x we can approximate these decimals to arbitrary precision by scaling by some 10^k), it is clear that we must either have only one of the values increasing by 1, or both of them increasing by 1, where increasing by 1 in this context also includes the looping over from 999 to 100. In particular, this looping over occurs at the first three digits of powers of $\sqrt[4]{10}$ between 1 and 10 (i.e. $177, 316, 562$), which are precisely the values of a_1, a_2, a_3 that we claimed to exist.

Therefore, our taxicab paths have the same total length as one going from $(100, 100)$ up to $(999 + 900 + 900 + 900, 999)$, by stacking our four segments to continue from each other vertically. It remains to compute the number of times both sides of the tuple increased simultaneously, which correspond to fourth powers in the interval $(1, 1000)$. There are four of these corresponding to $2^4, 3^4, 4^4, 5^4$, which are at $(199, 159)$ to $(200, 160)$, $(299, 809)$ to $(300, 810)$, $(399, 255)$ to $(400, 256)$, and $(499, 624)$ to $(500, 625)$. So, our taxicab path is only missing these four holes.

Our final count is equal to the total taxidistance of the path, minus 4, and then finally adding back 1 to account for a starting point.

$$2 \cdot 899 + 3 \cdot 900 - 4 + 1 = 4495.$$