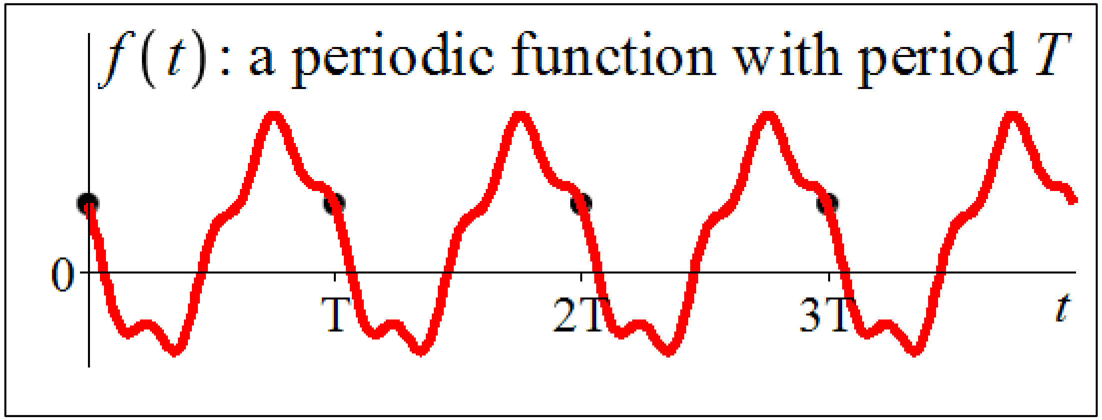
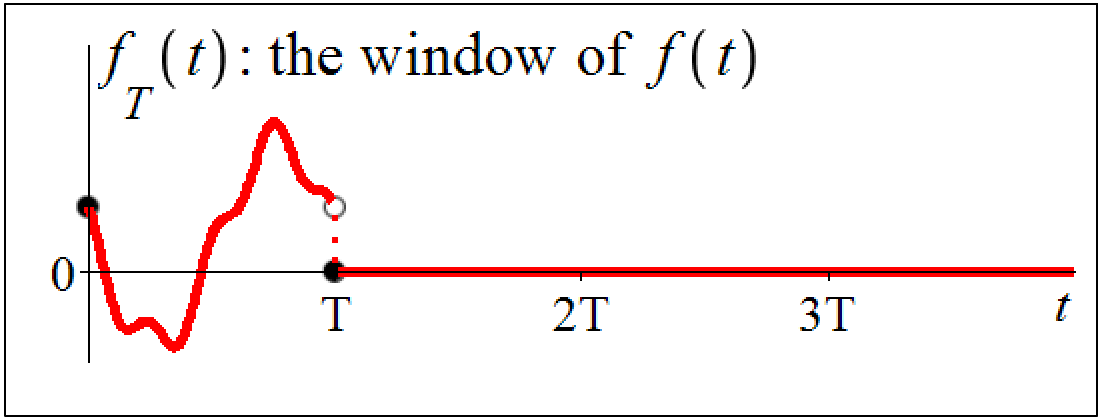
Laplace Transform of Periodic Functions



$f(t+T) = f(t)$



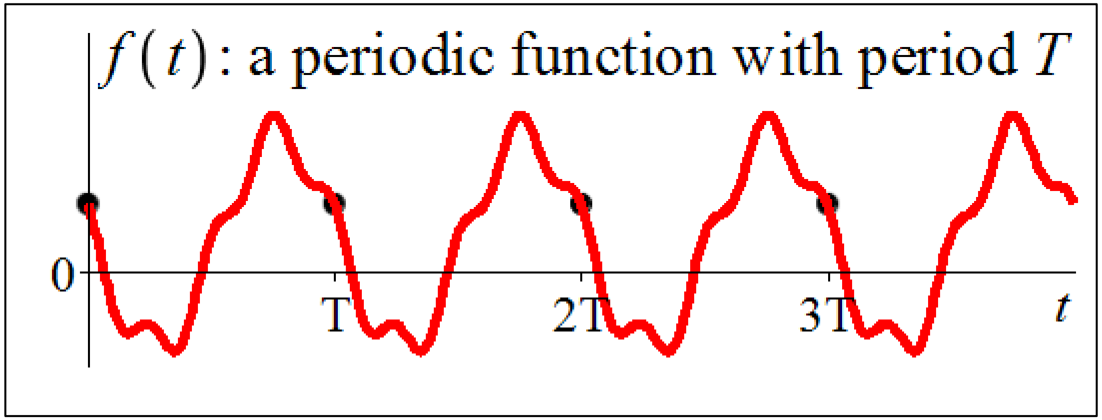
$$f_T(t) = \begin{cases} f(t) & 0 \le t < T \\ 0 & T \le t < \infty \end{cases}$$

How to find the Laplace transform $F(s)$ of a periodic function $f(t)$?

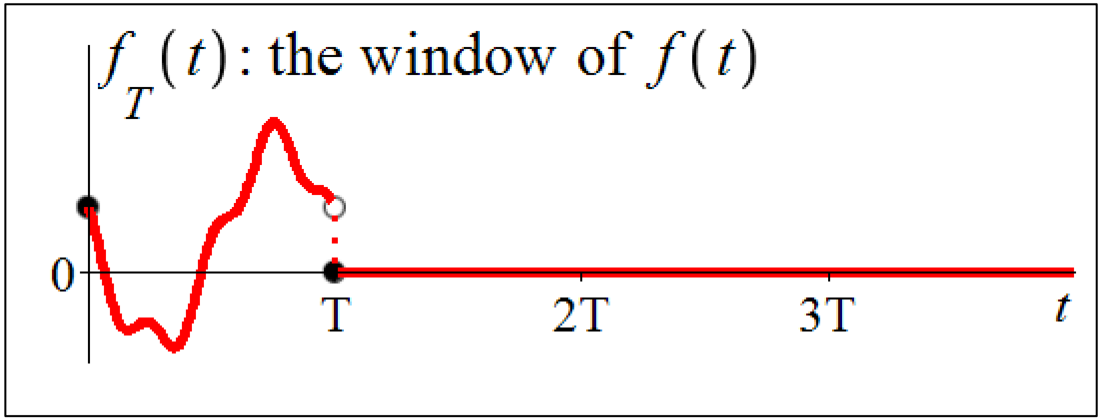
First, find the Laplace transform $F_T(s)$ of the window function $f_T(t)$

Then, use the formula:

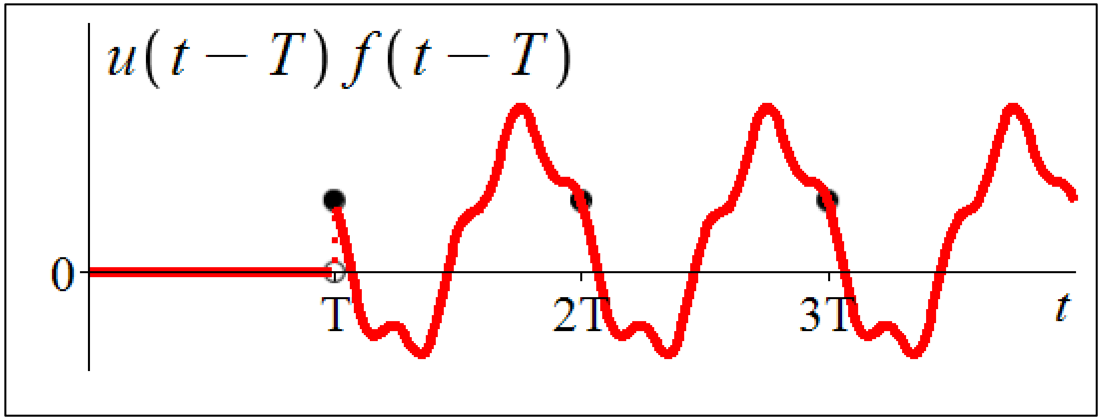
$$F(s) = \frac{F_T(s)}{1 - e^{-Ts}}$$



$$f(t+T) = f(t)$$

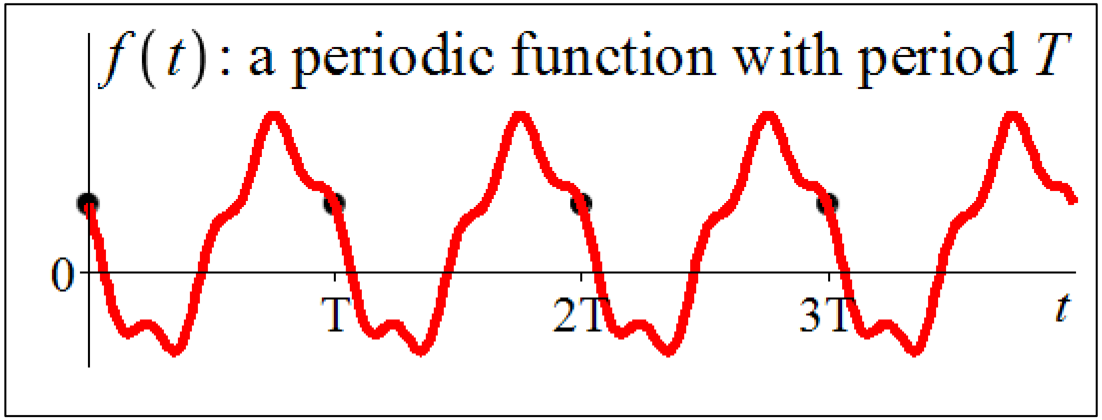


$$f_T(t) = \begin{cases} f(t) & 0 \le t < T \\ 0 & T \le t < \infty \end{cases}$$

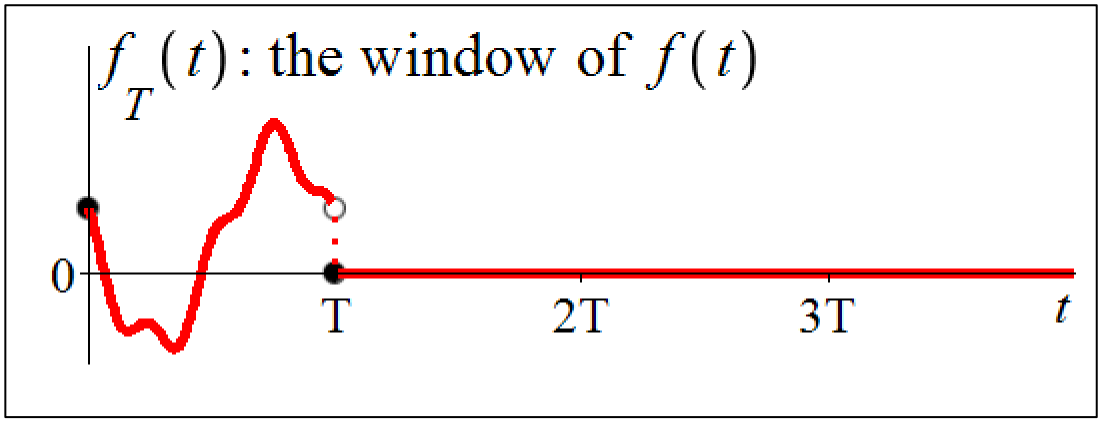


$$u(t-T)f(t-T) = \begin{cases} 0 & 0 \le t < T \\ f(t-T) & T \le t < \infty \end{cases}$$

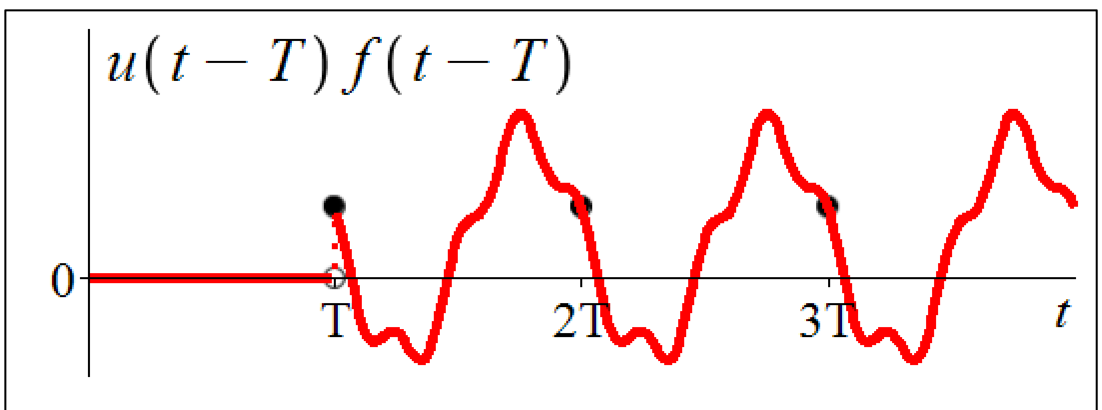
$$f(t) = f_T(t) + u(t-T)f(t-T)$$



$$f(t+T) = f(t)$$

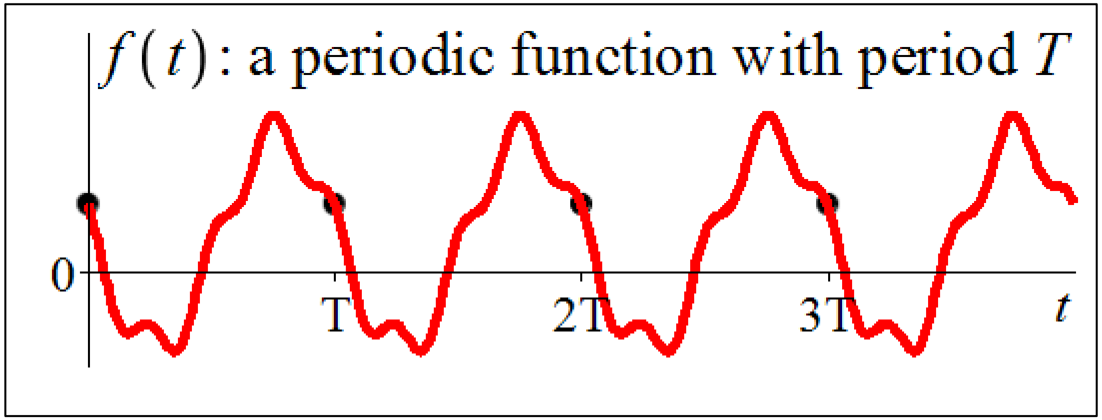


$$f_T(t) = \begin{cases} f(t) & 0 \le t < T \\ 0 & T \le t < \infty \end{cases}$$

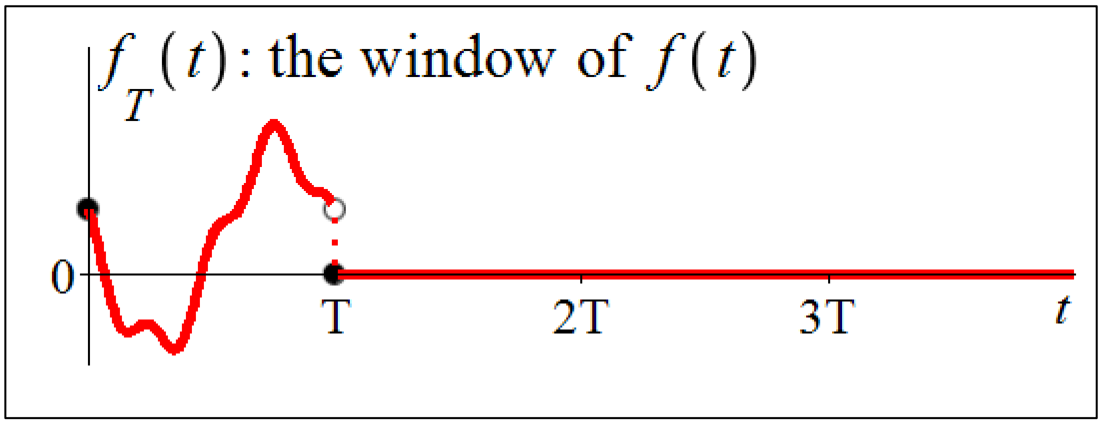


$$u(t-T)f(t-T) = \begin{cases} 0 & 0 \le t < T \\ f(t-T) & T \le t < \infty \end{cases}$$

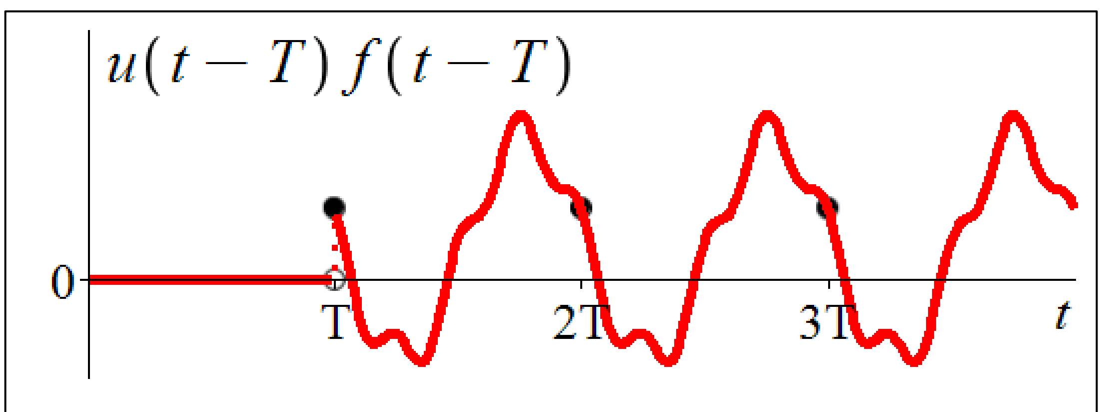
$$f(t) = f_T(t) + u(t-T)f(t-T) \quad \Rightarrow \quad F(s) = F_T(s) + e^{-Ts}F(s)$$



$$f(t+T) = f(t)$$



$$f_T(t) = \begin{cases} f(t) & 0 \le t < T \\ 0 & T \le t < \infty \end{cases}$$



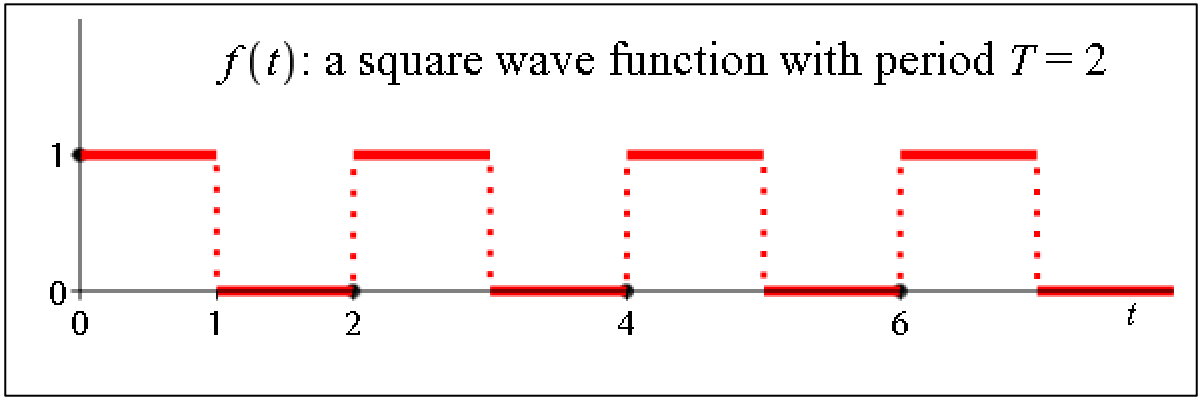
$$u(t-T)f(t-T) = \begin{cases} 0 & 0 \le t < T \\ f(t-T) & T \le t < \infty \end{cases}$$

$$f(t) = f_T(t) + u(t-T)f(t-T) \quad \Rightarrow \quad F(s) = F_T(s) + e^{-Ts}F(s) \quad \Rightarrow \quad F(s) = \frac{F_T(s)}{1 - e^{-Ts}}$$

Example 1.

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 2 \end{cases}$$

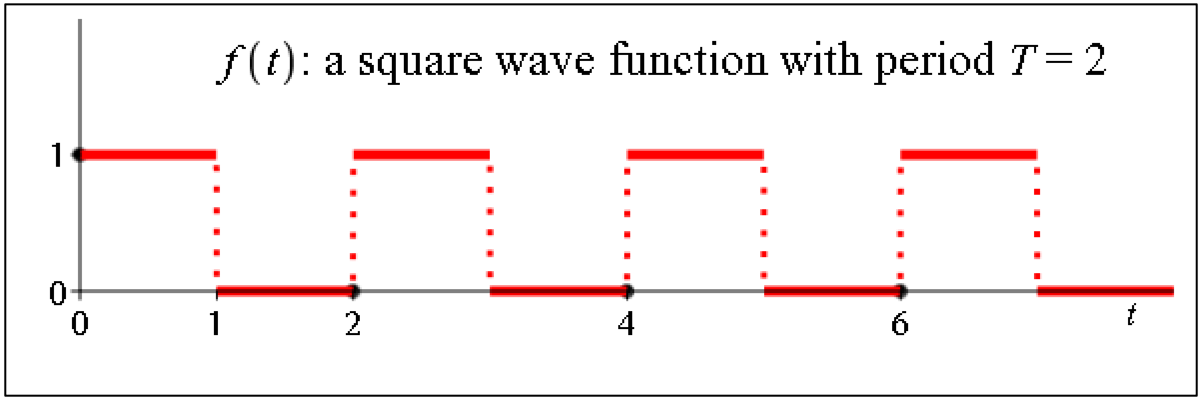
Find the Laplace transform of *f*(*t*).



Example 1.

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 2 \end{cases}$$

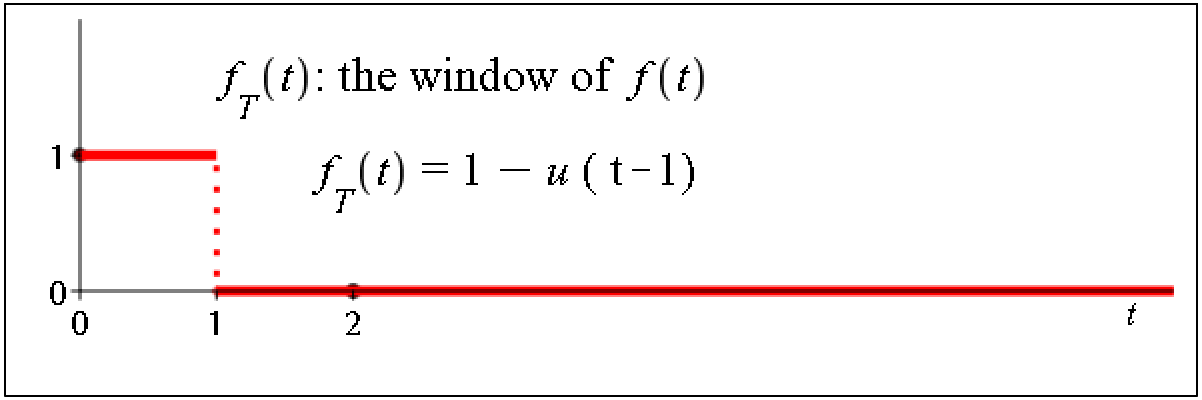
Find the Laplace transform of *f*(*t*).



Solution:

$$f_T(t) = \begin{cases} f(t) & 0 \le t < 2 \\ 0 & 2 \le t < \infty \end{cases}$$

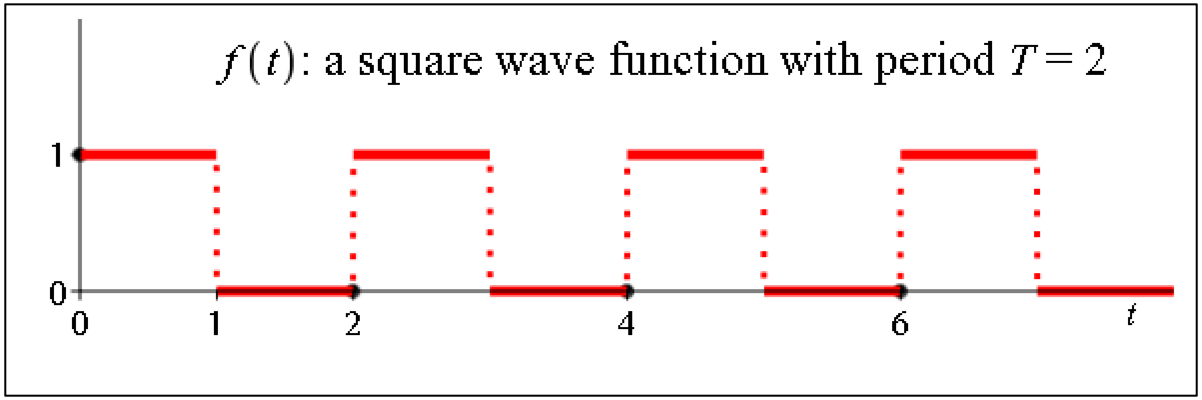
$$f_T(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 2 \\ 0 & 2 \le t < \infty \end{cases}$$



Example 1.

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 2 \end{cases}$$

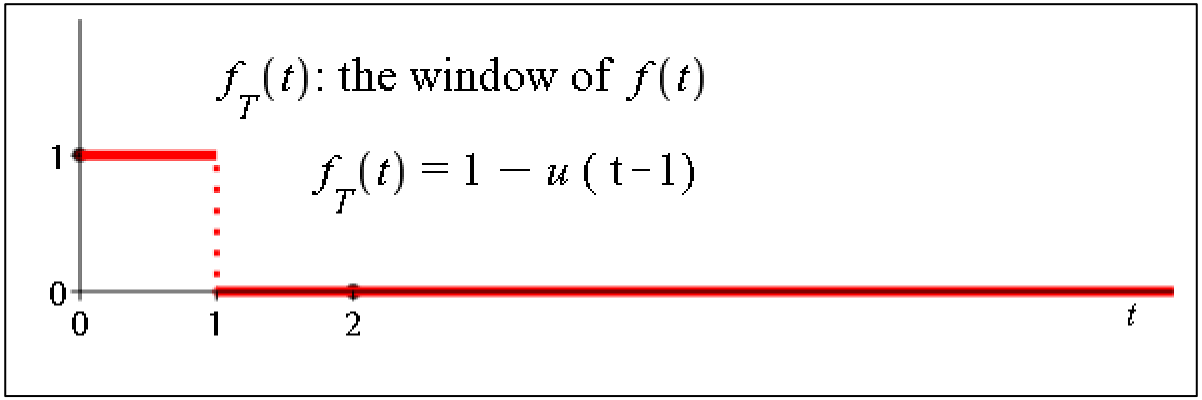
Find the Laplace transform of *f*(*t*).



Solution:

$$f_T(t) = \begin{cases} f(t) & 0 \le t < 2 \\ 0 & 2 \le t < \infty \end{cases}$$

$$f_T(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 2 \\ 0 & 2 \le t < \infty \end{cases}$$



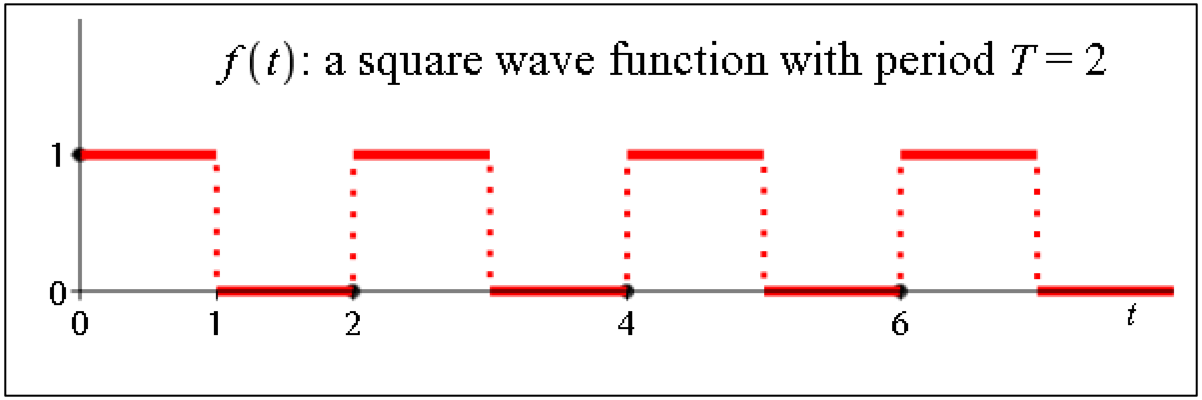
$$f_T(t) = 1 - u(t-1),$$

$$F_T(s) = \frac{1}{s} - e^{-s}\left(\frac{1}{s}\right) = \frac{1 - e^{-s}}{s}.$$

Example 1.

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 2 \end{cases}$$

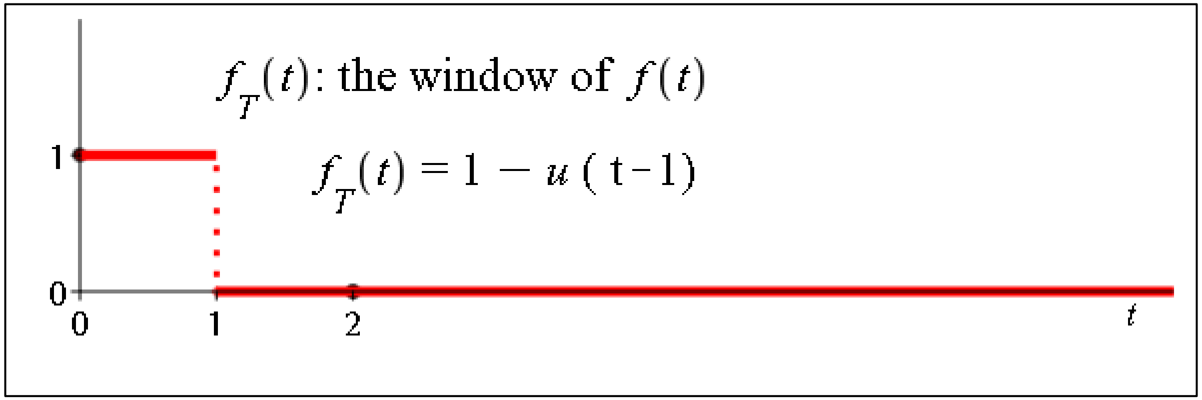
Find the Laplace transform of $f(t)$.



Solution:

$$f_T(t) = \begin{cases} f(t) & 0 \le t < 2 \\ 0 & 2 \le t < \infty \end{cases}$$

$$f_T(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 2 \\ 0 & 2 \le t < \infty \end{cases}$$



$$f_T(t) = 1 - u(t-1),$$

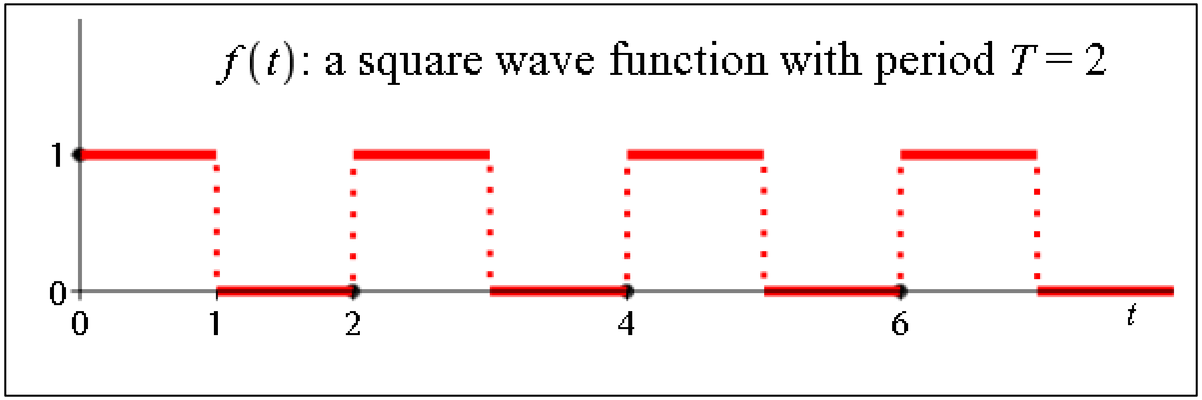
$$F_T(s) = \frac{1}{s} - e^{-s}\left(\frac{1}{s}\right) = \frac{1-e^{-s}}{s}.$$

$$F(s) = \frac{F_T(s)}{1-e^{-Ts}} = \frac{1-e^{-s}}{s(1-e^{-2s})} = \frac{1-e^{-s}}{s(1-e^{-s})(1+e^{-s})} = \frac{1}{s(1+e^{-s})}.$$

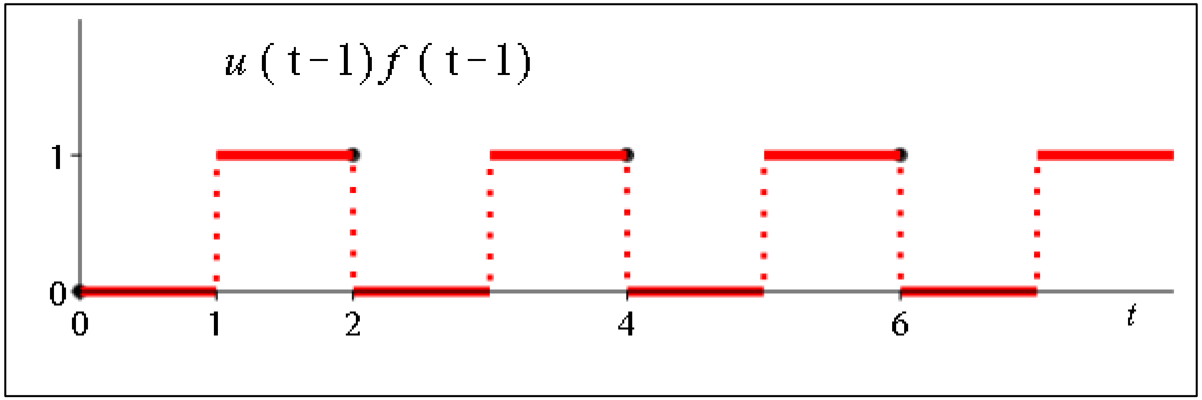
Example 1.

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 2 \end{cases}$$

Find the Laplace transform of $f(t)$.



Alternative Solution:



Observe

$$u(t-1)f(t-1) = 1 - f(t).$$

It follows

$$e^{-s}F(s) = \frac{1}{s} - F(s),$$

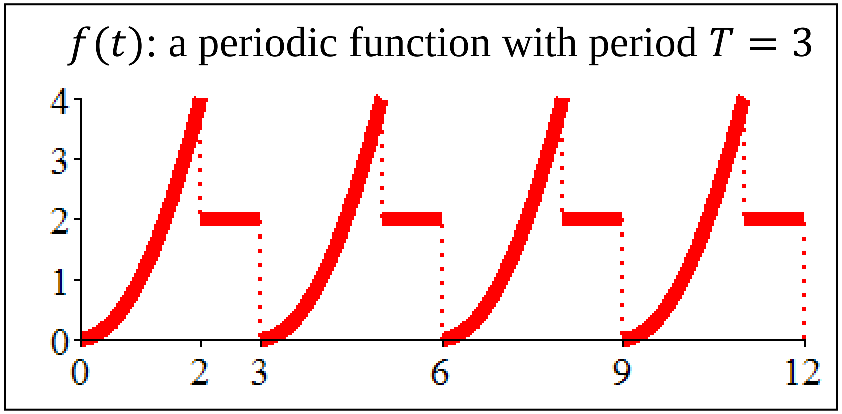
$$(1 + e^{-s})F(s) = \frac{1}{s},$$

$$F(s) = \frac{1}{s(1 + e^{-s})}.$$

Example 2. $f(t)$: a periodic function with period $T = 3$ such that

$$f(t) = \begin{cases} t^2 & 0 \le t < 2 \\ 2 & 2 \le t < 3 \end{cases}$$

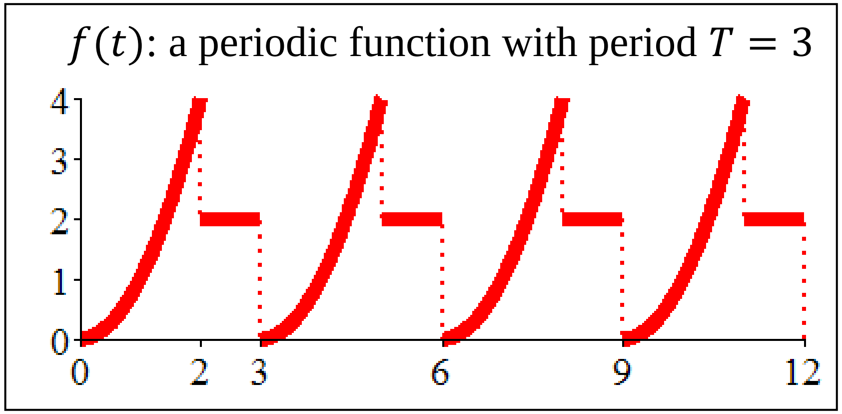
Find the Laplace transform of *f*(*t*).



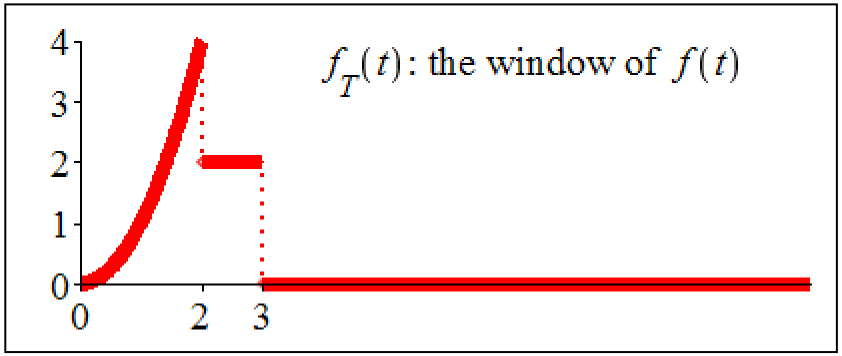
Example 2. $f(t)$: a periodic function with period $T = 3$ such that

$$f(t) = \begin{cases} t^2 & 0 \le t < 2 \\ 2 & 2 \le t < 3 \end{cases}$$

Find the Laplace transform of *f*(*t*).



Solution:

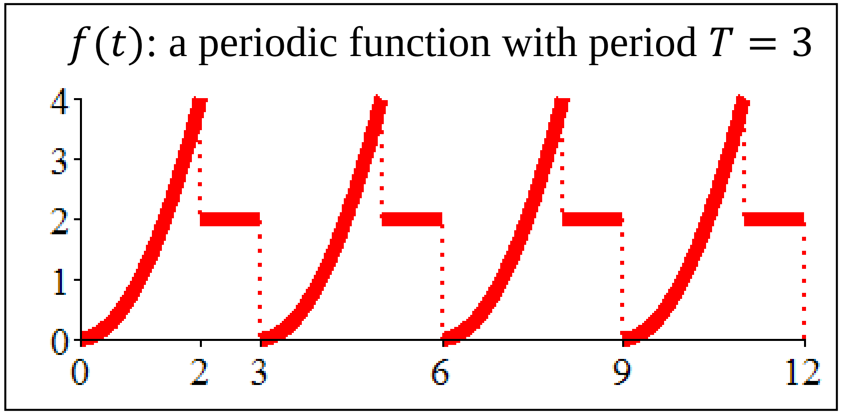


$$f_T(t) = \begin{cases} f(t) & 0 \le t < 3 \\ 0 & 3 \le t < \infty \end{cases} = \begin{cases} t^2 & 0 \le t < 2 \\ 2 & 2 \le t < 3 \\ 0 & 3 \le t < \infty \end{cases}$$

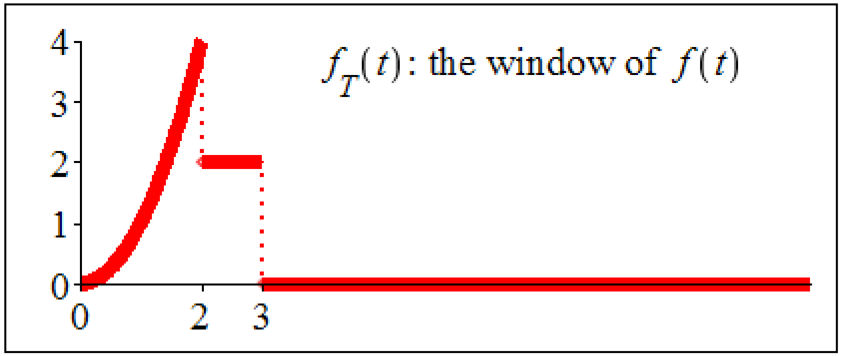
Example 2. $f(t)$: a periodic function with period $T = 3$ such that

$$f(t) = \begin{cases} t^2 & 0 \le t < 2 \\ 2 & 2 \le t < 3 \end{cases}$$

Find the Laplace transform of *f*(*t*).



Solution:



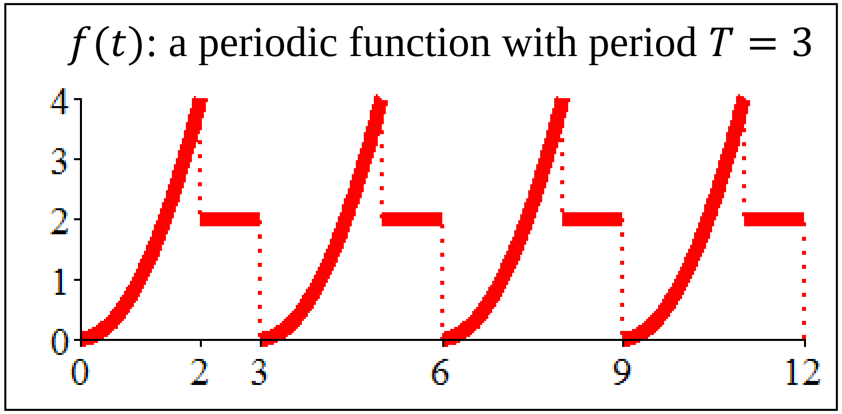
$$f_T(t) = \begin{cases} f(t) & 0 \le t < 3 \\ 0 & 3 \le t < \infty \end{cases} = \begin{cases} t^2 & 0 \le t < 2 \\ 2 & 2 \le t < 3 \\ 0 & 3 \le t < \infty \end{cases}$$

$$\begin{aligned} f_T(t) = \; & [u(t) - u(t-2)](t^2) \\ & + [u(t-2) - u(t-3)](2) \end{aligned}$$

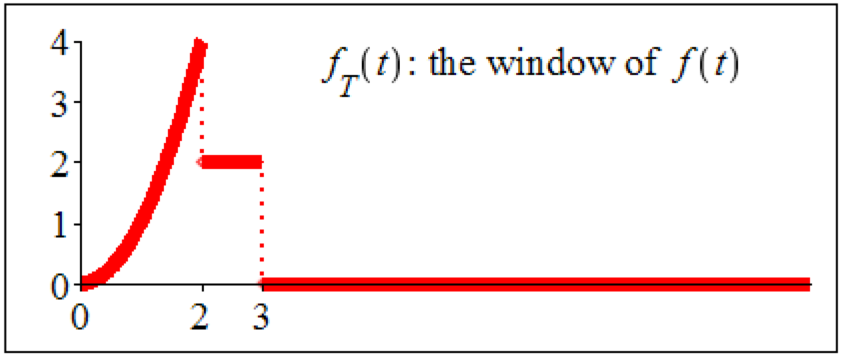
Example 2. $f(t)$: a periodic function with period $T = 3$ such that

$$f(t) = \begin{cases} t^2 & 0 \le t < 2 \\ 2 & 2 \le t < 3 \end{cases}$$

Find the Laplace transform of *f*(*t*).



Solution:



$$f_T(t) = \begin{cases} f(t) & 0 \le t < 3 \\ 0 & 3 \le t < \infty \end{cases} = \begin{cases} t^2 & 0 \le t < 2 \\ 2 & 2 \le t < 3 \\ 0 & 3 \le t < \infty \end{cases}$$

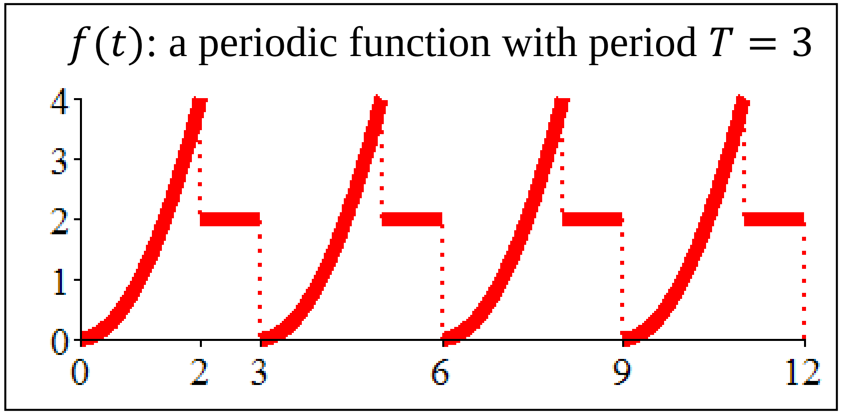
$$f_T(t) = [u(t) - u(t-2)](t^2) + [u(t-2) - u(t-3)](2)$$

$$= u(t)(t^2) + u(t-2)(-t^2 + 2) + u(t-3)(2),$$

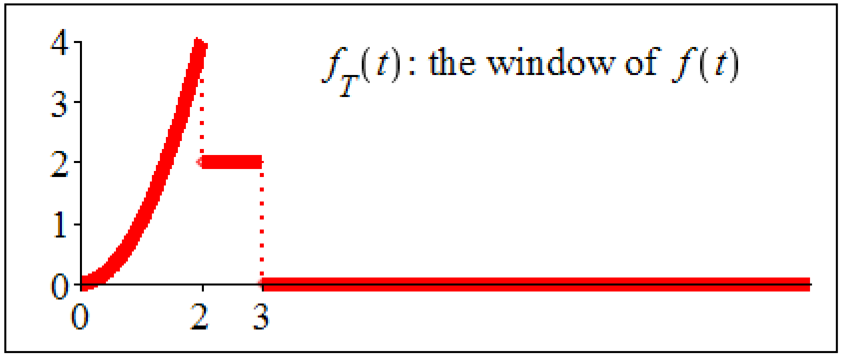
Example 2. $f(t)$: a periodic function with period $T = 3$ such that

$$f(t) = \begin{cases} t^2 & 0 \le t < 2 \\ 2 & 2 \le t < 3 \end{cases}$$

Find the Laplace transform of $f(t)$.



Solution:



$$f_T(t) = \begin{cases} f(t) & 0 \le t < 3 \\ 0 & 3 \le t < \infty \end{cases} = \begin{cases} t^2 & 0 \le t < 2 \\ 2 & 2 \le t < 3 \\ 0 & 3 \le t < \infty \end{cases}$$

$$f_T(t) = [u(t) - u(t-2)](t^2) + [u(t-2) - u(t-3)](2)$$

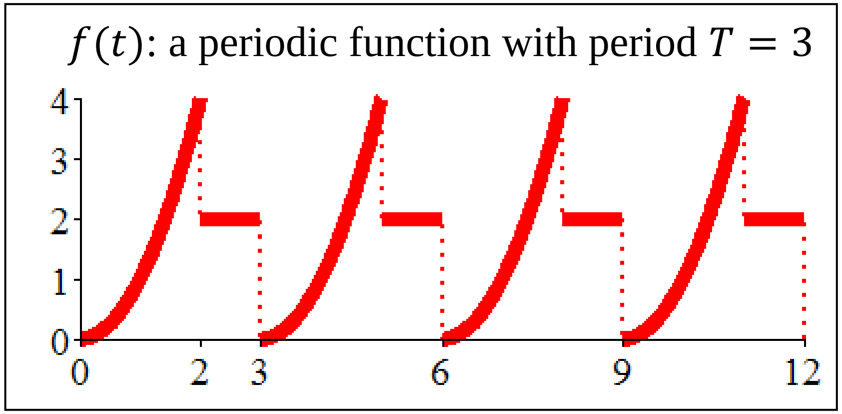
$$= u(t)(t^2) + u(t-2)(-t^2 + 2) + u(t-3)(2),$$

$$F_T(s) = e^{-0 \cdot s}\mathcal{L}\{t^2\} + e^{-2s}\mathcal{L}\{-(t+2)^2 + 2\} + e^{-3s}\mathcal{L}\{2\}$$

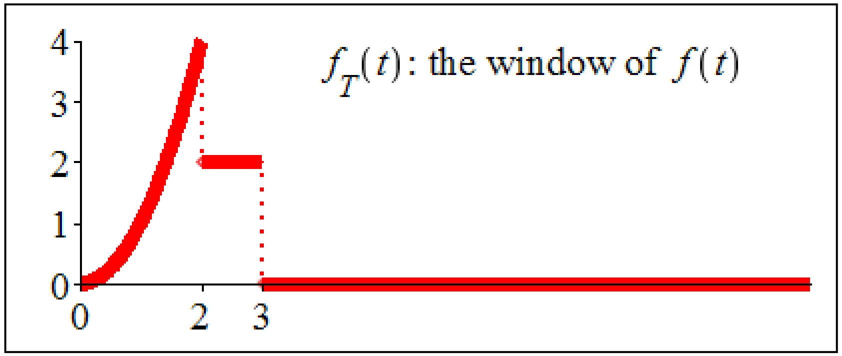
Example 2. $f(t)$: a periodic function with period $T = 3$ such that

$$f(t) = \begin{cases} t^2 & 0 \le t < 2 \\ 2 & 2 \le t < 3 \end{cases}$$

Find the Laplace transform of *f*(*t*).



Solution:



$$f_T(t) = \begin{cases} f(t) & 0 \le t < 3 \\ 0 & 3 \le t < \infty \end{cases} = \begin{cases} t^2 & 0 \le t < 2 \\ 2 & 2 \le t < 3 \\ 0 & 3 \le t < \infty \end{cases}$$

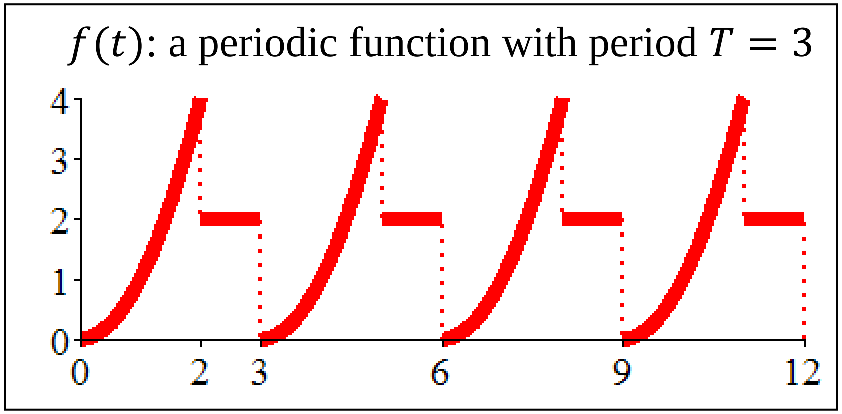
$$\begin{aligned} f_T(t) = &\ [u(t) - u(t-2)](t^2) \\ &+ [u(t-2) - u(t-3)](2) \\ = &\ u(t)(t^2) \ + u(t-2)(-t^2+2) \ + u(t-3)(2), \end{aligned}$$

$$\begin{aligned} F_T(s) &= e^{-0\cdot s}\mathcal{L}\{t^2\} \ + e^{-2s}\mathcal{L}\{-(t+2)^2+2\} \ + e^{-3s}\mathcal{L}\{2\} \\ &= \mathcal{L}\{t^2\} \ + e^{-2s}\mathcal{L}\{-t^2-4t-2\} \ + e^{-3s}\mathcal{L}\{2\} \ = \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{2}{s}\right) + e^{-3s}\left(\frac{2}{s}\right) \end{aligned}$$

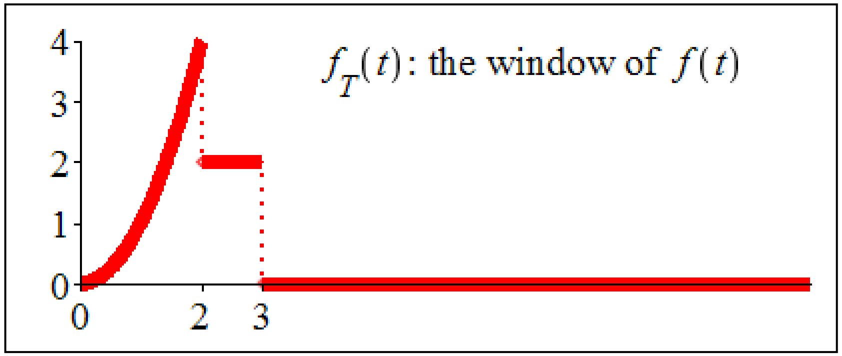
Example 2. $f(t)$: a periodic function with period $T = 3$ such that

$$f(t) = \begin{cases} t^2 & 0 \le t < 2 \\ 2 & 2 \le t < 3 \end{cases}$$

Find the Laplace transform of *f*(*t*).



Solution:



$$f_T(t) = \begin{cases} f(t) & 0 \le t < 3 \\ 0 & 3 \le t < \infty \end{cases} = \begin{cases} t^2 & 0 \le t < 2 \\ 2 & 2 \le t < 3 \\ 0 & 3 \le t < \infty \end{cases}$$

$$\begin{aligned} f_T(t) = & \ [u(t) - u(t-2)](t^2) \\ & + [u(t-2) - u(t-3)](2) \\ = & \ u(t)(t^2) \ + u(t-2)(-t^2+2) \ + u(t-3)(2), \end{aligned}$$

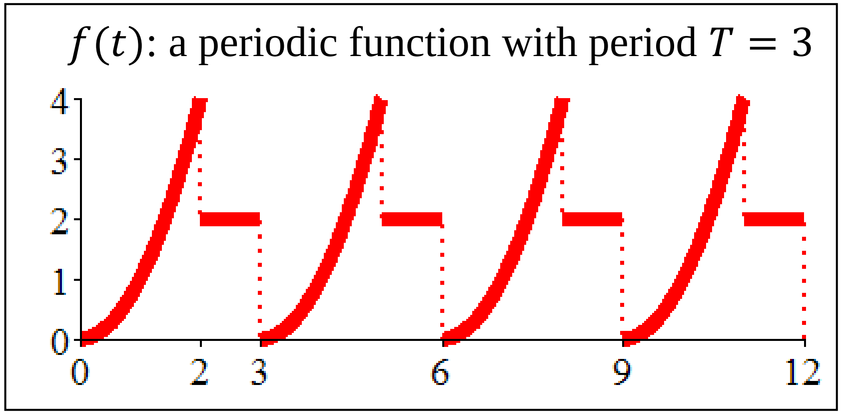
$$\begin{aligned} F_T(s) &= e^{-0 \cdot s}\mathcal{L}\{t^2\} \ + e^{-2s}\mathcal{L}\{-(t+2)^2 + 2\} \ + e^{-3s}\mathcal{L}\{2\} \\ &= \mathcal{L}\{t^2\} \ + e^{-2s}\mathcal{L}\{-t^2 - 4t - 2\} \ + e^{-3s}\mathcal{L}\{2\} \ = \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{2}{s}\right) + e^{-3s}\left(\frac{2}{s}\right) \end{aligned}$$

$$\boxed{F(s) = \frac{F_T(s)}{1 - e^{-Ts}} = \frac{1}{1 - e^{-3s}}\left[\frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{2}{s}\right) + e^{-3s}\left(\frac{2}{s}\right)\right].}$$

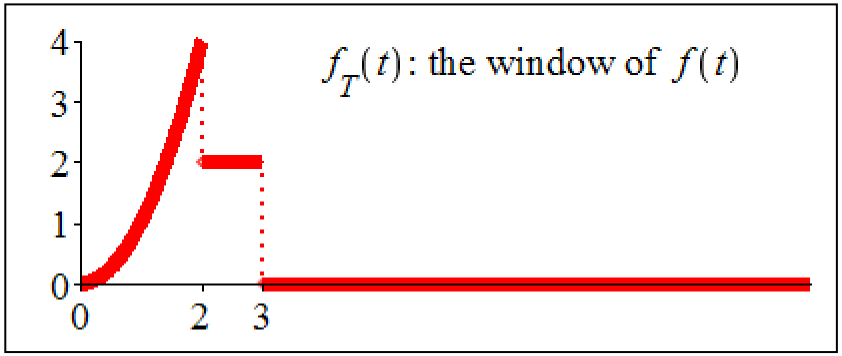
Example 2. $f(t)$: a periodic function with period $T = 3$ such that

$$f(t) = \begin{cases} t^2 & 0 \le t < 2 \\ 2 & 2 \le t < 3 \end{cases}$$

Find the Laplace transform of *f*(*t*).



Solution:



$$f_T(t) = \begin{cases} f(t) & 0 \le t < 3 \\ 0 & 3 \le t < \infty \end{cases} = \begin{cases} t^2 & 0 \le t < 2 \\ 2 & 2 \le t < 3 \\ 0 & 3 \le t < \infty \end{cases}$$

$$\begin{aligned} f_T(t) = &\ [u(t) - u(t-2)](t^2) \\ &+ [u(t-2) - u(t-3)](2) \\ = &\ u(t)(t^2) \quad + u(t-2)(-t^2 + 2) \quad + u(t-3)(2), \end{aligned}$$

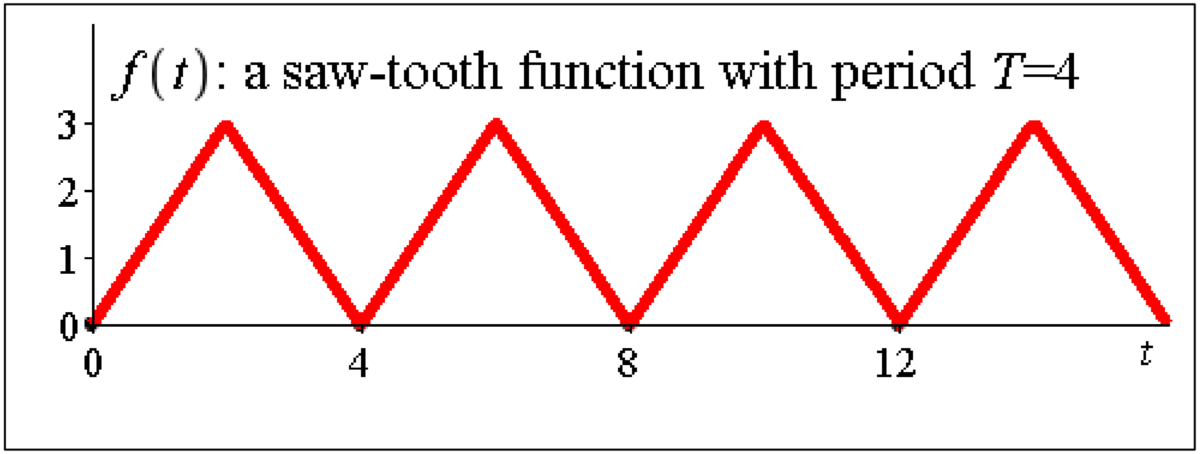
$$\begin{aligned} F_T(s) &= e^{-0 \cdot s}\mathcal{L}\{t^2\} \quad + e^{-2s}\mathcal{L}\{-(t+2)^2 + 2\} \quad + e^{-3s}\mathcal{L}\{2\} \\ &= \mathcal{L}\{t^2\} \quad + e^{-2s}\mathcal{L}\{-t^2 - 4t - 2\} \quad + e^{-3s}\mathcal{L}\{2\} \quad = \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{2}{s}\right) + e^{-3s}\left(\frac{2}{s}\right) \end{aligned}$$

$$F(s) = \frac{F_T(s)}{1 - e^{-Ts}} = \frac{1}{1 - e^{-3s}}\left[\frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{2}{s}\right) + e^{-3s}\left(\frac{2}{s}\right)\right].$$

Example 3.

$$f(t) = \begin{cases} 3t/2 & 0 \le t < 2 \\ 6 - 3t/2 & 2 \le t < 4 \end{cases}$$

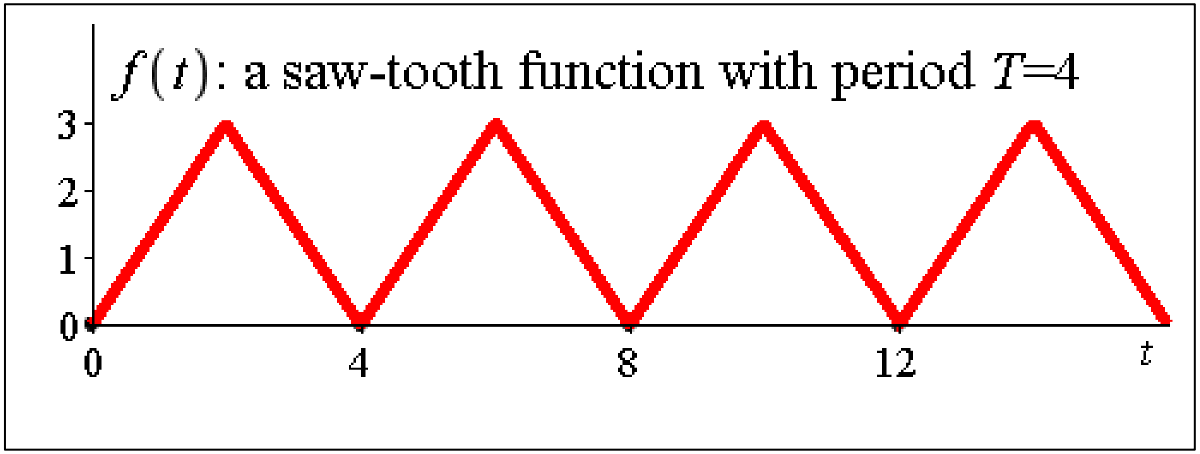
Find the Laplace transform of *f*(*t*).



Example 3.

$$f(t) = \begin{cases} 3t/2 & 0 \le t < 2 \\ 6 - 3t/2 & 2 \le t < 4 \end{cases}$$

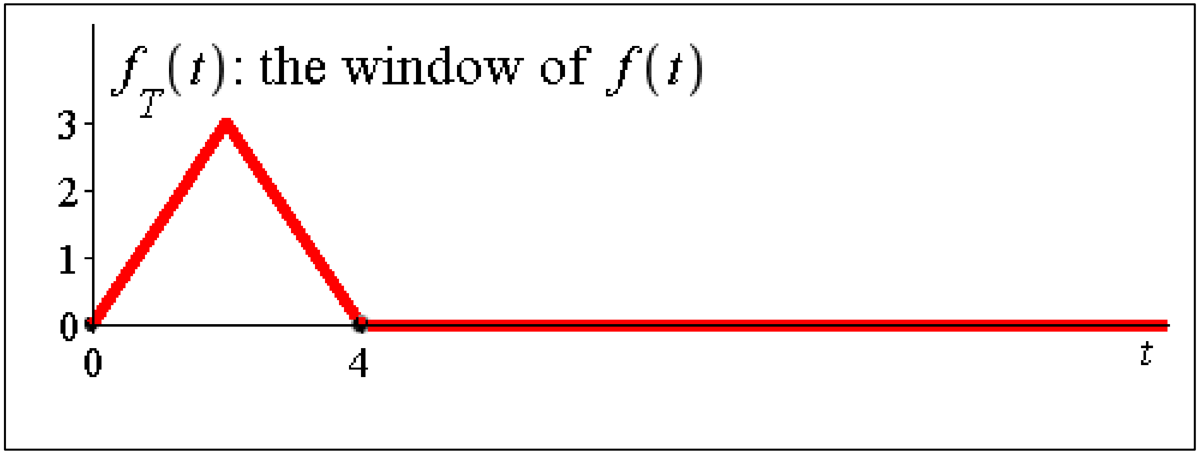
Find the Laplace transform of *f*(*t*).



Solution:

$$f_T(t) = \begin{cases} f(t) & 0 \le t < 4 \\ 0 & 4 \le t < \infty \end{cases}$$

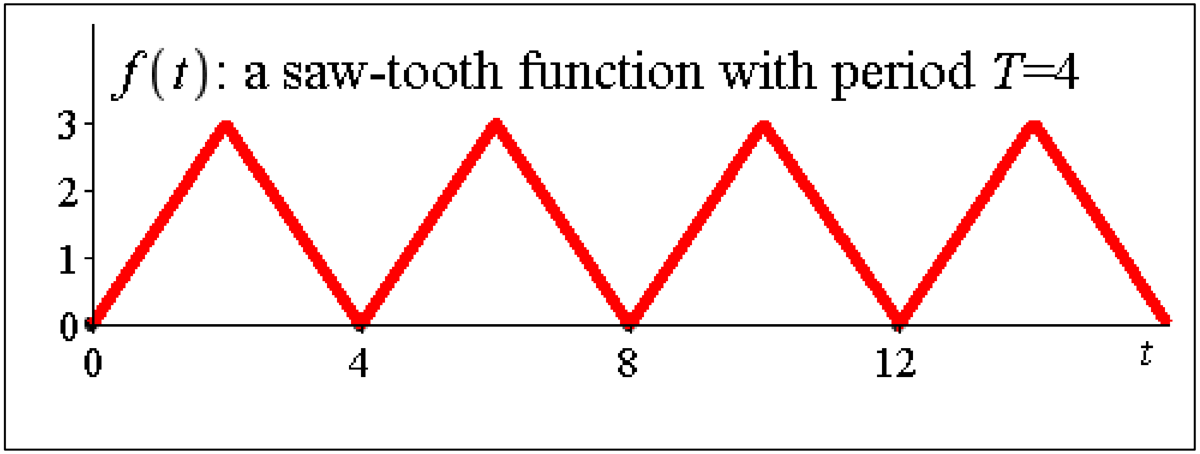
$$f_T(t) = \begin{cases} 3t/2 & 0 \le t < 2 \\ 6 - 3t/2 & 2 \le t < 4 \\ 0 & 4 \le t < \infty \end{cases}$$



Example 3.

$$f(t) = \begin{cases} 3t/2 & 0 \le t < 2 \\ 6 - 3t/2 & 2 \le t < 4 \end{cases}$$

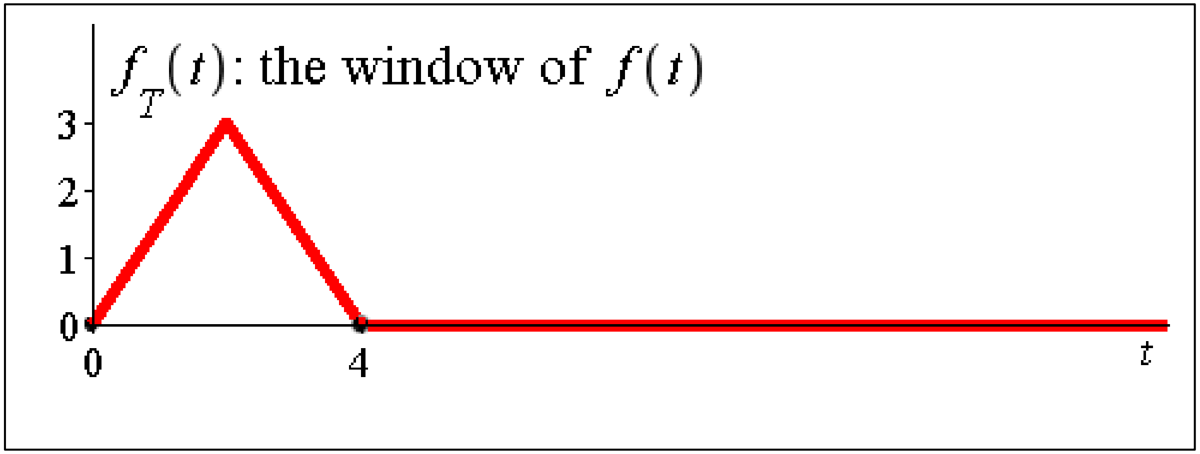
Find the Laplace transform of $f(t)$.



Solution:

$$f_T(t) = \begin{cases} f(t) & 0 \le t < 4 \\ 0 & 4 \le t < \infty \end{cases}$$

$$f_T(t) = \begin{cases} 3t/2 & 0 \le t < 2 \\ 6 - 3t/2 & 2 \le t < 4 \\ 0 & 4 \le t < \infty \end{cases}$$



$$f_T(t) = [1 - u(t-2)](3t/2) + [u(t-2) - u(t-4)](6 - 3t/2)$$

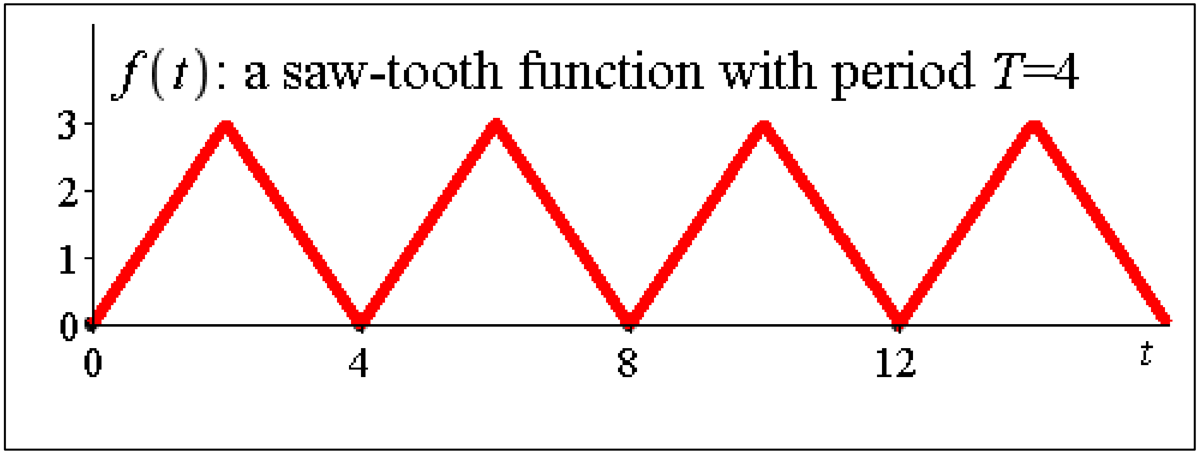
$$f_T(t) = 3t/2 + u(t-2)(6-3t) + u(t-4)(-6 + 3t/2)$$

$$f_T(t) = 3t/2 - u(t-2)3(t-2) + u(t-4)3(t-4)/2,$$

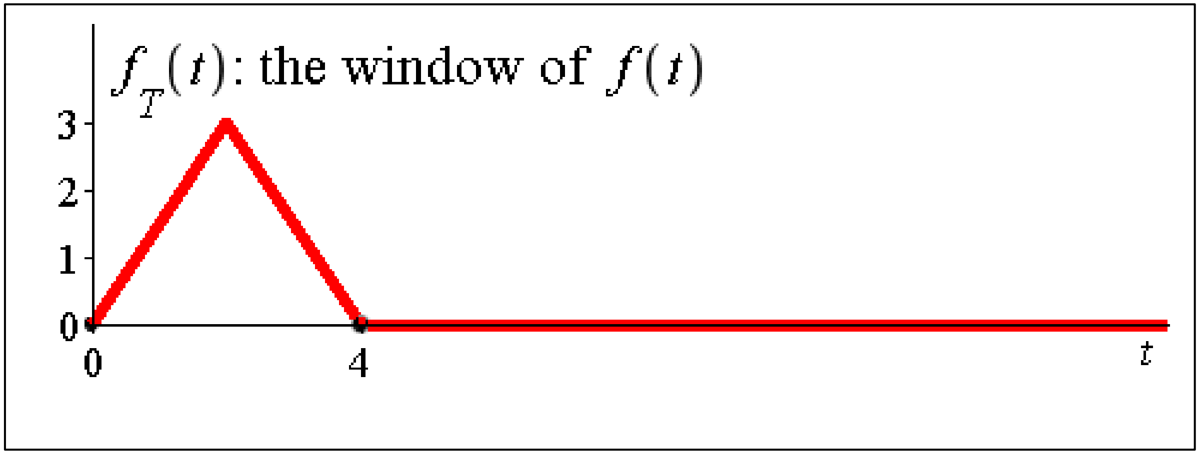
Example 3.

$$f(t) = \begin{cases} 3t/2 & 0 \le t < 2 \\ 6 - 3t/2 & 2 \le t < 4 \end{cases}$$

Find the Laplace transform of *f*(*t*).



Solution:



$$f_T(t) = \begin{cases} f(t) & 0 \le t < 4 \\ 0 & 4 \le t < \infty \end{cases}$$

$$f_T(t) = \begin{cases} 3t/2 & 0 \le t < 2 \\ 6 - 3t/2 & 2 \le t < 4 \\ 0 & 4 \le t < \infty \end{cases}$$

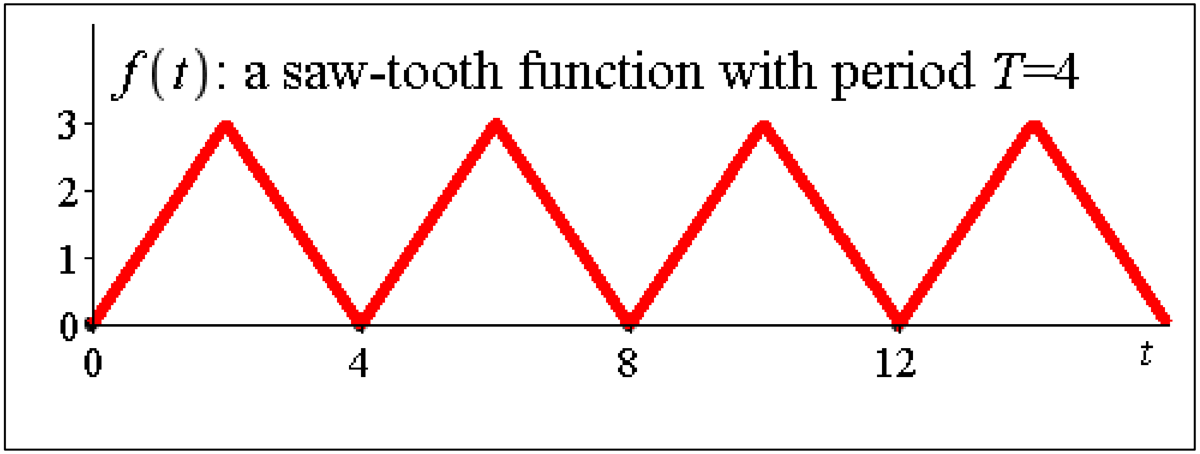
$$f_T(t) = [1 - u(t-2)](3t/2) + [u(t-2) - u(t-4)](6 - 3t/2)$$
$$f_T(t) = 3t/2 + u(t-2)(6 - 3t) + u(t-4)(-6 + 3t/2)$$
$$f_T(t) = 3t/2 - u(t-2)3(t-2) + u(t-4)3(t-4)/2,$$

$$F_T(s) = \frac{3}{2s^2} - e^{-2s}\left(\frac{3}{s^2}\right) + e^{-4s}\left(\frac{3}{2s^2}\right) = \frac{3(1 - 2e^{-2s} + e^{-4s})}{2s^2} = \frac{3(1 - e^{-2s})^2}{2s^2}.$$

Example 3.

$$f(t) = \begin{cases} 3t/2 & 0 \le t < 2 \\ 6 - 3t/2 & 2 \le t < 4 \end{cases}$$

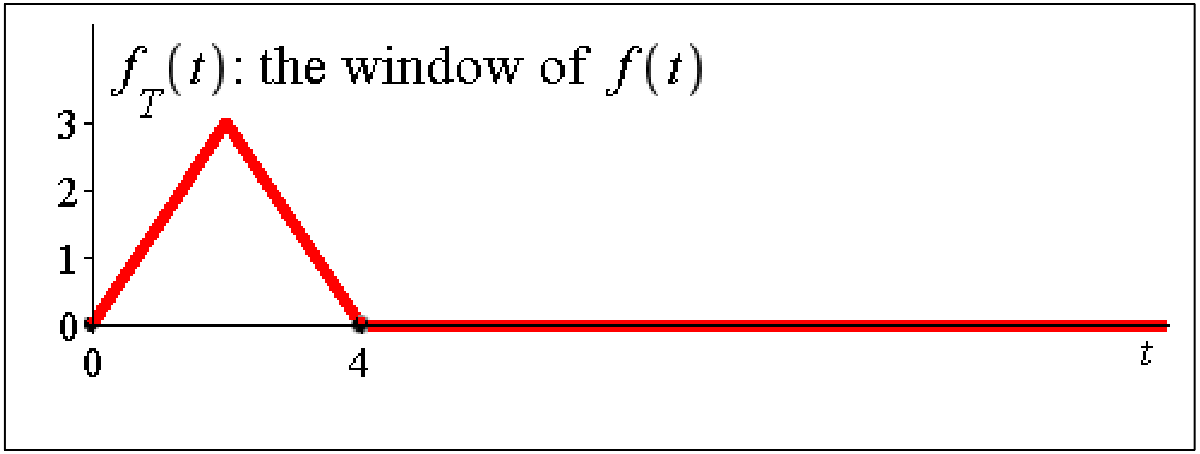
Find the Laplace transform of *f*(*t*).



Solution:

$$f_T(t) = \begin{cases} f(t) & 0 \le t < 4 \\ 0 & 4 \le t < \infty \end{cases}$$

$$f_T(t) = \begin{cases} 3t/2 & 0 \le t < 2 \\ 6 - 3t/2 & 2 \le t < 4 \\ 0 & 4 \le t < \infty \end{cases}$$



$$F_T(s) = \frac{3}{2s^2} - e^{-2s}\left(\frac{3}{s^2}\right) + e^{-4s}\left(\frac{3}{2s^2}\right) = \frac{3(1 - 2e^{-2s} + e^{-4s})}{2s^2} = \frac{3(1 - e^{-2s})^2}{2s^2}.$$

$$F(s) = \frac{F_T(s)}{1 - e^{-Ts}} = \frac{3(1 - e^{-2s})^2}{2s^2(1 - e^{-4s})} = \frac{3(1 - e^{-2s})^2}{2s^2(1 - e^{-2s})(1 + e^{-2s})} = \frac{3(1 - e^{-2s})}{2s^2(1 + e^{-2s})}.$$