Math 353 Lecture Notes
Fourier series

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Topics covered

- Function spaces: introduction to L^2
- Fourier series (introduction, convergence)

Before returning to PDEs, we explore a particular orthogonal basis in depth - the **Fourier series**. This theory has deep implications in mathematics and physics, and is one of the cornerstones of applied mathematics (not just a tool for solving PDEs!).

1 Periodic functions

Recall that we have considered the space of L^2 functions on the interval $[-\ell, \ell]$,

$$L^2[-\ell, \ell] = \{f : [-\ell, \ell] \to \mathbb{R} : \int_{-\ell}^{\ell} |f(x)|^2 \, dx < \infty\}. \tag{1}$$

This space can be identified with a different space (same elements, **different meaning**).

Definition (periodic functions): A function $f(x)$ is T-**periodic** if

$$f(x+T) = f(x) \text{ for all } x.$$

'The period' of the function, if it exists is the smallest $T > 0$ for which this is true.

For example, $\sin(x)$ is 2π-periodic, but also 4π periodic, 6π-periodic and so on.

A 2ℓ periodic function is determined by its values on any interval of that length - say, $[-\ell, \ell]$.

Motivated by this, given a function

$$f(x), \quad x \in [-\ell, \ell]$$

defined **in the interval** $[-\ell, \ell]$, we define its **periodic extension** to be the 2ℓ-periodic function taking on those values:

$$f_{\text{per}}(x) = f(x) \text{ for } x \in [-\ell, \ell]$$

$$f_{\text{per}}(x + 2\ell n) = f(x) \text{ for all } x \in [-\ell, \ell] \text{ and integers } n.$$

That is, we 'copy' the values of f over one period.

Conversely, every periodic function can be regarded as a function in $[-\ell, \ell]$ by restricting it to those values. There is no loss of information here!

Important note (endpoints): If the values at the endpoints do not match, then this just means there is a discontinuity at that point (see examples below) and we assign some value at the discontinuity.

We'll shortly deal with whether that value matters.

For example the figure below shows the periodic extensions for

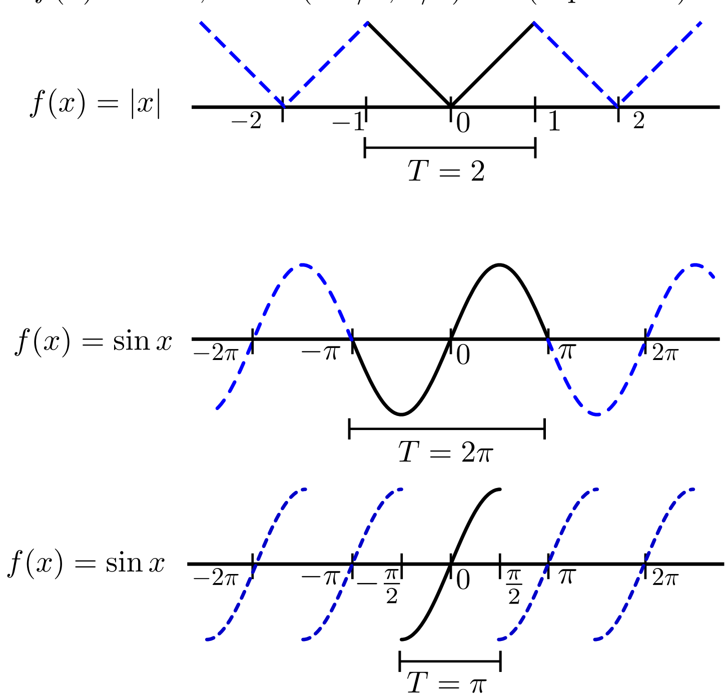
$$f(x) = |x|, \quad x \in (-1, 1) \qquad \text{(2-periodic)}$$

and

$$f(x) = \sin x, \quad x \in (-\pi, \pi) \qquad (2\pi\text{-periodic})$$

and

$$f(x) = \sin x, \quad x \in (-\pi/2, \pi/2) \qquad (\pi\text{-periodic}).$$



Note that the last two examples show that the periodic extension of a function **will depend on the interval chosen and/or period specified.** The π-periodic extension of $\sin x$ from $[-\pi/2, \pi/2]$ to $\mathbb{R}$ is different than the periodic extension from $[-\pi, \pi]$ to $\mathbb{R}$.

1.1 Continuity, L^2

We say that a 2ℓ-periodic function is L^2 if

$$\int_{-\ell}^{\ell} |f(x)|^2\,dx < \infty.$$

That is, its L^2-ness is imposed over one period, which is reasonable because the function behaves the same way over other periods.

The L^2 inner product is also integrated over one period:

$$\langle f, g\rangle = \int_{-\ell}^{\ell} f(x)g(x)\,dx \quad \text{ for } 2\ell\text{-periodic } f, g$$

Note that it doesnn't matter which interval is chosen, so long as its length is 2ℓ (why?).

This set of 2ℓ-periodic functions is almost the same as just L^2 functions defined on the interval $[-\ell, \ell]$'. However, there is one key difference!

To be **continuous**, the periodic function must be continuous for all x, which means:

i) Continuous in the interval $(-\ell, \ell)$

ii) The endpoints 'match' ($f(-\ell) = f(\ell)$)

For a function defined on $[-\ell, \ell]$, only (i) has to hold, of course. A periodic exdtension can only be continuous if the endpoints line up when we glue the pieces together.

For instance, as in the previous example,

$$f(x) = \sin x, \quad x \in (-\pi, \pi), \quad 2\pi\text{-periodic}$$

is continuous (as a periodic function), since $f(-\pi) = f(|pi)$. Howevefr,

$$f(x) = \sin x, \quad x \in (-\pi/2, \pi/2), \quad \pi\text{-periodic}$$

is **not** continuouus, because even though $\sin(x)$ is continuous in the defining interval, the endpoints do not match, leading to a jump at $x = \pm\pi/2$, $\pm 3\pi/2$, and so on.

2 Fourier series: the main result

The starting point here is the eigenvalue problem for

$$L = -d^2/dx^2, \quad x \in [-\ell, \ell]$$

with **periodic boundary conditions**. We want to solve

$$-\phi'' = \lambda\phi, \quad x \in [-\ell, \ell]$$

$$\phi = 2\ell\text{-periodic.}$$

Now, since $\phi(x)$ and its derivative must be continuous, the values must match at $\pm\ell$, so we can replace the 'periodic' constraint with **periodic boundary conditions**

$$\phi(-\ell) = \phi(\ell), \quad \phi'(-\ell) = \phi'(\ell).$$

Solving this eigenvalue problem (see homework), we obtain the eigenvalue/eigenfunctions

$$\lambda_n = (n\pi/\ell)^2, \ \phi_n = \cos(n\pi x/\ell), \quad \psi_n = \sin(n\pi x/\ell), \quad n \geq 1$$

along with a constant eigenfunction for the zero eigenvalue,

$$\lambda_0 = 0, \quad \phi_0 = \text{const.}.$$

To be precise, for each $n \geq 1$, the solution to the eigenvalue problem is

$$\phi(x) = a_n\phi_n(x) + b_n\psi_n(x)$$

i.e. it is spanned by ϕ_n and ψ_n. The theorem on eigenfunction bases applies here, and this particular basis has a special name:

Theorem (Fourier basis): The set of functions

$$\phi_0 = \text{const.}, \quad \phi_n = \cos\frac{n\pi x}{\ell}, \psi_n = \sin\frac{n\pi x}{\ell} \text{ for } n = 1, 2, \cdots \tag{2}$$

is an **orthogonal basis** for 2ℓ periodic L^2 functions. That is, every such function has a unique representation (the **Fourier series**)

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell} \tag{3}$$

with equality in the sense to be described in subsection 4.1.

The value 1/2 is a convention that simplifies bookkeeping.

Interpretation: Every periodic function with period T can be decomposed into a sum of sines and cosines whose frequencies are integer multiples of the 'fundamental' frequency $1/T$ (in period/time).

Explicitly, the orthogonality relations are

$$\int_{-\ell}^{\ell} \phi_m \psi_n \, dx = 0, \qquad \text{for all } m, n, \tag{4}$$

$$\int_{-\ell}^{\ell} \phi_m \phi_n \, dx = \begin{cases} 0 & m \neq n \\ \ell & m = n \text{ and } m \neq 0, \end{cases} \tag{5}$$

$$\int_{-\ell}^{\ell} \psi_m \psi_n \, dx = \begin{cases} 0 & m \neq n \\ \ell & m = n \end{cases} \tag{6}$$

and finally, for the $m = n = 0$ case,

$$\int_{-\ell}^{\ell} \phi_0 \phi_0 \, dx = \frac{\ell}{2}.$$

Because the basis is orthogonal, it is straightforward to compute the coefficients, by projecting the series onto each basis function, e.g. for the ϕ_n coefficients,

$$a_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

The results are given below.

Computing the Fourier series: The coefficients of the Fourier series (3) are given by

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell} \, dx \tag{7}$$

$$b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} \, dx \tag{8}$$

for $n \geq 1$, and

$$a_0 = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \, dx.$$

Note that the formula (7) works for $n = 0$ as well. This is the reason why $\phi_0 = 1/2$ was chosen as the basis function.

The meaning of the 'equals' in the Fourier series

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell}$$

is subtle, because this is an infinite series of *functions*. Before addressing this important point, we should first construct a few Fourier series to explore what they look like.

3 Computing Fourier series

Here we compute some Fourier series to illustrate a few useful computational tricks and to illustrate why convergence of Fourier series can be subtle. Because the integral is over a symmetric interval, some symmetry can be exploited to simplify calculations.

3.1 Even/odd functions:

A function $f(x)$ is called **odd** if

$$f(x) = -f(-x) \text{ for all } x$$

and **even** if

$$f(x) = f(-x) \text{ for all } x.$$

Due to the odd/even symmetry, integrals over intervals symmetric around zero are nice:

$$\text{if } f \text{ is odd, } \int_{-\ell}^{\ell} f(x)\,dx = 0,$$

$$\text{if } f \text{ is even, } \int_{-\ell}^{\ell} f(x)\,dx = 2\int_0^{\ell} f(x)\,dx.$$

Products of even/odd functions are even or odd (hence the name):

$$\text{odd} \cdot \text{odd} = \text{even}, \qquad \text{odd} \cdot \text{even} = \text{odd}.$$

Some common even/odd functions (m is an integer):

$$\begin{aligned} \text{odd: } & x^{2m+1}, \ \sin kx, \cdots \\ \text{even: } & x^{2m}, \quad \cos kx, \cdots \end{aligned}$$

As an example,

$$\int_{-1}^{1} x^6 \sin 2x + 3x^2\,dx = \int_0^1 3x^2 dx = x^3\Big|_0^1 = 1.$$

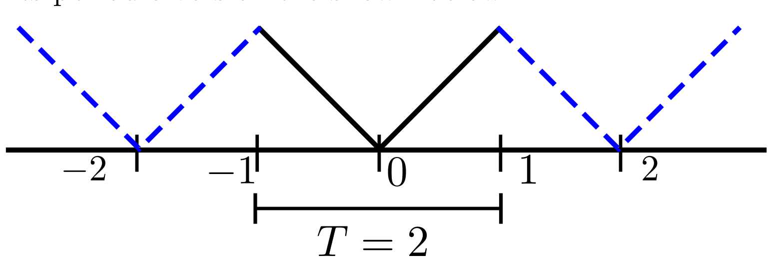
For the first term: since x^6 is even, $\sin 2x$ is odd so $x^6 \sin 2x$ is odd.

3.2 Triangle wave

Define a function $f \in L^2[-1,1]$ as

$$f(x) = |x| \text{ for } x \in [-1,1].$$

The function and its periodic version are shown below:



To compute the Fourier series, use (4)-(6) with $\ell = 1$. First, observe that $f(x)$ is an even function, so

$$f(x)\cos n\pi x \text{ is an even function}, \quad f(x)\sin n\pi x \text{ is an odd function} \tag{9}$$

for all n (note that the product of an odd and even function is odd).

For the cosine coefficients, we have

$$a_0 = \int_{-1}^{1} f(x)\,dx = 2\int_0^1 x\,dx = 1,$$

and for $n \geq 1$,

$$\begin{aligned}
a_n &= \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos(n\pi x)\,dx \\
&= 2\int_0^1 x\cos(n\pi x)\,dx && \text{(since the integrand is even)} \\
&= \left[\frac{2}{n\pi}x\sin(n\pi x) + \frac{2}{n^2\pi^2}\cos(n\pi x)\right]\Big|_0^1 \\
&= \frac{2}{n^2\pi^2}\cos(n\pi x)\Big|_0^1 && \text{(since } \sin(n\pi) = 0 \text{ for all n)} \\
&= \frac{2}{n^2\pi^2}((-1)^n - 1).
\end{aligned}$$

Thus the cosine coefficients are

$$a_0 = 1, \qquad a_n = \begin{cases} -\frac{4}{n^2\pi^2} & \text{for odd } n \\ 0 & \text{for even } n > 0 \end{cases}.$$

For the sine coefficients, the integrand is odd due to (9), so

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin(n\pi x)\,dx = 0 \text{ for all } n.$$

The Fourier series for f is therefore

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos((2n-1)\pi x).$$

The first few partial sums $S_N(x)$ (with modes up to N) are

$$S_1 = \frac{1}{2} - \frac{4}{\pi^2}\cos\pi x, \quad S_3 = \frac{1}{2} - \frac{4}{\pi^2}(\cos\pi x + \frac{1}{9}\cos 3\pi x), \cdots$$

A plot shows that agreement is quite good, even with only a few terms (Figure 1). The error is worst at the peaks of the function, where it has a sharp corner.

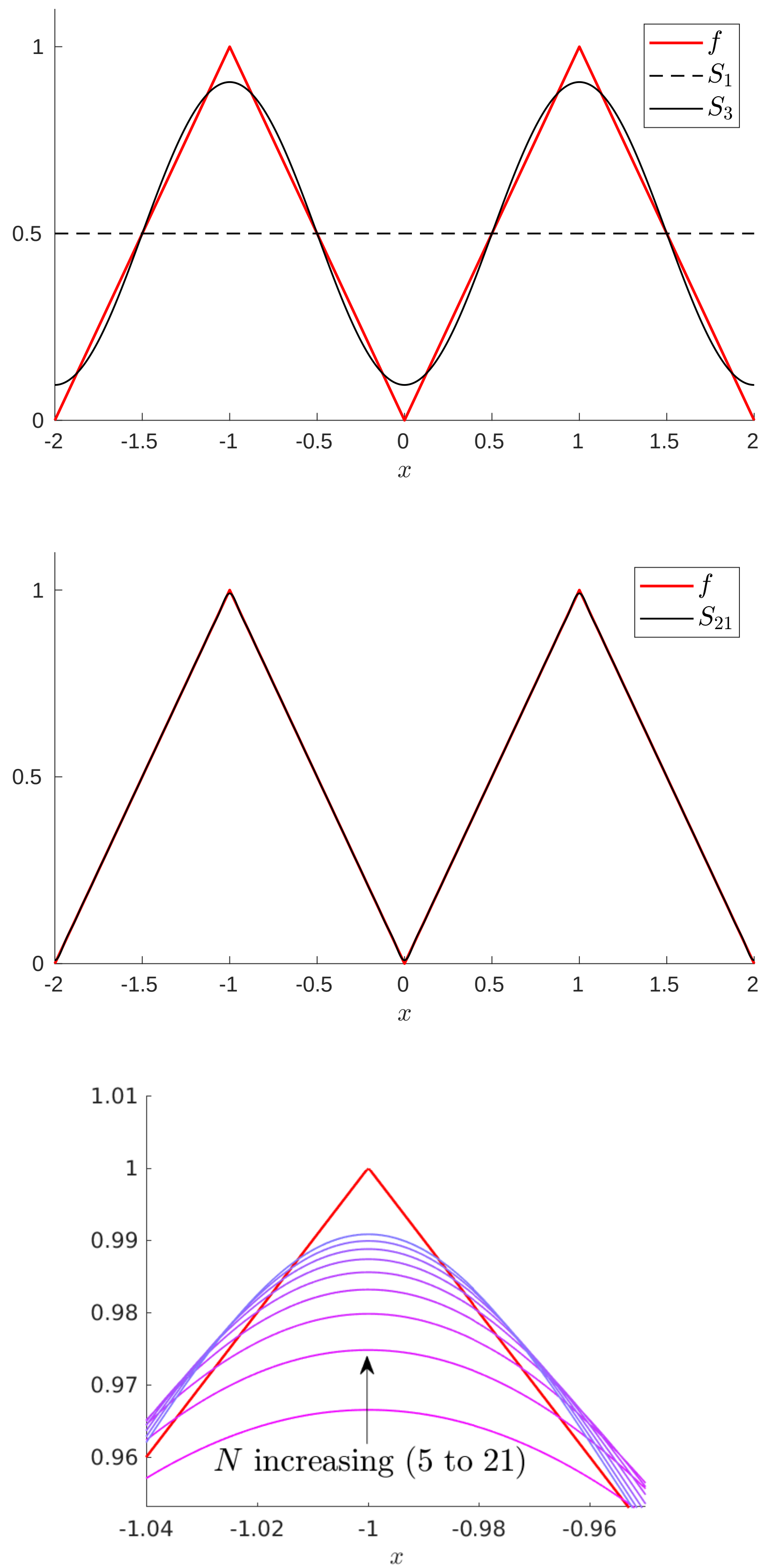


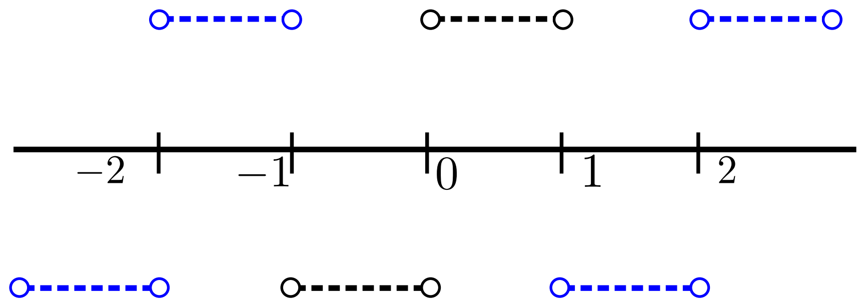
Figure 1: Partial sums S_1, S_3 and S_{21} for the triangle wave. Zoomed in plot shows the convergence at a peak of the triangle.

3.3 Square wave

Let

$$f(x) = \begin{cases} -1 & -1 \le x < 0 \\ 1 & 0 < x \le 1 \end{cases}$$

and $f(x) = f(x+2)$ when $x \notin [-1, 1]$ as shown below:



Note that $f(x)$ is an **odd function** so

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = 0 \text{ for all } n.$$

For the sine coefficients, use the fact that $f(x)\sin n\pi x$ is an even function:

$$\begin{aligned} b_n &= \int_{-1}^{1} f(x)\sin(n\pi x)\,dx \\ &= 2\int_{0}^{1} \sin(n\pi x)\,dx && \text{(since the integrand is even)} \\ &= -\frac{2}{n\pi}\cos(n\pi x)\Big|_0^1 \\ &= \begin{cases} 4/(n\pi) & \text{for odd } n \\ 0 & \text{for even } n \end{cases}. \end{aligned}$$

Thus the Fourier series for $f(x)$ is

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\sin((2n-1)\pi x).$$

The first few partial sums are

$$S_1(x) = \frac{4}{\pi}\sin \pi x, \quad S_3(x) = \frac{4}{\pi}\left(\sin \pi x + \frac{1}{3}\sin 3\pi x\right), \cdots .$$

Error: A plot of the approximation (Figure 2) shows that the partial sums converge nicely where f is continuous, but do not perform well at all near the discontinuity. The partial sums tend to oscillate and overshoot the discontinuity by a significant amount. This overshoot - by about 0.18 - is typical at discontinuities, and is called **Gibbs' phenomenon**.

The oscillations suggest we must be careful with the infinite series - the convergence is not so straightforward. In the next sections, we develop the theory in more detail.

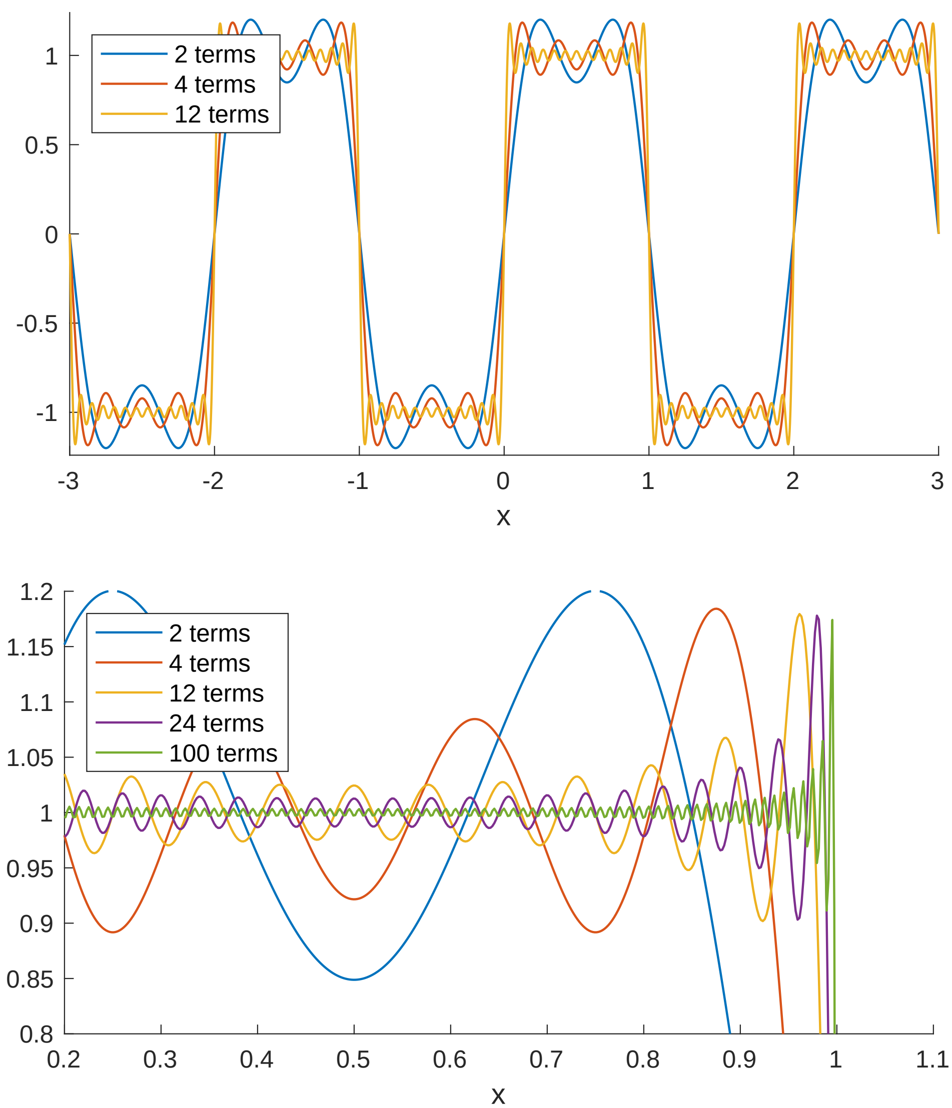


Figure 2: Partial sums for the square wave. Zoomed in figure shows the behavior near the corner at $x = 1$. Note that the oscillations around the discontinuity do not decrease in size as $N \to \infty$.

4 Types of convergence

It is worth clarifying the ways the series can converge (or fail to converge), in order to better understand **what equality means** in the Fourier series (3). There are several ways we can measure the error between the partial sum and the function. There are three main notions of convergence that are important here. In this section, we consider a function $f \in L^2[-\ell, \ell]$ and its Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right)$$

The N-th partial sum is defined to be the sum of terms up to $n = N$:

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right).$$

Convergence (definitions): Let f_n be the sequence of functions in $L^2[-\ell, \ell]$.

The sequence is said to converge **in norm** (or 'in L^2') to a limit f if

$$\|f_n - f\|_2 \to 0 \quad \text{as } n \to \infty$$

That is,

$$\int_{-\ell}^{\ell} |f_n(x) - f(x)|^2 \, dx \to 0 \text{ as } n \to \infty.$$

The sequence converges **pointwise** to f if

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ for all } x \in [-\ell, \ell].$$

L^2 **convergence** means that the mean-square error goes to zero; the weighted average of the area between the partial sums and the function goes to zero. However, it does not require convergence at each point (for instance, the square wave in the previous section).

Pointwise convergence is simpler: it says that at each point x, the value of the partial sums at x will converge to the value of $f(x)$. It does not, however, require that the partial sums converge at the same rate at each x. It could be that at some points, $S_N(x) \to f(x)$ quickly, but at other points, it converges (arbitrarily) slowly.

4.1 mean-square convergence

When f is an L^2 function, the partial sum is always a 'good' approximation to f in a particular sense. We use this to precisely define the equality in (2):

Convergence (mean square): Let $f \in L^2[-\ell, \ell]$. Then the partial sums $S_n(x)$ of its Fourier series converge to f in the L^2 norm; that is,

$$\lim_{N\to\infty} \|f - S_N(x)\|_2 = 0.$$

Explicitly, the 'mean square' (or L^2) distance between the S_N and f goes to zero:

$$\lim_{N\to\infty} \int_{-\ell}^{\ell} |S_N(x) - f(x)|^2 \, dx = 0. \tag{10}$$

If $f \in L^2[-\ell, \ell]$ we simply write that it is 'equal' to its Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell}$$

where the equality is meant in the sense of (10).

The theorem is important because it says that a nice function $f(x)$ on $[-\ell, \ell]$ (or a periodic function) can be approximated by its first N Fourier modes. Adding more modes (more terms to the sum) improves the approximation, at least in the mean-square sense.

However, it is critical to note that the convergence result (10) **does not mean** that plugging in a specific value of x gives an equality. That is, it is not necessarily true that

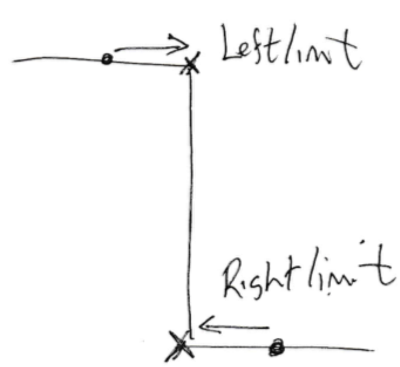
$$\lim_{n\to\infty} S_n(x) = f(x) \quad \text{for any particular } x.$$

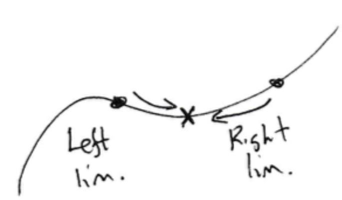
To have good convergence, we need much more than 'mean-square' - we want the series to actually converge at specific values of x.

4.2 Pointwise convergence for the Fourier series

The Fourier series is defined for functions in L^2, which allows for discontinuities. For a function with jump discontinuities, define the 'right' and 'left' limits

$$f(x^+) = \lim_{\xi \searrow 0} f(\xi), \quad f(x^-) = \lim_{\xi \nearrow 0} f(\xi).$$





If f is continuous at x then $f(x^+) = f(x^-) = x$.

It is important to note that these definitions apply for f **as a periodic function**. To be continuous f has to **match at endpoints**.

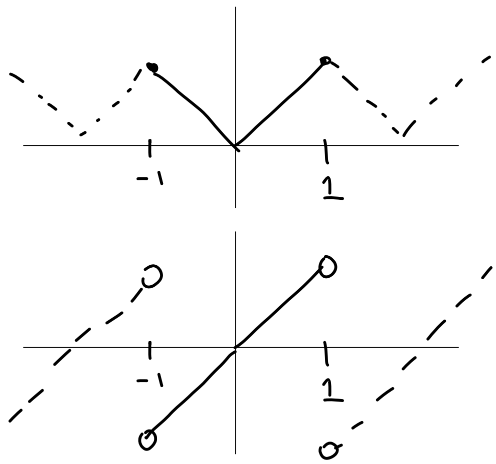
For example, the period 2 function

$$f(x) = |x|, \quad x \in [-1, 1], \qquad f(x) = f(x+2)$$

is continuous because $f(-1) = f(1)$. However,

$$f(x) = x, \quad x \in [-1, 1], \quad f(x) = f(x+2)$$

has a discontinuity at $x = \pm 1$ since $f(-1) \neq f(1)$.



The following theorem tells us about Fourier series convergence at each point:

Theorem (Pointwise convergence): Let $S_n(x)$ be the n-th partial sum of the Fourier series for a periodic function $f \in L^2[-\ell, \ell]$.

If f and f' are continuous except at some jump discontinuities, then

$$\lim_{n\to\infty} S_n(x) = \begin{cases} f(x) & \text{if } f \text{ is continuous at } x \\ \frac{1}{2}(f(x_-) + f(x_+)) & \text{if } f \text{ has a jump at } x. \end{cases}$$

That is, the partial sums converge to the average of the left and right limits.

In particular, if f and f' are continuous then

$$\lim_{n\to\infty} S_n(x) = f(x) \text{ for all } x \in [-\ell, \ell]$$

5 Examples of convergence

5.1 Triangle wave

Consider again the triangle wave

$$f(x) = |x| \text{ for } x \in [-1,1], \qquad f(x) = f(x+2).$$

Note that since $f(-1) = f(1) = 1$, the endpoints match, so the periodic extension will be continuous at these points. Since $f(x)$ is otherwise continuous, we see that f (as a 2-periodic function) is continuous.

Similarly, we have

$$f'(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}$$

and $f'(x)$ undefined at $x = 0, \pm 1$. Thus f' is piecewise continuous. It follows from the convergence theorem that the partial sums converge **pointwise** to $f(x)$.

The situation we observed in Figure 1 agrees with the theorem; The error is largest at the corner, and that error decreases to zero as $N \to \infty$ (albeit slowly).

5.2 Square wave

We may now finish discussing convergence the square wave.

Recall that we found the Fourier series for the square wave defined by

$$f(x) = \begin{cases} -1 & -1 \le x < 0 \\ 1 & 0 < x \le 1 \end{cases} \tag{11}$$

and $f(x) = f(x+2)$ when $x \notin [-1,1]$. Let us consider $f(x)$ on the interval $[-1,1]$. The Fourier series for $f(x)$ is

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin((2n-1)\pi x).$$

As a periodic function in $L^2[-1,1]$, $f(x)$ is continuous except for discontinuities at $x = 0$ and $x = \pm 1$. Note that $f(-1) = -1$ and $f(1) = 1$; the endpoints do not match, so the periodic version has discontinuities at $\pm 1, \pm 3, \cdots$ and so on.

Pointwise convergence: For the purposes of the convergence theorem, f defined on $[-1,1]$ has discontinuities at the endpoints, and

$$f(1^-) = -1, \quad f(1^+) = -1.$$

We also have

$$f(0^-) = 1, \quad f(0^+) = 1.$$

It follows that the partial sums converge to $f(x)$ when $x \neq -1, 0, 1$ and converge to 0 at all the discontinuities (the average is always $\frac{1}{2}(-1+1) = 0$). Define

$$\tilde{f}(x) = \frac{1}{2}(f(x^-) + f(x^+)) = \begin{cases} f(x) & \text{if } x \neq 0, \pm 1 \\ 0 & x = 0, \pm 1 \end{cases}.$$

Then the convergence theorem says that

$$\tilde{f}(x) = \lim_{N\to\infty} S_N(x) \text{ for } x \in [-1, 1].$$

In particular, it does not matter how $f(x)$ in (11) is defined at the discontinuities; the Fourier series will converge to $\tilde{f}(x)$ regardless of the values chosen for $f(\pm 1)$ and $f(0)$.

Remark: Note that in this case, the theorem is not needed to see what happens at the discontinuities since

$$S_N(0) = S_N(\pm 1) = 0 \text{ for all } N$$

as all the terms are zero individually.

5.3 Gibbs phenomenon (persistent oscillations):

However, $f(x)$ is not continuous, so we cannot conclude that the *maximum* error goes to zero. Indeed, it is not, as we have seen by direct inspection. The overshoot at the discontinuities never goes away, and it is true that

$$\max_{x\in[-1,1]} |f(x) - S_m(x)| \approx 0.18 \text{ as } m \to \infty.$$

Proving this requires some work (not done here). Below, the partial sums near $x =$ are shown. They tend to oscillate and overshoot the discontinuity by a significant amount, no matter how large we make N. The persistent overshoot here is called **Gibbs' phenomenon**, and is how Fourier series generically behave at discontinuities.