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The period function of potential systems of polynomials with real zeros

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Abstract

We consider some analytic behaviors (convexity, monotonicity and number of critical points) of the period function of period annuli of the potential system $\ddot{x} + g(x) = 0$ and focus on the case when $g(x)$ is a polynomial whose roots are all real. The main contributions of this paper are twofold: (i) analytic behaviors are given for the period functions of period annuli surrounding one or more and simple or degenerate equilibria; (ii) as a nontrivial application of the general conclusions in (i), a purely analytical and shorter proof is provided for a result for the case $\deg g = 4$ recently obtained by Chengzhi Li and Kening Lu with some help of computer algebra [Chengzhi Li, Kening Lu, The period function of hyperelliptic Hamiltonian of degree 5 with real critical points, Nonlinearity 21 (2008) 465–483].

Keywords: Period function; Period annulus; Convexity; Monotonicity; Critical points

1. Introduction and the main results

We consider potential systems of the form $\ddot{x} + g(x) = 0$ or its equivalent planar system

$$\dot{x} = y, \quad \dot{y} = -g(x). \tag{1.1}$$

The system (1.1) is Hamiltonian with the Hamiltonian function $H(x, y) = y^2/2 + G(x)$, where G is a primitive function of g, i.e., $G'(x) = g(x)$. The orbits of the system (1.1) are determined by the level curves γ_c: $y^2/2 + G(x) = c$, the parameter c is called energy as usual. We are interested

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in the case that there is a continuous family of periodic orbits (called period annulus) of (1.1). In this case a function called period function is well defined on the period annulus and assigns each periodic orbit in the annulus its minimal positive period. If γ_c is a periodic orbit, then its minimal period $T(c)$ can be given by

$$T(c) = \sqrt{2} \int_{x_1}^{x_2} \frac{dx}{\sqrt{c - G(x)}}, \tag{1.2}$$

where x_1, x_2 are determined by $G(x_1) = c = G(x_2)$, $x_1 < x_2$. We are interested in analytic behaviors of the period function $T(c)$, such as the convexity, the monotonicity, and the number of critical points of $T(c)$.

There is a lot of work on the study of the period functions, for example, Chicone [3], Chicone and Dumortier [5], Chicone and Jacobs [6], Chouikha and Cuvelier [7], Chow and Sanders [8], Chow and Wang [9], Cima et al. [10], Coppel and Gavrilov [11], Gasull et al. [12], Gasull and Zhao [13], Hsu [14], Chengzhi Li and Kening Lu [15], Mardesic et al. [16], Rothe [17,18], Sabatini [19,20], Schaaf [21,22], Villadelprat [24], Waldvogel [25], Wang Duo [26], Yulin Zhao [28] and many others.

The problem on the monotonicity and critical points of the period functions occurs in the study of subharmonic bifurcations from a perturbed system of the Hamiltonian system (1.1), also in the study of bifurcations of reaction–diffusion equations in one space variable, see Smoller and Wasserman [23], Chicone [4]. When g is a polynomial, the period function $T(c)$ in (1.2) is a special Abelian integral, and the problem of determining the number of the critical points of $T(c)$ in this case is closely related to the weakened Hilbert 16th problem, see Arnold [1, p. 313].

To state the main results of this paper, let us fix some notations. If x_0 is a zero of g of k-multiplicity, the corresponding equilibrium $(x_0, 0)$ of the system (1.1) is said to be k-multiple. Specially if x_0 is a zero of k-multiplicity and the equilibrium $(x_0, 0)$ is a center, then the center is said to be k-multiple, and in this case k must be odd and $g^{(k)}(x_0) > 0$.

We will in this paper study analytical behaviors of the period function of the system (1.1) with a focus on the case when g is a polynomial of degree $\geqslant 2$, mostly with real zeros, i.e. all zeros of g are real. The main results are the following four theorems and an application of them.

Theorem 1.1. *Let g be a polynomial of degree $\geqslant 2$ with real zeros, and there exists a period annulus of* (1.1) *surrounding only one center* $(x_0, 0)$ *of* $(2k+1)$*-multiplicity, no other equilibrium. Let $T(c)$ denote the corresponding period function. Then for a suitably chosen energy parameter c, the following conclusions hold.*

1. *If $k = 0$, that is the center is simple, then $T(c)$ is strictly monotone increasing* ($T'(c) > 0$) *on* $(0, c_1)$, *and* $\lim_{c \to c_1^-} T(c) = +\infty$, $\lim_{c \to 0^+} T(c) = 2\pi/\sqrt{g'(x_0)}$, *where $c_1 < +\infty$.*
2. *If $k \geqslant 1$ and* $\deg g = 2k+1$, *then $T(c)$ has an explicit expression $T(c) = \mu c^{-k/(2k+2)}$ on* $(0, +\infty)$, *where $\mu > 0$. Obviously $T(c)$ is strictly convex and strictly monotone decreasing on* $(0, +\infty)$, *and* $\lim_{c \to 0^+} T(c) = +\infty$, $\lim_{c \to +\infty} T(c) = 0$.
3. *If $k \geqslant 1$ and* $\deg g > 2k+1$, *then $T(c)$ has exactly one critical point on* $(0, c_1)$ *where $T(c)$ reaches its unique minimum, and $c_1 < +\infty$,* $\lim_{c \to 0^+} T(c) = +\infty = \lim_{c \to c_1^-} T(c)$.

The first conclusion of Theorem 1.1 is an improvement of a result of Schaaf, see Theorem 2 in [21], where an additional condition is required on $g(x)$ that each real root of g is simple.

Theorem 1.2. *Let $g(x)$ be a polynomial of degree $2n+1$ ($n \geqslant 1$) (not necessarily with real zeros) and have a positive leading coefficient $L_g > 0$, and let $\widehat{T}(A)$ denote the period of the periodic orbit $y^2/2 + G(x) = G(A)$, where $A > 0$ is large enough. Then there is $A_0 > 0$ such that $\widehat{T}(A)$ has the expansion with respect to $1/A$ as follows*

$$\widehat{T}(A) = \frac{\lambda}{A^n}\left(\alpha_0 + \frac{\alpha_1}{A} + \frac{\alpha_2}{A^2} + \cdots\right), \quad \forall A > A_0, \tag{1.3}$$

where

$$\lambda = 4\sqrt{\frac{n+1}{L_g}}, \qquad \alpha_0 = \int_0^1 \frac{du}{\sqrt{1-u^{2n+2}}}.$$

We note that $\widehat{T}(A) = T(c) = T(G(A))$, where $T(c)$ is defined in (1.2) and $A > 0$ is usually called the amplitude of the closed orbit. One obvious consequence of Theorem 1.2 is that $\lim_{c\to+\infty} T(c) = \lim_{A\to+\infty} \widehat{T}(A) = 0$.

Theorem 1.3. *Let g be a polynomial of degree $2n+1$ ($n \geqslant 1$) with real zeros and have a positive leading coefficient. Then the period function $T(c)$ of the period annulus surrounding all equilibria of* (1.1) *is strictly convex ($T''(c) > 0$) and strictly monotone decreasing ($T'(c) < 0$) on $(c_0, +\infty)$, and $\lim_{c\to+\infty} T(c) = 0$, where c_0 is finite.*

Theorem 1.4. *Let g be a polynomial of degree $2n$ ($n \geqslant 1$) with real zeros and have a negative leading coefficient. Let $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_{2n-1} \leqslant a_{2n}$ denote the zeros of g. If* (1) *the maximal zero a_{2n} of g is simple, i.e. $a_{2n-1} < a_{2n}$;* (2) *$g''(a_{2n-1}) \geqslant 0$;* (3) *$\max\{G(a_i),\ i < 2n\} < G(a_{2n})$, then there is a period annulus surrounding the equilibria $(a_i, 0)$, $i < 2n$, and the corresponding period function $T(c)$ on (c_0, c_1) is strictly convex ($T''(c) > 0$), has exactly one critical point where $T(c)$ reaches its minimum, and $\lim_{c\to c_0^+} T(c) = +\infty = \lim_{c\to c_1^-} T(c)$, where $c_0 = \max\{G(a_i),\ i < 2n\}$, $c_1 = G(a_{2n})$.*

Clearly a parallel result of Theorem 1.4 holds when g has a positive leading coefficient. A case of Theorem 1.4 for $n = 2$ is illustrated in Fig. 2.2.

As an application of the above theorems, we provide an alternative and shorter proof of a theorem recently obtained by Chengzhi Li and Kening Lu [15] that if $g(x)$ is a polynomial of degree 4 whose zeros are all real, then the period function of any period annulus of the system (1.1) has at most one critical point, and it has one if and only if the period annulus surrounds three equilibria, taking multiplicity into account. Our approach is purely analytical, different from that of Chengzhi Li and Kening Lu with some help of computer algebra.

Additionally, we will give new expressions of the period function (Proposition 3.2), and an estimate on the number of the critical points of the period function (Theorem 3.7).

Let us now outline the paper. In the next section we apply the four theorems to prove the result of Chengzhi Li and Kening Lu mentioned above. In Section 3 new expressions of the period function and its derivatives are established, these expressions are useful to study the period function. In Section 4 we present some conditions which are relatively easier to check and from which follow the monotonicity and the number of critical points of the period function. Finally we give detailed proofs of the four theorems in Section 5.

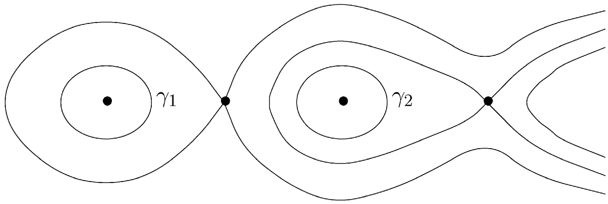

Fig. 2.1. Phase portrait for the case $G(b) \leqslant 0$.

2. Application to the case $\deg g = 4$

As an application of the general conclusions stated in the four theorems in Section 1, we will in this section provide a purely analytical proof of the following theorem for the case $\deg g = 4$ due to Chengzhi Li and Kening Lu [15].

Theorem 2.1. *If g is a polynomial of degree 4 with real zeros, then the period function of any period annulus of the system $\dot{x} = y$, $\dot{y} = -g(x)$ has at most one critical point; and it has one if and only if the annulus surrounds three equilibria, taking multiplicity into account.*

Proof. If g has a 4-multiple zero or two double zeros, then the system

$$\dot{x} = y, \qquad \dot{y} = -g(x) \tag{2.1}$$

has no closed orbit. Therefore there remain three cases to be considered.

Case I: g has 4 simple zeros;
Case II: g has one double zero and two simple zeros;
Case III: g has one triple zero and one simple zero.

In the following we separately discuss the three cases.

Case I: g has 4 simple zeros. We may assume that $g(x) = -(x+1)x(x-a)(x-b)$, $0 < a < b$, after a scaling change. Let $G(x) = \int_0^x g(s)\,ds$. In this case the system (2.1) has two period annuli surrounding the two simple centers $(-1, 0)$ and $(a, 0)$ respectively. Let $T_1(c)$ and $T_2(c)$ denote the corresponding period functions respectively. Then $T_1(c)$ is defined on $(G(-1), 0)$ and $T_2(c)$ on $(G(a), \min\{0, G(b)\})$. It follows from Theorem 1.1 that $T_1'(c) > 0$ and $T_2'(c) > 0$ on their definition interval respectively.

If $G(b) \leqslant 0$, then there are only the two period annuli of (2.1), and its phase portrait is shown in Fig. 2.1.

If $G(b) > 0$, the system (2.1) has an additional period annulus surrounding three equilibria $(-1, 0)$, $(0, 0)$ and $(a, 0)$, and the phase portrait in this case is shown in Fig. 2.2. We claim that the condition $G(b) > 0$ implies that $g''(a) > 0$. Therefore it follows from Theorem 1.4 that the period function $T_3(c)$ of the additional annulus has exactly one critical point on $(0, G(b))$.

Let us prove the claim that the condition $G(b) > 0$ implies that $g''(a) > 0$. By elementary computation we obtain

$$g''(a) = (2+4a)b - \left(6a^2 + 4a\right), \qquad G(b) = b^3\left(3b^2 - 5b(a-1) - 10a\right)/60.$$

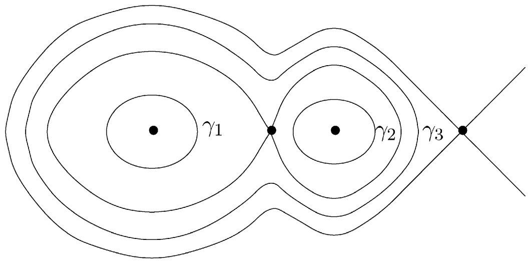

Fig. 2.2. Phase portrait for the case $G(b) > 0$.

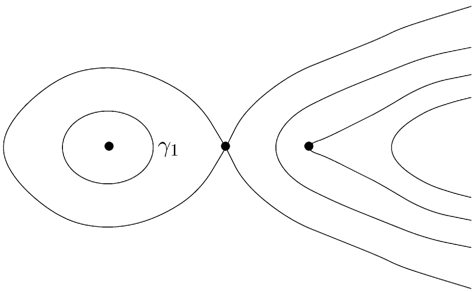

Fig. 2.3. Phase portrait for the subcase (i) $g(x) = -(x+1)x(x-a)^2$.

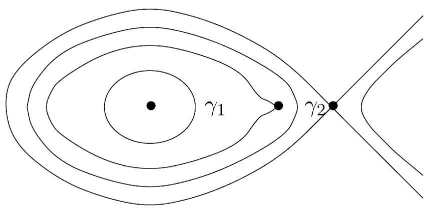

Fig. 2.4. Phase portrait for the subcase (ii) $g(x) = -(x+1)x^2(x-a)$.

So the claim holds if and only if

$$\frac{5}{6}\big((a-1)+\sqrt{(a-1)^2+24/5}\big) \geqslant \frac{2a+3a^2}{1+2a} \quad \text{for } a \in (0,+\infty).$$

Solving the above inequality we obtain $36(a^2+3a+1) \geqslant 0$ for $a \in (0,+\infty)$, which holds obviously. The claim is proved.

Case II: g has one double zero and two simple zeros. By scaling we may assume that g has one of the three forms (i) $g(x) = -(x+1)x(x-a)^2$; (ii) $g(x) = -(x+1)x^2(x-a)$; (iii) $g(x) = -(x+1)^2x(x-a)$, where $a > 0$. Let $G(x) = \int_0^x g(s)\,ds$ as before. For the subcase (i) $g(x) = -(x+1)x(x-a)^2$, the phase portrait of the system (2.1) is shown in Fig. 2.3, and the system has only one period annulus surrounding the simple center $(-1,0)$, and the corresponding period function is strictly monotone increasing on the interval $(G(-1),0)$ due to Theorem 1.1.

For the subcase (ii) $g(x) = -(x+1)x^2(x-a)$, the system (2.1) has two period annuli, one surrounding the other, and the inner annulus surrounds the simple center $(-1,0)$. The phase portrait is shown in Fig. 2.4. Let $T_1(c)$ and $T_2(c)$ denote the period function corresponding the

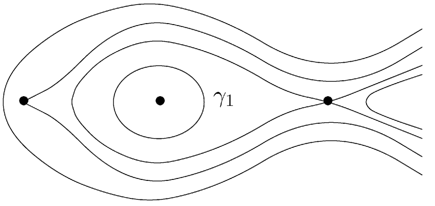

Fig. 2.5. Phase portrait for the case $G(-1) \geqslant G(a)$ in the subcase (iii).

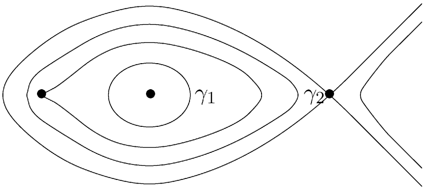

Fig. 2.6. Phase portrait for the case $G(-1) < G(a)$ in the subcase (iii).

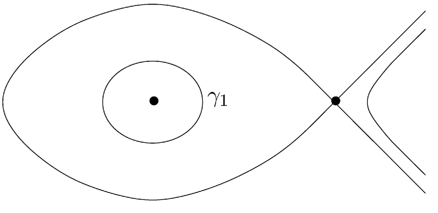

Fig. 2.7. Phase portrait for Case III.

inner and outer annulus respectively. It follows from Theorem 1.1 that $T_1(c)$ is strictly monotone increasing on the interval $(G(-1), 0)$. Noting $g''(0) = 2a > 0$ and applying Theorem 1.4 to $T_2(c)$ we conclude that $T_2(c)$ has exactly one critical point.

Consider the subcase (iii) $g(x) = -(x+1)^2 x(x-a)$. If $G(-1) \geqslant G(a)$, there is only one period annulus surrounding the simple center $(0,0)$, and corresponding period function is strictly monotone increasing on $(0, G(a))$ due to Theorem 1.1. The phase portrait in this case is shown in Fig. 2.5.

If $G(-1) < G(a)$, then the system (2.1) has an additional period annulus surrounding the double equilibrium $(-1,0)$ and the simple center $(0,0)$. Elementary computation shows that $G(-1) < G(a)$ for $a > 0$ holds if only if $a > 2/3$. Since $g''(0) = 2(2a-1) > 0$, the corresponding period function has exactly one critical point due to Theorem 1.4. The phase portrait in this case is shown in Fig. 2.6.

Case III: g has one triple zero and one simple zero. By scaling as before we may assume that g has one of two forms (i) $g(x) = x^3(1-x)$; (ii) $g(x) = x(1-x)^3$. For either subcase, the system (2.1) has only one period annulus. For the subcase (i) the annulus surrounds the triple center $(0,0)$ and corresponding period function has exactly one critical point by Theorem 1.1. For the subcase (ii) the annulus surrounds the simple center $(0,0)$, and corresponding period function is strictly monotone increasing also by Theorem 1.1. The phase portrait for Case III is shown in Fig. 2.7.

Summarizing the above discussion, we have completed a proof of Theorem 2.1. □

3. Expressions for the period function and its derivatives

The expression of the period function $T(c)$ given in (1.2) is sometimes analytically not convenient. For the proof of the four theorems stated in Section 1 we need analytically more suitable forms for $T(c)$. In this section we derive some other expressions for the period function.

Let g be smooth enough on $(-\infty,\infty)$ (not necessarily polynomial at this moment). Assume that the system (1.1) has a period annulus

$$\Gamma=\{(x,y),\ y^2/2+G(x)=c,\ c\in(c_0,c_1)\}=\bigcup_{c\in(c_0,c_1)}\gamma_c.$$

Obviously the intersection of the annulus Γ with x-axis is the union of two open intervals with empty intersection. We denote the two intervals by (a^*,a), (b,b^*), and $g(x)\neq 0$ on $(a^*,a)\cup(b,b^*)$, where $-\infty\leqslant a^*<a\leqslant b<b^*\leqslant+\infty$. We will always take the annulus Γ to be maximal. Consequently the limits

$$\gamma_{c_0^+}=\lim_{c\to c_0^+}\gamma_c,\qquad \gamma_{c_0^+}=\lim_{c\to c_1^-}\gamma_c$$

consist of equilibria, homoclinic or heteroclinic orbits, or empty, and $x=a$ or $x=b$ is a zero of g. Now we write the period function $T(c)$ of Γ given in (1.2) as $T(c)=\sqrt{2}(I_1(c)+I_2(c)+I_3(c))$, where

$$I_1(c)=\int_{x_1}^{a}\frac{dx}{\sqrt{c-G(x)}},\qquad I_2(c)=\int_{a}^{b}\frac{dx}{\sqrt{c-G(x)}},\qquad I_3(c)=\int_{b}^{x_2}\frac{dx}{\sqrt{c-G(x)}},\tag{3.1}$$

and $a^*<x_1<a\leqslant b<x_2<b^*$. Since both a and b are constants with respect to c. The integral $I_2(c)$ is obviously convex ($I_2''(c)>0$) and monotone decreasing ($I_2'(c)<0$) on (c_0,c_1). To deal with $T_1(c)$ and $T_3(c)$, it suffices to consider the integral

$$I(c)=\int_0^A\frac{dx}{\sqrt{c-G(x)}},\qquad G(A)=c,\quad A>0,\ c\in(0,c^*),\tag{3.2}$$

where $G(x)=\int_0^x g(s)\,ds$, $c^*=G(b^*)$ and g satisfies

$$g(0)=g'(0)=\cdots=g^{(m-1)}(0)=0,\qquad g^{(m)}(0)>0,\qquad g(x)>0\quad\text{on }(0,b^*).\tag{3.3}$$

Lemma 3.1. *The integral $I(c)$ given in* (3.2) *can be expressed in the following way*

$$I(c)=(m+1)\int_0^{\pi/2}\frac{\sin^m\theta\sqrt{1+\sin\theta}}{\sqrt{1+\sin\theta+\cdots+\sin^m\theta}}\frac{\sqrt{c}}{g(x)}\,d\theta,\tag{3.4}$$

$$I(c)=(m+1)\int_0^{\pi/2}\sqrt{\frac{\sin^{m-1}\theta(1+\sin\theta)}{1+\sin\theta+\cdots+\sin^m\theta}}\sqrt{G/g^2}(x)\,d\theta,\tag{3.5}$$

where $x=x(c,\theta)$ is implicitly determined by

$$G(x)=c\sin^{m+1}\theta,\quad 0\leqslant\theta\leqslant\pi/2,\ c\in(0,c^*).\tag{3.6}$$

Proof. The integral expression (3.2) of $I(c)$ turns to (3.4) under the change of variable $x \to \theta$ defined by (3.6). The other expression (3.5) directly follows from (3.6) and (3.4). $\square$

The function G/g^2 and its derivative $(G/g^2)' = (g^2 - 2Gg')/g^3$ play an important roll in the study of the period function, and appear in many papers in this field, see Bonorino et al. [2], Chicone [3], Cima et al. [10], Coppel and Gavrilov [11], Schaaf [21] and others.

It easily follows from (3.4) and (3.5) that the expressions for the derivatives of $I(c)$ of any order. For later use in Section 5 we list some of these derivatives:

$$\sqrt{c}I'(c) = \frac{m+1}{2}\int_0^{\pi/2} \frac{\sin^m\theta\sqrt{1+\sin\theta}}{\sqrt{1+\sin\theta+\cdots+\sin^m\theta}}\big(G/g^2\big)'(x)\,d\theta, \tag{3.7}$$

$$\big(\sqrt{c}I'(c)\big)' = \frac{m+1}{2}\int_0^{\pi/2} \frac{\sin^{2m+1}\theta\sqrt{1+\sin\theta}}{\sqrt{1+\sin\theta+\cdots+\sin^m\theta}}\frac{1}{g(x)}\big(G/g^2\big)''(x)\,d\theta, \tag{3.8}$$

$$I'(c) = \frac{m+1}{2}\int_0^{\pi/2} \sqrt{\frac{\sin^{3m+1}\theta(1+\sin\theta)}{1+\sin\theta+\cdots+\sin^m\theta}}\big[\big(G/g^2\big)'/\sqrt{G}\big](x)\,d\theta, \tag{3.9}$$

$$I''(c) = \frac{m+1}{2}\int_0^{\pi/2} \sqrt{\frac{\sin^{3m+1}\theta(1+\sin\theta)}{1+\sin\theta+\cdots+\sin^m\theta}}\frac{\sin^{m+1}\theta}{g(x)}\big[\big(G/g^2\big)'/\sqrt{G}\big]'(x)\,d\theta, \tag{3.10}$$

where $x = x(c,\theta)$ is implicitly determined by (3.6).

In the rest of this section we consider the case that the annulus Γ surrounds only one center and no other equilibrium, that is the case $a = b$. We assume that the center locates at $(0,0)$. It is known that the multiplicity m of $x = 0$ as zero of g must be odd, say $m = 2k+1$ and $g^{(2k+1)}(0) > 0$ in this case. Therefore

$$g(0) = g'(0) = \cdots = g^{(2k)}(0) = 0, \qquad g^{(2k+1)}(0) > 0, \qquad xg(x) > 0,$$
$$\forall x \in (a^*, b^*)\setminus\{0\}. \tag{H}$$

In other words the center $(0,0)$ is $(2k+1)$-multiple. Similar to Lemma 3.1 we have the following proposition.

Proposition 3.2. *Assume that g is smooth enough on $(-\infty,\infty)$ and satisfies the condition (H). Then the period function $T(c)$ of the period annulus surrounding the center $(0,0)$ has the following two expressions*

$$T(c) = 2\sqrt{2}(k+1)\int_{-\pi/2}^{\pi/2} \frac{\sin^{2k+1}\theta}{\sqrt{1+\sin^2\theta+\cdots+\sin^{2k}\theta}}\frac{\sqrt{c}}{g(x)}\,d\theta, \tag{3.11}$$

and

$$T(c) = 2\sqrt{2}(k+1)\int_{-\pi/2}^{\pi/2} \frac{|\sin^k\theta|}{\sqrt{1+\sin^2\theta+\cdots+\sin^{2k}\theta}}\sqrt{G/g^2(x)}\,d\theta, \tag{3.12}$$

where $x = x(\theta, c)$ is the implicit function uniquely determined by

$$G(x) = c\sin^{2k+2}\theta, \qquad x\sin\theta \geqslant 0, \quad \theta \in [-\pi/2, \pi/2],\ c \in (0, c^*), \tag{3.13}$$

where $c^* = \min\{G(a^*), G(b^*)\}$.

Proof. The proof of Lemma 3.1 also holds here. □

We would like to emphasize that the two expressions above hold for an arbitrary nonnegative integer $k \geqslant 0$, while most work on the period functions of the potential system (1.1) focus on the case $k = 0$, that is the center $(0, 0)$ is simple.

Taking the derivative with respect to c in (3.11) and (3.12), we obtain expressions of the derivative of $T(c)$ of the first and the second order below.

Proposition 3.3. *Under the assumptions on the function $g(x)$ in Proposition* 3.2*, the derivatives of $T(c)$ of the first and the second order are given by*

$$\sqrt{c}T'(c) = \sqrt{2}(k+1)\int_{-\pi/2}^{\pi/2} \frac{|\sin^{2k+1}\theta|\, sgn(\theta)}{\sqrt{1+\sin^2\theta+\cdots+\sin^{2k}\theta}}\big(G/g^2\big)'(x)\,d\theta, \tag{3.14}$$

$$\big(\sqrt{c}T'(c)\big)' = \sqrt{2}(k+1)\int_{-\pi/2}^{\pi/2} \frac{|\sin^{4k+3}\theta|}{\sqrt{1+\sin^2\theta+\cdots+\sin^{2k}\theta}}\frac{sgn(\theta)}{g(x)}\big(G/g^2\big)''(x)\,d\theta, \tag{3.15}$$

$$T'(c) = \sqrt{2}(k+1)\int_{-\pi/2}^{\pi/2} \frac{|\sin\theta|^{3k+2}\, sgn(\theta)}{\sqrt{1+\sin^2\theta+\cdots+\sin^{2k}\theta}}\big[\big(G/g^2\big)'/\sqrt{G}\,\big](x)\,d\theta, \tag{3.16}$$

$$T''(c) = \sqrt{2}(k+1)\int_{-\pi/2}^{\pi/2} \frac{|\sin\theta|^{5k+4}}{\sqrt{1+\sin^2\theta+\cdots+\sin^{2k}\theta}}\frac{sgn(\theta)}{g(x)}\big[\big(G/g^2\big)'/\sqrt{G}\,\big]'(x)\,d\theta, \tag{3.17}$$

where $x = x(\theta, c)$ is the implicit function determined by (3.13)*.*

Now we consider the limit behaviors of $T(c)$ and $T'(c)$ as $c \to 0^+$. We note that the implicit function $x = x(\theta, c)$ defined in (3.13) is continuously differentiable on $(\theta, c) \in [-\pi/2, \pi/2] \times (0, c^*)$. We collect some properties on the function $x(\theta, c)$ in the following lemma.

Lemma 3.4. *The implicit function $x(\theta, c)$ defined in* (3.13) *has following properties.*

1. $x(\theta, c) = \mu_k c^{\frac{1}{2k+1}}\sin\theta + O(c^{\frac{2}{2k+1}}\sin^2\theta)$, $\mu_k = (\frac{2k+2}{g^{2k+1}(0)})^{\frac{1}{2k+2}}$.
2. $\partial x/\partial\theta = (2k+2)c\cos\theta\sin^{2k+1}\theta/g(x)$, $\lim_{\theta\to 0}\partial x/\partial\theta = \mu_k c^{\frac{1}{2k+1}}$.
3. $\partial x/\partial c = \sin^{2k+2}\theta/g(x)$, $\lim_{\theta\to 0}\partial x/\partial c = 0$.

Proof. The proof easily follows from Eq. (3.13). □

Proposition 3.5. *Under the assumptions on the function $g(x)$ in Proposition* 3.2*, the following conclusions hold*:

$$\text{for } k=0, \quad \lim_{c\to 0^+} T(c)=\frac{2\pi}{\sqrt{g'(0)}}, \qquad \lim_{c\to 0^+} T'(c)=\frac{\pi(5(g'')^2-3g'g''')}{12(g')^{7/2}}\bigg|_{x=0}, \tag{3.18}$$

$$\text{for } k\geqslant 1, \quad \lim_{c\to 0^+} c^{\frac{k}{2k+2}} T(c)=\lambda_k, \qquad \lim_{c\to 0^+} c^{\frac{3k+2}{2k+2}} T'(c)=\frac{-k\lambda_k}{2k+2}, \tag{3.19}$$

where

$$\lambda_k=2\sqrt{2}\left(\frac{(2k+2)!}{g^{(2k+1)}(0)}\right)^{\frac{1}{2k+2}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1+\sin^2\theta+\cdots+\sin^{2k}\theta}}.$$

Proof. Taking limits in (3.11) and (3.14), we easily obtain the conclusions. □

We note that the limits in (3.18) are known, see for example, Schaaf [22], and the limits in (3.19) seem to be new. Using Proposition 3.3 we further obtain the expressions for higher order derivatives of $\sqrt{c}T'(c)$ and $T(c)$ in the following proposition. We rewrite $\sqrt{c}T'(c)$ in (3.14) and $T'(c)$ in (3.16) in the following way

$$\sqrt{c}T'(c)=\int_{-\pi/2}^{\pi/2} R_1(\theta)\frac{sgn(\theta)}{g(x)}w_0(x)\,d\theta,$$

$$T'(c)=\int_{-\pi/2}^{\pi/2} S_1(\theta)\frac{sgn(\theta)}{g(x)}(\sqrt{G}/g)'(x)\,d\theta,$$

where

$$R_1(\theta)=\frac{\sqrt{2}(k+1)|\sin\theta|^{2k+1}}{\sqrt{1+\sin^2\theta+\cdots+\sin^{2k}\theta}},$$

$$S_1(\theta)=\frac{2\sqrt{2}(k+1)|\sin\theta|^{3k+2}}{\sqrt{1+\sin^2\theta+\cdots+\sin^{2k}\theta}},$$

$$w_0(x)=\frac{g^2-2Gg'}{g^2}(x),$$

and define Zhang's operator $Z_g: C^n(I)\to C^{n-1}(I)$, $I=(a^*,b^*)\setminus\{0\}$ as follows

$$Z_g(h)(x):=\frac{d}{dx}[h/g](x), \quad \forall h\in C^n(I). \tag{3.20}$$

In 1958 Zhang Zhifen introduced the operator to study the uniqueness of limit cycles and established an important uniqueness theorem, see [27].

Proposition 3.6. *Let the assumptions on $g(x)$ in Proposition* 3.2 *hold. Then the derivatives of $\sqrt{c}T'(c)$ and $T'(c)$ of higher order are given by*

$$\left(\sqrt{c}T'(c)\right)^{(n-1)} = \int_{-\pi/2}^{\pi/2} R_n(\theta)\frac{sgn(\theta)}{g(x)} Z_g^{n-1}(w_0)(x)\,d\theta, \tag{3.21}$$

$$T^{(n)}(c) = \int_{-\pi/2}^{\pi/2} S_n(\theta)\frac{sgn(\theta)}{g(x)} Z_g^{n}(\sqrt{G})(x)\,d\theta, \tag{3.22}$$

where n is an arbitrary nonnegative integer, $x = x(\theta, c)$ is determined by (3.13)*, Z_g^n denotes the n times composition of the operator Z_g, and $R_n(\theta)$, $S_n(\theta)$ are positive for $\theta \neq 0$ and given by*

$$R_n(\theta) = \frac{\sqrt{2}(k+1)|\sin\theta|^{2n(k+1)-1}}{\sqrt{1+\sin^2\theta+\cdots+\sin^{2k}\theta}},$$
$$S_n(\theta) = \frac{2\sqrt{2}(k+1)|\sin\theta|^{2n(k+1)+k}}{\sqrt{1+\sin^2\theta+\cdots+\sin^{2k}\theta}}.$$

Proof. We only prove the formula (3.21), and a proof of (3.22) completely similarly follows. The formula (3.14) shows that (3.21) holds for $n = 1$. We assume (3.21) holds for n. Taking once again the derivative of the both sides of (3.21) and noting $\partial x/\partial c = \sin^{2k+2}\theta/g(x)$ (see Lemma 3.4), we obtain

$$\begin{aligned}\left(\sqrt{c}T'(c)\right)^{(n)} &= \int_{-\pi/2}^{\pi/2} R_n(\theta)\,sgn(\theta) Z_g^n(w_0)(x)\sin^{2k+2}\theta/g(x)\,d\theta \\ &= \int_{-\pi/2}^{\pi/2} R_{n+1}(\theta)\frac{sgn(\theta)}{g(x)} Z_g^n(w_0)(x)\,d\theta,\end{aligned}$$

i.e. (3.21) holds for $n+1$. □

One consequence of Proposition 3.6 the following theorem on the number of the critical points of the period function $T(c)$.

Theorem 3.7. *Let the assumptions on $g(x)$ in Proposition* 3.2 *hold. If there is a positive number n such that $Z_g^n(w_0)(x)$ or $Z_g^{n+1}(\sqrt{G})(x)$ does not change sign in the $(a^*, b^*) \setminus \{0\}$, then the period function $T(c)$ has at most n critical points on $(0, c^*)$, taking multiplicity into account.*

Proof. We only prove the conclusion for case $Z_g^n(w_0)(x)$ does not change sign on $(a^*, b^*) \setminus \{0\}$. The proof is similar for the other case. We note that $R_n(\theta)$ and $sgn(\theta)/g(x)$ are positive on $(-\pi/2, 0)$ and $(0, \pi/2)$. It follows from (3.21) that $(\sqrt{c}T'(c))^{(n)}$ does not change sign on $(0, c^*)$. If $T'(c)$ has more than n zeros, then $(\sqrt{c}T'(c))^{(n)}$ has at least one zero in $(0, c^*)$ (using Rolle theorem n times), a contradiction. The conclusion holds. □

It is in general not easy for a given function g to check if $Z_g^n(w_0)(x)$ or $Z_g^{n+1}(\sqrt{G})(x)$ does change sign on $(a^*, b^*) \setminus \{0\}$ for positive integer n. Let us consider the case $n = 1$. Note that $Z_g(w_0)(x) = (G/g^2)''(x)$, $Z_g^2(\sqrt{G})(x) = [(G/g^2)'/(2\sqrt{G})]'(x)$. Consequently if $(G/g^2)''$ or

$[(G/g^2)'/\sqrt{G}\,]'$ does not change sign on $(a^*, b^*) \setminus \{0\}$, then $(\sqrt{c}T'(c))'$ or $T''(c)$ also does not change sign on $(0, c^*)$, respectively.

As we mentioned above, the function G/g^2 and its derivative appear in many papers on the period function. Coppel and Gavrilov [11] obtained a nice expression for the derivative $T'(c)$ in terms of the function G/g^2 by

$$T'(c) = \frac{1}{2c} \iint\limits_{\sigma(c)} \left(G/g^2\right)''(x)\,dx\,dy$$

in the case $k = 0$, that is the center is simple, where $\sigma(c)$ denotes the compact region of bounded by γ_c. See also Chicone [3], Schaaf [21].

In the next section we will present some more manageable conditions on g which guarantee $(G/g^2)''(x) > 0$ and $[(G/g^2)'/\sqrt{G}\,]'(x) > 0$ on $(a^*, b^*) \setminus \{0\}$.

4. Conditions for $(G/g^2)'' > 0$ and $[(G/g^2)'/\sqrt{G}\,]' > 0$

We will in this section show that the conditions $(g''/g')'(x) < 0$ and $(g''/g)'(x) < 0$ together with others imply $(G/g^2)''(x) > 0$ and $[(G/g^2)'/\sqrt{G}\,]'(x) > 0$. The former are in general easier to check and satisfied at least by polynomials with real roots due to the following lemma.

Lemma 4.1. *If g is a polynomial with real zeros, then*

$$\left(\frac{g''}{g'}\right)'(x) < 0, \quad \forall x \in \mathbb{R} \setminus \{x \mid g'(x) = 0\}, \tag{4.1}$$

$$\left(\frac{g''}{g}\right)'(x) < 0, \quad \forall x > \textit{the maximal zero of } g. \tag{4.2}$$

Proof. Since all the zeros of g are real, so are all the zeros of its derivative $g'(x)$. Let $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n$ denote the n real roots of $g(x)$ and $q_1 \leqslant q_2 \leqslant \cdots \leqslant q_{n-1}$ denote the $n-1$ real roots of $g'(x)$. Then $g(x) = \lambda \prod_{i=1}^n (x - p_i)$, $g'(x) = \mu \prod_{i=1}^{n-1} (x - q_i)$, where $\lambda \neq 0$, $\mu \neq 0$. Hence

$$\left(\frac{g'}{g}\right)'(x) = \left(\sum_{i=1}^n \frac{1}{x - p_i}\right)' = -\sum_{i=1}^n \frac{1}{(x - p_i)^2} < 0, \quad \forall x \neq p_1, \ldots, p_n,$$

$$\left(\frac{g''}{g'}\right)'(x) = \left(\sum_{i=1}^{n-1} \frac{1}{x - q_i}\right)' = -\sum_{i=1}^{n-1} \frac{1}{(x - q_i)^2} < 0, \quad \forall x \neq q_1, \ldots, q_{n-1}.$$

Note that

$$\left(\frac{g''}{g}\right)' = \left(\frac{g''}{g'}\right)'\left(\frac{g'}{g}\right) + \left(\frac{g''}{g'}\right)\left(\frac{g'}{g}\right)',$$

and

$$\frac{g'}{g}(x) > 0, \qquad \frac{g''}{g'}(x) > 0 \quad \text{for } \forall x > p_n.$$

The conclusion holds. $\square$

In the following three lemmata we will show that the condition $(g''/g')' < 0$ and others imply $(G/g^2)'' > 0$. We assume as before that $g(x)$ is smooth on $(-\infty, +\infty)$ (not necessarily polynomial) and satisfies

$$g(0) = g'(0) = \cdots = g^{(m-1)}(0) = 0, \qquad g^{(m)}(0) > 0, \qquad g(x) > 0, \quad \forall x \in (0, b^*). \tag{H'}$$

Lemma 4.2. *Assume that $g(x)$ satisfies (H') and the following additional conditions*

1. $g'(x) > 0$ *on* $(0, b^*)$, *where* $0 < b^* \leqslant +\infty$,
2. $(g''/g')'(x) < 0$ *on* $(0, b^*)$,

then $(G/g^2)''(x) > 0$ *on* $(0, b^*)$.

Proof. Let $r(x) = (g^2 - 2Gg')(x)$. Then $r' = -2Gg''$ and

$$\big(G/g^2\big)'(x) = \big(r/g^3\big)(x), \qquad \big(G/g^2\big)''(x) = \frac{r'g - 3rg'}{g^4}(x). \tag{4.3}$$

So it is sufficient to show that

$$(r'g - 3rg')(x) > 0, \quad \forall x \in (0, b^*). \tag{4.4}$$

To do this, we write $3r(x)g'(x)$ as

$$3r(x)g'(x) = 3g'(x)\int_0^x -2G(s)g''(s)\,ds = 6g'(x)\int_0^x \left(\frac{G(s)}{g^2(s)}\right)\left(\frac{-g''(s)}{g'(s)}\right)g^2(s)g'(s)\,ds. \tag{4.5}$$

It follows from the second condition on $g(x)$ that $g''(x)$ has at most one zero on $(0, b^*)$. We will prove that the inequality (4.4) holds in three cases.

Case I: $g''(x) < 0$, $\forall x \in (0, b^*)$. Since $r' = -2Gg''$ and $r(0) = 0$ we have $r(x) > 0$, $\forall x \in (0, b^*)$. So the functions G/g^2 and $-g''/g'$ are monotone strictly increasing and positive on $(0, b^*)$. It follows from (4.5) that

$$3r(x)g'(x) < 6g'(x)\frac{G(x)}{g^2(x)}\left(\frac{-g''(x)}{g'(x)}\right)g^3(x)/3 = -2G(x)g''(x)g(x) = r'(x)g(x).$$

Hence the inequality (4.4) holds.

Case II: $g''(x) > 0$, $\forall x \in (0, b^*)$. We note that in this case $r'(x) = -2G(x)g''(x) < 0$, $r(0) = 0$ and hence $r(x) < 0$ for $\forall x \in (0, b^*)$. The functions G/g^2 and g''/g' are monotone strictly decreasing and positive on $(0, b^*)$. It follows from (4.5) that

$$3r(x)g'(x) < -6g'(x)\frac{G(x)}{g^2(x)}\frac{g''(x)}{g'(x)}g^3(x)/3 = -2G(x)g''(x)g(x) = r'(x)g(x).$$

The inequality (4.4) holds.

Case III: $g''(x)$ has one zero $x_1 \in (0, b^*)$. In this case we claim that $g''(x) > 0$ for $\forall x \in (0, x_1)$ and $g''(x) < 0$ for $\forall x \in (x_1, b^*)$. This is because that $g'''(x_1) < 0$ which follows from second condition on $g(x)$. Applying the conclusion of Case II to the interval $(0, x_1)$, we obtain the inequality (4.4) on $(0, x_1)$. Now let us show the inequality (4.4) holds on $[x_1, b^*)$. We note that

$r'(x) = -2G(x)g''(x) > 0$ on (x_1, b^*) and $r(x_1) < 0$. If $r(x) < 0$ for $\forall x \in [x_1, b^*)$, then (4.4) holds for $\forall x \in [x_1, b^*)$. We assume that $r(x)$ has one zero $\eta \in (x_1, b^*)$, $r(\eta) = 0$. The inequality (4.4) holds on $[x_1, \eta)$ since $r(x) < 0$ on $[x_1, \eta)$. Now we show (4.4) holds on (η, b^*). Similar to the relation (4.5) we have

$$3r(x)g'(x) = 3g'(x)\int_\eta^x r'(s)\,ds = 3g'(x)\int_\eta^x -2G(s)g''(s)\,ds$$
$$= 6g'(x)\int_\eta^x \left(\frac{G(s)}{g^2(s)}\right)\left(\frac{-g''(s)}{g'(s)}\right)g^2(s)g'(s)\,ds.$$

We note that the functions G/g^2 and $-g''/g'$ are strictly monotone increasing and positive on (η, b^*). Hence we obtain

$$3r(x)g'(x) < 6g'(x)\frac{G(x)}{g^2(x)}\left(\frac{-g''(x)}{g'(x)}\right)\left(g^3(x) - g^3(\eta)\right)/3 < -2G(x)g''(x)g(x)$$
$$= r'(x)g(x).$$

That is, the inequality (4.4) holds. $\square$

Remark 4.3. To show the inequality (4.4) we write $3r(x)g(x)$ in an integral form (4.5), from which we obtain (4.4) in different situations. This integral technique will be used later in this section several times.

The following lemma is a generalization of Lemma 4.2 to the case where $g'(x)$ has one zero.

Lemma 4.4. *Assume that $g(x)$ satisfies (H') and the following conditions:*

1. $0 < b^* < +\infty$, $g(b^*) = 0$,
2. $g'(x) > 0$ *on* $(0, x_1)$ *and* $g'(x) < 0$ *on* (x_1, b^*),
3. $(g''/g')'(x) < 0$ *on* $(0, b^*) \setminus \{x_1\}$,

then $(G/g^2)''(x) > 0$ *on* $(0, b^*)$.

Proof. It follows from Lemma 4.1 that $(G/g^2)''(x) > 0$ holds on $(0, x_1)$. We need only to show that $(G/g^2)''(x) > 0$ holds on $[x_1, b^*)$, or equivalently to show $(r'g - 3g'r)(x) > 0$ on $[x_1, b^*)$. It follows from the assumptions that $g''(x)$ has at most one zero on (x_1, b^*). Let us show $w'(x) > 0$ holds on $[x_1, b^*)$ in three cases.

Case I: $g''(x) > 0$, $\forall x \in (x_1, b^*)$. Since $g'(x_1) = 0$ we have $g'(x) > 0$ for $\forall x \in (x_1, b^*)$. But this contradicts the second condition. So this case is impossible.

Case II: $g''(x) < 0$, $\forall x \in (x_1, b^*)$. Since $r(x_1) = g^2(x_1) > 0$ and $r'(x) = -2G(x)g''(x) > 0$ for $\forall x \in (x_1, b^*)$, we have $r(x) > 0$ for $\forall x \in (x_1, b^*)$, therefore $3g'(x)r(x) < 0 < r'(x)g(x)$ for $\forall x \in (x_1, b^*)$. The conclusion $w'(x) > 0$ holds on $[x_1, b^*)$.

Case III: $g''(x)$ has one zero $\eta \in (x_1, b^*)$, $g''(\eta) = 0$. It follows from the conditions $(g''/g')' < 0$ and $g' < 0$ on (x_1, b^*) that $g'' < 0$ on (x_1, η) and $g'' > 0$ on (η, b^*). Now let us show $3rg' < r'g$ holds on the intervals (x_1, η) and (η, b^*). (i) On (x_1, η) we have $r = g^2 - 2Gg' > 0$ and $r' = -2Gg'' > 0$, and consequently $3r(x)g'(x) < 0 < r'(x)g(x)$. (ii) On (η, b^*) we have

$g''(x) > 0$, $r'(x) = -2G(x)g''(x) < 0$. We note that $r(b^*) > 0$. Hence $r(x) > 0$ on (η, b^*). By using the integral technique we show $3rg'(x) < r'g$ just as done in the proof of Lemma 4.2

$$3r(x)g'(x) = 3g'(x)\left(\int_\eta^x r'(s)\,ds + r(\eta)\right) = 3g'(x)\left(\int_\eta^x -2G(s)g''(s)\,ds + r(\eta)\right)$$
$$= 6g'(x)\int_\eta^x \left(\frac{G(s)}{g^2(s)}\right)\left(\frac{-g''(s)}{g'(s)}\right)g^2(s)g'(s)\,ds + 3r(\eta)g'(x).$$

Noting that the functions G/g^2 and $-g''/g'$ are strictly monotone increasing and positive on (η, b^*), and $3r(\eta)g'(x) < 0$ we obtain

$$3r(x)g'(x) < 6g'(x)\frac{G(x)}{g^2(x)}\left(\frac{-g''(x)}{g'(x)}\right)\big(g^3(x) - g^3(\eta)\big)/3 < -2G(x)g''(x)g(x)$$
$$= r'(x)g(x).$$

The lemma is proved. □

Now we consider the case where $g(x)$ is a polynomial whose zeros are all real.

Lemma 4.5. *Assume that $g(x)$ is a polynomial of degree $\geqslant 2$ with real zeros and the condition (H) holds, then*

$$\big(G/g^2\big)''(x) > 0, \quad x \in (a^*, b^*) \setminus \{0\}. \tag{4.6}$$

Proof. It follows from Lemma 4.1 that $g(x)$ satisfies the condition $(g''/g')' < 0$ on $(a^*, b^*) \setminus \{0\}$. Let us first show $(G/g^2)''(x) > 0$ holds on $(0, b^*)$.

Case I: $b^* = +\infty$, that is the case where $g(x)$ has no zero in $(0, +\infty)$, and so does $g'(x)$. Hence $g'(x) > 0$ in $(0, +\infty)$. Therefore $g(x)$ satisfies all conditions of Lemma 4.2, and so $(G/g^2)''(x) > 0$ holds $(0, b^*)$.

Case II: $b^* < +\infty$ is a positive zero of $g(x)$ and $g(x) > 0$ on $(0, b^*)$. It follows from Rolle's theorem that $g'(x)$ has just only one zero $x_1 \in (0, b^*)$, and $g'(x) > 0$ for $x \in (0, x_1)$, and $g'(x) < 0$ for $x \in (x_1, b^*)$. Therefore $g(x)$ satisfies all conditions of Lemma 4.4, and so $(G/g^2)''(x) > 0$ holds in $(0, b^*)$.

Now we show $(G/g^2)''(x) > 0$ holds also on $(a^*, 0)$. Let $\widehat{g}(x) = -g(-x)$. Then $\widehat{g}(x) > 0$ on $(0, \widehat{b^*})$, where $\widehat{b^*} = -a^*$. We make the same discussion as above if we replace $g(x)$ on $(0, b^*)$ by $\widehat{g}(x)$ on $(0, \widehat{b^*})$. Hence we obtain inequality $(\widehat{G}/\widehat{g}^2)''(x) > 0$ on $(0, \widehat{b^*})$, where $\widehat{G}(x) = \int_0^x \widehat{g}(s)\,ds$. It is quite easy to show that

$$\widehat{G}(x) = G(-x), \qquad \widehat{g}'(x) = g'(-x),$$

and so

$$\big(\widehat{G}/\widehat{g}^2\big)'(x) = -\big(G/g^2\big)'(-x), \qquad \big(\widehat{G}/\widehat{g}^2\big)''(x) = \big(G/g^2\big)''(-x), \quad \forall x \in (0, \widehat{b^*}).$$

Therefore we obtain inequality $(G/g^2)''(x) > 0$ on $(a^*, 0)$. □

In the rest of this section we will show that the condition $(g''/g')' < 0$, $(g''/g)' < 0$ and others imply $[(G/g^2)'/\sqrt{G}\,]' > 0$. Let us write down its derivative

$$\left[\frac{(G/g^2)'}{\sqrt{G}}\right]'(x) = \frac{2(G/g^2)''G - (G/g^2)'g}{2G^{3/2}}(x). \tag{4.7}$$

Under the assumptions on $g(x)$ in Lemma 4.2 we have $(G/g^2)''(x) > 0$ on $(0, b^*)$, and consequently $[(G/g^2)'/\sqrt{G}\,]'(x) > 0$ for those x where $(G/g^2)'(x) < 0$. We note that $(G/g^2)'/\sqrt{G}$ can be rewritten as

$$\frac{(G/g^2)'}{\sqrt{G}} = \frac{r}{\sqrt{G}g^3} = \frac{r}{G^2}\left(\frac{G}{g^2}\right)^{3/2}. \tag{4.8}$$

Therefore for those x where $(G/g^2)'(x) > 0$, we only need to consider the monotonicity of $(r/G^2)(x)$, whose derivative is given by

$$\left(r/G^2\right)'(x) = \frac{r'G - 2rg}{2G^3}(x). \tag{4.9}$$

Lemma 4.6. *Assume that $g(x)$ satisfies the condition (H') and the following ones*

1. $g'(x) > 0$ *on* $(0, b^*)$, *where* $0 < b^* \leqslant +\infty$,
2. $(g''/g')'(x) < 0$ *on* $(0, b^*)$,
3. $(g''/g)'(x) < 0$ *on* $(0, b^*)$,

then $[(G/g^2)'/\sqrt{G}\,]'(x) > 0$ *on* $(0, b^*)$.

Proof. We first note that $(G/g^2)''(x) > 0$ on $(0, b^*)$ due to Lemma 4.2, and that $g''(x)$ has at most one zero on $(0, b^*)$ due to the assumptions of Lemma 4.6. Let us prove the conclusion in three cases.

Case I: If $g''(x) > 0$ on $(0, b^*)$, then $r(x) < 0$ (recall $r(0) = 0$ and $r' = -2Gg''$) and so $(G/g^2)'(x) = r(x)/g(x)^3 < 0$ on $(0, b^*)$. It follows from (4.7) that $[(G/g^2)'/\sqrt{G}\,]'(x) > 0$ on $(0, b^*)$.

Case II: If $g''(x) < 0$ on $(0, b^*)$, then $r(x) > 0$ on $(0, b^*)$, and so $(G/g^2)'(x) = (r/g^3)(x) > 0$ on $(0, b^*)$. Using the integral technique as before we obtain

$$\begin{aligned} 2r(x)g(x) &= 2g(x)\int_0^x r'(s)\,ds = 2g(x)\int_0^x -2G(s)g''(s)\,ds < -2G(x)^2g''(x) \\ &= r'(x)G(x) \end{aligned}$$

and so $(r/G^2)'(x) > 0$ for $\forall x \in (0, b^*)$. Consequently $[(G/g^2)'/\sqrt{G}\,]'(x) > 0$ on $(0, b^*)$ due to (4.8) and (4.9).

Case III: Assume that $g''(x)$ has one zero $x_1 \in (0, b^*)$. By the assumptions of Lemma 4.6 one easily obtains that $g''(x) > 0$ on $(0, x_1)$ and $g''(x) < 0$ on (x_1, b^*). The conclusion in Case I implies that $[(G/g^2)'/\sqrt{G}\,]'(x) > 0$ on $(0, x_1)$ and that $r(x) < 0$ on $(0, x_1)$. If $r(x) < 0$ and so $(G/g^2)'(x) = r(x)/g(x)^3 < 0$ holds on the whole interval $(0, b^*)$, then the conclusion holds, see (4.7). We note $r(x)$ has at most one zero on $(0, b^*)$ due to $(G/g^2)''(x) > 0$ on $(0, b^*)$ and $(G/g^2)' = r/g^3$. Assume that $r(x)$ has a zero $\eta \in (x_1, b^*)$, then $r(x) < 0$ on $(0, \eta)$ and $r(x) > 0$

on (η, b^*). Therefore $[(G/g^2)'/\sqrt{G}\,]'(x) > 0$ on $(0, \eta)$. We claim that $(r/G^2)'(x) > 0$ on (η, b^*). This claim together with (4.8) implies that $[(G/g^2)'/\sqrt{G}\,]'(x) > 0$ on (η, b^*).

Let us prove the claim using the integral technique. It follows from $r(\eta) = 0$ and $g''(x) < 0$ on (η, b^*) that

$$\begin{aligned} 2r(x)g(x) &= 2g(x)\int_\eta^x r'(s)\,ds = 2g(x)\int_\eta^x -2G(s)g''(s)\,ds \\ &= 4g(x)\int_\eta^x \left(\frac{-g''}{g}\right)(s)G(s)g(s)\,ds < 4g(x)\left(\frac{-g''(x)}{g(x)}\right)\int_\eta^x G(s)g(s)\,ds \\ &= -2g''(x)\big(G(x)^2 - G(\eta)^2\big) < -2G(x)^2 g''(x) = r'(x)G(x) \end{aligned}$$

for $\forall x \in (\eta, b^*)$. The claim and so the conclusion hold. □

The following lemma is a generalization of Lemma 4.6 to the case that $g'(x)$ has a zero.

Lemma 4.7. *Let the assumption (H') hold. Furthermore we assume that*

1. *$g'(x)$ has a unique zero $x_1 \in (0, b^*)$ and $(g''/g')'(x) < 0$ on $(0, b^*) \setminus \{x_1\}$,*
2. *one of the following conditions holds*
   (2a) *$g''(x) < 0$ and $(g''/g)'(x) < 0$ on $(0, b^*)$,*
   (2b) *$g''(x)$ has a unique zero $x_2 \in (0, b^*)$ and $(g''/g)'(x) < 0$ on (x_2, b^*),*

then $[(G/g^2)'/\sqrt{G}\,]'(x) > 0$ on $(0, b^*)$.

Proof. First we note that $(G/g^2)''(x) > 0$ on $(0, b^*)$ due to Lemma 4.4. If the condition (2a) holds, then using the same reasoning just as in Case II of the proof Lemma 4.6 we obtain $[(G/g^2)'/\sqrt{G}\,]'(x) > 0$ on $(0, b^*)$. Assume the condition (2b) holds. It is obvious that $0 < x_2 < x_1$, $g''(x) > 0$ on $(0, x_2)$ and $g''(x) < 0$ on (x_2, b^*). Consequently $r(x) < 0$ on $(0, x_2]$ (note $r(0) = 0$ and $r' = -2Gg''$). So $r(x_2) < 0 < r(x_1) = g^2(x_1)$, which means that $r(x)$ has a unique zero $\eta \in (x_2, x_1)$ as $(r/g^3)'(x) = (G/g^2)''(x) > 0$ on $(0, b^*)$. Therefore we have $r(x) < 0$ on $(0, \eta)$ and $r(x) > 0$ on (η, b^*). Therefore we obtain $[(G/g^2)'/\sqrt{G}\,]'(x) > 0$ on $(0, \eta)$, see (4.7). Now using the integral technique just as doing in Case III in the proof of Lemma 4.6 we obtain $2r(x)g(x) < r'(x)G(x)$ and so $(r/G^2)'(x) > 0$ on (η, b^*), see (4.9). Finally it follows from (4.8) that $[(G/g^2)'/\sqrt{G}\,]'(x) > 0$ on (η, b^*). This completes the proof. □

Now we consider the case when g is a polynomial with real zeros.

Lemma 4.8. *Let $g(x)$ be a polynomial of degree $n \geqslant 2$ with a negative leading coefficient and all of its zeros be real, which are denoted by $a_1 \leqslant \cdots \leqslant a_n$. Further assume*:

1. *the maximal zero a_n of g is simple $(a_{n-1} < a_n)$;*
2. *$G(a_n) > \max\{G(a_1), \ldots, G(a_{n-1})\}$;*
3. *$g''(a_{n-1}) \geqslant 0$,*

where $G(x) = \int_{a_{n-1}}^x g(s)\,ds$, then $[(G/g^2)'/\sqrt{G}\,]'(x) > 0$ on (a_{n-1}, a_n).

Proof. Since $g(+\infty) = -\infty$ and the maximal zero a_n of g is simple, we have $g(x) > 0$ on (a_{n-1}, a_n). Let b_{n-1} denote the unique zero of $g'(x)$ in (a_{n-1}, a_n). Note b_{n-1} is simple and the maximal zero of $g'(x)$. So $g''(b_{n-1}) < 0$, $g'(x) < 0$ for $\forall x \in (b_{n-1}, +\infty)$. To prove Lemma 4.8, it suffices to verify the second condition of Lemma 4.7 for g. To this end, we write

$$(g''/g)' = \delta/g^2, \qquad \delta := g'''g - g'g''.$$

Let x_2 denote the maximal zero of $g''(x)$. Since $g''(a_{n-1}) \geqslant 0 > g''(b_{n-1})$, we obtain $a_{n-1} \leqslant x_2 < b_{n-1} < a_n$. If $a_{n-1} < x_2$, we have $g''(x) > 0$ for $\forall x \in (a_{n-1}, x_2)$ and $g''(x) < 0$ for $\forall x \in (x_2, a_n)$. Since x_2 is the maximal zero and g has a negative leading coefficient, we have

$$g^{(k)}(x) < 0 \quad \text{for } \forall x \in (x_2, +\infty),\ \forall k \geqslant 2.$$

Consequently

$$\delta(x_2) = g^{(3)}(x_2)g(x_2) < 0, \qquad \delta'(x) = \big(g^{(4)}g - (g'')^2\big)(x) < 0, \quad \forall x \in (x_2, a_n). \tag{4.10}$$

Therefore $\delta(x) < 0$ for $\forall x \in (x_2, a_n)$, and so

$$(g''/g)'(x) < 0, \quad \forall x \in (x_2, a_n). \tag{4.11}$$

The condition (2b) in Lemma 4.7 holds. If $a_{n-1} = x_2$, we have $g''(x) < 0$ for $\forall x \in (a_{n-1}, +\infty)$. So the relation (4.10) remains valid, and so does (4.11). The condition (2a) holds and so the proof is complete. □

5. Proofs of theorems

In this section we give proofs of the four theorems stated in Section 1. First let us mention a well-known fact in the following lemma. For the completeness we provide a proof of it.

Lemma 5.1. *Assume that* $g \in C^2(-\infty, +\infty)$ *and satisfies*

$$xg(x) > 0, \quad \forall x \in (a^*, b^*) \setminus \{0\},\ -\infty \leqslant a^* < 0 < b^* \leqslant +\infty. \tag{5.1}$$

(i) *If* $b^* < +\infty$ *and* $g(b^*) = 0$*, then*

$$\lim_{b \to b^{*-}} \int_0^b \frac{dx}{\sqrt{G(b) - G(x)}} = +\infty.$$

(ii) *If* $a^* > -\infty$ *and* $g(a^*) = 0$*, then*

$$\lim_{a \to a^{*+}} \int_a^0 \frac{dx}{\sqrt{G(a) - G(x)}} = +\infty.$$

Proof. We only prove the conclusion (i). The proof of (ii) is completely similar. We first show

$$\int_0^{b^*} \frac{dx}{\sqrt{G(b^*) - G(x)}} = +\infty. \tag{5.2}$$

The reasoning is as follows. We expand $G(x)$ at $x = b^*$ by

$$G(x) = G(b^*) + G'(b^*)(x - b^*) + G''(\xi)(x - b^*)^2/2, \quad x \in [0, b^*],$$

where $\xi \in (0, b^*)$. Note that $G'(b^*) = g(b^*) = 0$. So we have

$$G(b^*) - G(x) = -G''(\xi)(x - b^*)^2/2 = -g'(\xi)(b^* - x)^2/2 \leqslant M(b^* - x)^2,$$

where $M > 0$ is taken such that $|g'(x)| \leqslant 2M$ for $\forall x \in [0, b^*]$. Hence

$$\int_0^{b^*} \frac{dx}{\sqrt{G(b^*) - G(x)}} \geqslant \int_0^{b^*} \frac{dx}{(b^* - x)\sqrt{M}} = +\infty.$$

So (5.2) holds. This means that for $\forall N > 0$, $\exists \delta > 0$ such that

$$\int_0^{b} \frac{dx}{\sqrt{G(b^*) - G(x)}} \geqslant N, \quad \forall b \in (b^* - \delta, b^*),$$

which leads to

$$\int_0^{b} \frac{dx}{\sqrt{G(b) - G(x)}} \geqslant \int_0^{b} \frac{dx}{\sqrt{G(b^*) - G(x)}} \geqslant N, \quad \forall b \in (b^* - \delta, b^*).$$

This is, the conclusion (i) holds. □

Proof of Theorem 1.1. We assume without loss of generality that the center locates at $(0, 0)$ and the condition (H) holds for some nonnegative integer $k \geqslant 0$. It follows from Lemma 4.5 that $(G/g^2)''(x) > 0$ on $(a^*, b^*) \setminus \{0\}$, and from (3.15) that $(\sqrt{c}T'(c))' > 0$ on $(0, c^*)$, where $c^* = \min\{G(a^*), G(b^*)\}$.

If $k = 0$, then it follows from (3.18) that $\lim_{c\to 0^+} T(c) = 2\pi/\sqrt{g'(0)}$ and

$$\lim_{c\to 0^+} T'(c) = \frac{\pi(5(g'')^2 - 3g'g''')}{12(g')^{7/2}}\bigg|_{x=0}. \tag{5.3}$$

It follows from $(g''/g')'(0) < 0$ that the limit above is positive. Consequently $T'(c) > 0$ on $(0, c^*)$. The condition $\deg g \geqslant 2$ implies that a^* or b^* is a zero of g, and so $c^* = \min\{G(a^*), G(b^*)\} < +\infty$. It follows from Lemma 5.1 that $\lim_{c\to c^*-} T(c) = +\infty$. So the first conclusion of Theorem 1.1 holds.

If $k \geqslant 1$ and $\deg g = 2k+1$, then $g(x) = \lambda x^{2k+1}$, where $\lambda > 0$. In this case we have an explicit expression $T(c) = \mu c^{-k/(2k+2)}$ on $(0, +\infty)$ from (3.11), where $\mu > 0$. So $T''(c) > 0$, $T'(c) < 0$ on $(0, +\infty)$.

Consider the last case that $k \geqslant 1$ and $\deg g > 2k + 1$. Since $k \geqslant 1$ we have $\lim_{c\to 0^+} T(c) = +\infty$, see (3.19). On the other hand, the condition $\deg g > 2k+1$ implies that a^* or b^* is a zero of g. It follows from Lemma 5.1 that $\lim_{c\to c^*-} T(c) = +\infty$. So $T(c)$ reaches its minimum at some point in $(0, c^*)$ which is a critical point of $T(c)$. This shows that $T(c)$ has at least one critical point on $(0, c^*)$. But $T(c)$ has at most one critical point because $(\sqrt{c}T'(c))' > 0$ on $(0, c^*)$. This means that $T(c)$ has exactly one critical point on $(0, c^*)$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let $g(x) = a_{2n+1}x^{2n+1} + a_{2n}x^{2n} + \cdots + a_1x + a_0$, where $L_g = a_{2n+1} > 0$. For convenience we write $\widehat{T}(A)$ as $T(A)$ and split up it into two parts:

$$
\begin{aligned}
T(A) &= \sqrt{2}\int_0^A \frac{dx}{\sqrt{G(A)-G(x)}} + \sqrt{2}\int_{-B}^0 \frac{dx}{\sqrt{G(-B)-G(x)}} \\
&= \sqrt{2}\int_0^1 \frac{A\,du}{\sqrt{G(A)-G(Au)}} + \sqrt{2}\int_0^1 \frac{B\,du}{\sqrt{G(-B)-G(-Bu)}} \\
&= \sqrt{2}\big(AI(A)+BI(-B)\big), \qquad I(C) := \int_0^1 \frac{du}{\sqrt{G(C)-G(Cu)}},
\end{aligned}
$$

where $G(x)$ is a primitive of g and can be taken as $G(x)=\int_0^x g(s)\,ds$. So we have

$$
\begin{aligned}
G(C)-G(Cu) &= \sum_{i=1}^{2n+1} \frac{a_i C^{i+1}(1-u^{i+1})}{i+1} \\
&= \frac{a_{2n+1}C^{2n+2}}{2n+2}\big(1-u^{2n+1}\big)\left(1+\sum_{i=1}^{2n} \frac{(2n+2)a_i}{(i+1)a_{2n+1}} e_i(u) C^{i-2n-1}\right),
\end{aligned}
$$

where

$$
e_i(u) = \frac{1-u^{i+1}}{1-u^{2n+1}}, \quad u\in[0,1],\ 1\leqslant i\leqslant 2n.
$$

It is easy to see that $0\leqslant e_i(u)\leqslant 1$ for $u\in[0,1]$ and for $1\leqslant i\leqslant 2n$. Let $M=\max\{|a_i/a_{2n+1}|, 1\leqslant i\leqslant 2n\}$. Then we obtain

$$
\begin{aligned}
\left|\sum_{i=1}^{2n} \frac{(2n+2)a_i}{(i+1)a_{2n+1}} e_i(u) C^{i-2n-1}\right| &< (2n+2)M\left(\frac{1}{|C|}+\frac{1}{|C|^2}+\cdots+\frac{1}{|C|^{2n}}\right) \\
&< \frac{(2n+2)M}{|C|-1}.
\end{aligned}
$$

Taking C_0 sufficiently large such that $(2n+2)M/(C_0-1)<1$, that is $C_0>(2n+2)M+1$, we have the following expansion

$$
\frac{1}{\sqrt{G(C)-G(Cu)}} = \sqrt{\frac{2n+2}{a_{2n+1}C^{2n+2}(1-u^{2n+2})}}\left(1+\frac{f_1(u)}{C}+\frac{f_2(u)}{C^2}+\cdots\right), \tag{5.4}
$$

where

$$
f_1(u) = \frac{-1}{2}\frac{2n+2}{2n+1}\frac{a_{2n}}{a_{2n+1}}\frac{1-u^{2n+1}}{1-u^{2n+2}}.
$$

We note that the series in (5.4) is uniformly convergent for $u\in[0,1]$ and for $|C|\geqslant C_0$. Therefore

$$
\begin{aligned}
I(C) &= \int_0^1 \sqrt{\frac{2n+2}{a_{2n+1}C^{2n+2}(1-u^{2n+2})}}\left(1+\frac{f_1(u)}{C}+\frac{f_2(u)}{C^2}+\cdots\right)du \\
&= \frac{\sqrt{2}\lambda_n}{|C|^{n+1}}\left(\alpha_0+\frac{\alpha_1}{C}+\cdots\right),
\end{aligned}
$$

where

$$\lambda_n=\sqrt{\frac{n+1}{a_{2n+1}}},\qquad \alpha_0=\int_0^1\frac{du}{\sqrt{1-u^{2n+2}}},\qquad \alpha_1=\int_0^1\frac{f_1(u)\,du}{\sqrt{1-u^{2n+2}}}.$$

Therefore

$$\begin{aligned}T(A)&=\sqrt{2}\big(AI(A)+BI(-B)\big)\\&=2\lambda_n\left(\left(\frac{1}{A^n}+\frac{1}{B^n}\right)\alpha_0+\left(\frac{1}{A^{n+1}}-\frac{1}{B^{n+1}}\right)\alpha_1+\cdots\right).\end{aligned}\tag{5.5}$$

The following lemma shows that the implicit function $B=B(A)$ determined by $G(A)=G(-B)$ has the expansion

$$B(A)=A\left(1+\frac{b_1}{A}+\frac{b_1}{A^2}+\cdots\right),\quad \forall A\gg 1.\tag{5.6}$$

Putting the expansion into (5.5), we obtain (1.3). Theorem 1.2 is proved. □

Lemma 5.2. *There exists $A_0>0$ such that the equation $G(A)=G(-B)$ has a unique solution*

$$B=B(A)=A\left(1+\frac{b_1}{A}+\frac{b_2}{A^2}+\cdots\right),\quad \forall A>A_0.\tag{5.7}$$

Proof. It follows from $G(A)=G(-B)$ that

$$\begin{aligned}&\frac{a_{2n+1}}{2n+2}\big(A^{2n+2}-B^{2n+2}\big)+\frac{a_{2n}}{2n+1}\big(A^{2n+1}+B^{2n+1}\big)+\cdots+\frac{a_1}{2}\big(A^2-B^2\big)\\&\quad+a_0(A+B)=0.\end{aligned}$$

Dividing by $1/A^{2n+2}$ the above equation and denoting $\tau=B/A$, $\varepsilon=1/A$, we have

$$\frac{a_{2n+1}}{2n+2}\big(1-\tau^{2n+2}\big)+\frac{a_{2n}\varepsilon}{2n+1}\big(1+\tau^{2n+1}\big)+\cdots+\frac{a_1\varepsilon^{2n}}{2}\big(1-\tau^2\big)+a_0\varepsilon^{2n+1}(1+\tau)=0.\tag{5.8}$$

Let $h(\tau,\varepsilon)$ denote the left-hand side of (5.8). Then we have $h(1,0)=0$, $h_\tau(1,0)=-a_{2n+1}\neq 0$. By the implicit function theorem we get a unique solution $\tau=\tau(\varepsilon)$ with $\tau(0)=1$ of the equation $h(\tau,\varepsilon)=0$ near $(1,0)$ and $|\varepsilon|<\varepsilon_0$. Since $\tau(\varepsilon)$ is smooth enough, we have $\tau(\varepsilon)=1+b_1\varepsilon+b_2\varepsilon^2+\cdots$, that is, the relation (5.7) holds for $|A|>1/\varepsilon_0$. □

Proof of Theorem 1.3. Let $a_1\leqslant a_2\leqslant\cdots\leqslant a_{2n+1}$ denote the zeros of g. Then the period function $T(c)$ of the period annulus surrounding all the equilibria $(a_i,0)$, $i=1,\ldots,2n+1$, is given by

$$T(c)=\sqrt{2}\int_{x_1}^{x_2}\frac{dx}{\sqrt{c-G(x)}},\quad c\in(c_0,+\infty),\tag{5.9}$$

where $G(x_1)=c=G(x_2)$, $x_1<a_1\leqslant a_{2n+1}<x_2$, $c_0=\max\{G(a_1),\ldots,G(a_{2n+1})\}$, and $G(x)=\int_{a_1}^x g(s)\,ds$. We will show that $T(c)$ is convex on $(c_0,+\infty)$, that is $T''(c)>0$ on $(c_0,+\infty)$. The convexity of $T(c)$ together with $\lim_{c\to+\infty} T(c)=0$ (due to Theorem 1.2) implies that $T'(c)<0$ on $(c_0,+\infty)$, and so Theorem 1.3 holds.

To prove $T''(c)>0$, we split up the integral (5.9) into three parts $T(c)=\sqrt{2}(I_1(c)+I_2(c)+I_3(c))$, where

$$I_1(c)=\int_{x_1}^{a_1}\frac{dx}{\sqrt{c-G(x)}},\qquad I_2(c)=\int_{a_1}^{a_{2n+1}}\frac{dx}{\sqrt{c-G(x)}},\qquad I_3(c)=\int_{a_{2n+1}}^{x_2}\frac{dx}{\sqrt{c-G(x)}}.$$

We will show that $I_i''(c)>0$ on $(c_0,+\infty)$, $i=1,2,3$. The conclusion $I_2''(c)>0$ is obvious. Since the proof of $I_3(c)>0$ is similar to that of $I_1''(c)>0$. We only prove $I_1''(c)>0$ in the following.

Let $g_1(u)=-g(a_1-u)$. Then $g_1(u)$ is also a polynomial with a positive leading coefficient and has $2n+1$ real zeros with the maximal zero $u=0$. Using a change $x=a_1-u$ of variable in $I_2(c)$ we obtain another form of it as follows

$$I_1(c)=\int_0^{u_1}\frac{du}{\sqrt{c-G_1(u)}},\qquad G_1(u)=\int_0^{u_1}g_1(s)\,ds,\qquad G_1(u_1)=c,\quad u_1=a_1-x_1>0.$$

It is easy to check that $g_1(x)$ satisfies all conditions of Lemma 4.6 with $b^*=+\infty$. It follows from (3.10) and Lemma 4.6 that $I_1''(c)>0$ on $(c_0,+\infty)$. □

Proof of Theorem 1.4. Just as in the proof of Theorem 1.3 we split up the integral $T(c)$ defined in (1.2) into three parts $T(c)=\sqrt{2}(I_1(c)+I_2(c)+I_3(c))$, where

$$I_1(c)=\int_{x_1}^{a_1}\frac{dx}{\sqrt{c-G(x)}},\qquad I_2(c)=\int_{a_1}^{a_{2n-1}}\frac{dx}{\sqrt{c-G(x)}},\qquad I_3(c)=\int_{a_{2n-1}}^{x_2}\frac{dx}{\sqrt{c-G(x)}},$$

and $x_1<a_1\leqslant a_{2n-1}<x_2<a_{2n}$, $c\in(c_0,c_1)$. We will prove that $I_i''(c)>0$ for $i=1,2,3$ and so $T''(c)>0$ on (c_0,c_1). It follows from Lemma 5.1 $\lim_{c\to c_0^+} T(c)=+\infty=\lim_{c\to c_1^-} T(c)$. Therefore $T(c)$ has at least one critical point on (c_0,c_1) where $T(c)$ reaches its minimum. Since $T''(c)>0$, $T(c)$ has at most one critical point. The conclusion is proved.

In the following let us prove $I_i''(c)>0$ on (c_0,c_1) for $i=1,2,3$. Obviously $I_2''(c)>0$ holds. The proof of $I_1''(c)>0$ here is completely the same as that in the proof of Theorem 1.3. It remains to prove $I_3''(c)>0$. Applying Lemma 4.8 we obtain $[(G/g^2)'/\sqrt{G}\,]'(x)>0$ on (a_{2n-1},a_{2n}). Let us recall the one-side integral $I(c)$ defined in (3.2) and its second derivative $I''(c)$ given in (3.10), and note the equivalency of the two conditions

$$\bigl[\bigl(G/g^2\bigr)'/\sqrt{G}\,\bigr]'(x)>0\quad\text{on }(a_{2n-1},a_{2n})$$

and

$$\bigl[\bigl(\widehat{G}/\widehat{g}^2\bigr)'/\sqrt{\widehat{G}}\,\bigr]'(u)>0\quad\text{on }(0,h),$$

where

$$\widehat{g}(u)=g(u+a_{2n-1}),\qquad \widehat{G}(u)=G(u+a_{2n-1})=\int_0^u\widehat{g}(s)\,ds,\qquad h=a_{2n}-a_{2n-1}.$$

Hence we obtain $I_2''(c)>0$ on (c_0,c_1). □

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