The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

# Transformations of the Sine and Cosine Functions

## Introduction

### Two Parent Sinusoidal Functions

The sine function and the cosine function are referred to as **sinusoidal curves**.

Sinusoidal curves have the property that they oscillate above and below a central horizontal line.

For both $y = \sin(x)$ and $y = \cos(x)$, this central horizontal line or axis is $y = 0$.

The equation of the horizontal axis is $y = \dfrac{\text{maximum value} + \text{minimum value}}{2}$.

The sinusoidal functions are cyclic. That is, the function values repeat over regular intervals of the domain.

We say that these functions are **periodic**. The horizontal length of one cycle is called the **period**.

The period of both $y = \sin(x)$ and $y = \cos(x)$ is $2\pi$ radians or $360°$.

The **amplitude** is the perpendicular distance from the horizontal axis to either a maximum or minimum point on the curve.

We can calculate the amplitude with the formula

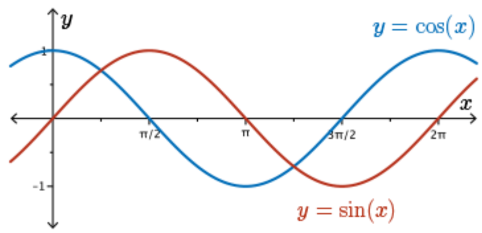
amplitude = $\dfrac{\text{maximum value} - \text{minimum value}}{2}$

For both functions, $y = \sin(x)$ and $y = \cos(x)$:

The domain is $\{x \mid x \in \mathbb{R}\}$ and

the range is $\{y \mid -1 \le y \le 1, y \in \mathbb{R}\}$.

We will refer to $y = \sin(x)$ and $y = \cos(x)$ as the parent sinusoidal functions.



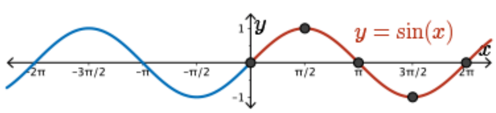

## Five Point Sketches

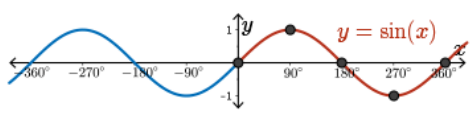
For $y = \sin(x)$, we identified five key points in one period starting on the $y$-axis.

In radian measure, these points are $(0, 0)$, $\left(\dfrac{\pi}{2}, 1\right)$, $(\pi, 0)$, $\left(\dfrac{3\pi}{2}, -1\right)$, and $(2\pi, 0)$.

In degrees, the five key points are $(0, 0)$, $(90°, 1)$, $(180°, 0)$, $(270°, -1)$, and $(360°, 0)$.

Once a five point sketch is complete, we can sketch as many periods as we require.



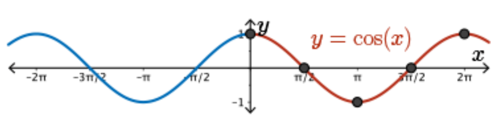



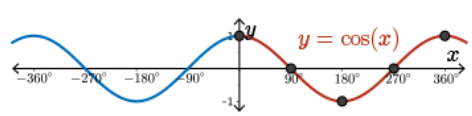

For $y = \cos(x)$, we identified five key points in one period starting on the $y$-axis.

In radian measure, these points are $(0, 1)$, $\left(\dfrac{\pi}{2}, 0\right)$, $(\pi, -1)$, $\left(\dfrac{3\pi}{2}, 0\right)$, and $(2\pi, 1)$.

In degrees, the five key points are $(0, 1)$, $(90°, 0)$, $(180°, -1)$, $(270°, 0)$, and $(360°, 1)$.

Once the five point sketch is complete, we can sketch as many periods as we require.



## Transformational Form

In an earlier module, we looked at transformations. Transformations on a function $y = f(x)$ can be identified when the function is written in the form $y = af[b(x - h)] + k$.

**The Sine Function**

$y = a\sin[b(x - h)] + k$

**The Cosine Function**

$y = a\cos[b(x - h)] + k$

We will review the role of the parameters $a, b, h$ and $k$ in transforming the sinusoidal functions.
But first, use the Maple Worksheet to look at the effect of changing the values of $a, b, h$ and $k$.

## Transformational Form

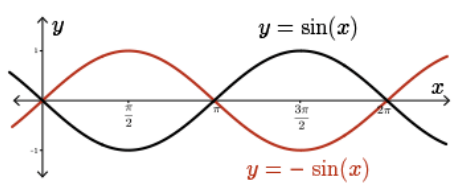
**The Sine Function**

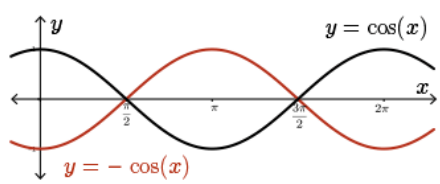
$y = a\sin[b(x - h)] + k$

**The Cosine Function**

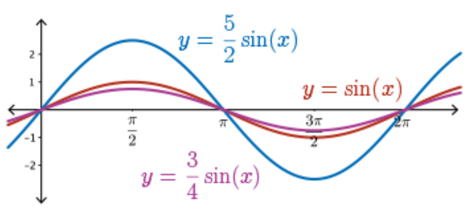
$y = a\cos[b(x - h)] + k$

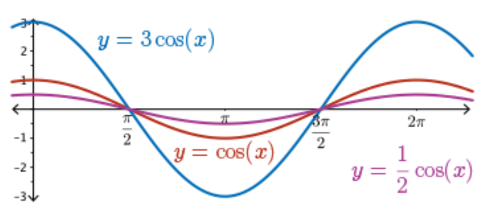
Let's review the role of the parameters $a$, $b$, $h$, and $k$ in transforming these functions(s).
Effect of $a$: If $a < 0$, the sinusoidal function is reflected in the $x$-axis.




The sinusoidal function is stretched vertically from the $x$-axis by a factor of $|a|$.




The amplitude of a sinusoidal function is affected by a vertical stretch.

## Transformational Form

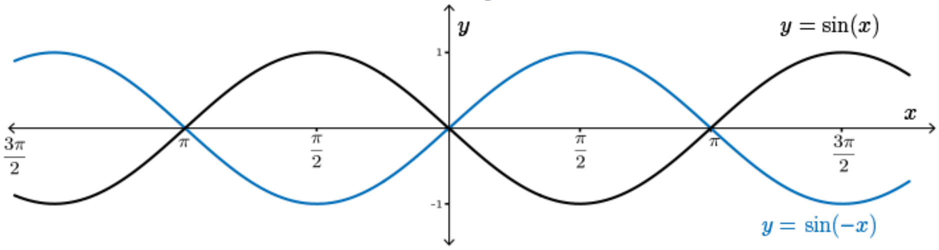
**The Sine Function**

$y = a\sin[b(x-h)]+k$

**The Cosine Function**

$y = a\cos[b(x-h)]+k$

Effect of $b$: If $b<0$, the sinusoidal function is reflected in the $y$-axis.




The graph of $y=\sin(-x)$ appears to be the same as the graph of $y=-\sin(x)$ from earlier in the module.

Since $y=\cos(x)$ is symmetric about the $y$-axis, a reflection in the $y$-axis has no effect. That is, $\cos(x)=\cos(-x)$.

## Transformational Form

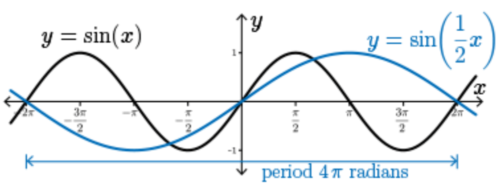
**The Sine Function**

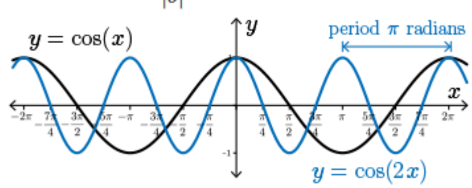
$y = a\sin[b(x-h)]+k$

**The Cosine Function**

$y = a\cos[b(x-h)]+k$

The sinusoidal function is stretched horizontally from the $y$-axis by a factor of $\dfrac{1}{|b|}$.




The period of a sinusoidal function is affected by a horizontal stretch and can be obtained by multiplying the original period by the horizontal stretch factor $\dfrac{1}{|b|}$.

The length of one period of the horizontally stretched function is shown on each graph.

In radians, the period is $\dfrac{1}{|b|}(2\pi)=\dfrac{2\pi}{|b|}$. In degrees, the period is $\dfrac{1}{|b|}(360^\circ)=\dfrac{360^\circ}{|b|}$.

## Transformational Form

**The Sine Function**

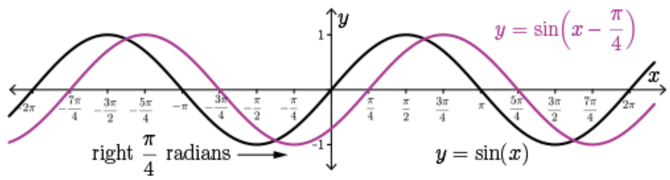
$y = a\sin[b(x-h)]+k$

**The Cosine Function**

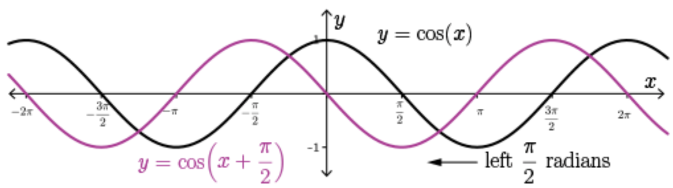
$y = a\cos[b(x-h)]+k$

Effect of $h$: The sinusoidal function is translated horizontally $h$ units.

If $h > 0$, the function moves to the right.




If $h < 0$, the function moves to the left.




A horizontal translation affects the $x$-coordinate of every point on a sinusoidal function.

The $y$-coordinates stay the same.

When sketching sinusoidal functions, the horizontal translation is called the **phase shift**.

## Transformational Form

**The Sine Function**

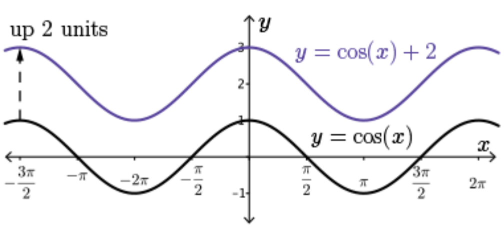
$y = a\sin[b(x-h)]+k$

**The Cosine Function**

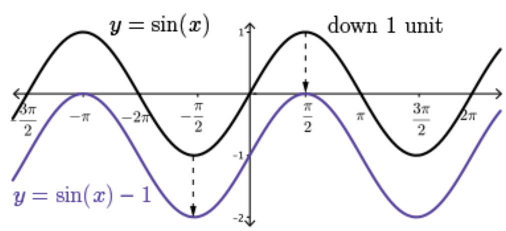
$y = a\cos[b(x-h)]+k$

Effect of $k$: The sinusoidal function is translated vertically $k$ units.

If $k > 0$, the functions moves up.




If $k < 0$, the function moves down.




A vertical translation affects the $y$-coordinate of every point on a sinusoidal function.

The $x$-coordinates stay the same.

The central horizontal axis is translated up or down depending on the value of $k$.

This vertical movement is often referred to as **vertical displacement**.

## Transformational Form

**The Sine Function**

$y = a\sin[b(x-h)] + k$

**The Cosine Function**

$y = a\cos[b(x-h)] + k$

Recall the general form of a transformed function: $y = af[b(x-h)] + k$

**Mapping Notation**

As discussed in earlier modules, the image of each point after the transformation is applied is defined by the mapping:

$$(x, y) \to \left(\frac{1}{b}x + h, ay + k\right)$$

There are several ways in which we could sketch sinusoidal functions.

No matter which method of sketching we use, we should first identify the amplitude, the period, the phase shift, and the vertical displacement.

Then, apply the reflections, stretches and translations to sketch the image.

Or, using our knowledge of the general parent function, we could determine the location of five key points in one period to sketch the graph.

Or, apply the general mapping to each of five key points in one period of the parent function to find their respective images in order to sketch the curve.

## Examples

### Example 1

Sketch two periods of the function $y = 4\sin\left[3\left(x - \frac{\pi}{3}\right)\right] + 1$.

**Solution**

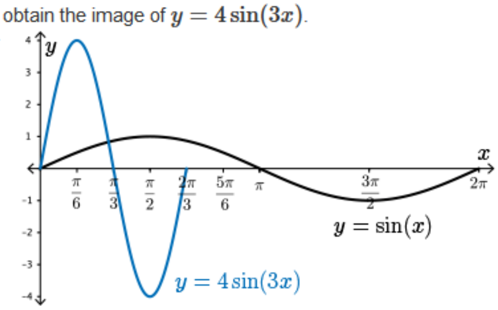
Identify the transformations applied to the parent function, $y = \sin(x)$, to obtain $y = 4\sin\left[3\left(x - \frac{\pi}{3}\right)\right] + 1$.

Since $a = 4$, there is a vertical stretch about the $x$-axis by a factor of $4$. It follows that the amplitude of the image is $4$.

Since $b = 3$, there is a horizontal stretch about the $y$-axis by a factor of $\frac{1}{3}$.

It follows that the period of the image is $\frac{2\pi}{|b|} = \frac{2\pi}{3}$. No reflections have been applied. We will sketch one period of $y = \sin(x)$ and apply the horizontal and vertical stretches to obtain the image of $y = 4\sin(3x)$.

Notice that three of the five key points in one period of each of the five-point sketches lie on the horizontal axis: one at the beginning, one at the half-period mark, and one at the end. A maximum point occurs on the curve at the halfway point between the first two points on the horizontal axis and a minimum occurs on the curve at the halfway point between the next two points on the horizontal axis.



## Examples

### Example 1

Sketch two periods of the function $y = 4\sin\left[3\left(x - \frac{\pi}{3}\right)\right] + 1$.

**Solution**

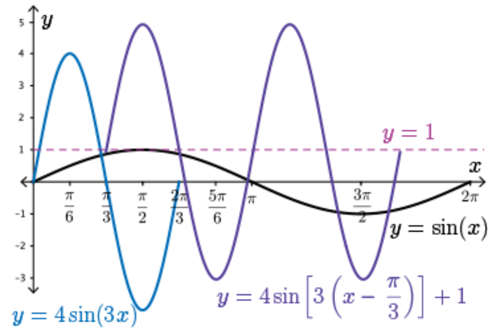
Now we identify $h = \frac{\pi}{3}$ and $k = 1$.

There is a phase shift of $\frac{\pi}{3}$ radians to the right and a vertical displacement of $1$ unit up.

Each of the $x$-coordinates of the five key points will increase by $\frac{\pi}{3}$ and each of the $y$-coordinates of the respective points will increase by $1$. Note, that the horizontal axis of the function moves up $1$ unit to $y = 1$.

At this point, we can sketch as many periods of the image function as we like.

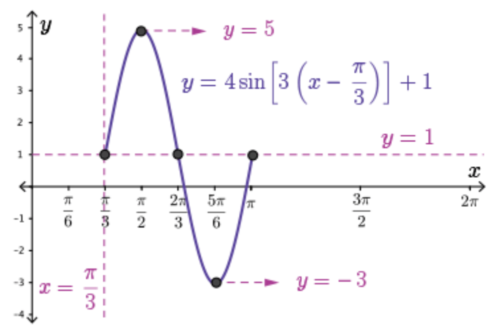
The required graph is shown. We will use this result to show a more efficient method of getting to the final image.




## Examples

### Example 1 - A Second Look

Let's look at the final sketch of one period of the image.
From the equation, we know that $a = 4, b = 3, h = \frac{\pi}{3}$, and $k = 1$.
The vertical displacement, $k = 1$, allows us to find the equation of the horizontal axis, $y = 1$.
The amplitude is $|a| = 4$. The maximum $y$-value is $k + |a| = 5$ and the minimum $y$-value is $k - |a| = -3$.
Since $h = \frac{\pi}{3}$, the phase shift is $\frac{\pi}{3}$, and the $x$-coordinate of the first point on the five-point sketch is $x = \frac{\pi}{3}$.




Since $b = 3$, the period is $\frac{2\pi}{|b|} = \frac{2\pi}{3}$ and the $x$-coordinate of the last point on the five-point sketch is

$x = \frac{\pi}{3} + \frac{2\pi}{3} = \pi$.

## Examples

### Example 1 - A Second Look

The $2^{\text{nd}}$ point on the five-point sketch has $x$-coordinate

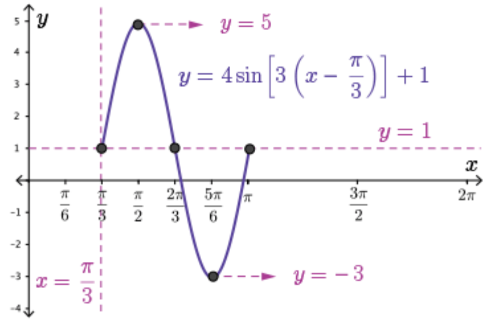
$$\frac{\pi}{3}+\frac{1}{4}\times(\text{period})=\frac{\pi}{3}+\frac{1}{4}\left(\frac{2\pi}{3}\right)=\frac{2\pi}{2}.$$

The $3^{\text{rd}}$ point on the five-point sketch has $x$-coordinate

$$\frac{\pi}{3}+\frac{1}{2}\times(\text{period})=\frac{\pi}{3}+\frac{1}{2}\left(\frac{2\pi}{3}\right)=\frac{2\pi}{3}.$$

The $4^{\text{th}}$ point on the five-point sketch has $x$-coordinate

$$\frac{\pi}{3}+\frac{3}{4}\times(\text{period})=\frac{\pi}{3}+\frac{3}{4}\left(\frac{2\pi}{3}\right)=\frac{5\pi}{6}.$$




### Example 2

Sketch one period of the function $y=3\cos\left[\frac{1}{2}\left(x+\frac{\pi}{4}\right)\right]-4$.

## Examples

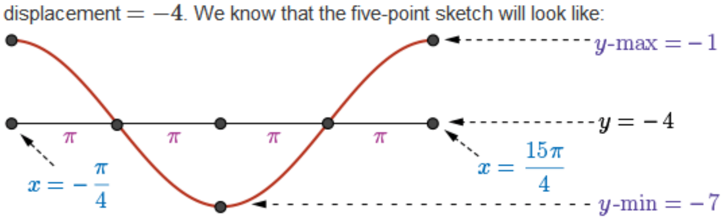
### Example 2

Sketch one period of the function $y=3\cos\left[\frac{1}{2}\left(x+\frac{\pi}{4}\right)\right]-4$.

**Solution**

From the equation, we know that $a=3, b=\frac{1}{2}, h=-\frac{\pi}{4}$, and $k=-4$. There are no reflections.

We can list the attributes of the image: amplitude $=3$, period $=\frac{2\pi}{\frac{1}{2}}=4\pi$, phase shift $=-\frac{\pi}{4}$, and vertical displacement $=-4$. We know that the five-point sketch will look like:




The minimum and maximum values of $y$ are $-7$ and $-1$, respectively.

In the five-point sketch the $x$-coordinates of the first and last points are $-\frac{\pi}{4}$ and $-\frac{\pi}{4}+4\pi=\frac{15\pi}{4}$.

Horizontal distance between each of the points on the sketch is $4\pi\div4=\pi$. Five points on the sketch are

$$A\left(-\frac{\pi}{4},-1\right), B\left(-\frac{3\pi}{4},-4\right), C\left(-\frac{7\pi}{4},-7\right), D\left(-\frac{11\pi}{4},-4\right), \text{ and } E\left(-\frac{15\pi}{4},-1\right)$$

## Examples

### Example 2

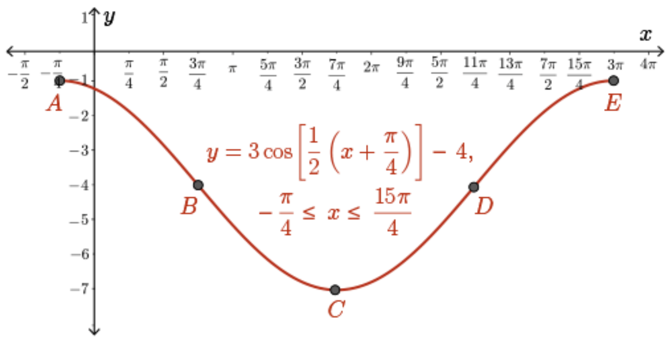
Sketch one period of the function $y = 3\cos\left[\frac{1}{2}\left(x + \frac{\pi}{4}\right)\right] - 4$.

**Solution**

Five points on the sketch are

$$A\left(-\frac{\pi}{4}, -1\right), B\left(-\frac{3\pi}{4}, -4\right), C\left(-\frac{7\pi}{4}, -7\right), D\left(-\frac{11\pi}{4}, -4\right), \text{ and } E\left(-\frac{15\pi}{4}, -1\right)$$

The final sketch is shown. Once we have one period, we can sketch as many periods as we like.




## Examples

### Example 3

Sketch two periods of the function $y = -5\cos\left(2x - \frac{\pi}{3}\right) + 4$.

**Solution**

The equation must be written in the form $y = a\cos[b(x - h)] + k$.

Rewriting the equation, we obtain $y = -5\cos\left[2\left(x - \frac{\pi}{6}\right)\right] + 4$.

From the equation, we see that $a = -5, b = 2, h = \frac{\pi}{6}$, and $k = 4$.

There is a reflection about the $x$-axis since $a < 0$.

We can list attributes of the image: amplitude $= 5$, period $= \frac{2\pi}{2} = \pi$, phase shift $= \frac{\pi}{6}$, and vertical displacement $= 4$.

## Examples

### Example 3

Sketch two periods of the function $y = -5\cos\left(2x - \frac{\pi}{3}\right) + 4$.

**Solution**

We know that the five-point sketch will look like:

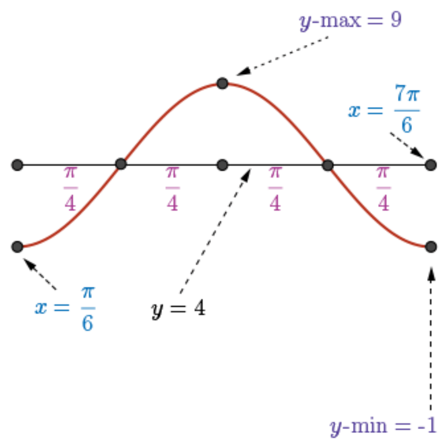
The minimum and maximum values of $y$ are $-1$ and $9$, respectively.

In the five-point sketch the $x$-coordinates of the first and last points are $\frac{\pi}{6}$ and $\frac{\pi}{6} + \pi = \frac{7\pi}{6}$.

The horizontal distance between each of the points on the sketch is $\pi \div 4 = \frac{\pi}{4}$.

Five points on the sketch are

$$A\left(\frac{\pi}{6}, -1\right), B\left(\frac{5\pi}{12}, 4\right), C\left(\frac{2\pi}{3}, 9\right),$$
$$D\left(\frac{11\pi}{12}, 4\right), \text{ and } E\left(\frac{7\pi}{6}, -1\right)$$





## Examples

### Example 3

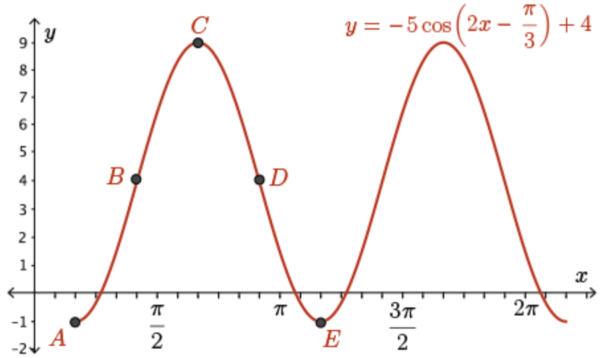
Sketch two periods of the function $y = -5\cos\left(2x - \frac{\pi}{3}\right) + 4$.

**Solution**

Five points on the sketch are

$$A\left(\frac{\pi}{6}, -1\right), B\left(\frac{5\pi}{12}, 4\right), C\left(\frac{2\pi}{3}, 9\right), D\left(\frac{11\pi}{12}, 4\right), \text{ and } E\left(\frac{7\pi}{6}, -1\right)$$


The final sketch is shown. The horizontal scale is $\frac{\pi}{12}$ radians.



## Summary

In this module, we have developed approaches for sketching transformations of the sinusoidal functions $y = \sin(x)$ and $y = \cos(x)$.

In upcoming modules, we will look at applications which can be modelled with sinusoidal functions.

The ability to efficiently and accurately sketch is a crucial component in examining various applications.