The Relation Between Acceleration and Time Dilation in Special Relativity

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Let two clocks originate at a single point in flat space-time with relative velocity v_0 between the clocks. Let the first clock follow an inertial path (no acceleration.) Denote time on the inertial clock by the variable s, with $s = 0$ at the original point. Let the second clock follow a general path (allowing accelerations.) Denote time on the second clock by the variable t, with $t = 0$ at the original point where/when the clocks are together. At any time t let the function $v(t)$ represent the relative velocity between the two clocks at time t. So $v(0) = v_0$.

As the clocks progress through space-time, any standard textbook on Special Relativity tells you that the relation between the variables s and t at any given instant is

$$\frac{ds}{dt} = \left(1 - \left(\frac{v(t)}{c}\right)^2\right)^{-\frac{1}{2}}$$

This makes s, the elapsed time on the inertial path, a function $s(t)$ of the elapsed time t on the general path followed by the second clock. For any elapsed time T on the second path let $S = s(T)$ be the corresponding elapsed time on the inertial path. Integrating both sides of the textbook relationship,

$$S = \int_0^T \left(1 - \left(\frac{v(t)}{c}\right)^2\right)^{-\frac{1}{2}} dt$$

Use the binomial expansion to expand the the integrand:

$$\begin{aligned} S &= \int_0^T \left(1 + \frac{1}{2}\left(\frac{v(t)}{c}\right)^2 + \frac{3}{8}\left(\frac{v(t)}{c}\right)^4 + \frac{5}{16}\left(\frac{v(t)}{c}\right)^6 + \ ...\right) dt \\ &= T + \int_0^T \left(\frac{1}{2}\left(\frac{v(t)}{c}\right)^2 + \frac{3}{8}\left(\frac{v(t)}{c}\right)^4 + \frac{5}{16}\left(\frac{v(t)}{c}\right)^6 + \ ...\right) dt \\ &= T + \int_0^T \frac{1}{2c} v(t) \left(\left(\frac{v(t)}{c}\right) + \frac{3}{4}\left(\frac{v(t)}{c}\right)^3 + \frac{5}{8}\left(\frac{v(t)}{c}\right)^5 + \ ...\right) dt \end{aligned}$$

Now integrate by parts:

$$\begin{aligned} S &= T+\frac{1}{2c}\left(\int_0^T v\left(t\right)dt\right)\left(\left(\frac{v\left(T\right)}{c}\right)+\frac{3}{4}\left(\frac{v\left(T\right)}{c}\right)^3+\frac{5}{8}\left(\frac{v\left(T\right)}{c}\right)^5+\ \ldots\right) \\ &\quad -\frac{1}{2c^2}\int_0^T\left(\int_0^t v\left(r\right)dr\right)\frac{dv\left(t\right)}{dt}\left(1+\frac{9}{4}\left(\frac{v\left(t\right)}{c}\right)^2+\frac{25}{8}\left(\frac{v\left(t\right)}{c}\right)^4+\ \ldots\right)dt \\ &= T+\frac{1}{2c^2}x\left(T\right)v\left(T\right)\left(1+\frac{3}{4}\left(\frac{v\left(T\right)}{c}\right)^2+\frac{5}{8}\left(\frac{v\left(T\right)}{c}\right)^4+\ \ldots\right) \\ &\quad -\frac{1}{2c^2}\int_0^T x\left(t\right)a(t)\left(1+\frac{9}{4}\left(\frac{v\left(t\right)}{c}\right)^2+\frac{25}{8}\left(\frac{v\left(t\right)}{c}\right)^4+\ \ldots\right)dt \end{aligned} \qquad \text{Equation A}$$

where

$$x(t)=\int_0^t v\left(r\right)dr$$

can be interpretted to be the spacelike distance from the position of the second clock at time t to the inertial path of the first clock, measured on a perpendicular in the reference frame of another inertial path with velocity $\frac{1}{t}\int_0^t v\left(r\right)dr$ relative to the inertial path of the first clock;

and where

$$a(t)=\frac{dv\left(t\right)}{dt}$$

is, naturally, the acceleration undergone by the second clock at time t. This acceleration is not dependent on any inertial reference frame, it is a physical invariant, proportional to the actual physical force undergone by the second clock at time t, detectible entirely locally by an accelerometer.

One can use Equation A to model a lot of different time dilation situations in flat space-time, and to derive a lot of interesting ways to look at them. It includes, for example, the usual Special Relativity time dilation expression because in that case $v\left(t\right)=v_0$, a constant for all t, meaning $a\left(t\right)=0$ for all t and $x(t)=tv_0$ for all t. With these values, Equation A becomes after a little algebra the binomial expansion for

$$S=T\left(1-\left(\frac{v_0}{c}\right)^2\right)^{-\frac{1}{2}}$$

which is the textbook Special Relativity value.

For this discussion, we now focus on the case when the first (inertial) clock meets the second clock again at time T on the second clock. This is a special case of the Twin Paradox, namely the case when the first twin maintains an inertial path, undergoing no acceleration. In this case, $x\left(T\right)=0$ because the clocks have met back up again. Now Equation A reads

$$S=T-\frac{1}{2c^2}\int_0^T x\left(t\right)a(t)\left(1+\frac{9}{4}\left(\frac{v\left(t\right)}{c}\right)^2+\frac{25}{8}\left(\frac{v\left(t\right)}{c}\right)^4+\ \ldots\right)dt. \qquad \text{Twin Equation}$$

This gives two different insights into this special case of the Twin Paradox, each of which embodies the role of the accelerations undergone by the second twin.

First, notice that the age S registered on the first (inertial) clock when they meet up differs from age T registered on the second clock by an expression that is simply the accumulation over all points t on the second clock's journey of the acceleration $a(t)$ undergone by the second clock at time t, multiplied by the space-like distance $x(t)$ from the second clock at time t to the path of first clock (measured along a well-defined perpendicular as described above), multiplied by a relativistic correction factor $\left(1+\frac{9}{4}\left(\frac{v(t)}{c}\right)^2+\frac{25}{8}\left(\frac{v(t)}{c}\right)^4+\ ...\right)$ that will be negligble unless the velocity gets to be a respectable fraction of the speed of light.

The first two pieces, $x(t)\,a(t)$, could be called "the moment of acceleration of the second clock with respect to the first clock" where the word "moment" here has nothing to do with time, but is used in the way the phrase "moment of inertia" is used in classical physics. It is an exertion (in this case the acceleration) multiplied by the (spatial) distance from the point where the exertion occurs (in this case the second clock) to a point where the exertion is having an effect (in this case, the difference in the rate of time passage on the first clock compared to the rate of time passage on the second clock.)

In this reading, the time dilation observed upon the two clocks meeting up again is an accumulation, over all points t on the second clock's journey, of the second clock's moment of acceleration relative to the inertial path of the first clock, with a relativistic adjustment governed by the relation of the velocities to the speed of light at each point along the way. In particular, the moments of acceleration undergone by the second clock, adjusted for relativity to the speed of light, completely determine the observed time dilation.

For a second way to look at the same equation

$$S=T-\frac{1}{2c^2}\int_0^T x(t)\,a(t)\left(1+\frac{9}{4}\left(\frac{v(t)}{c}\right)^2+\frac{25}{8}\left(\frac{v(t)}{c}\right)^4+\ ...\right)dt. \qquad \text{Twin Equation}$$

notice that at each time t we could insert $v(t)=v_0+\int_0^t a(r)\,dr$ because $a(t)=\frac{dv(t)}{dt}$. In other words, every occurance of $v(t)$ in the time dilation formula is just the initial relative velocity plus an accumulation of the accelerations undergone by the second clock up to time t.

At each time t we can also insert $x(t)=\int_0^t v(r)\,dr$ because $v(t)=\frac{dx(t)}{dt}$; which extends to $x(t)=tv_0+\int_0^t\int_0^r a(p)\,dpdr$. In other words, every occurance of $x(t)$ in the time dilation formula is just t times the initial relative velocity plus an accumulation of the accumulations of the accelerations undergone by the second clock up to time t.

This is a true physical effect, not just mathematics. The second twin needs only a clock, an accelerometer to record accelerations, and a recording device that can accumulate records of acceleration and accumulate the accumulations in order to know the rate of time passing on the first clock, without ever observing the first clock again after noting their relative velocity at the point of departing from each other.

The entire time dilation, even the relativistic correction portion, is determined by the intitial relative velocity plus accumulations of, and accumulations of accumulations of, the accelerations undergone by the second clock. There's nothing else there in the physical experience except the second clock's local time, local accelerations and initial velocity.

This explains the Twin Paradox in the special case when the first twin follows an inertial path, one with no accelerations. However, this solution easilty extends to the general Twin Paradox in flat space-time.

Consider two twins,two clocks, both following general time-like paths through flat space-time, each one allowed to undergo accelerations, but the two start out together, with some relative velocity, and then come together again in the future with some relative velocity.

Since they start together and come back together again, there is a unique inertial path from their common starting point to their common meeting point in the future. (Uniqueness follows from the assumption of flat space-time.) Now just apply the Twin Equation above two times: first get the time dilation between the first twin and the unique inertial path, then get the time dilation between the second twin and the unique inertial path. The time dilation between the two twins will be the difference between their respective time dilations relative to the unique inertial path. Since accelerations are real physical phenomena, not relative, and since the initial relative velocity of the two twins will be the difference between their respecting initial velocities relative to the unique inertial path we can conclude:

The entire time dilation between the twins, even the relativistic correction portion, is determined by their intitial relative velocity plus accumulations of, and accumulations of accumulations of, the accelerations undergone by the two twins. There's nothing else there except accelerations and initial relative velocity.

Some might prefer to think about this in terms of differences between their accumulated moments of acceleration (with relativistic corrections) with respect to the unique inertial path.