Velocity, Acceleration and Curvature

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Introduction

Most of the definitions of velocity and acceleration from functions of one variable carry over to vectors without change except for notation. The interesting part comes when we introduce the ideas of unit tangents, normals and curvature, along with parametrization by arc length.

In all of the following, we will assume that we are dealing with a curve $\mathsf{x} = \mathsf{x}(t)$ in R^n. When we are interested in velocity and acceleration, we will assume that a particle is moving along the curve and $\mathsf{x}(t)$ represents its position at time t.

Definitions

De nition 1 (Velocity)

$$\mathsf{v}(t) = \mathsf{x}'(t).$$

De nition 2 (Acceleration)

$$\mathsf{a}(t) = \mathsf{v}'(t).$$

These definitions are direct translations of the corresponding definitions for ordinary functions.

For ordinary functions, you may recall that the derivative may be used to obtain the slope of a line tangent to the graph of the function. Thus, it is natural to expect that, when dealing with vector functions, the derivative will give a vector whose direction is tangent to the graph of the function. However, since the same curve may have different parametrizations, each of which will yield a different derivative at a given point, we don't get the same uniqueness that we get for ordinary functions. To cope with that problem, we *normalize* the tangent by dividing by its length to get what's called the *unit tangent*.

De nition 3 (Unit Tangent)

$$\mathsf{T} = \frac{\mathsf{x}'(t)}{|\mathsf{x}'(t)|}.$$

Since T has unit length, it is orthogonal to its derivative and we may say that its derivative it orthogonal to the curve. If we *normalize* it, we get what's called the *unit normal*.

De nition 4 (Unit Normal)

$$\mathsf{N} = \frac{\mathsf{T}'(t)}{|\mathsf{T}'(t)|}.$$

Since velocity is a vector whose magnitude is speed and whose direction is tangent to the path of the particle, we see that we may write $\mathsf{v} = (speed) \cdot \mathsf{T}$. We introduce the following notation for speed.

Notation 5 (Speed) *We denote the speed of a particle by* $\frac{ds}{dt} = |\mathsf{v}|$.

With that notation, we may write

$$\mathsf{v} = \frac{ds}{dt} \cdot \mathsf{T}. \tag{1}$$

Using this formulation, we can get another way of looking at acceleration. If we take (1) and calculate the derivative, using the product rule and the definition of the unit normal, we obtain the following.

$$\mathsf{a} = \frac{d^2 s}{dt^2}\mathsf{T} + \frac{ds}{dt} \cdot |\mathsf{T}'| \cdot \mathsf{N}. \tag{2}$$

This formula breaks down the acceleration into two components, one tangent to the direction of motion and one normal to the direction of motion. The component tangent to the direction of motion is simply the derivative of speed; in other words, it is simply the rate at which the speed is changing. That should seem eminently reasonable. The other component is a bit more complicated; we will need to introduce the concepts of curvature and parametrization by arc length before analyzing it further.

Curvature and Arc Length

Suppose a particle starts traveling at a time t_0 along a path $\mathsf{x}(t)$ at a speed $|\mathsf{x}'(t)|$. Then, at time t, it will have travelled a distance

$$s = \int_{t_0}^{t} |\mathsf{x}'(u)|\, du.$$

Note that, except for notation, this is exactly the same formula used in single variable calculus to calculate the arc length of a curve. Also, note that, by the second part of the fundamental theorem of calculus, $s' = |\mathsf{x}'(t)|$, which justifies our notation $\frac{ds}{dt}$ for speed.

Note that if we assume, as we now will, that $\mathsf{x}'(t) \neq \mathsf{0}$, then it follows that s is a one-to-one function of t, thus invertible, and we may therefore also consider t to be a function of s. To make that explicit, we will often write $t = t(s)$. If we define a new vector function $\mathsf{y}(s)$ by $\mathsf{y}(s) = \mathsf{x}(t(s))$, then the graph of $\mathsf{y}(s)$ is clearly the same as the graph of $\mathsf{x}(t)$. In other words, we have another parametrization of the same graph. Since the new parameter is s, which represents arc length, we call it a parametrization of the curve by arc length.

Certain *nice* things happen when we use a parametrization by arc length. For example, consider formula (1) for velocity. Changing the notation slightly, we may write $\frac{d\mathsf{x}}{dt} = \frac{ds}{dt} \cdot \mathsf{T}$ or

$$\mathsf{T} = \frac{\frac{d\mathsf{x}}{dt}}{\frac{ds}{dt}} = \frac{d\mathsf{x}}{dt} \cdot derts = \frac{d\mathsf{y}}{ds}$$

In other words, when we have a parametrization by arc length, we don't have to normalize in order to get the unit tangent.

The parametrization, because it is unique, gives us an unambiguous way of defining another property, curvature. If we consider a curve and its unit tangent, we observe that, although the length of the tangent does not change, its direction does. Therefore, the magnitude of its derivative gives us a measure of how fast the curve is *curving*. Although its derivative will quite naturally depend on the parametrization, we can eliminate any ambiguity by defining *curvature* in terms of the parametrization by arc length.

De nition 6 (Curvature)

$$K = |\frac{d\mathsf{T}}{ds}|.$$

The question immediately arises about what to do when we have a different parametrization. However, since the chain rule implies that $\frac{d\mathsf{T}}{ds} = \frac{d\mathsf{T}}{dt} \cdot \frac{dt}{ds} = \frac{\frac{d\mathsf{T}}{dt}}{\frac{ds}{dt}}$, we immediately get the formula

$$K = \frac{|\frac{d\mathsf{T}}{dt}|}{\frac{ds}{dt}} = \frac{|\mathsf{T}'|}{|\mathsf{x}'|} \tag{3}$$

Another way of looking at (3) is $|\mathsf{T}'| = K\frac{ds}{dt}$ and, if we plug that into the formula (2) for acceleration we obtain a more meaningful formula

$$\mathsf{a} = \left(\frac{d^2s}{ds^2}\right) \cdot \mathsf{T} + K \cdot \left(\frac{ds}{dt}\right)^2 \mathsf{N}. \tag{4}$$

The improvement here is that we can see, in a meaningful way, the normal component of acceleration. It has two factors. One factor is curvature, while the other is the square of the speed. For anyone who has studied some physics, both these factors should seem to make sense.

In practice, since it is relatively easy to calculate a directly and, as a byproduct, it is also easy to find T, we can usually find the tangential and normal components to the acceleration vector without resorting to formula (4).

For convenience, let us use the following notations.

Notation 7 (Tangential Component of Acceleration)

$$\mathsf{a}_T = a_T\mathsf{T} = \frac{d^2s}{dt^2}\mathsf{T} \tag{5}$$

Notation 8 (Normal Component of Acceleration)

$$\mathsf{a}_N = a_N\mathsf{N} = K\left(\frac{ds}{dt}\right)^2 \mathsf{N} \tag{6}$$

Using the fact that the tangential and normal vectors are orthogonal, we easily see that $\mathsf{a} \cdot \mathsf{T} = \mathsf{a}_T$, which gives us an easy way of getting the tangential component of acceleration. We can then use the fact that $\mathsf{a}_N = \mathsf{a} - \mathsf{a}_T$ to get the normal component. Once we have the normal component, we can normalize it to find N and use its length to find curvature, since $K = \frac{|\mathsf{a}_N|}{\left(\frac{ds}{dt}\right)^2}$.

An Example

Let's consider the function $\mathsf{x} = (\cos t, \sin t, t^2)$. We will calculate all the relevant quantities mentioned above, both in general and at the specific point $t = 0$.

Follow the calculations carefully and keep your eyes open and your pencils sharp. There are some errors in the calculations below. See if you can find them.

$$\begin{aligned}
\mathsf{v} &= \mathsf{x}' = (-\sin t, \cos t, 2t) \\
\frac{ds}{dt} &= |\mathsf{v}| = \sqrt{1+4t^2} \\
\mathsf{T} &= \mathsf{v}/|\mathsf{v}| = \frac{(-\sin t, \cos t, 2t)}{\sqrt{1+4t^2}} \\
\mathsf{a} &= \mathsf{v}' = (-\cos t, -\sin t, 2)
\end{aligned}$$

Since the normal vector is difficult to get directly (because T will be messy to differentiate), we approach it indirectly by first calculating the tangential and normal components of a.

$$
\begin{aligned}
\mathsf{a}_T &= (\mathsf{a}\cdot\mathsf{T})\mathsf{T} = \left((-\cos t, -\sin t, 2)\cdot\frac{(-\sin t,\cos t, 2t)}{\sqrt{1+4t^2}}\right)\frac{(-\sin t,\cos t, 2t)}{\sqrt{1+4t^2}} \\
&= \frac{4t}{1+4t^2}(-\sin t, \cos t, 2t) \\
|a_{\mathbf{T}}| &= \mathsf{a}_{\mathbf{T}} = \frac{4t}{\sqrt{1+4t^2}}
\end{aligned}
$$

Based on a_T, we can now find a_N.

$$
\begin{aligned}
\mathsf{a}_N &= \mathsf{a} - \mathsf{a}_T \\
&= \left(\frac{4t\sin t}{1+4t^2} - \cos t, \frac{-4t\cos t}{1+4t^2} - \sin t, \frac{-8t^2}{1+4t^2} + 2\right) \\
|\mathsf{a}_N| &= \frac{\sqrt{272t^4+88t^2+5}}{1+4t^2} \\
\mathsf{N} &= \frac{\mathsf{a}_{\mathbf{N}}}{|\mathsf{a}_{\mathbf{N}}|} \\
&= \frac{\left(4t\sin t - (1+4t^2)\cos t, -4t\cos t - (1+4t^2)\sin t, -2-16t^2\right)}{\sqrt{272t^4+88t^2+5}} \\
K &= \frac{\mathsf{a}_N}{\left(\frac{ds}{dt}\right)^2} \\
&= \frac{\sqrt{272t^4+88t^2+5}}{(1+4t^2)^2}
\end{aligned}
$$

If we plug in the value $t = 0$, we obtain

$$
\begin{aligned}
\mathsf{x} &= (1,0,1) \\
\mathsf{v} &= (0,1,0) \\
\frac{ds}{dt} &= 1 \\
\mathsf{a} &= (-1,0,2) \\
\mathsf{a}_T &= \mathsf{0} \\
\mathsf{a}_N &= (-1,0,2) \\
|\mathsf{a}_N| &= \sqrt{5} \\
\mathsf{N} &= (-1,0,2)/\sqrt{5} \\
K &= \sqrt{5}
\end{aligned}
$$