Solving the Simple Harmonic Oscillator

1. The harmonic oscillator solution: displacement as a function of time

We wish to solve the equation of motion for the simple harmonic oscillator:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x\,, \tag{1}$$

where k is the spring constant and m is the mass of the oscillating body that is attached to the spring. We impose the following initial conditions on the problem. At $t = 0$, the initial displacement is denoted by x_0 and the corresponding velocity is denoted by v_0. That is,

$$x(t=0) \equiv x_0\,, \qquad \text{and} \quad \frac{dx}{dt}(t=0) = v_0\,. \tag{2}$$

These initial conditions then uniquely specify the problem.

The method we shall employ for solving this differential equation is called the method of inspired guessing. In class, we argued that the motion of the oscillating body was periodic. Since the sine and cosine functions are periodic, we propose the following solution for the displacement x as a function of the time t:

$$x(t) = x_0 \cos\omega t + b \sin\omega t\,, \tag{3}$$

where b and ω are to be determined. If we set $t = 0$, we find $x(t = 0) = x_0$ as required, so one of the two initial conditions is automatically satisfied. The initial condition for the velocity will determine b. That is, take the derivative of eq. (3) with respect to t. This yields:

$$v(t) \equiv \frac{dx}{dt} = -\omega x_0 \sin\omega t + \omega b \cos\omega t\,. \tag{4}$$

Setting $t = 0$ in eq. (4) then yields $v_0 = \omega b$; hence

$$b = \frac{v_0}{\omega}\,. \tag{5}$$

Thus, our proposed solution now has the following form:

$$\boxed{x(t) = x_0 \cos\omega t + \frac{v_0}{\omega}\sin\omega t}\,. \tag{6}$$

This solution clearly satisfies the two initial conditions specified by eq. (2).

We have not yet proved that eq. (6) is in fact a solution to eq. (1). To do this, we compute the second derivative of x with respect to t. Taking two derivatives of eq. (6) with respect to t yields:

$$\begin{aligned}\frac{d^2x}{dt^2} &= -\omega^2 x_0 \cos\omega t - \omega v_0 \sin\omega t \\ &= -\omega^2\left(x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega\right) \\ &= -\omega^2 x\,. \end{aligned} \tag{7}$$

Thus, we have verified that

$$\frac{d^2x}{dt^2} = -\,\omega^2 x\,. \tag{8}$$

Comparing with eq. (1), we conclude that $\omega^2 = k/m$, or

$$\omega = \sqrt{\frac{k}{m}}\,. \tag{9}$$

2. The amplitude/phase form of the harmonic oscillator solution

Giancoli writes the solution to eq. (1) in another form:

$$\boxed{x(t) = A\cos(\omega t + \phi)}\,, \tag{10}$$

where A is a positive constant. Two questions immediately come to mind. First, why would one prefer a solution of the form of eq. (10)? Second, is the solution given by eq. (10) identical to the one given by eq. (6)? One advantage of eq. (10) is that it satisfies the inequality:

$$|x(t)| \leq A\,, \tag{11}$$

which follows from the well known result $|\cos\theta| \leq 1$. This allows us to immediately identify A as the *amplitude* of the oscillation. The angle ϕ is called the *phase* of the oscillation, and I will have more to say about this phase angle in section 4. To show that eqs. (10) and (6) are equivalent, we make use of the trigonometric identity:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta\,. \tag{12}$$

Identifying $\alpha = \omega t$ and $\beta = \phi$, we can rewrite eq. (10) as:

$$x(t) = A\cos\omega t\cos\phi - A\sin\omega t\sin\phi\,. \tag{13}$$

This result is precisely equivalent to eq. (6) if we identify:

$$x_0 = A\cos\phi\,, \qquad \text{and} \qquad \frac{v_0}{\omega} = -A\sin\phi\,. \tag{14}$$

Thus, we have demonstrated that Giancoli's solution is equivalent to that of eq. (6).

3. The amplitude A of the oscillator

We can now use eq. (14) to relate the amplitude A and the phase ϕ to the initial conditions for the displacement and the velocity. Let us rewrite eq. (14) as:

$$\boxed{\cos\phi = \frac{x_0}{A}\,, \qquad \text{and} \qquad \sin\phi = -\frac{v_0}{\omega A}}\,. \tag{15}$$

Using the trigonometric identity $\sin^2\phi + \cos^2\phi = 1$, it follows that:

$$\frac{x_0^2}{A^2} + \frac{v_0^2}{\omega^2 A^2} = 1\,. \tag{16}$$

Cross-multiplying by A^2, we can solve for A:

$$\boxed{A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}}\,. \tag{17}$$

4. The phase ϕ of the oscillator

The phase angle ϕ can be determined from eq. (15). However, there is a slight complication—namely, ϕ is not uniquely defined, since I can add 2π to the phase angle without changing any physical results.* Thus, we say that the phase angles ϕ and $\phi+2\pi$ are "equivalent," meaning that they lead to the same physical results. Thus, I can establish a convention in which the value of ϕ is chosen to lie within some interval of length 2π. The most common convention is to choose ϕ such that:

$$\boxed{-\pi < \phi \leq \pi}\,. \tag{18}$$

That is, ϕ lies between $-180°$ and $180°$. Note that I have included one of the endpoints, π, in the interval specified by eq. (18), but not the other endpoint $-\pi$. I have made this choice since the phase angle $-\pi$ is "equivalent" to the phase angle $-\pi + 2\pi = \pi$ (using the meaning of "equivalent" explained above).

It is tempting to divide the two equations given in eq. (15), which yields:†

$$\tan\phi \equiv \frac{\sin\phi}{\cos\phi} = -\frac{v_0}{\omega x_0}\,. \tag{19}$$

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*Noting that $x(t) = \cos(\omega t + \phi) = \cos(\omega t + \phi + 2\pi)$ [i.e., the cosine is a periodic function with period 2π], we wee that the phase angle ϕ and $\phi + 2\pi$ yield the same solution for $x(t)$.

†Eq. (19) would seem to imply that $\phi = -\tan^{-1}[v_0/(\omega x_0)]$. However, this form is not especially convenient as the inverse tangent (also called the arctangent) is often defined to lie in an interval between $-\pi/2$ and $\pi/2$. See the discussion following eq. (19).

However, in doing so, we have lost some information, since changing the sign of both x_0 and v_0 changes the phase angle by π but does not change the value of $\tan\phi$. This is not surprising since $\tan(\phi+\pi) = \tan\phi$, i.e. the tangent is a periodic function with period π. The simplest way to avoid the ambiguity of eq. (19) is to notice that eq. (15) implies that:

$$\begin{aligned} v_0 < 0 \qquad &\Longrightarrow \qquad 0 < \phi < \pi\,, \\ v_0 > 0 \qquad &\Longrightarrow \qquad -\pi < \phi < 0\,, \\ v_0 = 0 \text{ and } x_0 > 0 \qquad &\Longrightarrow \qquad \phi = 0\,, \\ v_0 = 0 \text{ and } x_0 < 0 \qquad &\Longrightarrow \qquad \phi = \pi\,. \end{aligned} \tag{20}$$

Then, eqs. (19) and (20) determine the phase angle ϕ uniquely within its defined interval as specified by eq. (18).

The significance of the phase angle ϕ is that it determines by how much the displacement, $x(t) = A\cos(\omega t + \phi)$ leads or lags behind as compared to a pure cosine curve, $A\cos\omega t$. The pure cosine curve has a maximum at $t = 0$, whereas $x(t) = A\cos(\omega t+\phi)$ has a maximum when $\cos(\omega t+\phi) = 1$. The latter implies that the maximum of the displacement closest to time $t = 0$ occurs when $\omega t + \phi = 0$ or $t = -\phi/\omega$. If $\phi < 0$, then this maximum occurs for positive t, as shown in fig. 14.7 of Giancoli. In this case, we say that the phase of $x(t)$ *lags behind* the pure cosine curve, since we have to wait a positive time interval ($\Delta t = -\phi/\omega$ in fig. 14.7) before reaching the maximum. If $\phi > 0$, then this maximum occurs for negative t. In this case, the maximum has been achieved *before* the maximum of the pure cosine curve; and thus the phase of $x(t)$ *leads* that of the pure cosine curve.

5. Two simple examples

We review the two simple examples introduced by Giancoli on page 373.

1. In the first example, we start the pendulum with zero velocity ($v_0 = 0$) at a displacement $x_0 = A$. Indeed, eq. (17) confirms that for $v_0 = 0$ we have $x_0 = A$. That is, for zero initial velocity the displacement of the pendulum at $t = 0$ is the amplitude of the oscillation. Moreover, eq. (20) yields $\phi = 0$ [note that eq. (19) is consistent with this result]. Hence, eq. (10) yields

$$x(t) = A\cos\omega t\,. \tag{21}$$

2. In the second example, we start the pendulum at its equilibrium point ($x_0 = 0$) with initial positive velocity $v_0 > 0$. In this case, eq. (17) yields $A = v_0/\omega$, and eq. (20) implies that $-\pi < \phi < 0$. Finally, we use eq. (19) to conclude that $\phi = -\pi/2$. Hence, eq. (10) yields

$$x(t) = A\cos\left(\omega t - \frac{\pi}{2}\right) = A\sin\omega t\,. \tag{22}$$