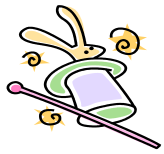
Algebra

1. From the Specific to the General

INTRODUCTION

How do you react when you see the word algebra? Many people find the concept of algebra difficult, so if you are one of them, please relax, as you have plenty of company! The main thing to remember about algebra is that it is really a way of generalising all the operations you do with numbers, for example, how to express operations such as $+$, $-$, $\times$ and $\div$ so that they work IN GENERAL, rather than just between 2 known numbers.



To illustrate this, let's do a MAGIC TRICK.

Please think of any number. We'll do some calculations with it, so to keep it simple, maybe think of a whole number between 1 and 10 (you can include 1 or 10 if you want).

Now, I'm going to ask you to do a series of calculations. You can use a calculator if you wish. Write your answer for each calculation on the space.

First, please DOUBLE your number ______________________

Now, ADD 5 to the result ______________________

Next, MULTIPLY that result by 3 ______________________

Now, TAKE AWAY 9 ______________________

DIVIDE the result by 6 ______________________

TAKE AWAY your original number ______________________

You should have ended with 1, no matter what number you began with.

Now, imagine you and a number of other people had done this. Each of you would have started with a different number, yet you would all have finished with 1. But I had no idea what number each of you started with, yet I know the answer is 1. How did I do that? The trick is – I used ALGEBRA!

Let's illustrate this using a different kind of representation. Suppose you let the number you started with be represented by some sort of concrete object, such as… a MATCHBOX! This will help you visualise what happens to your number. Have a look at the table on the next page.

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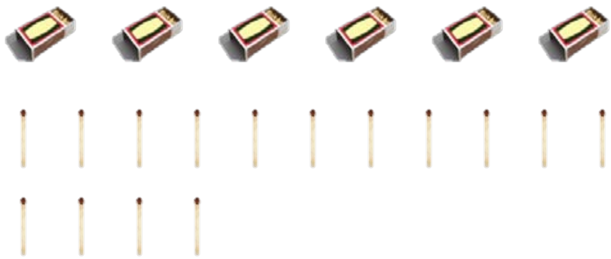
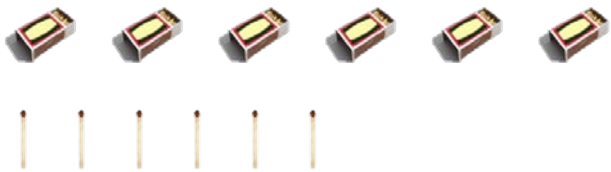
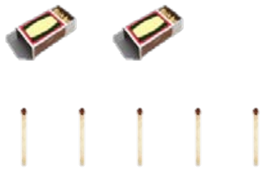
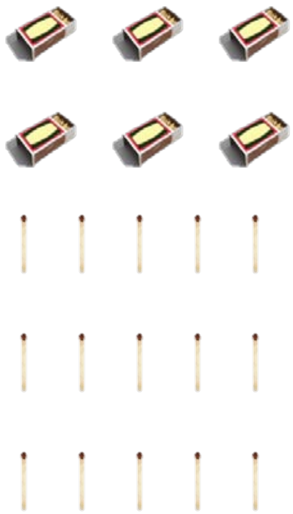
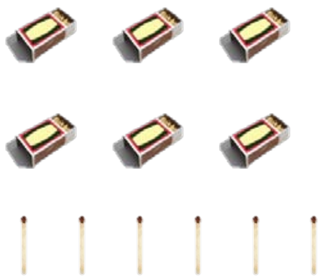
CALCULATION	Your result	REPRESENTATION
Your original number		
DOUBLE the number		 (If we double 1 matchbox, we just get 2 matchboxes)
ADD 5 to the result		 (We don't know what the matchbox represents so we can't add anything to it, so the easiest way is just to add 5 single objects – I've used matches.)
MULTIPLY that result by 3		 (We had to multiply EVERYTHING by 3.)
TAKE AWAY 9		 (We still don't know what the matchbox represents so couldn't take anything away from there – so we take it away from the 15 single matches.)
DIVIDE the result by 6		 (Imagine dividing that lot among 6 people – everyone would get 1 matchbox and 1 match.)
TAKE AWAY your original number		(Your original number was the matchbox!)

Now, let's do that again but this time we'll introduce the algebra part.

Let x represent your original number.

CALCULATION	Your result	REPRESENTATION	ALGEBRA
Your original number			x
DOUBLE the number			$2x$
ADD 5 to the result			$2x + 5$
MULTIPLY that result by 3			$3(2x + 5) = 6x + 15$
TAKE AWAY 9			$6x + 15 - 9 = 6x + 6$
DIVIDE the result by 6			$\frac{6x + 6}{6} = x + 1$
TAKE AWAY your original number			1

When we "DOUBLE the number", in algebra, we just get 2 xs, or 2 matchboxes, or 2 cups, or 2 books or 2… and that's what we write. In other words, we leave out the multiplication sign, so we write $2x$. So, from now on, you know that $2x$ means 2 times x, $3x$ means 3 times x, and so on.

For "ADD 5", we don't know what x represents, so we can't really add anything to it. All we can do is write the symbols for "ADD 5" which are "$+5$", so we get $2x + 5$. (If we use the matchboxes, then right now we have 2 matchboxes, but we don't know what they represent, so to add 5 we just use the single matches to add on the 5.)

Next, we have to "MULTIPLY that result by 3". This means we need to multiply everything we have by 3. We have 2 matchboxes and 5 matches, so we would get 6 matchboxes and 15 matches. We have to multiply $2x + 5$ by 3, so we would get $6x + 15$.

TAKE AWAY 9. Remember we don't know what number x (or the matchbox) stands for, so we can't take 9 away from that, so we have to take 9 away from the number 15. We still have $6x$ though, so we get $6x + 6$.

Next, we have to DIVIDE the whole lot by 6. This means we are sharing out our $6x$ and our 6 (or our 6 matchboxes and our 6 matches). Each share would be one matchbox and one match, or $x + 1$.

Lastly, TAKE AWAY your original number, which was x, and we would be left with 1. Here is just the algebra version:

CALCULATION	**Your result**	**ALGEBRA**
Your original number		x
DOUBLE the number		$2x$
ADD 5 to the result		$2x + 5$
MULTIPLY that result by 3		$6x + 15$
Take away 9		$6x + 6$
Divide the result by 6		$x + 1$
Take away your original number		1

Now to illustrate the calculation itself, using algebra.

CALCULATION	ALGEBRA NOTATION	ALGEBRA RESULT
Your original number	x	x
DOUBLE the number	$2x$ This is the same as multiplying x by 2, or adding x and x.	$2x$
ADD 5 to the result	$2x + 5$	$2x + 5$
MULTIPLY that result by 3	$3(2x + 15)$ Everything in the brackets has to be multiplied by 3.	$6x + 15$
Take away 9	$6x + 15 - 9$	$6x + 6$
Divide the result by 6	$\frac{6x + 6}{6}$ Everything in the numerator (top) has to be divided by 6.	$x + 1$
TAKE AWAY your original number	$x + 1 - x$	1

What has been illustrated so far has been ways of writing operations such as $+$, $-$, $\times$, $\div$ and ( ) using algebra. What has also been illustrated is that algebra is a way of GENERALISING rather than using only specific numbers. The trick we used will always work using any number you like.

Lastly, let's do the trick again, but this time we'll use the algebra result. Say we thought of the number 11. (It's bigger than 10 so we're sure it is not your original number.) We'll use the algebra result and SUBSTITUTE the value of 11 for every x and see if we get the same result as if we just performed the calculation.

CALCULATION	USING 11 to CALCULATE	ALGEBRA RESULT	SUBSTITUTING 11 into ALGEBRA RESULT
Your original number	11	x	11
DOUBLE the number	$11 \times 2 = \underline{22}$	$2x$	$2 \times 11 = \underline{22}$
ADD 5 to the result	$22 + 5 = \underline{27}$	$2x + 5$	$2 \times 11 + 5 = 22 + 5 = \underline{27}$
MULTIPLY that result by 3	$27 \times 3 = \underline{81}$	$6x + 15$	$6 \times 11 + 15 = 66 + 15 = \underline{81}$
Take away 9	$81 - 9 = \underline{72}$	$6x + 6$	$6 \times 11 + 6 = 66 + 6 = \underline{72}$
Divide the result by 6	$72 \div 6 = \underline{12}$	$x + 1$	$11 + 1 = \underline{12}$
TAKE AWAY your original number	$12 - 11 = \underline{1}$	1	$\underline{1}$

Notice that the results in the SUBSTITUTION column are exactly the same as they are in the "USING 11 to calculate" column. Please try this experiment with ANY other number. You can choose decimals, fractions, even negative numbers will work.

CALCULATION	**USING your new number to CALCULATE**	**ALGEBRA RESULT**	**SUBSTITUTING your new number into ALGEBRA RESULT**
Your original number		x	
DOUBLE the number		$2x$	
ADD 5 to the result		$2x + 5$	
MULTIPLY that result by 3		$6x + 15$	
Take away 9		$6x + 6$	
Divide the result by 6		$x + 1$	
TAKE AWAY your original number		1	

This brings us to the next part of algebra:

SUBSTITUTION

You might have come across *formulas* in all sorts of situations. Some examples could be: calculating repayments on a mortgage; finding the area of the floor of a room to be carpeted or tiled; working out the amount of tax to pay. For each of these types of calculations you can use a formula, no matter what the size of the mortgage, the dimensions of the room, or whatever amount you have earned. This is because formulas are simply a way of generalising from specific numbers, so that we can apply the same formula, or method, or procedure, to the same concept every time.

Probably the simplest example from the ones above is to find the area of the rectangular floor of a room. You might know that we multiply the length of the floor by the width of the floor. (If you want to know more, please refer to *Area and Perimeter – 1. Areas of Plane Shapes*.)

Let's say the length of a floor is 3 metres and the width is 2.5 metres. We multiply 3×2.5 and get 7.5 m^2 as the area. But we are not going to always have a length of 3 and a width of 2.5. Maybe we'll need to calculate the area of a floor that is 7.2 metres by 5.1 metres and someone else wants to calculate…

This is where algebra comes in. We can generalise and say that the length of the floor can be represented by l and the width can be represented by w. Then the area, which we can represent using A, is the result of multiplying l by w, which can be written as $A = l \times w$, or just $A = lw$. Whatever the length and width might be, we can simply SUBSTITUTE their values into our formula and find the answer.

(To be able to substitute correctly into a formula, it helps to know about the order of operations for calculating and how to use your calculator. If you are unsure about either of these, please refer to *Calculations – Order of Operations in Mathematics* and *Calculators – Getting to Know Your Scientific Calculator*.)

Other Common Formulas

The area of a circle: $A = \pi r^2$

A represents the value of the area of a circle and r represents the value of the radius of that circle. These values will change depending on the size of the circle. We say that A and r are variables because they can change or vary.

With this formula, we also meet a new concept, π. π is what is called a constant, that is, unlike other letters for which we can substitute any value, π always remains the same. You can pick up any calculator and press the button for π and you will always see the same result. It is not a variable like A, r, (or l and w in the rectangle formula). That is, π does not vary, it is always the same.

Using the formula $A = \pi r^2$, we can find the area of any circle we like. The radius, r, can be any value, and by *substituting* that value into our formula, we can find the area of that circle. For example, let's find the area of a circle with radius 4.1 cm. We multiply π by the result of squaring 4.1 (which is 16.81) so we get $\pi \times 16.81$, which is equal to 52.81 cm^2 (rounded).

We would write this calculation as follows:

$$\begin{aligned} A &= \pi r^2 \\ &= \pi \times 4.1^2 \\ &= \pi \times 16.8^{*} \\ &\approx 52.81 \end{aligned}$$

The $\approx$ means the values has been rounded and is approximate. So the area of the circle is approximately 52.81 cm^2.

Speed

We can calculate the average speed with which we travelled on a trip. We do this by recognising that speed is just the distance travelled in a certain amount of time, example, kilometres per hour (km/h), metres per second (m/s) and so on (*per hour* means *during each hour* or *during 1 hour*). For example, you might have travelled 100 km in 1 hour, which gives you an average speed of 100 km/h. If you had taken 2 hours to do the 100 km, you would have had an average of 50 km/h. If you had taken 4 hours to do the 100 km, you would have had an average of 25 km/h.

All we have done each time has been to divide the distance by the time taken, so the average speed can be calculated by using the formula $v = \frac{d}{t}$ where v is the speed (v stands for velocity, which is equal to the speed and direction of travel), d is the distance travelled and t is the time taken. The units of distance and time will determine the units of speed.

Example: Calculate the average speed on a trip of 350 km which took 5 hours.

Here, $d = 350$ and $t = 5$, so using $v = \frac{d}{t}$, we get

$$v = \frac{350}{5} = 70$$

So the average speed was 70 km/h.

Many people like to use the units as part of their calculations, so we could also write:

$$v = \frac{350\ km}{5\ h} = 70\ km/h$$

Perimeter of a rectangle

Another common formula is to find the *perimeter* of a rectangle. This means we are finding the measurement of the outside of the rectangle. You can imagine walking on the outside of the rectangle – this is the perimeter. If we measure

* We can leave this line out because we can just enter the previous line into the calculator and get the answer straight away

the outside of the rectangle, it would mean we have measured the length twice (top and bottom, say) and the width twice (left and right) and have added them together to find the total. So we have 2 lengths and 2 widths altogether.

If the length is l and the width is w, then the perimeter must be

$$P = 2l + 2w$$

For the rectangular floor with length 3 m and width 2.5 m, the perimeter is

$$\begin{aligned} P &= 2 \times 3 + 2 \times 2.5 \\ &= 6 + 5 \\ &= 11 \end{aligned}$$

So the perimeter is 11 m.

Temperature conversion

Another commonly used formula, especially when people travel to the USA, is the one which converts temperature between Fahrenheit (°F) and Celsius (°C). Fahrenheit is still used in the USA, Australia uses Celsius.

The formula to convert from °F to °C is:

$$C = \frac{5}{9}(F - 32)$$

So, if the temperature in the US is 95 °F, then the Celsius would be:

$$\begin{aligned} C &= \frac{5}{9}(95 - 32) \\ &= \frac{5}{9}(63) \\ &= (5 \times 63) \div 9 \\ &= 35 \end{aligned}$$

So the Celsius equivalent of 95 °F is 35 °C.

Mortgage repayments

And finally, putting everything together, a more difficult one. To find the monthly repayment on a fixed rate mortgage, one formula is:

$$C = \frac{\Pr(1 + r)^N}{(1 + r)^N - 1},$$

where C is the monthly payment on a fixed rate mortgage, r is the monthly interest rate, N is the number of monthly payments, and P is the amount borrowed.

Example: Let's borrow \$500 000 over 30 years, at the annual interest rate of 4.95%.

Before we substitute into the formula, we need to do a few calculations to work out the values of N and r.

Because N is the number of *monthly* repayments, we need to know how many months there are in 30 years. So, $N = 30 \times 12 = 360$.

The interest rate, 4.95%, is given for a year. We need to work out how much that will be for a month, so we divide 4.95% by 12. It is easiest to work in decimals from the start, so $4.95\% = 0.0495$ and $0.0495 \div 12 = 0.004125$.

We now know: $P = \$500\ 000$, $r = 0.004125$, and $N = 360$ so we are ready to substitute.

$$\begin{aligned} C &= \frac{\Pr(1 + r)^N}{(1 + r)^N - 1} \\ &= \frac{500000 \times 0.004125 \times (1 + 0.004125)^{360}}{(1 + 0.004125)^{360} - 1} \end{aligned}$$

So, our monthly repayment rate would be \$2 668.20.

(If you had any problems reaching that answer, you might like to refer to *Calculators – Getting to Know Your Scientific Calculator*.)

If you need help with any of the maths covered in this resource (or any other maths topics), you can make an appointment with Learning Development through reception: phone (02) 4221 3977, or Level 2 (top floor), Building 11, or through your campus.

Acknowledgement: Matches and matchboxes idea from TAFENSW Career Education for Women course 1989. Author(s) unknown.