CHAPTER 2

INFINITE SUMS (SERIES)

Lecture Notes

We extend the notion of a finite sum $\Sigma_{k=1}^{n} a_k$ to an INFINITE SUM which we write as

$$\Sigma_{n=1}^{\infty} a_n$$

as follows.

DEFINITION 1

For a given sequence

$\{a_n\}_{n \in N - \{0\}}$, i.e the sequence

$$a_1, a_2, a_3, ...a_n, ...$$

we form a following (infinite) sequence

$$S_1 = a_1, S_2 = a_1 + a_2, ..., S_n = \Sigma_{k=1}^{n} a_k, ...$$

We use it to define the infinite sum as follows.

DEFINITION 1

If the limit of the sequence $\{S_n\}$ exists we call it an INFINITE SUM of the sequence $\Sigma_{k=1}^{n} a_k$.

We write it as

$$\Sigma_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \Sigma_{k=1}^{n} a_k.$$

The sequence $\{S_n = \Sigma_{k=1}^{n} a_k\}$ is called its sequence of **partial sums**.

DEFINITION 2

If the limit $\lim_{n\to\infty} S_n$ exists and is finite, i.e.

$$\lim_{n\to\infty} S_n = S,$$

then we say that the infinite sum $\Sigma_{n=1}^{\infty} a_n$ CONVERGES to S and

we write it as

$$\Sigma_{n=1}^{\infty} a_n = S,$$

otherwise the infinite sum DIVERGES.

In a case that

$$\lim_{n\to\infty} S_n$$

exists and is infinite, i.e.

$$\lim_{n\to\infty} S_n = \infty,$$

we say that the infinite sum

$$\Sigma_{n=1}^{\infty}\, a_n$$

DIVERGES to ∞ and

we write it as

$$\Sigma_{n=1}^{\infty}\, a_n = \infty.$$

In a case that $\lim_{n\to\infty} S_n$ does not exist we say that the infinite sum $\Sigma_{n=1}^{\infty}\, a_n$ DIVERGES.

Observation 1 In a case when all elements of the sequence $\{a_n\}$ are equal 0 starting from a certain $k \geq 1$ the infinite sum becomes a finite sum, hence the infinite sum is a generalization of the finite one, and this is why we keep the similar notation.

EXAMPLE 1 The infinite sum of a geometric sequence $a_n = x^k$ for $x \geq 0$, i.e.

$$\Sigma_{n=1}^{\infty}\ x^n$$

converges if and only if $\mid x \mid < 1$ because

$$\Sigma_{k=1}^{n}\ x^k = S_n = \frac{x^{n+1} - x}{x-1}, \quad \text{and}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{x}{x-1}(x^n - 1) = \frac{x}{x-1} \quad \text{iff } \mid x \mid < 1,$$

hence

$$\Sigma_{n=1}^{\infty} x^k = \frac{x}{x-1}.$$

EXAMPLE 2 The series $\Sigma_{n=1}^{\infty} 1$ DIVERGES to ∞ as $S_n = \Sigma_{k=1}^{n} 1 = n$ and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty.$$

EXAMPLE 3 The infinite sum $\Sigma_{n=1}^{\infty} (-1)^n$ DIVERGES.

EXAMPLE 4 The infinite sum

$$\Sigma_{n=1}^{\infty} \frac{1}{(k+1)(k+2)} \text{ CONVERGES and}$$

$$\Sigma_{n=1}^{\infty} \frac{1}{(k+1)(k+2)} = 1.$$

Proof: first we evaluate $S_n = \Sigma_{k=1}^{n} \frac{1}{(k+1)(k+2)}$ as follows.

$$S_n = \Sigma_{k=1}^{n} \frac{1}{(k+1)(k+2)} = \Sigma_{k=1}^{n} k^{\underline{-2}} =$$

$$-\frac{1}{x+1}\Big|_0^{n+1} = -\frac{1}{n+2} + 1 \text{ and}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} -\frac{1}{n+2} + 1 = 1.$$

DEFINITION 3 For any infinite sum (series) $\Sigma_{n=1}^{\infty} a_n$ a series $r_n = \Sigma_{m=n+1}^{\infty}\ a_m$ is called its n-th REMINDER.

FACT If $\Sigma_{n=1}^{\infty} a_n$ converges, then so does its n-th REMINDER $r_n = \Sigma_{m=n+1}^{\infty}\ a_m$.

Proof: first, observe that if $\Sigma_{n=1}^{\infty} a_n$ converges, then for any value on n so does $r_n = \Sigma_{m=n+1}^{\infty}\ a_m$ because

$$r_n = \lim_{n\to\infty}(a_{n+1} + ... + a_{n+k}) =$$

$$\lim_{n\to\infty} S_{n+k} - S_n = \Sigma_{m=1}^{\infty} a_m\ - S_n.$$

So we get

$$\lim_{n\to\infty}\ r_n = \Sigma_{m=1}^{\infty} a_m\ - \lim_{n\to\infty} S_n =$$

$$\Sigma_{m=1}^{\infty} S_m\ -\ \Sigma_{n=1}^{\infty} a_n = S - S = 0.$$

General Properties of Infinite Sums

THEOREM 1

$$\text{If } \Sigma_{n=1}^{\infty} a_n \text{ converges, then}$$

$$\lim_{n\to\infty} a_n = 0.$$

Proof: observe that $a_n = S_n - S_{n-1}$ and hence

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n\, The - \lim_{n\to\infty} S_{n-1} = 0,$$

as $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1}$.

REMARK The reverse statement to the theorem 1

$$\text{If } \lim_{n\to\infty} a_n = 0. \text{ then } \Sigma_{n=1}^{\infty} a_n \text{ converges}$$

is not always true. There are infinite sums with the term converging to zero that are not convergent.

EXAMPLE 5 The infinite HARMONIC sum

$$H = \Sigma_{n=1}^{\infty} \frac{1}{n}$$

DIVERGES to ∞, i.e.

$$\Sigma_{n=1}^{\infty} \frac{1}{n} = \infty$$

but $\lim_{n \to \infty} \frac{1}{n} = 0$.

DEFINITION 4 Infinite sum

$$\Sigma_{n=1}^{\infty} a_n$$

is BOUNDED if its sequence of partial sums

$$S_n = \Sigma_{k=1}^{n} \ a_k$$

is BOUNDED; i.e. there is a number M such that

$$|S_n| < M, \text{for all} n \leq 1, n \in N.$$

FACT 2 Every convergent infinite sum is bounded.

THEOREM 2 If the infinite sums

$$\Sigma_{n=1}^{\infty} a_n,\ \Sigma_{n=1}^{\infty} b_n$$

CONVERGE, then the following properties hold.

$$\Sigma_{n=1}^{\infty}(a_n + b_n) = \Sigma_{n=1}^{\infty} a_n \ + \ \Sigma_{n=1}^{\infty} b_n,$$

$$\Sigma_{n=1}^{\infty} c a_n = c \Sigma_{n=1}^{\infty} a_n,\ c \in R.$$

Alternating Infinite Sums

DEFINITION 5 An infinite sum

$$\Sigma_{n=1}^{\infty}(-1)^{n+1}a_n, \quad \text{for} \quad a_n \geq 0$$

is called ALTERNATING infinite sum (alternating series).

EXAMPLE 6 Consider

$$\Sigma_{n=1}^{\infty}(-1)^{n+1} = 1 - 1 + 1 - 1 + ...$$

If we group the terms in pairs, we get

$$(1-1) + (1-1) + ... = 0$$

but if we start the pairing one step later, we get

$$1-(1-1)-(1-1)-..... = 1-0-0-0-... = 1.$$

It shows that grouping terms in a case of infinite sum can lead to inconsistencies (contrary to the finite case). Look also example on page 59. We need to develop some strict criteria for manipulations and convergence/divergence of alternating series.

THEOREM 3 The alternating infinite sum

$$\Sigma_{n=1}^{\infty}(-1)^{n+1}a_n, (a_n \geq 0)$$

such that

$$a_1 \geq a_2 \geq a_3 \geq .... \text{ and } \lim_{n\to\infty} a_n = 0$$

always CONVERGES.

Moreover, its partial sums

$$S_n = \Sigma_{k=1}^{n}(-1)^{n+1}a_n$$

fulfil the condition

$$S_{2n} \leq \Sigma_{n=1}^{\infty}(-1)^{n+1}a_n \leq S_{2n+1}.$$

Proof: observe that the sequence of S_{2n} is increasing as

$$S_{2n+2} = S_{2n} + (a_{2n+1} - a_{2n+2}$$

and

$$a_{2n+1} - a_{2n+2} \geq 0,$$

i.e.

$$S_{2n+2} \geq S_{2n}.$$

The sequence of S_{2n} is also bounded as

$$S_{2n} = a_1 - ((a_2 - a_3) + (a_4 - a_5) + ...a_{2n}) \leq a_1.$$

We know that any bounded and increasing sequence is is convergent, so we proved that S_{2n} converges.

Let denote $\lim_{n\to\infty} S_{2n} = g$.

To prove that

$$\Sigma_{n=1}^{\infty}(-1)^{n+1}a_n = \lim_{n\to\infty} S_n$$

converges we have to show now that

$$\lim_{n\to\infty} S_{2n+1} = g.$$

Observe that

$$S_{2n+1} = S_{2n} + a_{2n+2}$$

and we get

$$\lim_{n\to\infty} S_{2n+1} =$$

$$\lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+2} = g$$

as we assumed that

$$\lim_{n\to\infty} a_n = 0.$$

We proved that the sequence $\{S_{2n}\}$ is increasing.

We prove in a similar way that the sequence $\{S_{2n+1}\}$ is decreasing.

Hence we get

$$S_{2n} \leq \lim_{n\to\infty} S_{2n} = g = \Sigma_{n=1}^{\infty}(-1)^{n+1}a_n$$

and

$$S_{2n+1} \geq \lim_{n\to\infty} S_{2n+1} = g$$

and

$$\Sigma_{n=1}^{\infty}(-1)^{n+1}a_n = g,$$

i.e

$$S_{2n} \leq \Sigma_{n=1}^{\infty}(-1)^{n+1}a_n \leq S_{2n+1}.$$

EXAMPLE 7

Consider the ANHARMONIC series

$$AH = \Sigma_{n=1}^{\infty} \ (-1)^{n+1} \ \frac{1}{n} =$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} .....$$

Observe that $a_n = \frac{1}{n}$, and

$$\frac{1}{n} \geq \frac{1}{n+1}$$

i.e. $a_n \geq a_{n+1}$ for all n.

This proves that the assumptions of the theorem 3 are fulfilled for AH and hence AH CONVERGES.

In fact, it is proved (by analytical methods) that

$$AH = \Sigma_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln 2.$$

EXAMPLE 8 A series (infinite sum)

$$\Sigma_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}$$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\ldots\ldots$$

CONVERGES, by Theorem 3.

Proof is similar to the one in the example 7).

It also is proved that

$$\Sigma_{n=0}^{\infty}(-1)^n\frac{1}{2n+1} = \frac{\pi}{4}.$$

THEOREM 4 (ABEL Theorem)

IF a sequence $\{a_n\}$ fulfils the assumptions of the theorem 3 i.e.

$$a_1 \geq a_2 \geq a_3 \geq .... \text{ and}$$

$$\lim_{n\to\infty} a_n = 0$$

and an infinite sum (converging or diverging)

$$\Sigma_{n=1}^{\infty} b_n \text{ is bounded,}$$

THEN the infinite sum

$$\Sigma_{n=1}^{\infty} a_n b_n$$

always converges.

Observe that Theorem 3 is a special case of theorem 4 when $b_n = (-1)^{n+1}$.

Convergence of Infinite Sums with Positive Terms

We consider now infinite sums with all its terms being positive real numbers, i.e.

$$S = \Sigma_{n=1}^{\infty} a_n,$$

$$\text{for } a_n \geq 0, a_n \in R.$$

Observe that if all $a_n \geq 0$,
then the sequence $\{S_n\}$ of partial sums is increasing; i.e.

$$S_1 \leq S_2 \leq .... \leq S_n...$$

and hence the limit

$$\lim_{n \to \infty} S_n$$

exists and is finite or is ∞. This proves the following theorem.

THEOREM 5

The infinite sum

$S = \Sigma_{n=1}^{\infty} a_n$, where $a_n \geq 0, a_n \in R$

always CONVERGES, or DIVERGES to ∞.

THEOREM 6 (Comparing the series)

Let $\Sigma_{n=1}^{\infty} a_n$ be an infinite sum and $\{b_n\}$ be a sequence such that for all $n \in N$

$$0 \leq b_n \leq a_n.$$

If the infinite sum $\Sigma_{n=1}^{\infty} a_n$ converges then the sum $\Sigma_{n=1}^{\infty} b_n$ also converges and

$$\Sigma_{n=1}^{\infty} b_n \ \leq \ \Sigma_{n=1}^{\infty} a_n.$$

Proof: we denote

$$S_n = \Sigma_{k=1}^{n} a_k,\ T_n = \Sigma_{k=1}^{n} b_k.$$

As $0 \leq b_n \leq a_n$ we get that also

$$S_n \leq T_n.$$

But

$$S_n \leq \lim_{n \to \infty} S_n = \Sigma_{n=1}^{\infty} a_n$$

so also

$$T_n \leq \Sigma_{n=1}^{\infty} a_n = S.$$

The inequality

$$T_n \leq \Sigma_{n=1}^{\infty} a_n = S$$

means that the sequence $\{T_n\}$ is a bounded sequence with positive terms,

hence by theorem 5, it converges.

By the assumption that

$$\Sigma_{n=1}^{\infty} a_n$$

we get that

$$\Sigma_{n=1}^{\infty} a_n = \lim_{n \to \infty} \Sigma_{k=1}^{n} a_k =$$

$$\lim_{n \to \infty} S_n = S.$$

We just proved that

$$T_n = \Sigma_{k=1}^{n} b_k$$

converges, i.e.

$$\lim_{n \to \infty} T_n = T = \Sigma_{n=1}^{\infty} b_n.$$

But also we proved that

$$S_n \leq T_n,$$

hence

$$\lim_{n \to \infty} S_n \leq \lim_{n \to \infty} T_n$$

what means that

$$\Sigma_{n=1}^{\infty} b_n \ \leq \ \Sigma_{n=1}^{\infty} a_n.$$

EXAMPLE 9

Let's use Theorem 5 to prove that the series

$$\Sigma_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

converges.

We prove by analytical methods that it converges to $\frac{\pi^2}{6}$.

Here we prove only that it does converge.

First observe that the series below converges to 1, i.e.

$$\Sigma_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Consider

$$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} .... + \frac{1}{n(n+1)} =$$

$$(1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + ...(\frac{1}{n} - \frac{1}{n+1}) =$$

$$1 - \frac{1}{n+1}$$

so we get

$$\Sigma_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} S_n =$$

$$\lim_{n\to\infty} (1 - \frac{1}{n+1}) = 1.$$

Now we observe (easy to prove) that

$$\frac{1}{2^2} \le \frac{1}{1 \cdot 2}, \quad \frac{1}{3^2} \le \frac{1}{1 \cdot 3}, \ \ldots\ldots$$

$$\ldots \frac{1}{(n+1)^2} \le \frac{1}{n(n+1)}, \ldots\ldots$$

i.e. we proved that all assumptions if Theorem 5 hold, hence

$$\Sigma_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

converges and moreover

$$\Sigma_{n=1}^{\infty} \frac{1}{(n+1)^2} \ \le \ \Sigma_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

THEOREM 7 (D'Alambert's Criterium)

Any series with all its terms being positive real numbers, i.e.

$$\Sigma_{n=1}^{\infty} a_n, \text{ for } a_n \geq 0, a_n \in R$$

converges if the following condition holds:

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} < 1.$$

Proof: let h be any number such that

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} < h < 1.$$

It means that there is k such that for any $n \geq k$ we have

$$\frac{a_n}{a_{n+1}} < h \text{ and} a_{n+1} < h a_n.$$

Hence

$$a_{k+1} < a_k h, \quad a_{k+2} = a_{k+1} h < a_k h^2, ...$$

i.e. all terms a_n of

$$\Sigma_{n=k}^{\infty} a_n$$

are smaller then the terms of a converging (as $0 < h < 1$) geometric series

$$\Sigma_{n=0}^{\infty} a_k h^n = a_k + a_k h + a_k h^2 + ...$$

By Theorem 5 the series

$$\Sigma_{n=1}^{\infty} a_n$$

must converge.

THEOREM 7 (Cauchy's Criterium)

Any series with all its terms being positive real numbers, i.e.

$$\Sigma_{n=1}^{\infty} a_n, \text{ for } a_n \geq 0, a_n \in R$$

CONVERGES if the following condition holds:

$$\lim_{n \to \infty} \sqrt[n]{a_n} < 1.$$

Proof: we carry the proof in a similar way as the proof of theorem 6.

Let h be any number such that

$$\lim_{n\to\infty} \sqrt[n]{a_n} < h < 1.$$

It means that there is k such that for any $n \geq k$ we have $\sqrt[n]{a_n} < h$, i.e. $a_n < h^n$.

This means that all terms a_n of $\Sigma_{n=k}^{\infty} a_n$ are smaller then the terms of a converging (as $0 < h < 1$) geometric series

$$\Sigma_{n=k}^{\infty} h^n = h^k + h^{k+1} + h^{k+2} + ...$$

By Theorem 5 the series $\Sigma_{n=1}^{\infty} a_n$ must converge.

THEOREM 7 (Divergence Criteria)

Any series with all its terms being positive real numbers, i.e.

$$\Sigma_{n=1}^{\infty} a_n, \text{ for } a_n \geq 0, a_n \in R$$

DIVERGES if

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} > 1$$

$$\text{or} \quad \lim_{n\to\infty} \sqrt[n]{a_n} > 1$$

Proof:

observe that if

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} > 1,$$

then for sufficiently large n we have that

$$\frac{a_n}{a_{n+1}} > 1, \text{and hence } \; a_{n+1} > a_n.$$

This means that the limit of the sequence $\{a_n\}$ can't be 0.

By theorem 1 we get that $\Sigma_{n=1}^{\infty} a_n$ diverges.

Similarly, if

$$\lim_{n\to\infty} \sqrt[n]{a_n} > 1,$$

then then for sufficiently large n we have that

$$\sqrt[n]{a_n} > 1 \text{ and hence } \quad a_n > 1,$$

what means that the limit of the sequence $\{a_n\}$ can't be 0.

By theorem 1 we get that $\Sigma_{n=1}^{\infty} a_n$ diverges.

Remark It can happen that for a certain infinite sum $\Sigma_{n=1}^{\infty} a_n)$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1 = \lim_{n\to\infty} \sqrt[n]{a_n}.$$

In this case our Divergence Criteria do not decide whether the infinite sum converges or diverges.

In this case we say that the infinite sum **DOES NOT React** on the criteria.

EXAMPLE 10

The Harmonic series

$$H = \Sigma_{n=1}^{\infty} \frac{1}{n}$$

does not react on D'Alambert's Criterium (Theorem 7) because

$$\lim_{n \to \infty} \frac{n}{n+1} = \lim_{n \to \infty} \frac{1}{(1 + \frac{1}{n})} = 1.$$

EXAMPLE 11

The series from example 9

$$\Sigma_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

does not react on D'Alambert's Criterium (Theorem 7) because

$$\lim_{n\to\infty} \frac{(n+1)^2}{(n+2)^2} = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+4n+1} =$$

$$\lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{1+\frac{4}{n}+\frac{4}{n^2}} = 1.$$

Remark

Both series

$$\Sigma_{n=1}^{\infty}\frac{1}{n}$$

and

$$\Sigma_{n=1}^{\infty}\frac{1}{(n+1)^2}$$

do not react on D'Alambert's, but first in divergent and the second is convergent.

There are more criteria for convergence, most known are Kumer's criterium and Raabe criterium.