11.3: Infinite Series

An infinite series is a "sum" of the form

$$a_1 + a_2 + a_3 + \cdots = \sum_{i=1}^{\infty}$$

Examples

- $1 + 1 + 1 + 1 + 1 + 1 + \cdots = \sum_{n=1}^{\infty} 1$
- $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}$
- $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n}$
- $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$
- $1 - 1 + 1 - 1 + 1 - 1 + \cdots = \sum_{n=1}^{\infty} (-1)^n$

Partial Sums

Given a sequence $a_1, a_2, a_3, \ldots$ of numbers, the N^{th} partial sum of this sequence is

$$S_N := \sum_{n=1}^{N} a_n$$

We *define* the infinite series $\sum_{n=1}^{\infty} a_n$ by

$$\sum_{n=1}^{\infty} a_n = \begin{cases} \lim_{N\to\infty} S_N \text{ if this limit exists} \\ \text{divergent, otherwise} \end{cases}$$

Examples of partial sums

For the sequence $1, 1, 1, 1, \ldots$, we have $\sum_{n=1}^{N} 1 = N \to \infty$. Thus, $\sum_{n=1}^{\infty} 1$ is divergent.

For the sequence $1, -1, 1, -1, \ldots$, we have $S_1 = 1$, $S_2 = 0$, $S_3 = 1$, *etc.* In general, $S_N = 1$ for N odd and $S_N = 0$ for N even. Thus, $\sum_{n=1}^{\infty}(-1)^n$ is divergent.

The harmonic series

If one computes the partial sums for $\sum_{n=1}^{\infty} \frac{1}{n}$ one finds

$S_1 = 1$, $S_2 = \frac{3}{2} = 1.5$, $S_3 = \frac{11}{6} \approx 1.87$, $S_{10} \approx 2.93$, $S_{20} \approx 3.40$, $S_{1000} \approx 7.49$, $S_{100,000} \approx 12.09$. In fact, $S_N \to \infty$, so that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, though we will see why only later.

$\zeta(2)$

If one computes the partial sums for $\sum_{n=1}^{\infty} \frac{1}{n^2}$, then one obtains

$S_1 = 1$, $S_2 = \frac{5}{4} = 1.25$, $S_3 = \frac{49}{36} \approx 1.36$, $S_{10} \approx 1.55$, $S_{100} \approx 1.63$, $S_{1000} \approx 1.64$.

In fact,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} \approx 1.644934068$$

Geometric series

The series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is an example of a geometric series. Computing, we find $S_1 = 0.5$, $S_2 = 0.75$, $S_3 = 0.875$, $S_4 = 0.9375$, $S_{10} = .9990234375$. In fact, $S_N \to 1$.

A geometric series is a series of the form

$$\sum_{n=1}^{\infty} r^n$$

In the above case $r = \frac{1}{2}$.

Computing partial geometric sums

If

$$S_N = \sum_{n=1}^{N} r^n = (r + r^2 + r^3 + \cdots + r^N)$$

then

$$rS_N = \sum_{n=1}^{N} r^{n+1} = (r^2 + r^3 + \cdots + r^{N+1}) = S_N - r + r^{N+1}$$

Computation, continued

Subtracting S_N from both sides, we obtain

$$(r-1)S_N = r^{N+1} - r$$

Hence,

$$S_N = \frac{r^{N+1} - r}{r - 1}$$

Computing infinite geometric sums

So

$$\sum_{n=1}^{\infty} r^n = \lim_{N\to\infty} \frac{r^{N+1} - r}{r - 1}$$

provided that the limit on the right exists.

If $|r| < 1$, then $\lim_{N\to\infty} r^{N+1} = 0$, so that

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$$

If $|r| > 1$, then $\lim_{N\to\infty} r^{N+1}$ does not exist, so $\sum_{n=1}^{\infty} r^n$ diverges.

Finally, in the case that $|r| = 1$, we have already seen that the series diverges.

Using the geometric sum formula

Compute the following sums

- $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots$
- $-\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} + \cdots$

Solution

In the first case, we may write the sum as $1 + \sum_{n=1}^{\infty} (\frac{2}{3})^n$. So, the sum is $1 + \frac{\frac{2}{3}}{1-\frac{2}{3}} = 1 + \frac{\frac{2}{3}}{\frac{1}{3}} = 1 + 2 = 3$.

In the second case, we may write the sum as $\sum_{n=1}^{\infty} (\frac{-1}{2})^n$ so that this sum is $\frac{\frac{-1}{2}}{1-\frac{-1}{2}} = \frac{\frac{-1}{2}}{\frac{3}{2}} = \frac{-1}{3}$.