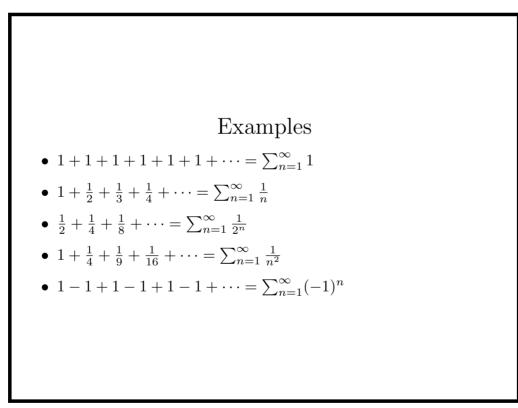
11.3: Infinite Series

An infinite series is a "sum" of the form

$$a_1 + a_2 + a_3 + \dots = \sum_{i=1}^{\infty}$$

1



 $\mathbf{2}$

Partial Sums

Given a sequence a_1, a_2, a_3, \ldots of numbers, the N^{th} partial sum of this sequence is

$$S_N := \sum_{n=1}^N a_n$$

We define the infinite series $\sum_{n=1}^{\infty} a_n$ by

$$\sum_{n=1}^{\infty} a_n = \begin{cases} \lim_{N \to \infty} S_N \text{ if this limit exists} \\ \text{divergent, otherwise} \end{cases}$$

3

Examples of partial sums

For the sequence 1, 1, 1, 1, ..., we have $\sum_{n=1}^{N} 1 = N \to \infty$. Thus, $\sum_{n=1}^{\infty} 1$ is divergent.

For the sequence $1, -1, 1, -1, \ldots$, we have $S_1 = 1, S_2 = 0, S_3 = 1$, etc. In general, $S_N = 1$ for N odd and $S_N = 0$ for N even. Thus, $\sum_{n=1}^{\infty} (-1)^n$ is divergent.

The harmonic series

If one computes the partial sums for $\sum_{n=1}^{\infty} \frac{1}{n}$ one finds $S_1 = 1, S_2 = \frac{3}{2} = 1.5, S_3 = \frac{11}{6} \approx 1.87, S_{10} \approx 2.93, S_{20} \approx 3.40,$ $S_{1000} \approx 7.49, S_{100,000} \approx 12.09$. In fact, $S_N \to \infty$, so that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, though we will see why only later.

 $\zeta(2)$

5

If one computes the partial sums for $\sum_{n=1}^{\infty} \frac{1}{n^2}$, then one obtains $S_1 = 1, S_2 = \frac{5}{4} = 1.25, S_3 = \frac{49}{36} \approx 1.36, S_{10} \approx 1.55, S_{100} \approx 1.63, S_{1000} \approx 1.64.$

In fact,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} \approx 1.644934068$$

Geometric series

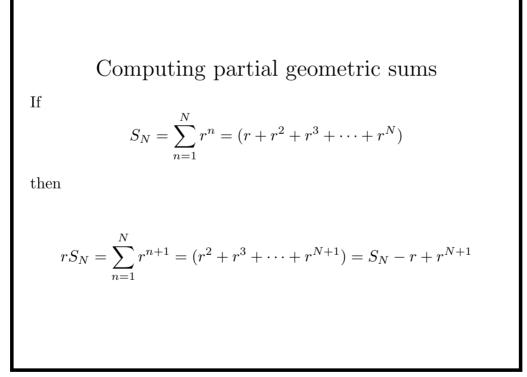
The series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is an example of a geometric series. Computing, we find $S_1 = 0.5, S_2 = 0.75, S_3 = 0.875, S_4 = 0.9375, S_{10} = .9990234375$. In fact, $S_N \to 1$.

A geometric series is a series of the form

$$\sum_{n=1}^{\infty} r^n$$

In the above case $r = \frac{1}{2}$.

7



Computation, continued

Subtracting S_N from both sides, we obtain

$$(r-1)S_N = r^{N+1} - r$$

Hence,

$$S_N = \frac{r^{N+1} - r}{r - 1}$$

9

Computing infinite geometric sums

 So

$$\sum_{n=1}^{\infty} r^n = \lim_{N \to \infty} \frac{r^{N+1} - r}{r - 1}$$

provided that the limit on the right exists.

If |r| < 1, then $\lim_{N \to \infty} r^{N+1} = 0$, so that

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$$

If |r| > 1, then $\lim_{N\to\infty} r^{N+1}$ does not exist, so $\sum_{n=1}^{\infty} r^n$ diverges. Finally, in the case that |r| = 1, we have already seen that the series diverges.

Using the geometric sum formula

Compute the following sums

- $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots$ $-\frac{1}{2} + \frac{1}{4} \frac{1}{8} + \frac{1}{16} + \cdots$

11

Solution

In the first case, we may write the sum as $1 + \sum_{n=1}^{\infty} (\frac{2}{3})^n$. So, the sum is $1 + \frac{\frac{2}{3}}{1 - \frac{2}{3}} = 1 + \frac{\frac{2}{3}}{\frac{1}{3}} = 1 + 2 = 3$. In the second case, we may write the sum as $\sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^n$ so that this sum is $\frac{\frac{-1}{2}}{1-\frac{-1}{2}} = \frac{\frac{-1}{2}}{\frac{3}{2}} = \frac{-1}{3}$.