Infinite Series and the Residue Theorem

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Abstract

After a brief review o Residue Calculus and the Residue Theorem we will investigate an application of the Residue Theorem to evaluating Infinite Series. From this we will derive a summation formula for particular infinite series and consider several series of this type along with an extension of our technique.

1 Introduction

The Residue Theorem (Theorem 2.1) proves invaluable in complex integration and even in the evaluation of particularly trouble some real integrals. But this is not where this theorems use ends, in face it can be applied in the opposite direction and use integrals to evaluate infinite sums. Using this fact we can develop a formula for evaluating series of the form

$$\sum_{n=-\infty}^{\infty} f(n)$$

with a given function f (see Section 3). We will begin with a short review of Residue Calculus, develop the summation formula previously mentioned, and then apply it to a few series, in particular,

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

of Euler fame. We will conclude with an extension and variation of the summation formula.

2 The Residue Calculus

The technique we will develop relies heavily on the Residue Theorem, so before considering any infinite series, let us briefly review a few aspects of Residue Calculus (Note: Very briefly, for a full discussion on Residue Calculus see [G], [HM] or [SS]).

Recall (as seen in [G]) if we have a function f with an isolated singularity at z_0 and Laurent Series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad 0 < |z - z_0| < \rho,$$

we define the ***residue*** of $f(z)$ at z_0 to be the coefficient a_{-1} of $\dfrac{1}{z-z_0}$ in this Laurent expansion. That is

$$\text{Res}\,[\,f(z),\, z_0\,] = a_{-1} = \frac{1}{2\pi i}\oint_{|z-z_0|=r} f(z)\,dz$$

where r is any fixed radius satisfying $0 < r < \rho$.

And now, the Residue Theorem taken from [G]:

Theorem 2.1. *(Residue Theorem) Let D be a bounded domain in $\mathbb{C}$ with piecewise smooth boundary. Suppose $f(z)$ is meromorphic on $D \cup \partial D$ such that $f(z)$ has finitely many singularities at $\{z_1, \ldots, z_2\} \in D$. Then*

$$\oint_{\partial D} f(z)\,dz = 2\pi i \sum \{\textit{residues of } f \textit{ at the poles of } f \textit{ contained by } \partial D\}\,.$$

An intuitive proof of this theorem can be found in [HM]. Our technique will involve calculating specific residues, these are only two of the many methods to calculate a residue (notice that you can always expand the Laurent series to find a_{-1}) but will prove valuable in the examples we cover later. First for a simple pole at z_0,

Remark 2.2. *If $f(z)$ has a simple pole at z_0, then*

$$\text{Res}\,[\,f(z),\, z_0\,] = \lim_{z\to z_0} (z-z_0)f(z)$$

where $\text{Res}\,[\,f(z),\, z_0\,]$ *is residue of f at $z = z_0$.*

And for a double pole at z_0,

Remark 2.3. *If $f(z)$ has a double pole at z_0, then*

$$\text{Res}\,[\,f(z),\, z_0\,] = \lim_{z\to z_0} \frac{d}{dz}\left\{(z-z_0)^2 f(z)\right\}$$

.

3 Evaluating Infinite Series

We will now develop a general technique to evaluate infinite series of the form

$$\sum_{n=-\infty}^{\infty} f(n)$$

where $f(n)$ is a given function. First let us restrict $f(n)$ to be a meromorphic function (i.e. analytic in $\mathbb{C}$ except for some subset of $\mathbb{C}$), that is f has a finite number of poles, further let f be such that none of these poles are integers. Suppose G(z) is a meromorphic function whose poles are all simple at $z \in \mathbb{Z}$, and that the residues are all 1. Therefore the residues of $G(z)f(z)$ are $f(n)$ [HM]. Consider the closed curve C_N, a square that encloses the points $-N, -N+1, \ldots, -1, 0, 1, \ldots, N-1, N$, as seen in Figure 1. (Note: C_N can be any closed curve enclosing these points) [SO].

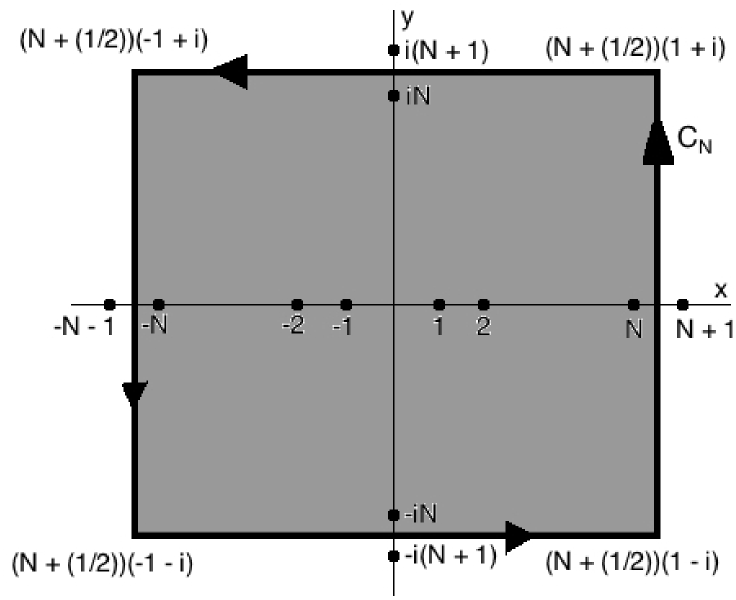

Figure 1, the curve C_N

By Theorem 2.1 we know,

$$\oint_{C_N} G(z)f(z)\,dz = 2\pi i \sum \{\text{residues of } G(z)f(z) \text{ within } C_N\}$$

That is to say

$$\oint_{C_N} G(z)f(z)\,dz = 2\pi i \sum \{\text{residues of } G(z)f(z) \text{ within } C_N\}$$

$$= 2\pi i \sum \{\text{residues of } G(z)f(z) \text{ within } C_N \text{ at poles of } G(z \in \mathbb{Z})\}$$
$$+ 2\pi i \sum \{\text{residues of } G(z)f(z) \text{ within } C_N \text{ at poles of } f\}$$
$$= 2\pi i \sum_{n=-N}^{N} f(n) + 2\pi i \sum \{\text{residues of } G(z)f(z) \text{ within } C_N \text{ at poles of } f\}$$

So, if $\oint_{C_N} G(z)f(z)\,dz$ has a convergent limit as C_N gets large, that is as $N \to \infty$, we will be able to conclude things regarding

$$\lim_{N\to\infty} \sum_{n=-N}^{N} f(n) = \sum_{n=-\infty}^{\infty} f(n)$$

Note: If some of f's poles are at integers then we can reorder terms such that (from [HM]):

$$\oint_{C_N} G(z)f(z)\,dz = 2\pi i \sum_{n=-N}^{N} \{f(n)\,|\,n \text{ is not a singularity of } f\}$$
$$+2\pi i \sum \{\text{residues of } G(z)f(z) \text{ within } C_N \text{ at poles of } f\}$$

$\pi\cot(\pi z)$ satisfies the restrictions on $G(z)$ wonderfully, so let $\pi\cot(\pi z) = G(z)$. Following from this we have the summation formula [HM]:

$$\sum_{n=-\infty}^{\infty} \{f(n)|n \text{ is not a singularity of } f\} = -\sum \{\text{residues of } \pi cot(\pi z)f(z) \text{ at singularities of } f\},$$

the very tool we wished to develop. Proving this equality requires a slight bit of machinery so, in that effort first let us consider $\cot(\pi z)$ on C_N, as in [SO]:

Lemma 3.1. *Let C_N be a square with vertices at*

$$(N+\tfrac{1}{2})(1+i), \qquad (N+\tfrac{1}{2})(-1+i), \qquad (N+\tfrac{1}{2})(-1-i), \qquad (N+\tfrac{1}{2})(1-i)$$

as can be seen in Figure 1, then on C_N, $|\cot(\pi z)| < A$ where A is a constant.

This proof can be seen in [SO].

Proof: We will consider the parts of C_N where $y > \frac{1}{2}$, $-\frac{1}{2} \leq y \leq \frac{1}{2}$ and $y < -\frac{1}{2}$.

Case 1: $y > \frac{1}{2}$. Let $z = x + iy$, then

$$|\cot(\pi z)| = \left|\frac{e^{\pi iz} + e^{-\pi iz}}{e^{\pi iz} - e^{-\pi iz}}\right| = \left|\frac{e^{\pi ix - \pi y} + e^{-\pi ix + \pi y}}{e^{\pi ix - \pi y} - e^{-\pi ix + \pi y}}\right| \leq \frac{\left|e^{\pi ix - \pi y}\right| + \left|e^{-\pi ix + \pi y}\right|}{\left|e^{-\pi ix + \pi y}\right| - \left|e^{\pi ix - \pi y}\right|}$$
$$= \frac{e^{-\pi y} + e^{\pi y}}{e^{\pi y} - e^{-\pi y}} = \frac{1 + e^{-2\pi y}}{1 - e^{-2\pi y}} \leq \frac{1 + e^{-\pi}}{1 - e^{-\pi}} = A_1$$

Case 2: $y < -\frac{1}{2}$. Here, similar to Case 1, we have

$$|\cot(\pi z)| \leq \frac{\left|e^{\pi ix - \pi y}\right| + \left|e^{-\pi ix + \pi y}\right|}{\left|e^{\pi ix - \pi y}\right| - \left|e^{-\pi ix + \pi y}\right|} = \frac{e^{-\pi y} + e^{\pi y}}{e^{-\pi y} - e^{\pi y}} = \frac{1 + e^{2\pi y}}{1 - e^{2\pi y}} \leq \frac{1 + e^{-\pi}}{1 - e^{-\pi}} = A_1$$

Case 3: $-\frac{1}{2} \leq y \leq \frac{1}{2}$. This time, consider $z = N + \frac{1}{2} + iy$. Then we have

$$|\cot(\pi z)| = \left|\cot(\pi(N + \frac{1}{2} + iy))\right| = |\cot(\pi/2 + \pi iy)| = |\tanh(\pi y)| \leq \tanh\left(\frac{\pi}{2}\right) = A_2$$

And if $z = -N - \frac{1}{2} + iy$, we have similarly that

$$|\cot(\pi z)| = \left|\cot(\pi(-N - \frac{1}{2} + iy))\right| = |\tanh(\pi y)| \leq \tanh\left(\frac{\pi}{2}\right) = A_2$$

So choose A such that $A > max\{A_1, A_2\}$. Then we have $|\cot(\pi z)| < A$ on C_N with an A independent of N. $\square$

Now, equipped with Lemma 3.1 we can strive to prove the ***Summation Theorem***, the necessary and sufficient conditions for our formula to hold (statement and proof adapted from [HM] and [SO]).

Theorem 3.2. *(Summation Theorem) Let $f(z)$ be analytic in $\mathbb{C}$ except for some finite set of isolated singularities. Also, let $|f(z)| < \frac{M}{|z|^k}$ along the path C_N(Figure 1), where $k > 1$ and M are constants independent of N. Then we have the* ***summation formula****:*

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum\{\textit{residues of } \pi\cot(\pi z)f(z) \textit{ at } f\textit{'s poles}\}$$

Proof: Since $f(z)$ has finitely many singularities we will begin by choosing a large enough N such that C_N(Figure 1) contains all of the poles of $f(z)$. Assume $f(z)$ has no poles at n, $\forall n \in \mathbb{Z}$ since the given series would diverge otherwise. The poles of $\cot(\pi z)$ are simple and occur at $z = 0, \pm 1, \pm 2, \ldots$ (i.e. $z \in \mathbb{Z}$).
Thus, using L'Hospital's rule, the residues of $\pi\cot(\pi z)f(z)$ at $z = n, n \in \mathbb{Z}$ are

$$\lim_{z\to n}(z-n)\pi\cot(\pi z)f(z) = \lim_{z\to n}\pi\left(\frac{z-n}{\sin(\pi z)}\right)\cos(\pi z)f(z) = f(n)$$

By the residue theorem (Theorem 2.1),

$$\oint_{C_N} \pi\cot(\pi z)f(z)\,dz = \sum_{n=-N}^{N} f(n) + \mathcal{S}$$

Where $\mathcal{S}$ is the sum of the residues of $\pi\cot(\pi z)f(z)$ at the poles of $f(z)$. By the assumption on $f(z)$ and Lemma 3.1 we can see that

$$\left| \oint_{C_N} \pi \cot(\pi z) f(z)\, dz \right| \leq \frac{\pi AM}{N^k}(8N+4)$$

because $(8N+4)$ is the length of our curve C_N. Now, consider the limit as $N \to \infty$, we have

$$\begin{aligned} \lim_{N\to\infty} \oint_{C_N} \pi \cot(\pi z) f(z)\, dz &= \lim_{N\to\infty} \sum_{n=-N}^{N} f(n) + \mathcal{S} \\ 0 &= \sum_{n=-\infty}^{\infty} f(n) + \mathcal{S} \end{aligned}$$

Thus, as we wished, we have

$$\sum_{n=-\infty}^{\infty} f(n) = -\mathcal{S}$$

That is,

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum \{\text{residues of } \pi \cot(\pi z) f(z) \text{ at } f\text{'s poles}\}$$

□

So we have obtained a formula to evaluate a common form of the infinite series; Theorem 3.2 will prove to be a very useful armament in it's simplicity by greatly simplifying these evaluations.

4 Examples

To begin, a couple of simple examples from [SO]:

Example 4.1. *Prove that* $\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\coth(\pi a)$ *where* $a > 0$.

Proof: *Let* $f(z) = \frac{1}{z^2+a^2}$, *which has simple poles at* $z = \pm ai$.
Using Remark 2.2, the residue of $\frac{\pi\cot(\pi z)}{z^2+a^2}$ *at* $z = ai$ *is*

$$\lim_{z\to ai}(z-ai)\frac{\pi\cot(\pi z)}{z^2+a^2} = \lim_{z\to ai}(z-ai)\frac{\pi\cot(\pi z)}{(z-ai)(z+ai)} = \frac{\pi\cot(\pi ai)}{2ai} = -\frac{\pi}{2a}\coth(\pi a)$$

Similarly, the residue at $z = -ai$ *is* $-\frac{\pi}{2a}\coth(\pi a)$.
Therefore, the sum of the residues is $-\frac{\pi}{a}\coth(\pi a)$. *So, by the Summation Theorem we have*

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = -\left(-\frac{\pi}{2a}\coth(\pi a)\right) = \frac{\pi}{2a}\coth(\pi a)$$

□

Example 4.2. *Prove that* $\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{2a}\coth(\pi a) - \frac{1}{2a^2}$ *where* $a > 0$.

Proof: *Consider the following rewrite of Example 4.1 where* $\frac{1}{a^2} = f(0)$*:*

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} &= \frac{\pi}{a}\coth(\pi a) \\ \sum_{n=-\infty}^{-1} \frac{1}{n^2+a^2} + \frac{1}{a^2} + \sum_{n=1}^{\infty} \frac{1}{n^2+a^2} &= \frac{\pi}{a}\coth(\pi a) \\ 2\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} + \frac{1}{a^2} &= \frac{\pi}{a}\coth(\pi a) \end{aligned}$$

since f *is even. Therefore, we have*

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{1}{2}\left(\frac{\pi}{a}\coth(\pi a) - \frac{1}{a^2}\right) = \frac{\pi}{2a}\coth(\pi a) - \frac{1}{2a^2}$$

□

And now a concrete example involving a double pole, taken from an exercise in [SS]

Example 4.3. *Verify that* $\sum_{n=-\infty}^{\infty} \frac{1}{(n-\frac{1}{2})^2} = \pi^2$.

Proof: *Let* $fz = \frac{1}{(z-\frac{1}{2})^2}$*;* f *has a double pole at* $z = \frac{1}{2}$.

Using Remark 2.3, the residue of $\frac{\pi\cot(\pi z)}{(z-\frac{1}{2})^2}$ *at* $z = \frac{1}{2}$ *is*

$$\lim_{z\to\frac{1}{2}} \frac{d}{dz}\left\{(z-\frac{1}{2})^2\frac{\pi\cot(\pi z)}{(z-\frac{1}{2})^2}\right\} = \lim_{z\to\frac{1}{2}} \frac{d}{dz}\{\pi\cot(\pi z)\} = \lim_{z\to\frac{1}{2}} -\pi^2(\csc(\pi z))^2 = -\pi^2$$

So, by the summation theorem,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n-\frac{1}{2})^2} = -(-\pi^2) = \pi^2$$

□

To conclude this section, we will consider the famous series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which Leonhard Euler proved to converge to $\frac{\pi^2}{6}$ in 1741 using the Taylor series expansion of $\sin(x)$. Using the summation formula, we can verify this equality another way. This approach is found in [HM].

Example 4.4. *Prove that* $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Proof: *Let* $f(z) = \frac{1}{z^2}$. $\cot(z)$ *has a simple pole at* $z = 0$ *because* $\tan(z)$ *has a simple zero there. If the Laurent expansion is* $\cot(z) = \frac{b_1}{z} + a_0 + a_1 z + \cdots$, *then*

$$\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right) = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\left(\frac{b_1}{z} + a_0 + a_1 z + \cdots\right)$$

If we multiply, collect terms and then equate coefficients we find that $b_1 = 0$, $a_0 = 0$ *and* $a_1 = -\frac{1}{3}$. *Thus,*

$$\frac{\pi\cot(\pi z)}{z^2} = \frac{\pi(\frac{1}{\pi z} - \frac{\pi z}{3} + \cdots)}{z^2} = \frac{1}{z^3} - \frac{\pi^2}{3z} + \cdots$$

Hence the residue of $\frac{\pi\cot(\pi z)}{z^2}$ *at* $z = 0$ *is* $\frac{-\pi^2}{3}$. $z = 0$ *is the only singularity of* f *so the summation formula tells us*

$$\begin{aligned} \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{n^2} &= \frac{\pi^2}{3} \\ \lim_{N\to\infty} \left(\sum_{n=-N}^{-1} \frac{1}{n^2} + \sum_{n=1}^{N} \frac{1}{n^2}\right) &= \frac{\pi^2}{3} \end{aligned}$$

and because f *is even, i.e.* $\frac{1}{(-n)^2} = \frac{1}{n^2}$, *we see that*

$$\begin{aligned} \lim_{N\to\infty} 2\sum_{n=1}^{N} \frac{1}{n^2} &= \frac{\pi^2}{3} \\ \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^2} &= \frac{\pi^2}{6} \end{aligned}$$

So we can conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

□

5 An Extension

The summation formula can be extended for different trigonometric functions, allowing us to solve more infinite series. For instance, this form taken from [SO] chooses $\pi\csc(\pi)$ as $G(z)$ rather than $\pi\cot(\pi z)$

Corollary 5.1. *Let $f(z)$ be such that the hypothesis of Theorem 3.2 is satisfied. Then we have*

$$\sum_{n=-\infty}^{\infty}(-1)^n f(n) = -\sum\{\textit{residues of } \pi\csc(\pi z)f(z) \textit{ at the poles of } f\}$$

Again, to prove this we need a slight bit of back up; let us first investigate $\csc(\pi z)$ on C_N:

Lemma 5.2. *Consider again the curve C_N from Section 4, as in Figure 1. For all z on C_N, $|\csc(\pi z)| < A$ where A is some constant.*

Proof: In an effort similar to the proof of Lemma 3.1, let us consider the parts of C_N which lie in the regions $y > \frac{1}{2}$, $-\frac{1}{2} \leq y \leq \frac{1}{2}$ and $y < -\frac{1}{2}$.

Case 1: Let $y > \frac{1}{2}$ and $z = x + iy$, then

$$|\csc(\pi z)| = \left|\frac{2i}{e^{\pi iz} - e^{-\pi iz}}\right| = \left|\frac{2i}{e^{\pi ix-\pi y} - e^{-\pi ix+\pi y}}\right| \leq \frac{|2i|}{|e^{-\pi ix+\pi y}| - |e^{\pi ix-\pi y}|}$$
$$= \frac{2}{e^{\pi y} - e^{-\pi y}} = \frac{2}{1 - e^{-2\pi y}} \leq \frac{2}{1 - e^{-\pi}} = A_1$$

Case 2: Let $y < -\frac{1}{2}$ and again, $z = x + iy$. Then similar to Case 1

$$|\csc(\pi z)| \leq \frac{|2i|}{|e^{-\pi ix+\pi y}| - |e^{\pi ix-\pi y}|}$$
$$= \frac{2}{e^{-\pi y} - e^{\pi y}} = \frac{2}{1 - e^{2\pi y}} \leq \frac{2}{1 - e^{-\pi}} = A_1$$

Case 3: $-\frac{1}{2} \leq y \leq \frac{1}{2}$. As before, consider $z = N + \frac{1}{2} + iy$. Then we have

$$|\csc(\pi z)| = \left|\csc(\pi(N + \frac{1}{2} + iy))\right| = |\csc(\pi/2 + \pi iy)| = |\operatorname{sech}(\pi y)| \leq \operatorname{sech}\left(\frac{\pi}{2}\right) = A_2$$

And if $z = -N - \frac{1}{2} + iy$,

$$|\csc(\pi z)| = \left|\csc(\pi(-N - \frac{1}{2} + iy))\right| = |\text{sech}(\pi y)| \leq \text{sech}\left(\frac{\pi}{2}\right) = A_2$$

So choose A such that $A > max\{A_1, A_2\}$. Then $|\csc(\pi z)| < A$ on C_N with an A independent of N. $\square$

Now with this tool, we can prove Corollary 5.1. This proof, outlined in [SO], follows very close to the proof for Theorem 3.2.

Proof: Let us approach this in a similar manner to the proof of Theorem 3.2. The poles of $\csc(\pi z)$ are simple at $\{z | z \in \mathbb{Z}\}$.
The residues of $\pi \csc(\pi z) f(z)$ at $z = n, n = 0, \pm 1, \pm 2, ...$, are

$$\lim_{z \to n} (z - n)\pi \csc(\pi z) f(z) = \lim_{z \to n} \pi \left(\frac{z - n}{\sin(\pi z)}\right) f(z) = (-1)^n f(n)$$

and by the residue theorem,

$$\oint_{C_N} \pi \csc(\pi z) f(z)\, dz = \sum_{n=-N}^{N} (-1)^n f(n) + \mathcal{R}$$

Where $\mathcal{R}$ is the sum of the residues of $\pi \csc(\pi z) f(z)$ at the poles of $f(z)$. $|f(z)| \leq \frac{M}{|z|^k}$ so by Lemma 5.2 we can see that

$$\left|\oint_{C_N} \pi \csc(\pi z) f(z)\, dz\right| \leq \frac{\pi A M}{N^k}(8N + 4)$$

where the length of C_N is $(8N + 4)$. Now, as in the proof of Theorem 3.2, if we the limit as $N \to \infty$, we have

$$\oint_{C_N} \pi \csc(\pi z) f(z)\, dz \Longrightarrow 0$$

Therefore,

$$0 = \lim_{N \to \infty} \sum_{n=-N}^{N} (-1)^n f(n) + \mathcal{R}$$

And as we wished, we have

$$\sum_{n=-\infty}^{\infty} f(z) = -\mathcal{R}$$

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum \{\text{residues of } \pi \cot(\pi z) f(z) \text{ at } f\text{'s poles}\}$$

□

And now one quick example using this new tool, found in [SO].

Example 5.3. *Show that* $\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)^2} = \frac{\pi^2 \cos(\pi a)}{(\sin(\pi a))^2}$ *where* $a \in \mathbb{R} \setminus \mathbb{Z}$

Proof: *Let* $f(z) = \frac{1}{(z+a)^2}$ *which has a double pole at* $z = -a$.
The residue of $\frac{\pi \csc(\pi z)}{(z+a)^2}$ *at* $z = -a$ *is*

$$\lim_{z \to -a} \frac{d}{dz} \left\{ (z+a)^2 \frac{\pi \csc(\pi z)}{(z+a)^2} \right\} = \lim_{z \to -a} \frac{d}{dz} \{\pi \csc(\pi z)\}$$
$$= \lim_{z \to -a} -\pi^2 \csc(\pi z) \cot(\pi z) = -\pi^2 \csc(\pi a) \cot(\pi a)$$

Then, by Corollary 5.1

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)^2} = -(-\pi^2 \csc(\pi a) \cot(\pi a)) = \pi^2 \csc(\pi a) \cot(\pi a) = \frac{\pi^2 \cos(\pi a)}{(\sin(\pi a))^2}$$

□

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