INFINITE SERIES
Elizabeth Wood

SERIES AND PARTIAL SUMS

Consider the sequence $\{a_n\}$ where $a_n = \left(\frac{1}{2}\right)^n$. The first 10 terms of this sequence starting with n = 0 are $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}, \frac{1}{512}$.

What if we wanted to sum up the terms of this sequence, how many terms would I have to use? 1, 2, 3, . . . 10, . . . ∞ ? Well, we could start creating sums of a finite number of terms, called partial sums, and determine if the sequence of partial sums converge to a number.

$$S_1 = 1 \qquad\qquad 2-1$$

$$S_2 = 1+\frac{1}{2} \qquad\qquad 2-\frac{1}{2}$$

$$S_3 = 1+\frac{1}{2}+\frac{1}{4} \qquad\qquad 2-\frac{1}{4}$$

$$S_4 = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8} \qquad\qquad 2-\frac{1}{8}$$

$$\vdots$$

$$S_n = 1+\frac{1}{2}+\cdots+\frac{1}{2^n} \qquad\qquad 2-\frac{1}{2^{n-1}}$$

What do you think that this sequence of partial sums is converging to? It is approaching the value of 2. Therefore, we can conclude that the sum of all the terms of this sequence is 2.

To discuss this topic fully, let us define some terms used in this and the following sets of supplemental notes.

DEFINITION: Given a sequence of numbers $\{a_n\}$, the sum of the terms of this sequence, $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$, is called an infinite series.

DEFINITION: The sequence $\{S_n\}$ defined by $S_1 = a_1, S_2 = a_1 + a_2, \ldots, S_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$ is the sequence of partial sums of the series, the number S_n being the nth partial sum.

FACT: If the sequence of partial sums converge to a limit L, then we can say that the series converges and its sum is L.

FACT: If the sequence of partial sums of the series does not converge, then the series diverges.

There are many different types of series, but we going to start with series that we might of seen in Algebra.

GEOMETRIC SERIES

DEFINITION: Geometric series is a series of the form $a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{k=1}^{\infty} ar^{k-1}$ which a and r fixed real numbers and $a \neq 0$.

FACT: If $|r| < 1$, then the geometric series will converge and its sum is $S = \dfrac{a}{1-r}$.

FACT: If | r | ≥ 1, then the geometric series will diverge.

EXAMPLE 1: Find the nth partial sum and determine if the series converges or diverges.

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots + \frac{1}{4^n} + \cdots$$

SOLUTION:

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots + \frac{1}{4^n} = \sum_{k=1}^{n}\left(\frac{1}{4}\right)^k$$

This is a geometric series with ratio r = $\frac{1}{4} < 1$, therefore it will converge.

Now to calculate the sum for this series.

$$a = \frac{1}{4} \quad r = \frac{1}{4} \qquad S = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

EXAMPLE 2: Find the nth partial sum and determine if the series converges or diverges.

$$1 - 3 + 9 - 27 + \ldots + (-1)^{n-1}(-3)^{n-1}$$

SOLUTION:

$$1-3+9-27+\ldots+(-1)^{n-1}(3)^{n-1}=\sum_{k=1}^{n}(-1)^{k-1}(3)^{k-1}$$

This is a geometric series with ratio | r | = |(-1)(3)| = | 3 | ≥ 1, therefore it will diverge.

EXAMPLE 3: Write out the first few terms or the following series to show how the series starts. Then find the sum of the series.

$$\sum_{n=0}^{\infty}\left(\frac{5}{2^n}-\frac{1}{3^n}\right)$$

SOLUTION:

$$\sum_{n=0}^{\infty}\left(\frac{5}{2^n}-\frac{1}{3^n}\right)=(5-1)+\left(\frac{5}{2}-\frac{1}{3}\right)+\left(\frac{5}{4}-\frac{1}{9}\right)+\left(\frac{5}{8}-\frac{1}{27}\right)+\cdots$$

Now to determine the sum of this series. To do this, I will split the original sum into a difference of two sums.

$$\sum_{n=0}^{\infty}\left(\frac{5}{2^n}-\frac{1}{3^n}\right)=\sum_{n=0}^{\infty}\left(\frac{5}{2^n}\right)-\sum_{n=0}^{\infty}\left(\frac{1}{3^n}\right)=10-\frac{3}{2}=\frac{17}{2}$$

$$\sum_{n=0}^{\infty}\left(\frac{5}{2^n}\right):\quad a=5\quad r=\frac{1}{2}\quad S=\frac{5}{1-\frac{1}{2}}=10$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{3^n}\right):\quad a=1\quad r=\frac{1}{3}\quad S=\frac{1}{1-\frac{1}{3}}=\frac{3}{2}$$

EXAMPLE 4: Does this series converge or diverge? If it converges, find its sum.

$$\sum_{n=0}^{\infty}e^{-2n}$$

SOLUTION:

$$\sum_{n=0}^{\infty}e^{-2n}=\sum_{n=0}^{\infty}\left(e^{-2}\right)^n\qquad r=e^{-2}<1\quad \textit{converges}$$

$$a=1\quad S=\frac{1}{1-\frac{1}{e^2}}=\frac{e^2}{e^2-1}$$

EXAMPLE 5: Does this series converge or diverge? If it converges, find its sum.

$$\sum_{n=0}^{\infty}\left(\frac{4}{3}\right)^n$$

SOLUTION:

$$\sum_{n=0}^{\infty}\left(\frac{4}{3}\right)^n \qquad r=\frac{4}{3}\geq 1 \qquad \textit{diverges}$$

EXAMPLE 6: Find the values of x for which the geometric series converges. Also, find the sum of the series (as a function of x) for those values of x.

$$\sum_{n=0}^{\infty}(-1)^n x^{-2n}$$

SOLUTION: For this geometric series to converge, the absolute value of the ration has to be less than 1.

$r = (-1)\ x^{2}$

$|-x^2| < 1 \rightarrow |x^2| < 1 \rightarrow 0 < x^2 < 1 \rightarrow |x| < 1$

It will converge for $-1 < x < 1$. Now to determine the sum.

$$a=1 \quad r=-x^2 \quad S=\frac{1}{1+x^2}$$

EXAMPLE 7: Find the values of x for which the geometric series converges. Also, find the sum of the series (as a function of x) for those values of x.

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n (x-3)^n$$

SOLUTION:

$$r=-\frac{1}{2}(x-3)$$

$$\left|-\frac{1}{2}(x-3)\right|<1 \rightarrow \left|\frac{1}{2}(x-3)\right|<1 \rightarrow -1<\frac{1}{2}(x-3)<1$$
$$\rightarrow -2<x-3<2 \rightarrow 1<x<5$$

This geometric series will converge for values of x that are in the interval (1, 5).

Now to determine the sum.

$$a = 1 \quad S = \frac{1}{1 + \frac{1}{2}(x-3)} = \frac{1}{\frac{2 + x - 3}{2}} = \frac{2}{x-1}$$

TELESCOPING SERIES

Now let us investigate the telescoping series. It is different from the geometric series, but we can still determine if the series converges and what its sum is. To be able to do this, we will use the method of partial fractions to decompose the fraction that is common in some telescoping series.

EXAMPLE 8: Use the method of partial fractions to find the sum of the following series.

$$\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$$

SOLUTION: First we will decompose this fraction using the method of partial fractions.

$$\frac{A}{2n-1} + \frac{B}{2n+1} = \frac{6}{(2n-1)(2n+1)}$$

2An + A + 2Bn - B = 6

2A + 2B = 0 → A = -B

A - B = 6 → -2B = 6 → **B = -3** → **A = 3**

$$S_k = \sum_{n=1}^{k} \frac{6}{(2n-1)(2n+1)} = \sum_{n=1}^{k} \left(\frac{3}{2n-1} - \frac{3}{2n+1} \right)$$

$$= 3 - 1 + 1 - \frac{3}{5} + \frac{3}{5} - \frac{3}{7} + \dots + \frac{3}{2k-1} - \frac{3}{2k+1} = 3 - \frac{3}{2k+1}$$

Now let us evaluate the limit of S_k as k goes to infinity.

$$\lim_{k\to\infty}\left(3-\frac{3}{2k+1}\right)=3$$

Therefore it converges and its sum is 3.

EXAMPLE 9: Use partial fractions to find the sum of this series.

$$\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}$$

SOLUTION: $\frac{A}{n}+\frac{B}{n^2}+\frac{C}{n+1}+\frac{D}{(n+1)^2}=\frac{2n+1}{n^2(n+1)^2}$

$An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2 = 2n + 1$

$An^3 + 2An^2 + An + Bn^2 + 2Bn + B + Cn^3 + Cn^2 + Dn^2 = 2n + 1$

$A + C = 0 \rightarrow$ **$C = 0$**

$2A + B + C + D = 0 \rightarrow 0 + 1 + 0 + D = 0 \rightarrow$ **$D = -1$**

$A + 2B = 2 \rightarrow A + 2 = 2 \rightarrow$ **$A = 0$**

$B = 1$

$$\text{So } S_k=\sum_{n=1}^{k}\frac{2n+1}{n^2(n+1)^2}=\sum_{n=1}^{k}\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)$$

$$=1-\frac{1}{4}+\frac{1}{4}-\frac{1}{9}+\cdots+\frac{1}{k^2}-\frac{1}{(k+1)^2}=1-\frac{1}{(k+1)^2}$$

Now let us evaluate the limit of S_k as k goes to infinity.

$$\lim_{k\to\infty}\left(1-\frac{1}{(k+1)^2}\right)=1$$

Therefore, this series converges and it sum is 1.

So far we have been able to determine that the following types of series converge: geometric series with $|r| < 1$ and telescoping series. We also know that the geometric series with ratio $|r| \geq 1$ diverges, but not all series fit into these two types. Therefore, this makes determining if the series converges or diverges harder. To help us in this task, there are several tests we can use. The first on is the **nth term test for divergence**.

Nth TERM TEST FOR DIVERGENCE

Before I state the criterion for this test, I will state this basic fact that will make this test possible.

FACT: If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

Note that this fact is not saying $a_n \to 0$, then $\sum_{n=1}^{\infty} a_n$ converges. It could possibly diverge. All it is saying is that converging series have the trait tha t the nth term goes to zero.

Nth Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or different from zero.

This is the only test used to determine divergence of a series. If the nth term goes to zero, you cannot conclude that the series converges.

EXAMPLE 10: Does the following series converge or diverge?

$$\sum_{n=1}^{\infty} \ln \frac{1}{n}$$

SOLUTION:

$$a_n = \ln \frac{1}{n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln \frac{1}{n} = -\infty$$

$$\text{as } n \to \infty, \frac{1}{n} \to 0$$

The limit does not exist, therefore, the series diverges by the nth term test for divergence.

EXAMPLE 11: Does the following series converge or diverge.

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$$

SOLUTION:

$$a_n = \ln\left(\frac{n}{2n+1}\right)$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{n}{2n+1}\right) = \ln\frac{1}{2}$$

The limit exist, therefore, the series diverges by the nth term test for divergence.

EXAMPLE 12: Does the following series converge or diverge?

$$\sum_{n=0}^{\infty} \frac{e^{n\pi}}{\pi^{ne}}$$

SOLUTION:

This is a geometric series with ratio, $r = \dfrac{e^{\pi}}{\pi^{e}} > 1$, therefore this series diverges.

EXAMPLE 13: Does the following series converge or diverge?

$$\sum_{n=0}^{\infty} \frac{n!}{1000^n}$$

SOLUTION:

From the list of limits that arise frequently, the $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0$. Well,

the nth term, $a_n = \dfrac{n!}{1000^n}$ is the reciprocal of the above nth term.

So the $\lim\limits_{n\to\infty} \dfrac{n!}{1000^n} = \infty$. (n! will eventually grow faster tha n 1000^n.)

So this series diverges by the nth term test for divergence.

In summary, we have dealt with two specific types of series - geometric and telescoping series. We have learned how to determine if these series converge or diverge. We have also discussed a test that we can use to determine if a series diverges. Now we will build on these types of series by a investigating different types of series and test that we can use to determine their convergence or divergence. Work through these examples taking note of the types of series that you will encounter.