INFINITE SERIES

2.1 Sequences: A sequence of real numbers is defined as a function $f: \boldsymbol{N} \rightarrow \boldsymbol{R}$,where $\boldsymbol{N}$ is a set of natural numbers and **R** is a set of real numbers. A sequence can be expressed as$\langle f_1, f_2, f_3, \ldots \ldots, f_n, \ldots. \rangle$or $\langle f_n \rangle$. For example $\langle \frac{1}{n} \rangle = \langle \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots \ldots. \rangle$ is a sequence.

Convergent sequence: A sequence $\langle u_n \rangle$ converges to a number l, if for given $\varepsilon > 0$, there exists a positive integer m depending on ε , such that $|u_n - l| < \varepsilon \; \forall \; n \geq m$.

Then l is called the limit of the given sequence and we can write

$\lim_{n \to \infty} u_n = l$ or $u_n \rightarrow l$

2.2 Definition of an Infinite Series

An expression of the form $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ is known as the infinite series of real numbers, where each $u_n$is a real number. It is denoted by$\sum_{n=1}^{\infty} u_n$.

For example $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$is an infinite series.

Convergence of an infinite series

Consider an infinite series $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots$

Let us define$S_1 = u_1$, $S_2 = u_1 + u_2$, $S_3 = u_1 + u_2 + u_3 , \ldots \ldots..$,

$S_n = u_1 + u_2 + u_3 + \cdots + u_n$ and so on .

Then the sequence $\langle S_n \rangle$ so formed is known as the sequence of partial sums (S.O.P.S.) of the given series.

Convergent series: A series $u_1 + u_2 + u_3 + \cdots + u_n + \cdots = \sum_{n=1}^{\infty} u_n$ converges if the sequence $\langle S_n \rangle$ of its partial sums converges i.e. if $\lim_{n\to\infty} S_n$ exists. Also if $\lim_{n\to\infty} S_n = S$ then S is called as the sum of the given series .

Divergent series: A series $u_1 + u_2 + u_3 + \cdots + u_n + \cdots = \sum_{n=1}^{\infty} u_n$ diverges if the sequence $\langle S_n \rangle$ of its partial sums diverges i.e. if $\lim_{n\to\infty} S_n = +\infty \ or - \infty$.

Example 1 Show that the Geometric series $\sum_{n=1}^{\infty} r^{n-1} = 1 + r + r^2 + r^3 + \ldots..$, where $r > 0$, is convergent if $r < 1$ and diverges if $r \geq 1$.

Solution: Let us define $S_1 = 1$, $S_2 = 1 + r$, $S_3 = 1 + r + r^2$, … … . .,

$$S_n = 1 + r + r^2 + \cdots + r^{n-1}$$

Case 1: $r < 1$

Consider $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-r^n}{1-r} = \frac{1}{1-r} - \lim_{n\to\infty} \frac{r^n}{1-r}$

$= \frac{1}{1-r}$ (As $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$)

Since $\lim_{n\to\infty} S_n$ is finite ∴ the sequence of partial sums i.e. $\langle S_n \rangle$ converges and hence the given series converges.

Case2: $r > 1$

Consider $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{r^n-1}{r-1} = \lim_{n\to\infty} \frac{r^n}{1-r} - \frac{1}{r-1}$

$\to \infty$ (As $r^n \to \infty$ if $r > 1$)

Since $\langle S_n \rangle$ diverges and hence the given series diverges.

Case2: $r = 1$

Consider $S_n = 1 + r + r^2 + \cdots + r^{n-1}$

$= 1 + 1 + 1 + 1 + \cdots..+1 = n \Rightarrow \lim_{n\to\infty} S_n = \infty$

Since $\langle S_n \rangle$ diverges and hence the given series diverges.

Positive term series

An infinite series whose all terms are positive is called a positive term series.

p-series: An infinite series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots (p > 0)$ is called p-series.

It converges if $p > 1$ and diverges if $p \leq 1$.

For example:

1. $\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$ converges (As $p = 3 > 1$)

2. $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}} = \frac{1}{1^{5/2}} + \frac{1}{2^{5/2}} + \frac{1}{3^{5/2}} + \cdots$ converges $\left(\text{As } p = \frac{5}{2} > 1\right)$

3. $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \frac{1}{1^{1/2}} + \frac{1}{2^{1/2}} + \frac{1}{3^{1/2}} + \cdots$ converges $\left(\text{As } p = \frac{1}{2} < 1\right)$

Necessary condition for convergence:

If an infinite series $\sum_{n=1}^{\infty} u_n$ is convergent then $\lim_{n\to\infty} u_n = 0$. However, converse need not be true.

Proof: Consider the sequence $\langle S_n \rangle$ of partial sums of the series $\sum_{n=1}^{\infty} u_n$.

We know that $S_n = u_1 + u_2 + u_3 + \cdots\cdots\cdots\cdots + u_n$

$$= u_1 + u_2 + u_3 + \cdots + u_{n-1} + u_n$$

$$\Rightarrow S_{n-1} = u_1 + u_2 + u_3 + \cdots\cdots\cdots + u_{n-1}$$

Now $S_n - S_{n-1} = u_n$

Taking limit $n \to \infty$, we get

$$\lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} u_n$$

$\Rightarrow \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} u_n$ ....(1)

As $\sum_{n=1}^{\infty} u_n$ is convergent $\therefore$ sequence $\langle S_n \rangle$ of its partial sums is also convergent.

Let $\lim_{n\to\infty} S_n = l$, then $\lim_{n\to\infty} S_{n-1} = l$

Substituting these values in equation (1), we get $\lim_{n\to\infty} u_n = 0$.

To show that converse may not hold , let us consider the series$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n}$.

Here $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

But$\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series (As $p = 1$)

Corollary: If $\lim_{n\to\infty} u_n \neq 0$, then $\sum_{n=1}^{\infty} u_n$cannot converge.

Example 2 Test the convergence of the series $\sum_{n=1}^{\infty} cos\frac{1}{n}$

Solution: Here $u_n = cos\frac{1}{n} \Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} cos\frac{1}{n} = 1 \neq 0$

Hence the given series is not convergent.

Example 3 Test the convergence of the series $\sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}$

Solution: Here $u_n = \sqrt{\frac{n}{n+1}} \Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}}$

$$\Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n}}} = 1 \neq 0$$

Hence the given series is not convergent.

2.3 Tests for the convergence of infinite series

1. Comparision Test:

Let $\sum_{n=1}^{\infty} u_n$and $\sum_{n=1}^{\infty} v_n$be two positive term series such that

$u_n \leq kv_n \;\; \forall \;\; n$ (where k is a positive number)

Then (i) If $\sum_{n=1}^{\infty} v_n$converges then $\sum_{n=1}^{\infty} u_n$also converges.

(ii) If $\sum_{n=1}^{\infty} u_n$diverges then $\sum_{n=1}^{\infty} v_{\,n}$also diverges.

Example 4 Test the convergence of the following series

$(i)\sum_{n=1}^{\infty} \frac{1}{n^n}$ $\qquad$ $(ii)\sum_{n=2}^{\infty} \frac{1}{log\, n}$ $\qquad$ $(iii)\sum_{n=1}^{\infty} \frac{1}{2^n+x} \forall x > 0$

Solution: (i) Here $u_n = \frac{1}{n^n}$ We know that $n^n > 2^n$ for $n > 2$

Hence $\frac{1}{n^n} < \frac{1}{2^n}$ for $n > 2$

Now $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series $(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots)$whose common ratio is $\frac{1}{2}$.

Since $\frac{1}{2} < 1 \therefore \sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent series. Thus by comparision test $\sum_{n=1}^{\infty} \frac{1}{n^n}$ is also convergent.

(ii) Here $u_n = \frac{1}{\log n}$ We know that $\log n < n$ for $n \geq 2$

Hence $\frac{1}{\log n} > \frac{1}{n}$ for $n \geq 2$ $\Rightarrow \frac{1}{n} < \frac{1}{log\, n}$ for $n \geq 2$

Now $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series (As $\mathrm{p} = 1$) . Thus by comparision test $\sum_{n=2}^{\infty} \frac{1}{\log n}$ is also divergent.

(iii) Here $u_n = \frac{1}{2^n+x}$. Clearly $2^n + x > 2^n$ $(as\ x > 0)$

$\therefore \frac{1}{2^n+x} < \frac{1}{2^n}$

Now $\sum_{n=1}^{\infty}\frac{1}{2^n}$ is a geometric series $(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots)$whose common ratio is $\frac{1}{2}$.

Since $\frac{1}{2} < 1 \therefore \sum_{n=1}^{\infty}\frac{1}{2^n}$ is a convergent series. Thus by comparision test $\sum_{n=1}^{\infty}\frac{1}{2^n+x}$ is also convergent.

Example 4 Test the convergence of the series$\sum_{n=1}^{\infty}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}\right]$

Solution: Here $u_n = \left[\frac{1}{n^2}+\frac{1}{(n+1)^2}\right]$

Clearly $\frac{1}{n^2}+\frac{1}{(n+1)^2} < \frac{1}{n^2}+\frac{1}{n^2} < \frac{1}{n^2}$

Now $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent series (As p $= 2 > 1)$. Thus by comparision test $\sum_{n=1}^{\infty}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}\right]$ is also convergent.

2. Limit Form Test:

Let $\sum_{n=1}^{\infty}u_n$and $\sum_{n=1}^{\infty}v_n$be two positive term series such that

$\lim_{n\to\infty}\frac{u_n}{v_n} = l$ (where l is a finite and non zero number).

Then $\sum_{n=1}^{\infty}u_n$and $\sum_{n=1}^{\infty}v_n$behave in the same manner i.e. either both converge or both diverge.

Example 5 Test the convergence of the series $\frac{1}{3.7}+\frac{1}{4.9}+\frac{1}{5.11}+\ldots..$

Solution: Here $u_n = \frac{1}{(n+2)(2n+5)}$

Let $v_n = \frac{1}{n^2}$. Now consider $\frac{u_n}{v_n} = \frac{1}{(n+2)(2n+5)}n^2 = \frac{n^2}{2n^2+9n+10}$

$\Rightarrow \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{n^2}{2n^2+9n+10}$

$$=\lim_{n\to\infty}\frac{1}{2+\frac{9}{n}+\frac{10}{n^2}}=\frac{1}{2}$$ (which is a finite and non zero number)

Hence by Limit form test , $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ behave similarly.

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (as $p = 2 > 1$)

$\therefore \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty}\frac{1}{(n+2)(2n+5)}$ also converges.

Example 6 Test the convergence of the series

$(i)\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\frac{1}{\sqrt{4}+\sqrt{5}}+\ldots..(ii)\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

Solution:(i) Here $u_n = \frac{1}{\sqrt{n+1}+\sqrt{n+2}}$

Let $v_n = \frac{1}{\sqrt{n}}$. Now consider $\frac{u_n}{v_n} = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n+2}}$

$$\Rightarrow \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n+2}}$$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}}$$

$$= \frac{1}{\sqrt{2}}$$ (which is a finite and non zero number)

Hence by Limit form test , $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ behave similarly.

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges (as $p = \frac{1}{2} < 1$)

$\therefore \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}+\sqrt{n+2}}$ also diverges.

(ii) Here $u_n = \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

$$=\frac{\sqrt{n+1}-\sqrt{n-1}}{n}\cdot\frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n-1}} = \frac{(n+1)-(n-1)}{n\sqrt{n+1}+\sqrt{n-1}} = \frac{2}{n\sqrt{n+1}+\sqrt{n-1}}$$

Let $v_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$. Now consider $\frac{u_n}{v_n} = \frac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n-1}}$

$$\Rightarrow \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n-1}}$$

$$= \lim_{n\to\infty}\frac{2}{\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}}$$

= 1 (which is a finite and non zero number)

Hence by Limit form test , $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ behave similarly.

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges (as $p = \frac{3}{2} > 1$)

$\therefore \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n-1}}{n}$ also converges.

Example 7 Test the convergence of the series

(*i*) $\sum_{n=1}^{\infty}\left[(n^3+1)^{1/3}-n\right]$ (*ii*) $\sum_{n=1}^{\infty} sin\frac{1}{n}$

Solution:(*i*) Here $u_n = (n^3+1)^{1/3} - n = n\left(1+\frac{1}{n^3}\right)^{1/3} - n$

$$= n\left[1+\frac{1}{3n^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\cdot\frac{1}{n^6}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}\cdot\frac{1}{n^9}+\cdots\right]-n$$

$$= \frac{1}{3n^2}-\frac{1}{9n^5}$$

Let $v_n = \frac{1}{n^2}$.

$\Rightarrow \lim_{n\to\infty}\frac{u_n}{v_n} = \frac{1}{3}$ (which is a finite and non zero number)

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (as $p = 2 > 1$)

$\therefore \sum_{n=1}^{\infty} u_n$ also converges (by Limit form test).

(*ii*) Here $u_n = sin\frac{1}{n}$. Let $v_n = \frac{1}{n}$.

Then $\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{sin\frac{1}{n}}{\frac{1}{n}}$

=1 (which is a finite and non zero number)

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (as $p = 1$)

$\therefore \sum_{n=1}^{\infty} u_n$ also diverges (by Limit form test).

Exercise 2A

Test the convergence of the following series:

1. $\sum_{n=1}^{\infty} e^{-n^2}$ Ans. Convergent
2. $\sum_{n=1}^{\infty} \frac{1}{n^2 \log n}$ Ans. Convergent
3. $\sum_{n=1}^{\infty} \left(\sqrt{n^3+1} - \sqrt{n^3}\right)$ Ans. Convergent
4. $\frac{1}{5} + \frac{\sqrt{2}}{7} + \frac{\sqrt{3}}{9} + \frac{\sqrt{4}}{11} \ldots..$ Ans. Divergent
5. $\frac{1}{1.2^2} + \frac{1}{2.3^2} + \frac{1}{3.4^2} + \ldots..$ Ans. Convergent
6. $\sum_{n=1}^{\infty} \left((n^3+1)^{1/3} - n\right)$ Ans. Divergent
7. $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ Ans. Divergent
8. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} tan \frac{1}{n}$ Ans. Convergent
9. $\frac{1}{1+x} + \frac{1}{2+x} + \frac{1}{3+x} + \ldots..$ Ans. Divergent
10. $\sum_{n=1}^{\infty} \frac{1}{n-1}$ Ans. Divergent

3. D' Alembert's Ratio Test

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = l$$

Then (i) $\sum_{n=1}^{\infty} u_n$ converges if $l < 1$

(ii) $\sum_{n=1}^{\infty} u_n$ diverges if $l > 1$

(iii) Test fails if $l = 1$

Example 8 Test the convergence of the following series:

$(i)\frac{1}{3}+\frac{1}{2.3^2}+\frac{1}{3.3^3}+\frac{1}{4.3^4}\ldots..$ $(ii)\ \frac{1^2 2^2}{1}+\frac{2^2 3^2}{2!}+\frac{3^2 4^2}{3!}+\frac{4^2 5^2}{4!}\ldots..$ $(iii)\sum_{n=1}^{\infty}\frac{n!}{n^n}$

Solution: (i)Here $u_n=\frac{1}{n3^n}$ $\Rightarrow u_{n+1}=\frac{1}{(n+1)3^{n+1}}$

$$\text{Then } \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{n3^n}{(n+1)3^{n+1}}=\lim_{n\to\infty}\frac{n}{(n+1)3}$$

$$=\lim_{n\to\infty}\frac{1}{3\left(1+\frac{1}{n}\right)}\quad = 0<1$$

Hence by Ratio test ,the given series converges.

(ii) Here $u_n=\frac{n^2(n+1)^2}{n!}\Rightarrow u_{n+1}=\frac{(n+1)^2(n+2)^2}{(n+1)!}$

$$\text{Then } \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(n+1)^2(n+2)^2}{(n+1)!}\cdot\frac{n!}{n^2(n+1)^2}$$

$$=\lim_{n\to\infty}\frac{(n+2)^2}{(n+1)}\cdot\frac{1}{n^2}=\lim_{n\to\infty}\frac{1}{(n+1)}\cdot\left(\frac{1+\frac{2}{n}}{1}\right)^2=0\ <1$$

Hence by Ratio test , the given series converges.

(iii) Here $u_n=\frac{n!}{n^n}$ $\Rightarrow u_{n+1}=\frac{(n+1)!}{(n+1)^{(n+1)}}$

$$\text{Then } \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{(n+1)}}\cdot\frac{n^n}{n!}=\lim_{n\to\infty}\frac{n^n}{(n+1)^n}$$

$$=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}=\frac{1}{2.718}<1$$

Hence by Ratio test , the given series converges.

Example 9 Test the convergence of the following series:

$(i)\ \frac{1}{7}+\frac{2!}{7^2}+\frac{3!}{7^3}+\frac{4!}{7^4}\ldots..$ $(ii)\ \left(\frac{1}{3}\right)^2+\left(\frac{1.2}{3.5}\right)^2+\left(\frac{1.2.3}{3.5.7}\right)^2+\left(\frac{1.2.3.4}{3.5.7.9}\right)^2+\ldots..$

Solution: (i) Here $u_n=\frac{n!}{7^n}\Rightarrow u_{n+1}=\frac{(n+1)!}{7^{n+1}}$

$$\text{Then } \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{(n+1)!}{7^{(n+1)}}\cdot\frac{7^n}{n!}$$

$$= \lim_{n\to\infty}\frac{n+1}{7} = \infty > 1$$

Hence by Ratio test , the given series diverges.

(ii) Here $u_n = \left[\frac{1.2.3.4.....n}{3.5.7.9....(2n+1)}\right]^2 \Rightarrow u_{n+1} = \left[\frac{1.2.3.4.....n(n+1)}{3.5.7.9....(2n+1)(2n+3)}\right]^2$

$$\text{Then } \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(\frac{n+1}{2n+3}\right)^2 = \frac{1}{2^2} = \frac{1}{4} < 1$$

Hence by Ratio test , the given series converges.

Example 10 Test the convergence of the following series:

(i) $\frac{x}{\sqrt{5}} + \frac{x^3}{\sqrt{7}} + \frac{x^5}{\sqrt{9}} + \frac{x^7}{\sqrt{11}} + \ldots..$ (ii) $\frac{x}{1.3} + \frac{x^2}{2.4} + \frac{x^3}{3.5} + \frac{x^4}{4.6} + \cdots (x > 0)$

Solution: (i) Here $u_n = \frac{x^{2n-1}}{\sqrt{2n+3}} \Rightarrow u_{n+1} = \frac{x^{2n+1}}{\sqrt{2n+5}}$

$$\text{Then } \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{x^{2n+1}}{\sqrt{2n+5}}\frac{\sqrt{2n+3}}{x^{2n-1}} = x^2$$

Hence by Ratio test , the given series converges if $x^2 < 1$ and diverges if $x^2 > 1$.

Test fails if $x^2 = 1$. i.e. $x = 1$

When $x = 1$, $u_n = \frac{1}{\sqrt{2n+3}}$

Let $v_n = \frac{1}{\sqrt{n}}$. Now consider $\frac{u_n}{v_n} = \frac{\sqrt{n}}{\sqrt{2n+3}}$

$$\Rightarrow \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{2n+3}}$$

$$= \frac{1}{2} \text{ (which is a finite and non zero number)}$$

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges (as $p = \frac{1}{2} < 1$) $\therefore \sum_{n=1}^{\infty} u_n$ also diverges for x=1 (by Limit form test).

$\therefore$ the given series converges for $x < 1$ and diverges for $x \geq 1$.

(ii) Here $u_n = \frac{x^n}{n(n+2)} \Rightarrow u_{n+1} = \frac{x^{n+1}}{(n+1)(n+3)}$

Then $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n(n+2)}{(n+1)(n+3)} x = x$

Hence by Ratio test , the given series converges if $x < 1$ and diverges if $x > 1$

Test fails if $x = 1$

When $x = 1$, $u_n = \frac{1}{n(n+2)}$

Let $v_n = \frac{1}{n^2}$.

$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^2}{n(n+2)}$

$= 1$ (which is a finite and non zero number)

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (as $p = 2 > 1$)

$\therefore \sum_{n=1}^{\infty} u_n$ also converges for $x=1$ (by Limit form test).

$\therefore$ the given series converges for $x \leq 1$ and diverges for $x > 1$.

3. Cauchy's n th Root Test

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that

$$\lim_{n\to\infty} (u_n)^{1/n} = l$$

Then (i) $\sum_{n=1}^{\infty} u_n$ converges if $l < 1$

(ii) $\sum_{n=1}^{\infty} u_n$ diverges if $l > 1$

(iii) Test fails if $l = 1$

Example 11 Test the convergence of the following series:

(*i*) $1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}$….. (*ii*) $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$ (*iii*) $\sum_{n=1}^{\infty}5^{-n-(-1)^n}$

Solution: (*i*) Here $u_n=\frac{1}{n^n}$

$$\Rightarrow \lim_{n\to\infty}(u_n)^{1/n}=\lim_{n\to\infty}\frac{1}{n}=0<1$$

Hence by Cauchy's root test, the given series converges.

(*ii*) Here $u_n=\left(\frac{n}{n+1}\right)^{n^2} \Rightarrow \lim_{n\to\infty}(u_n)^{1/n}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n}$

$$=\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^{n}=\frac{1}{e}<1$$

Hence by Cauchy's root test, the given series converges.

(*iii*) Here $u_n=5^{-n-(-1)^n} \Rightarrow \lim_{n\to\infty}(u_n)^{1/n}=\lim_{n\to\infty}5^{-\{n+(-1)^n\}.1/n}$

$$=\lim_{n\to\infty}5^{-\left\{1+\frac{(-1)^n}{n}\right\}}=5^{-1}$$

$$=\frac{1}{5}<1$$

Hence by Cauchy's root test, the given series converges.

Example 12 Test the convergence of the following series:

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots$$

Solution: Here $u_n=\left[\left(\frac{n+1}{n}\right)^{n+1}-\frac{n+1}{n}\right]^{-n}$

$$\Rightarrow \lim_{n\to\infty}(u_n)^{1/n}=\lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)^{n+1}-\frac{n+1}{n}\right]^{-1}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-1} \left[\left(\frac{n+1}{n}\right)^{n} - 1\right]^{-1}$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{-1} \left[\left(1 + \frac{1}{n}\right)^{n} - 1\right]^{-1}$$

$$= \frac{1}{e-1} < 1$$

Hence by Cauchy's root test, the given series converges.

Exercise 2B

Test the convergence of the following series:

1. $\sum_{n=1}^{\infty} \frac{2^n}{n^2+2}$ Ans. Convergent
2. $\sum_{n=1}^{\infty} \frac{n!}{2^{2n-1}}$ Ans. Divergent
3. $\sum_{n=1}^{\infty} \frac{1.2.3....n}{7.10....(3n+4)}$ Ans. Convergent
4. $1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!}$ Ans. Convergent
5. $1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + ...$ Ans. Convergent if $x < 1$, divergent if $x \geq 1$
6. $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$ Ans. Convergent
7. $\sum_{n=1}^{\infty} \frac{n^{n^2}}{\left(n+\frac{1}{5}\right)^{n^2}}$ Ans. Convergent
8. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$ Ans. Convergent
9. $1+\frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + ...(p > 0)$ Ans. Convergent
10. $\sum_{n=1}^{\infty} \sqrt{\frac{n-1}{n^3+1}}\, x^n \ (x > 0)$ Ans.Convergent if $x < 1$, divergent if $x \geq 1$

4. Raabe's Test

Let $\sum_{n=1}^{\infty} u_n$be a positive term series such that

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$$

Then (i) $\sum_{n=1}^{\infty} u_n$converges if l> 1

(ii) $\sum_{n=1}^{\infty} u_n$ diverges if l < 1

(iii)Test fails if l = 1

Example 13 Test the convergence of the following series:

(i) $\frac{2}{3} + \frac{2.4}{3.5} + \frac{2.4.6}{3.5.7} + \frac{2.4.6.8}{3.5.7.9} + \ldots..$ (ii) $1 + \frac{3x}{7} + \frac{3.6\,x^2}{7.10} + \frac{3.6.9x^3}{7.10.13} + \cdots (x > 0)$

Solution:(i) Here $u_n = \frac{2.4.6....2n}{1.3.5....(2n+1)} \Rightarrow u_{n+1} = \frac{2.4.6....2n(2n+2)}{1.3.5....(2n+1)(2n+3)}$

Then $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2n+2}{2n+3} = 1$

Hence Ratio test fails.

Now applying Raabe's test, we have

$$\operatorname{Lim}_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{2n+3}{2n+2} - 1\right)$$

$$= \lim_{n\to\infty}\left(\frac{n}{2n+1}\right) = \frac{1}{2} < 1$$

Hence by Raabe's test , the given series diverges.

(ii) Ignoring the first term,$u_n = \frac{3.6.9...3n}{7.10.13...(3n+4)} x^n$

$$\Rightarrow u_{n+1} = \frac{3.6.9...3n(3n+3)}{7.10.13...(3n+4)(3n+7)} x^{n+1}$$

Then $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{3n+3}{3n+7} x = x$

Hence by Ratio test , the given series converges if $x < 1$ and diverges if $x > 1$

Test fails if $x = 1$

When $x = 1$, $\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}$

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{3n+7}{3n+3} - 1\right)$$

$$= \lim_{n\to\infty} \frac{4n}{3n+3} = \frac{4}{3} > 1$$

Hence by Raabe's test, the given series converges if $x = 1$

∴ the given series converges if $x \leq 1$ and diverges if $x > 1$.

4. Logarithmic Test

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that

$$\lim_{n\to\infty} n \frac{u_n}{u_{n+1}} = l$$

Then (i) $\sum_{n=1}^{\infty} u_n$ converges if $l > 1$

(ii) $\sum_{n=1}^{\infty} u_n$ diverges if $l < 1$

Example 14 Test the convergence of the series

$$x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \cdots$$

Solution: Here $u_n = \frac{n^n x^n}{n!} \Rightarrow u_{n+1} = \frac{(n+1)^{n+1}x^{n+1}}{(n+1)!}$

$$\text{Then } \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}x}{(n+1)n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n x}{n^n}$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n x = e.x$$

Hence by Ratio test , the given series converges if $ex < 1$ $i.e. x < \frac{1}{e}$ and diverges if $ex > 1$ $i.e. x > \frac{1}{e}$

Test fails if $ex = 1$ $i.e. x = \frac{1}{e}$

Since $\frac{u_{n+1}}{u_n}$ involves e ∴ applying logarithmic test.

$$\frac{u_n}{u_{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^n x}$$

∴ for $x = \frac{1}{e}$, $\frac{u_n}{u_{n+1}} = e.\frac{1}{\left(1+\frac{1}{n}\right)^n}$

$$log\left(\frac{u_n}{u_{n+1}}\right) = log\ e - log\left(1+\frac{1}{n}\right)^n = 1 - nlog\left(1+\frac{1}{n}\right)$$

$$=1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \cdots\right)$$

$$= \frac{1}{2n} - \frac{1}{3n^2} + \ldots$$

$$= \lim_{n\to\infty} nlog\left(\frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty} n\left(\frac{1}{2n} - \frac{1}{3n^2} + \cdots\right) = \frac{1}{2} < 1$$

∴ By logarithmic test , the series diverges for $x = \frac{1}{e}$.

Hence the given series converges for $x < \frac{1}{e}$ and diverges for $x \geq \frac{1}{e}$.

5. Cauchy's Integral Test

If $u(x)$ is non-negative , integrable and monotonically decreasing function such that $u(n) = u_n$, then if $\int_1^\infty u(x)\ d(x)$ converges then the series $\sum_{n=1}^{\infty} u_n$ also converges.

Example 15 Test the convergence of the following series

$(i)\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ $(ii)\ \sum_{n=2}^{\infty}\frac{1}{n(\log n)}$

Solution:(i) Here $u_n = \frac{1}{n^2+1}$.

Let $u(x) = \frac{1}{x^2+1}$

Clearly $u(x)$ is non-negative , integrable and monotonically decreasing function.

Consider $\int_1^{\infty}\frac{1}{x^2+1}\ d(x) = [tan^{-1}x]_1^{\infty}$

$$= tan^{-1}\infty - tan^{-1}1$$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ which is finite.

Hence $\int_1^{\infty}\frac{1}{x^2+1}\ d(x)$ converges so $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ also converges.

(ii) Here $u_n = \frac{1}{n(\log n)}$.

Let $u(x) = \frac{1}{x(\log x)}$

Clearly $u(x)$ is non-negative , integrable and monotonically decreasing function.

Consider $\int_2^{\infty}\frac{1}{x(\log x)}\ d(x) = \log(log\infty) - \log\ (log2) = \infty$

Hence $\sum_{n=2}^{\infty}\frac{1}{n(\log n)}$ diverges.

Exercise 2C

Test the convergence of the following series:

1. $\sum_{n=1}^{\infty} \frac{2.4.6....(2n+2)}{3.5.7......(2n+3)} x^{n-1} \quad (x > 0)$

   Ans. Convergent if $x < 1$, divergent if $x \geq 1$

2. $\sum_{n=1}^{\infty} \frac{(2n!)}{(n!)^2} x^n \quad (x > 0)$

   Ans. Convergent if $x < \frac{1}{4}$, divergent if $x \geq \frac{1}{4}$

3. $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

   Ans. Convergent

4. $\sum_{n=1}^{\infty} \frac{1.3.5....(2n-1)}{2.4.6......2n} x^n (x > 0)$

   Ans. Convergent if $x < 1$, divergent if $x \geq 1$

5. $x^2 + \frac{2^2}{3.4} x^4 + \frac{2^2 4^2}{3.4.5.6} x^6 + \frac{2^2 4^2 6^2}{3.4.5.6.7.8} x^8 + \ldots..$

   Ans. Convergent if $|x| \leq 1$, divergent if $|x| > 1$

6. $1 + \frac{x}{2} + \frac{2! x^2}{3^2} + \frac{3! x^3}{4^3} + \frac{4! x^4}{5^4} + \cdots$

   Ans. Convergent if $x < e$, divergent if $x \geq e$.

7. $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$

   Ans. Convergent

2.4 Alternating Series

An infinite series of the form $u_1 - u_2 + u_3 - u_4 + \cdots \; (u_i > 0 \; \forall \; i)$

is called an infinite series.

We write $u_1 - u_2 + u_3 - u_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} u_n$

Leibnitz's Test

The alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ converges if it satisfies the following conditions:

$(i)\ u_{n+1} \leq u_n$

$(ii)\ \lim_{n\to\infty} u_n = 0$

Example 16 Test the convergence of the following series

$(i) 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \qquad (ii)\ 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

Solution: (i) The given series is $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$. Here $u_n = \frac{1}{n}$

Since $\frac{1}{n+1} < \frac{1}{n} \quad \therefore\ u_{n+1} \leq u_n$

Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

Hence by Leibnitz's test , the given series converges.

(ii) The given series is $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$. Here $u_n = \frac{1}{n^2}$

Since $\frac{1}{(n+1)^2} < \frac{1}{n^2} \therefore\ u_{n+1} \leq u_n$

Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$

Hence by Leibnitz's test , the given series converges.

Absolute Convergence

A series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty}|u_n|$ is convergent.

For example $\sum_{n=1}^{\infty} u_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots$ is absolutely convergent as $\sum_{n=1}^{\infty}|u_n| = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$ is a convergent series (Since it is a geometric series whose common ratio $\frac{1}{2} < 1$).

Result: Every absolutely convergent series is convergent. But the converse may not be true.

Conditional Convergence

A series which is convergent but not absolutely convergent is called conditionally convergent series.

Example 17 Test the convergence and absolute convergence of the following series:

$(i) 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ $\quad (ii)\ 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

$(iii)\ \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{log\ n}$

Solution: (i) The given series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is convergent by Leibnitz's test.

Now, $\sum_{n=1}^{\infty} |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent (As p=1)

Hence the given series is not absolutely convergent. This is an example of conditionally convergent series.

(ii) The given series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$ is convergent by Leibnitz's test.

Also, $\sum_{n=1}^{\infty} |u_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (As p = 2 > 1)

Hence the given series is absolutely convergent.

(iii)The given series $\sum_{n=2}^{\infty} (-1)^{n+1} u_n$

Here $u_n = \frac{1}{log\ n}$. Now $log\ x$ is an increasing function $\forall x > 0$

$\therefore \log(n+2) > \log(n+1)$

$or\ \frac{1}{\log(n+2)} < \frac{1}{\log(n+1)}$

$\therefore u_{n+1} \leq u_n$

Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{log\ n} = 0$

Hence by Leibnitz's test , the given series is convergent.

Now for absolute convergence , consider $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{log\ n}$

It is a divergent series (as discussed earlier).

Hence the given series is not absolutely convergent. This is an example of conditionally convergent series.

Example 18 Test the convergence of the series:

$(i\ )\sum_{n=1}^{\infty}(-1)^{n-1}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}\right]$ $(ii)\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n2^n}$

Solution:(i) The given series is $\sum_{n=1}^{\infty}(-1)^{n-1} u_n$

Consider $\sum_{n=1}^{\infty}|u_n| = \sum_{n=1}^{\infty}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}\right]$

Now, $\frac{1}{n^2}+\frac{1}{(n+1)^2} < \frac{1}{n^2}+\frac{1}{n^2} = \frac{2}{n^2}$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$is convergent (As p = 2 > 1) $\therefore$ by Comparision test $\sum_{n=1}^{\infty}|u_n|$ is also convergent.

Hence the given series is absolutely convergent and so convergent.

(ii) The given series is $\sum_{n=1}^{\infty}(-1)^{n-1} u_n$

Consider $\sum_{n=1}^{\infty}|u_n| = \sum_{n=1}^{\infty}\frac{1}{n2^n}$

Here $|u_n| = \frac{1}{n2^n}$

Now $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{n2^n}{(n+1)2^{n+1}}\right| = \frac{1}{2} < 1$

$\therefore$ by Ratio test $\sum_{n=1}^{\infty}|u_n|$ is convergent or the given series is absolutely convergent and hence convergent.

Example 19 Find the values of x for which the series

$x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$ is absolutely convergent and conditionally convergent.

Solution: The given series is $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = \sum_{n=1}^{\infty}(-1)^{n-1}u_n$

Then $|u_n| = \left|\frac{x^{2n-1}}{2n-1}\right|$

Now, $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+1}}{2n+1}.\frac{2n-1}{x^{2n-1}}\right| = \frac{1}{x^2}$

Thus , by Ratio test $\sum_{n=1}^{\infty}|u_n|$ converges if $x^2 < 1$ $i.e.$ $|x| < 1$, diverges if $x^2 > 1$ $i.e.$ $|x| > 1$ and test fails if $|x| = 1$

When $|x| = 1$ $i.e.$ $x = 1$ or $x = -1$, we have

For $x = 1$,

the given series is $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$, which is convergent by Leibnitz's test but not absolutely convergent.

For $x = -1$,

the given series is $-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \cdots$, which is also convergent by Leibnitz's test but not absolutely convergent.

Hence the given series is absolutely convergent for $|x| < 1$ or

$-1 < x < 1$ and conditionally convergent for $|x| = 1$ $i.e.$

$x = 1$ or -1.

Exercise 2D

1. Show that the series $1 - \frac{(2!)^2}{4!} + \frac{(3!)^2}{6!} - \cdots$is convergent.
2. Show that the series $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$is absolutely convergent.
3. Test the convergence and absolute convergence of the series
   $1 - \frac{1}{2.3} + \frac{1}{2^2.5} - \frac{1}{2^3.7} \ldots$
   Ans. Absolutely convergent
4. Show that the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n+3}$ is conditionally convergent.
5. Test the absolute convergence of the series
   $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\sqrt{n^2+1} - n\right)$
   Ans. Not absolutely convergent
6. Show that the series $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \cdots$ converges absolutely.
7. Find the interval of convergence of the series $x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} \ldots$
   Ans. $0 < x \leq 1$

2.5 EXPANSION OF FUNCTIONS

Taylor Series:

If a function $f(x)$ is infinitely differentiable at the point *a* then $f(x)$ can be expanded about the point $'a'$ as

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \ldots\ldots$$

Also $f(a+h)$, where h is small, can be expanded as

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + \ldots\ldots$$

Malaurin Series:

It is the special case of Taylor series about the point 0 . Hence the Maclaurin series of $f(x)$ is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \ldots\ldots$$

Maclaurin series of standard functions:

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
3. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$
4. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, |x| < 1$
5. $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$
6. $(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$

Example20 Expand $e^x cos\ x$ by Maclaurin series.

Solution : By Maclaurin's expansion , we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f''(0) + \cdots \ldots\ldots ①$$

Here $f(x) = e^x cos\ x$

Now $f(0) = e^0 cos\ 0 = 1$

$$f'(x) = e^x cos\ x - e^x sin\ x$$

$$\Rightarrow f'(0) = e^0 cos\ 0 - e^0 sin\ 0 = 1$$

$$f''(x) = e^x cos\ x - e^x sin\ x - e^x sin\ x - e^x cos\ x$$

$$= -2e^x sin\ x$$

$$\Rightarrow f''(0) = -2e^0 sin\ 0 = 0$$

$f'''(x) = -2e^x cos\, x - 2e^x sin\, x$

$\Rightarrow f'''(0) = -2e^0 cos\, 0 - 2e^0 sin\, 0 = -2$

Similarly $f^{iv}(0) = -1$ and so on .

Putting these values in ①, we get

$$e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{4x^4}{4!} - \cdots$$

Example21 Expand $tan\, x$ in powers of $(x - \frac{\pi}{4})$ upto forst four terms.

Solution: By Taylor's expansion , we have

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f''(a) + \cdots \ldots ①$$

Here $f(x) = tan\, x$ and $a = \frac{\pi}{4}$

Now $f\left(\frac{\pi}{4}\right) = tan \frac{\pi}{4} = 1$

$f'(x) = sec^2 x$

$\Rightarrow f'\left(\frac{\pi}{4}\right) = sec^2 \frac{\pi}{4} = 2$

$f''(x) = 2sec^2 x\, tan\, x$

$\Rightarrow f''\left(\frac{\pi}{4}\right) = 2sec^2\left(\frac{\pi}{4}\right) tan\left(\frac{\pi}{4}\right) = 4$

$f'''(x) = 2sec^4 x + 4\, tan^2 x sec^2 x$

$\Rightarrow f'''\left(\frac{\pi}{4}\right) = 16$

Putting these values in ①, we get

$$tan\, x = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$$

Example22 Show that $log\ sec\ x = \frac{x^2}{2} + \frac{x^4}{12} + \cdots$

Solution: By Maclaurin's expansion , we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f''(0) + \cdots \ldots ①$$

Here $f(x) = log\ secx$

Now $f(0) = 0$

$f'(x) = tanx$

$\Rightarrow f'(0) = 0$

$f''(x) = sec^2x = 1 + tan^2x$

$\Rightarrow f''(0) = 1$

$f'''(x) = 2\sec x\ secx\ tanx$

$\Rightarrow f'''(0) = 0$

Similarly $f^{iv}(0) = 2$ and so on .

Putting these values in ①, we get

$$log\ sec\ x = \frac{x^2}{2} + \frac{x^4}{12} + \cdots$$

Example 23 Show that $sin\left(\frac{\pi}{4} + \theta\right) = \frac{1}{\sqrt{2}}\left(1 + \theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \cdots\right)$

Solution: By Taylor's expansion , we have

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f''(a) + \cdots \ldots ①$$

Here $f(x) = sin\ x,\ a = \frac{\pi}{4}$ and $h = \theta$

So ① becomes

$$sin(a + h) = sin(a) + h\ cos\ a + \frac{h^2}{2!}(-sina) + \frac{h^3}{3!}(-cosa) + \cdots$$

or $sin\left(\frac{\pi}{4}+\theta\right)=sin\left(\frac{\pi}{4}\right)+\theta cos\ \frac{\pi}{4}+\frac{\theta^2}{2!}\left(-sin\frac{\pi}{4}\right)+\frac{\theta^3}{3!}\left(-cos\frac{\pi}{4}\right)+\cdots$

or $sin\left(\frac{\pi}{4}+\theta\right)=\frac{1}{\sqrt{2}}\left(1+\theta-\frac{\theta^2}{2!}-\frac{\theta^3}{3!}+\cdots\right)$

Example 24 Estimate the value of $\sqrt{10}$ correct to four places of decimal.

Solution: Let $f(x)=\sqrt{x}$

By Taylor's theorem, we have

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f''(a)+\cdots$$

$\Rightarrow\ (a+h)^{1/2}=a^{1/2}+h\left(\frac{d}{dx}x^{1/2}\right)_{at\ x=a}+\frac{h^2}{2!}\left(\frac{d^2}{dx^2}x^{1/2}\right)_{at\ x=a}+$
$\frac{h^3}{3!}\left(\frac{d^3}{dx^3}x^{1/2}\right)_{at\ x=a}+\cdots$....①

Taking $a=9$ and $h=1$ in ①, we get

$10^{1/2}\ =\ 9^{1/2}+\left(\frac{1}{2}x^{-1/2}\right)_{at\ x=9}+\frac{1}{2!}\left(\frac{1(-1)}{2.2}x^{-3/2}\right)_{at\ x=9}+$
$\frac{1}{3!}\left(\frac{1(-1)(-3)}{2.2.2}x^{-5/2}\right)_{at\ x=9}+\cdots.$

$$=3+\frac{1}{2.3}-\frac{1}{8.27}+\cdots$$

=3.1623(approx.)

2.6 Approximate Error

Let y be a function of x i.e. y=f(x). If δx is a small change in x then the resulting change in y is denoted by δy and is given by

$\delta y=\frac{dy}{dx}\delta x$ approximately.

Example 25 Find the change in the total surface area of a right circular cone when

(i) the radius is constant but there is a small change in the altitude

(ii) the altitude is constant but there is a small change in the radius.

Solution: Let the radius of the base be r , altitude be h and the change in the altitude be the radius is constant but there is a small change in the altitude δh.

Let S be the total surface area of the cone , then

$$S = \pi r^2 + \pi r\sqrt{r^2 + h^2}$$

(i)If altitude changes then $\delta S = \frac{dS}{dh}\delta h$

$$\text{Now, } \frac{dS}{dh} = 0 + \frac{\pi r}{2}(r^2 + h^2)^{-1/2}.2h = \frac{\pi rh}{\sqrt{r^2+h^2}}$$

$$\therefore\ \delta S = \frac{dS}{dh}\delta h = \frac{\pi rh}{\sqrt{r^2+h^2}}\delta h \text{ approximately.}$$

(ii)If radius changes then $\delta S = \frac{dS}{dr}\delta r$

$$\text{Now, } \frac{dS}{dr} = 2\pi r + \pi\sqrt{r^2 + h^2} + \frac{2\pi r^2}{2\sqrt{r^2+h^2}} = 2\pi r + \frac{\pi(2r^2+h^2)}{\sqrt{r^2+h^2}}$$

$$\therefore\ \delta S = \frac{dS}{dr}\delta r = 2\pi r + \frac{\pi(2r^2+h^2)}{\sqrt{r^2+h^2}}\delta r \text{ approximately.}$$

Example 26 If a , b , c are the sides of the triangle ABC and S is the semi- perimeter , show that if there is a small error δcin the measurement of side c then the error $\delta\Delta$ in the area Δ of the triangle is given by

$$\delta\Delta = \frac{\Delta}{4}\left(\frac{1}{S} + \frac{1}{S-a} + \frac{1}{S-b} + \frac{1}{S-c}\right)\delta c$$

Solution: We know that $S = \frac{(a+b+c)}{2}$

$$\text{and } \Delta^2 = S(S-a)(S-b)(S-c)$$

$$or\ 2log\Delta = log\ S + \log(S-a) + \log(S-b) + \log\ (S-c)$$

On differentiating both the sides w.r.t. c , we get

$$\frac{2d\Delta}{\Delta dc} = \frac{1}{S}\frac{dS}{dc} + \frac{1}{S-a}\frac{d(S-a)}{dc} + \frac{1}{S-b}\frac{d(S-b)}{dc} + \frac{1}{S-c}\frac{d(S-c)}{dc}$$

$$=\frac{1}{S}\frac{1}{2} + \frac{1}{2(S-a)} + \frac{1}{2(S-b)} + \frac{1}{(S-c)}\left(\frac{1}{2} - 1\right)$$

$$\Rightarrow \frac{d\Delta}{dc} = \frac{\Delta}{4}\left(\frac{1}{S} + \frac{1}{(S-a)} + \frac{1}{(S-b)} + \frac{1}{(S-c)}\right)$$

$$\Rightarrow \delta\Delta = \frac{d\Delta}{dc}\delta c = \frac{\Delta}{4}\left(\frac{1}{S} + \frac{1}{(S-a)} + \frac{1}{(S-b)} + \frac{1}{(S-c)}\right)\delta c$$

Example 27 If $T = 2\pi\sqrt{\left({}^{l}/_{g}\right)}$ find the error in T corresponding to 2% error in l where g is constant.

Solution: Error in T is given by $\delta T = \frac{dT}{dl}\delta l$

$$\text{Now } \frac{dT}{dl} = \frac{2\pi}{\sqrt{g}}\frac{1}{2\sqrt{l}} \quad \therefore \quad \delta T = \frac{\pi}{\sqrt{g}}\frac{1}{\sqrt{l}}\delta l$$

$$\Rightarrow \frac{\delta T}{T} = \frac{\pi}{\sqrt{g}}\frac{\delta l}{\sqrt{l}}\frac{\sqrt{g}}{2\pi\sqrt{l}} = \frac{1}{2}\frac{\delta l}{l}$$

$$\Rightarrow \frac{\delta T}{T}.100 = \frac{1}{2}\frac{\delta l}{l}.100$$

As $\frac{\delta l}{l}.100 = 2 \;\therefore\; \frac{\delta T}{T}.100 = 1$ Hence error in T is 2%.

Exercise 2E

1. Expand $tan^{-1}x$ in powers of (x-1).

   Ans. $tan^{-1}x = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3 + \cdots$

2. Using Taylor's theorem find the approximate value of $f\left(\frac{11}{10}\right)$ where $f(x) = x^3 + 3^2 + 15x - 10$

   Ans. 11.461

3. Show that $\log(1 + sinx) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots$

4. Show that $tan\left(\frac{\pi}{4}+x\right)=1+2x+2x^2+\frac{8}{3}x^3+\frac{10}{3}x^4+\cdots$ and hence find *tan* 46^o
   Ans. 1.0355
5. A soap bubble of radius 2cm shrinks to radius 1.9 cm. Finf the decrease in volume and surface area.

   Ans. -5.024 cm^3 and -.5.024cm^2

6. If $\log_{10}4 = 0.6021$, find the approximate value of $\log_{10}404$.

   Ans. 2.61205

7. Let A , B and C be the angles of a triangle opposite to the sides a ,b and c respectively. If small errors $\delta a, \delta b$ and $\delta c$are made in the sides then show that $\delta A=\frac{a}{2\Delta}(\delta a-\delta b\ cosC-\delta c\ cosB)$ where Δ is the area of the triangle.