

Sequences and infinite series

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The lists of numbers you generate using a numerical method like Newton's method to get better and better approximations to the root of an equation are examples of (mathematical) sequences.

Sequences are infinite lists of numbers $a_1, a_2, a_3, \dots, a_n, \dots$

Sometimes it is useful to think of them as functions from the positive integers to the real numbers, in other words, $a(1) = a_1$, $a(2) = a_2$, and so forth.

Convergent and divergent

The feeling we have about numerical methods like Newton's method and the bisection method is that if we continue the iteration process more and more times, we would get numbers that are closer and closer to the actual root of the equation. In other words:

$$\lim_{n \rightarrow \infty} a_n = r \quad \text{where } r \text{ is the root.}$$

Sequences for which $\lim_{n \rightarrow \infty} a_n$ exists and is finite are called *convergent sequences*, and other sequences are called *divergent sequences*.

Examples

For example...

- The sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots, \frac{1}{2^n}, \dots$ is *convergent* (and *converges to zero*, since $\lim_{n \rightarrow \infty} \frac{1}{2^n} = 0$)
- The sequence $1, 4, 9, 16, \dots, n^2, \dots$ is *divergent*

Practice

- The sequence $\left\{ \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n+1}{n+2}, \dots \right\}$
 - A. Converges to 0
 - B. Converges to 1
 - C. Converges to n
 - D. Converges to e
 - E. Diverges
- The sequence $\left\{ -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \dots, (-1)^n \frac{n+1}{n+2}, \dots \right\}$
 - A. Converges to 0
 - B. Converges to 1
 - C. Converges to -1
 - D. Converges to e
 - E. Diverges



A powerful existence theorem

It is sometimes possible to assert that a sequence is convergent even if we can't find its limit directly. One way to do this is by using the *least upper bound property* of the real numbers.

If a sequence has the property that $a_1 < a_2 < a_3 < \dots$, then it is called a “*monotonically increasing*” sequence. Such a sequence either is **bounded** (all the terms are less than some fixed number) or else the terms increase without bound to infinity.

In the latter (unbounded) case, the sequence is divergent, and a *bounded, monotonically increasing sequence must converge to the least upper bound of the set of numbers $\{a_1, a_2, \dots\}$* . So if we can find *some* upper bound for a monotonically increasing sequence, we are *guaranteed* convergence, even if we can't find the least upper bound.

For example, consider the sequence. . .

$$\sqrt{2}, \sqrt{2 + \sqrt{2}}, \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}, \dots$$

This is a *recursively-defined sequence* — to get each term from the previous one, you add 2 and then take the square root, in other words $x_{n+1} = \sqrt{2 + x_n}$.

This is a monotonically increasing sequence (since another way to look at how to get from one term to the next is to add an extra $\sqrt{2}$ under the innermost radical, which makes it a little bigger).

We will show that all the terms are less than 2. For any x that satisfies $0 < x < 2$, we have

$$x^2 < 2x = x + x < 2 + x < 2 + 2, \quad \text{and so} \quad x < \sqrt{2 + x} < 2.$$

So by induction, all the x_n 's are less than 2 and so the sequence has a limit according to the theorem. **But what is the limit??**

Series of constants

We've looked at limits of sequences. Now, we look a specific kind of sequential limit, namely the limit (or sum) of a series.

Zeno's paradox

How can an infinite number of things happen in a finite amount of time?

(Zeno's paradox concerned Achilles and a tortoise.)

Discussion questions

- 1 Is Meg Ryan's reasoning correct? If it isn't what is wrong with it?
- 2 If a ball bounces an infinite number of times, how come it stops? How do you figure out the total distance traveled by the ball?

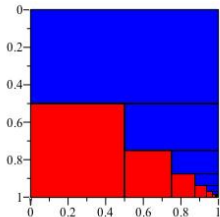
Resolving these problems

The resolution of these problems is accomplished by the use of limits.

In particular, each is resolved by understanding why it is possible to “add together” an infinite number of numbers and get a finite sum.

Meg Ryan worried about adding together

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$



The picture suggests that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

should be 1. This is in fact true, but requires some proof. We will provide the proof, but in a more general context.

The idea of a series

A *series* is any “infinite sum” of numbers. Usually there is some pattern to the numbers, so we can communicate the pattern either by giving the first few numbers, or by giving an actual formula for the n th number in the list. For example, we could write

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \quad \text{as} \quad \sum_{n=1}^{\infty} \frac{1}{2^n} \quad \text{or as} \quad \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n.$$

The things being added together are called the *terms* of the series.

Other series we will consider

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots, \text{ or } \sum_{n=1}^{\infty} \frac{1}{n}.$$

This is sometimes called the “harmonic series”.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \text{ or } \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

This is called the “alternating harmonic series”.

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \cdots,$$

which you could recognize as $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots, \text{ or } \sum_{n=0}^{\infty} \frac{1}{n!} \text{ (since } 0! = 1 \text{ by definition).}$$

Questions about a series

Two obvious questions to ask about a series:

1. Does the series have a sum? (Officially: “*Does the series converge?*”)
2. What is the sum? (Officially: “*What does the series converge to?*”)

A less obvious question:

3. How *fast* does the series converge? In other words, how many terms at the beginning of the series do you have to add together to get an approximation of the sum within a specified error?

Convergence defined

The word *convergence* suggests a limiting process. Fortunately, we don't have to invent a new kind of limit for series.

Think of series as a process of adding together the terms starting from the beginning. Then the n th *partial sum* of the series is simply the sum of the first n terms of the series.

For example, the partial sums of the Meg Ryan series

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ are:

- 1st partial sum = $\frac{1}{2}$
- 2nd partial sum = $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$
- 3rd partial sum = $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

and so forth.

It looks like the N th partial sum of this series is $\frac{2^N - 1}{2^N}$

Definition

It is only natural to define (and this is even the official definition!) the sum or limit of the series to be equal to the limit of the sequence of its partial sums, if the latter limit exists.

So for the Meg Ryan series, we really do have

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{1}{2^n} = \lim_{N \rightarrow \infty} \frac{2^N - 1}{2^N} = 1$$

which bears out our earlier intuition.

This presents a problem. . .

The problem is that it is often difficult or impossible to get an explicit expression for the partial sums of a series.

So, as with integrals, we'll learn a few basic examples, and then do the best we can — sometimes only answering question 1, other times managing 1 and 2, and still other times all three.

Geometric series

One kind of series for which we can find the partial sums is the *geometric series*. The Meg Ryan series is a specific example of a geometric series.

A *geometric series* has terms that are (possibly a constant times) the successive powers of a number. The Meg Ryan series has successive powers of $\frac{1}{2}$.

Other examples:

$$1 + 1 + 1 + 1 + \dots = \sum_{n=1}^{\infty} 1^n$$

$$0.333333\dots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \dots = \sum_{n=1}^{\infty} 3 \left(\frac{1}{10} \right)^n$$

$$3 + 12 + 48 + 192 + \dots = \sum_{n=0}^{\infty} 3(4^n)$$

$$5 - \frac{5}{7} + \frac{5}{49} - \frac{5}{343} + \dots = \sum_{n=0}^{\infty} 5 \left(\frac{-1}{7} \right)^n$$

$$\frac{3}{32} + \frac{3}{64} + \frac{3}{128} + \frac{3}{256} + \dots = \sum_{n=5}^{\infty} \frac{3}{2^n}$$

Partial sums of geometric series

Start (how else?) with partial sums:

A finite geometric sum is of the form:

$$S_N = a + ar + ar^2 + ar^3 + \cdots + ar^N$$

Multiply both sides by r to get:

$$rS_N = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{N+1}$$

Now subtract the second equation from the first (look at all the cancellation on the right side!) to get

$$(1 - r)S_N = a(1 - r^{N+1}) \quad \text{and so} \quad S_N = \frac{a(1 - r^{N+1})}{1 - r}$$

(unless $r = 1$, and if $r = 1$ then $S_N = aN$).

Convergence of geometric series

Since the N th partial sum of the geometric series $\sum_{n=0}^N ar^n$ is

$$S_N = \frac{a(1 - r^{N+1})}{1 - r} \quad \text{if } r \neq 1$$

we conclude that $\lim_{N \rightarrow \infty} S_N$ is equal to $\frac{a}{1 - r}$ if $|r| < 1$ and does not exist otherwise.

(If $r = 1$ and $a \neq 0$, then $\lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} aN = \pm\infty$).

Therefore the geometric series converges precisely when $|r| < 1$ and diverges otherwise.

Which of these geometric series converge?

What are the sums of the convergent ones?

$$1 + 1 + 1 + 1 + \dots = \sum_{n=1}^{\infty} 1^n$$

$$0.333333\dots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \dots = \sum_{n=1}^{\infty} 3 \left(\frac{1}{10}\right)^n$$

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Telescoping series



Another kind of series that we can sum: telescoping series

This seems silly at first, but it's not!

A series is said to *telescope* if almost all the terms in the partial sums cancel except for a few at the beginning and at the ending.

Example

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) + \cdots$$

Clearly the N th partial sum of this series is $1 - \frac{1}{N+1}$ and so the series converges to 1.

What's the big deal?

Well, you could rewrite the series as

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots + \frac{1}{n(n+1)} + \cdots$$

and now it's not so obvious that the sum is 1 (and recall that this was one of the examples given near the beginning of today's class).

You try one!

What is the sum of the series $\sum_{k=2}^{\infty} \frac{1}{k^2 - 1}$?

A. 1

B. 3/4

C. 1/2

D. 1/4

E. 1/8

Improper integrals as telescoping series

One important class of examples of telescoping series is provided by improper integrals.

Suppose $F'(x) = f(x)$. We can think of the following improper integral of $f(x)$ as being the sum of the (telescoping) series

$$\begin{aligned}\int_1^{\infty} f(x) dx &= \int_1^2 f(x) dx + \int_2^3 f(x) dx + \int_3^4 f(x) dx + \dots \\ &= (F(2) - F(1)) + (F(3) - F(2)) + (F(4) - F(3)) + \dots\end{aligned}$$

Since the N th partial sum of this series is $F(N + 1) - F(1)$, it's clear that the series converges to $\left(\lim_{N \rightarrow \infty} F(N)\right) - F(1)$, the same limit to which the improper integral would converge.

(But the series might converge even though the integral does not. Can you think of an example?)

The convergence question

Now, we'll spend some time concentrating on the question:

① Does the series converge?

One obvious property that convergent series must have is that their terms must get smaller and smaller in order for the limit of the partial sums to exist.

Fundamental **necessary** condition for convergence

A series $\sum_{n=1}^{\infty} a_n$ **cannot** converge unless $\lim_{n \rightarrow \infty} a_n = 0$.

This is a test you can use only to prove that a series does NOT converge.

For example,

$\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges, as does $\sum_{n=1}^{\infty} \arctan(n)$.

The harmonic series, an essential example

The converse of the previous statement is **false**. In other words, just because the n th term of a series goes to zero does **NOT** guarantee that the series converges. An important example

illustrating this is the *harmonic series*: $\sum_{n=1}^{\infty} \frac{1}{n}$.

We can show that the harmonic series diverges using the partial sums:

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > \frac{3}{2} + \frac{1}{4} + \frac{1}{4} = \frac{4}{2}$$

$$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{4}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{5}{2}$$

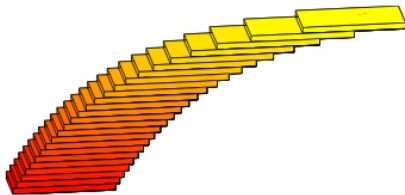
and so on – every time we double the number of terms, we add at least one more half, so $S_{2^n} > \frac{1}{2}(n - 2)$. This shows that

$\lim_{n \rightarrow \infty} S_n = \infty$ so the harmonic series **diverges**.

Cantilever tower

The proof of the divergence theorem on the preceding page was discovered and published by Oresme around 1350.

The divergence of the harmonic series makes the following trick possible. It is possible to stack books (or cards, or any other kind of stackable, identical objects) near the edge of a table so that the top object is completely off the table (and as far off as one wishes, provided you have enough objects to stack).



Tests for convergence of series of positive terms

Convergence questions for series of positive terms (i.e., series with all plus signs) are easiest to understand conceptually.

- Since all the terms a_n are assumed to be positive, the sequence of partial sums $\{S_n\}$ must be an increasing sequence.
- So the least upper bound property discussed earlier comes into play — either the sequence of partial sums has an upper bound or it doesn't.
- If the sequence of partial sums is bounded above, then it must converge and so will the series. If not, then the series diverges. That's it.

These upper bound observations give rise to several “tests” for convergence of series of positive terms. They all are based on showing that the partial sums of the series being tested is bounded are all less than those of a series that is known to converge. The names of the tests we will discuss are. . .

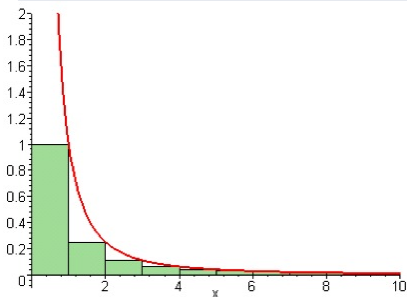
Convergence tests

- 1 The integral test
- 2 The (direct) comparison test
- 3 The ratio test
- 4 The limit comparison test (also called the ratio comparison test)
- 5 The root test

The integral test

Since improper integrals of the form $\int_1^{\infty} f(x) dx$ provide us with lots of examples of telescoping series whose convergence is readily determined, we can use integrals to determine the convergence of some series.

For example, consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$



From the picture, it is evident that the n th partial sum of this series is less than

$$1 + \int_1^n \frac{1}{x^2} dx.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

The sum of the series is equal to the sum of the areas of the shaded rectangles, and if we start integrating at 1 instead of 0, the improper integral $\int_1^{\infty} \frac{1}{x^2} dx$ converges.

(**Questions:** What is the value of the integral? So what bound do you infer for the sum of the series?)

Since the value of the improper integral (plus 1) provides us with an upper bound for all of the partial sums, the series must converge.

It is an interesting question as to exactly what the sum is. We will answer it next week.

The integral test

The integral test . . .

says that if the function $f(x)$ is bounded, positive and decreasing, then the series $\sum_{n=1}^{\infty} f(n)$ and the integral $\int_1^{\infty} f(x) dx$ either both converge or both diverge.

For example, this gives us an easier proof of the divergence of the harmonic series — because we already know the divergence of the integral $\int_1^{\infty} \frac{1}{x} dx$.

Question:

We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, . . .

So for which exponents p does the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

These are called p -series, for obvious reasons — and these together with the geometric series give us lots of useful examples of series whose convergence or divergence we know, which will come in handy when we discuss the various comparison tests below.

Error estimates

Using the picture that proves the integral test for convergent series, we can get an estimate of how far we are from the limit of the series if we stop adding after N terms for any finite value of N :

If we approximate the convergent series $\sum_{n=1}^{\infty} f(n)$ by the partial sum

$S_N = \sum_{n=1}^N f(n)$, then the error we commit is less than the value of

the integral $\int_N^{\infty} f(x) dx$, and it is greater than $\int_{N+1}^{\infty} f(x) dx$. In

other words:

$$\sum_{n=1}^N f(n) + \int_{N+1}^{\infty} f(x) dx < \sum_{n=1}^{\infty} f(n) < \sum_{n=1}^N f(n) + \int_N^{\infty} f(x) dx.$$

For instance. . .

If we take a closer look at the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$:

The sum $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} = \frac{5269}{3600}$, which is approximately 1.46.

This differs by less than $\int_5^{\infty} \frac{1}{x^2} dx = \frac{1}{5}$, or 0.2, from the sum of the series.

And as we shall see later, this estimate isn't far off — the actual sum is a little bigger than 1.6.

Questions

Does the series $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$ converge or diverge?

A. Converge

B. Diverge

Does the series $\sum_{n=1}^{\infty} \frac{\arctan(n)}{1+n^2}$ converge or diverge?

A. Converge

B. Diverge

For this latter series, find a bound on the error if we use the sum of the first 100 terms to approximate the limit.

(answer: it is less than about .015657444)

The comparison test

This convergence test is even more common-sensical than the integral test. It says that if all the terms of the series $\sum_{n=1}^{\infty} a_n$ are less than the corresponding terms of the series $\sum_{n=1}^{\infty} b_n$ and if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges also.

This test can also be used in reverse — if the b series diverges and the a 's are **bigger** than the corresponding b 's, then $\sum_{n=1}^{\infty} a_n$ diverges also.

Examples

$\sum_{n=1}^{\infty} \frac{1}{2^n + n}$ converges; compare with the convergent $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n + \sin(n)}$ diverges; compare with the divergent $\sum_{n=2}^{\infty} \frac{1}{n-1}$.

Does the series $\sum_{k=5}^{\infty} \frac{1}{\sqrt{k} - 2}$ converge or diverge?

A. Converge

B. Diverge

Does the series $\sum_{n=1}^{\infty} \frac{1}{2^n + n^2}$ converge or diverge?

A. Converge

B. Diverge

The ratio test

The ratio test is a specific form of the comparison test, where the comparison series is a geometric series. We begin with the observation that for geometric series, the ratio of consecutive terms $\frac{a_{n+1}}{a_n}$ is a constant (we called it r earlier).

For other series, even if the ratio of consecutive terms is not constant, it might have a limit as n goes to infinity. If this is the case, and the limit is not equal to 1, then the series converges or diverges according to whether the geometric series with the same ratio does. In other words:

The ratio test

The ratio test

If $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = r$, then the series $\sum_{n=1}^{\infty} a_n$:

- converges if $r < 1$
- diverges if $r > 1$
- If $r = 1$ or if the limit does not exist, then no conclusion can be drawn.

Example: For $\sum_{n=1}^{\infty} \frac{1}{n!}$ we have

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 < 1$$

so the series converges.

An anti-example

For $\sum_{n=1}^{\infty} \frac{1}{n^p}$, the ratio is 1 and the ratio test is inconclusive.

Of course, the integral test applies to these p -series.

Does the series $\sum_{n=1}^{\infty} \frac{n!}{5^n}$ converge or diverge?

A. Converge

B. Diverge

Does the series $\sum_{k=2}^{\infty} \frac{\ln k}{e^k}$ converge or diverge?

A. Converge

B. Diverge

The root test

The last test for series with positive terms that we have to worry about is the root test. This is another comparison with the geometric series. It's like the ratio test, except that it begins with the observation that for geometric series, the n th root of the n th term approaches the ratio r as n goes to infinity (because the n th term is ar^n and so the n th root of the n th term is $a^{1/n}r$ — which approaches r since the n th root of any positive number approaches 1 as n goes to infinity).

The root test

If $\lim_{n \rightarrow \infty} a_n^{1/n} = r$, then the series $\sum_{n=1}^{\infty} a_n$:

- converges if $r < 1$
- diverges if $r > 1$
- If $r = 1$ or if the limit does not exist, then no conclusion can be drawn.

Examples

The series

$$\sum_{n=1}^{\infty} \left(\frac{n}{2n+5} \right)^n$$

converges by the root test.

Does the series $\sum_{n=1}^{\infty} (1 - e^{-n})^n$ converge or diverge?

A. Converge

B. Diverge

Series whose terms are not all positive

- Now that we have series of positive terms under control, we turn to series whose terms can change sign.
- Since subtraction tends to provide cancellation which should “help” the series converge, we begin with the following observation:

A series with $+$ and $-$ signs will definitely converge if the corresponding series obtained by replacing all the $-$ signs by $+$ signs converges.

Absolute convergence

- Another way to say this is to think of the terms a_n of the series $\sum_{n=1}^{\infty} a_n$ as some being positive and others negative. Then the series obtained by changing all the minus signs to plus signs would be $\sum_{n=1}^{\infty} |a_n|$ — the “series of absolute values”

Absolute convergence

A series whose series of absolute values converges, which is itself then convergent, is called an *absolutely convergent series*.

Examples

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots$ is absolutely convergent.

$\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \dots$ is divergent, since a_n does not approach zero, and (of course) its series of absolute values is also divergent.

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$ is convergent (as we will see) even though its series of absolute values is divergent.

Series that are convergent although their series of absolute values diverge (convergent but not absolutely convergent) are called *conditionally convergent*.

Alternating series

A special case of series whose terms are of both signs that arises surprisingly often is that of **alternating series**. These are series whose terms *alternate in sign* (from some point on). There is a surprisingly simple convergence test that works for many of these:

Alternating series test:

Suppose a_0, a_1, \dots are all positive, so that

$\sum_{n=0}^{\infty} (-1)^n a_n = a_0 - a_1 + a_2 - a_3 + \dots$ is an alternating series.

If (the absolute values of) the terms are decreasing, that is $a_0 > a_1 > a_2 > \dots$ and if $\lim_{n \rightarrow \infty} a_n = 0$, then the series converges.

Moreover, the difference between the limit of the series and the partial sum S_n has the same sign and is less in absolute value than the first omitted term a_{n+1} ,

Example: The alternating harmonic series

The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

clearly satisfies the conditions of the test and is therefore convergent.

The error estimate tells us that the partial sum

$$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$$

is **less** than the limit, and within $1/5$ of it.

And just to practice using the jargon: *The alternating harmonic series, being convergent but not absolutely convergent, is an example of a conditionally convergent series.*

Your turn: Classify each of the following:

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$$

- A. Absolutely convergent B. Conditionally convergent
C. Divergent

$$\sum_{k=1}^{\infty} \frac{\sin k}{k^3}$$

- A. Absolutely convergent B. Conditionally convergent
C. Divergent

$$\sum_{n=1}^{\infty} \frac{n \cos n\pi}{n^2 + 5}$$

- A. Absolutely convergent B. Conditionally convergent
C. Divergent



Getting ready for power series

In the homework, you were to estimate the sum of a few series with your calculator or computer, and then try to identify the actual sums. The answers, correct to ten decimal places, are:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \approx 0.7853981635$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.718281828$$

$$\sum_{n=0}^{\infty} \frac{1}{n^2} \approx 1.6644934068$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} = 0.6931471806$$

We can recognize these numbers as $\frac{\pi}{4}$, e , $\frac{\pi^2}{6}$, and $\ln 2$, respectively.

This presents us with two directions of investigation:

- 1 Given a number, come up with a series that has the number as its sum, so we can use it to get decimal approximations.
- 2 Develop an extensive vocabulary of “known” series, so we can recognize “familiar” series more often.

Geometric series revisited

We begin with our old friend, the geometric series:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + ar^4 + \cdots = \frac{a}{1-r}$$

provided $|r| < 1$.

r as a variable

Changing our point of view for a minute (or a week, or a lifetime), let's think of r as a variable. We change its name to x to emphasize the point:

$$f(x) = \sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + ax^3 + ax^4 + \dots = \frac{a}{1-x} \quad (\text{for } |x| < 1)$$

So the series defines a **function** (at least for certain values of x).

Watch out...

We can identify the geometric series when we see it, we can calculate the function it represents and go back and forth between function values and specific series.

We must be careful, though, to avoid substituting values of x that are not allowed, lest we get nonsensical statements like

$$1 + 2 + 2^2 + 2^3 + \dots = 1 + 2 + 4 + 8 + \dots = -1 !!$$

Polynomials and series

If you look at the geometric series as a function, it looks rather like a polynomial, but of infinite degree.

Polynomials are important in mathematics for many reasons among which are:

- ① **Simplicity** — they are easy to express, to add, subtract, multiply, and occasionally divide
- ② **Closure under algebraic operations** — they stay polynomials when they are added, subtracted and multiplied.
- ③ **Calculus** – they are easy to differentiate and integrate, and they stay polynomials when differentiated or integrated

Infinite polynomials

So we'll think of power series as polynomials of infinite degree and write

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Some questions arise:

- 1 Given a function (other than $f(x) = \frac{a}{1-x}$), can it be expressed as a power series? If so, how?
- 2 For what values of x is a power series representation valid? This is a two-part question — if we start with a function and form “its” power series, then
 - (a) For which values of x does the series converge?
 - (b) For which values of x does the series converge **to** $f(x)$?
 - (There’s also the question of “how fast”)
- 3 Given a series, can we tell what function it came from?
- 4 What is all this good for?

As it turns out, the order of difficulty of the first three questions are 1, 2(a), 2(b) and then 3. So we’ll start with question 1.

The power series of a function $f(x)$

Suppose the function $f(x)$ has the power series:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

Q, How to calculate the coefficients a_n from a knowledge of $f(x)$?

A. One at a time — differentiate and plug in $x = 0$!!

Calculating coefficients

- First (or zeroth): $f(0) = a_0 + a_1 0 + a_2 0^2 + a_3 0^3 + \dots = a_0$.
So we have that $a_0 = f(0)$.
- Second (first?): It seems reasonable to write $f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$, so we should have $a_1 = f'(0)$.
- Another derivative:
 $f''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \dots$, so we should have $a_2 = \frac{1}{2} f''(0)$



Continuing in this way...

$$a_3 = \frac{f'''(0)}{6} \quad a_4 = \frac{f^{(4)}(0)}{24} \quad a_5 = \frac{f^{(5)}(0)}{120} \quad \text{etc.}$$

In general:

$$a_n = \frac{f^{(n)}(0)}{n!} = \text{the } n\text{th derivative of } f \text{ evaluated at } 0 \text{ divided by } n!$$

Example

Suppose we know for the function f that $f(0) = 1$ and $f'(x) = f(x)$. (From what we know about differential equations, this means that $f(x) = e^x$.) So $f'' = f'$, $f''' = f''$, etc., so $f(0) = f'(0) = f''(0) = \dots = 1$. Therefore:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Series for e^x

The power series for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Set $x = 1$ in the series (and in e^x) to get

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}$$

which is one of the series we considered earlier.

Your turn!

Can you find the series for $f(x) = \sin x$?