# USE OF LAPLACE TRANSFORMS TO SUM INFINITE SERIES

One of the more valuable approaches to summing certain infinite series is the use of Laplace transforms in conjunction with the geometric series. One starts with the basic definition for the Laplace transform of a function **f(t)** and treats the Laplace variable **s** as an integer **n**. That is-

$$F(n) = \int_0^\infty f(t)\exp(-nt)dt$$

Summing both sides from n=0 to n=∞, one finds-

$$\sum_{n=1}^{\infty} F(n) = \int_{t=0}^{\infty} \frac{f(t)}{\exp(t)-1}dt$$

after making use of the geometric series-

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r} \quad provided\ that \quad |r|<1$$

One thus has a way to sum certain infinite series when the function **F(n)** corresponds to a Laplace transform of a function **f(t)** provided one can evaluate the integral on the right side of the equality exactly or evaluate it numerically faster than summing the original infinite series.

Lets look at several examples. Consider first the infinite sum from n=0 to ∞ for the function **F(n)=(-1)$^{n}$ /(n+1)**. Here one finds-

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)} = \int_{t=0}^{\infty} \frac{\exp(-t)}{1+\exp(-t)}dt = -\ln(1+\exp(-t))\Big|_0^\infty = \ln(2)$$

It is interesting that this series converges to a finite value, while removing the alternating sign from it would convert it to the harmonic series which diverges. Take next the case of **f(t)=[exp(t)-1]exp-kt** , where **F(n) = 1/(n+k-1)-1/(n+k) = 1/[(n+k-1)(n+k)]** . Substituting we find-

$$\sum_{n=1}^{\infty} \frac{1}{[(n+k-1)(n+k)} = \int_{t=0}^{\infty} \exp(-(kt) = \frac{1}{k}$$

Thus if **k=½ ,** one has-

$$\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\frac{1}{63}+\frac{1}{99}+...+\frac{1}{4n^2-1}+...=\frac{1}{2}$$

As a third example consider **f(t)=sin(at)** with the corresponding **F(n)=a/[a²+n²]** . Here we have-

$$\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}=\left(\frac{1}{a}\right)\int_0^{\infty}\frac{\sin(at)}{[\exp(t)-1]}=\frac{1}{2a}[\pi\coth(\pi a)-\frac{1}{a}]$$

The integral was evaluated by complex variable methods involving a rectangular contour in the complex plane. For this example it is also possible to evaluate the sum directly by applying the residue theorem to the function π **cot(∞ z) F(z)** at the poles of **F(z)(see M.Spiegel Theory and Problems of Complex Variables”**). For the special case of **a=1** the value of the sum becomes **1.076674047**...Also,differentiating the above result two times with respect to **a** and then setting **a=1/π** leads to the interesting result-

$$\int_0^{\infty}\frac{t^2\sin(t)}{[\exp(\pi t)-1]}dt=[1+\coth(1)-\coth(1)^3]=0.049281490...$$

Next let us look at **f(t)=t$^{p}$/Γ(p+1)** where **p>0** but not necessarily integer. In this case **F(n)=1/n$^{p+1}$** whose sum from **n=1 to** ∞ gives the standard definition of the **Riemann Zeta function.** One has the identity-

$$\zeta(p+1)=\sum_{n=1}^{\infty}\frac{1}{n^{p+1}}=\int_0^{\infty}\frac{[t^p/\Gamma(p+1)]}{[\exp(t)-1]}dt$$

When **p=1** the zeta function ζ.**(2)=π$^{2}$/6,** a well known result first obtained by Leonard Euler. We note that the integral form of the zeta function shown above is also encountered in the Debye theory of specific heats for the case **p=3 (see Kittel,”Introduction to Solid State Physics”**).

Finally we give you a few more infinite sums converted to integrals and the solved-

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n} = \int_{t=0}^{\infty} \frac{Heaviside[t - \ln(2)]}{\exp(t) - 1} dt = \ln(2)$$

$$\int_{t=0}^{\infty} \frac{\ln(t+1)}{t(t+1)} dt = \int_0^{\infty} \frac{x}{\exp(x) - 1} dx = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{\exp(-n)}{n^2} = \int_{t=0}^{\infty} \frac{(t-1) \cdot [Heaviside(t-1)]}{\exp(t) - 1} dt = 0.4087542875...$$

*and*

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n + 2} = \int_{t=0}^{\infty} \frac{\sin(t)\exp(-t)}{\exp(t) - 1} dt = 0.576674047468...$$

Note that in the last two samples the integrals could not be solved analytically but the sums were easily summed. Thus the present discussion between infinite series and certain indefinite integrals can also be used in a reverse manner to obtain numerical values for integrals knowing the values of the corresponding sums.

October 19, 2004