Algorithms (CS:3330:0001 or 22C:031:001)
Homework 3

Each of the four questions counts for the same number of points. The homework is due in class on Friday, October 3rd. If you can't make it to class, drop it in my mailbox in the MacLean Hall mailroom. For this homework, you can use at most one late day. I'd like to post solutions by 1:20 pm on October 6, in time for the first midterm.

1. There is a desert with n oases, and I would like to walk from oasis s to oasis t. I have a map with the locations of the n oases, and this map also tells me the distance one needs to walk to go directly between any two oases. The catch is that I have only one water bottle, and I can travel at most k miles without refilling it. (That is, the water in the bottle runs out in k miles.) I can refill the bottle at any oasis, but nowhere else. Describe an efficient algorithm that, given the map, s, t, and k, computes a sequences of oases such that (a) the first oasis is s and the last one is t; and (b) the direct walking distance between any two consecutive oases in the sequence is at most k. What if I also wanted a smallest such sequence?

   As an illustration of the problem, suppose that the oases are s, t, and a. And the distances are $d(s,t) = 10$, $d(a,s) = 7$, and $d(a,t) = 5$. Suppose that $k = 8$. Then a sequence that works is $\langle s, a, t\rangle$, and this is also the smallest sequence that works. (Its length is 3.) On the other hand, if $k = 10$, the sequence $\langle s, t\rangle$ also works, and is the smallest sequence that works. (Its length is 2.)

2. Decide whether the following claim is true or not, and give either a proof of the claim or an example that shows it is not true.

   > Claim: Let G be an undirected graph on n nodes, where $n \geq 2$ is an even number. If every node of G has degree at least $n/2$, then G is connected.

   Recall that the degree of a node is the number of other nodes to which it has edges. (This is basically Exercise 7 of Chapter 3 of the text, where you can also find a scenario that motivates the problem.)

3. Decide whether the following claim is true or not, and give either a proof of the claim or an example that shows it is not true.

   > Claim: Let G be a directed graph on n nodes, where $n \geq 2$ is an even number. If every node of G has out-degree at least $n/2$, then G is strongly connected.

   The out-degree of a node u is the number of other nodes v such that (u, v) is an edge.

4. You're helping a group of ethnographers analyze some oral history data they've collected by interviewing members of a village to learn about the lives of people who've lived there over the past two hundred years.

From those interviews, they've learned about a set of n people (all of them now deceased), whom we'll denote $P_1, P_2, \ldots, P_n$. They've also collected facts about when these people lived relative to one another. Each fact has one of the two following forms:

- Person P_i died before person P_j was born.
- The life spans of P_i and P_j overlapped at least partially.

(Here P_i and P_j are two specific people whom the fact is about.)

Naturally, they're not sure that all these facts are correct; memories are not so good, and a lot of this was passed down by word of mouth. So what they'd like you to determine is whether the data they've collected is at least internally consistent, in the sense that there could have existed a set of people for which all the facts they've learned simultaneously hold.

Give an algorithm with polynomial running time to do this: either it should produce proposed dates of birth and death for each of the n people so that all the facts hold true, or it should report (correctly) that no such dates can exist – that is, the facts collected by the ethnographers are not internally consistent. (This is Exercise 12 of Chapter 3.)

For example, suppose we have these facts:

- Alice died before Bob was born.
- The life spans of Carol and Bob overlapped at least partially.

Given these facts, we can come up with dates of birth and death for Alice, Bob, and Carol that are consistent with these facts.

On the other hand, suppose we also have the fact:

- Carol died before Alice was born.

In this case, such dates cannot exist.

Let's review some policy from the first day handout. "For homework problems, collaboration is allowed, assuming each of you has first spent some time (about 30 minutes) working on the problem yourself. However, no written transcript (electronic or otherwise) of the collaborative discussion should be taken from the discussion by any participant. Furthermore, discussing ideas is okay but viewing solutions of others is not. It will be assumed that each of you is capable of orally explaining the solution that you turn in, so do not turn in something you don't understand. Students are responsible for understanding this policy; if you have questions, ask for clarification."

Please note that although the correctness of your solution counts for a significant fraction of your score, the quality of your writing and a demonstration that you understood the question and made a serious attempt at solving it also count for a significant fraction.