MAE 113, Summer Session 1, 2009
HW# 2 Solutions

1.17 An aircraft begins a cruise at a wing loading W/S_w of 100 lbf/ft^2 and Mach 0.8. The drag coefficients are $K_1 = 0.056$, $K_2 = -0.008$, $C_{D0} = 0.014$, and TSFC is constant at 0.8 (lbm/h)/lbf. For a weight fraction W_f / W_i of 0.9, determine the range and other parameters for two different types of cruise.

a) For a cruise climb (maximum C_L / C_D) flight path, determine C_L, C_D, initial and final altitudes, and range.

Here we'll use equations 1.47 and 1.48

$$C_L^* = \sqrt{\frac{C_{D0}}{K_1}} = \sqrt{\frac{0.014}{0.056}} = \boxed{0.5}$$

$$\left(\frac{C_L}{C_D}\right)^* = \frac{1}{2\sqrt{C_{D0} K_1} + K_2} = \frac{1}{2\sqrt{(0.014)(0.056)} + -0.008} = 20.83$$

$$C_D{}^* = \frac{C_L^*}{\left(\frac{C_L}{C_D}\right)^*} = \frac{0.5}{20.83} = \boxed{0.024}$$

by combining equations 1.29 and 1.30b, and solving for δ, we get

$$\delta = \frac{2\,n\,W}{\gamma\,P_{\text{ref}}\,M_0{}^2\,C_L\,S_w}$$

for the initial case,

$$\delta_{\text{intial}} = \frac{2\,n\,W_{\text{initial}}}{\gamma\,P_{\text{ref}}\,M_0{}^2\,C_L\,S_w} = \frac{2\,(1)\left(100\,\frac{\text{lbf}}{\text{ft}^2}\right)}{1.4\left(2116.8\,\frac{\text{lbf}}{\text{ft}^2}\right)(0.8)^2\,(0.5)} = 0.2109$$

this corresponds to an initial altitude of about $\boxed{37,360\text{ ft.}}$

now for the final case

$$\delta_{\text{final}} = \frac{2\,n\,W_{\text{inital}}\left(\frac{W_f}{W_i}\right)}{\gamma\,P_{\text{ref}}\,M_0{}^2\,C_L\,S_w} = \frac{2\,(1)\left(100\,\frac{\text{lbf}}{\text{ft}^2}\right)(0.9)}{1.4\left(2116.8\,\frac{\text{lbf}}{\text{ft}^2}\right)(0.8)^2\,(0.5)} = 0.1898$$

this corresponds to a final altitude of about $\boxed{39,560\text{ ft.}}$

For range factor, we use equation 1.43

$$\text{RF} = \frac{C_L}{C_D}\,\frac{V}{\text{TSFC}}\,\frac{g_c}{g_0} = \frac{0.5}{0.024}\,\frac{1116\,\frac{\text{ft}}{s}(0.8)\left(\sqrt{0.7519}\right)\frac{3600\,s}{\text{hr}}}{0.8\,\frac{\text{lbm/hr}}{\text{lbf}}}\,\frac{32.174\,\frac{\text{lbm ft}}{\text{lbf}\,s^2}}{32.174\,\frac{\text{lbm ft}}{\text{lbf}\,s^2}}\,\frac{\text{nm}}{6080\text{ ft}} = 11,937\text{ nm}$$

and now equation 1.45a

$$S = \text{RF}\ln\left(\frac{W_i}{W_f}\right) = 11\,937\text{ nm}\ln\left(\frac{1}{0.9}\right) = \boxed{1258\text{ nm}}$$

b) For a level cruise (maximum $\sqrt{C_L}\,/\,C_D$) flight path, determine C_L, C_D, initial and final altitudes, and range.

We start back at equation 1.32

$$C_D = K_1\,C_L{}^2 + K_2\,C_L + C_{D0}$$
$$\frac{C_D}{\sqrt{C_L}} = K_1\,C_L{}^{3/2} + K_2\,C_L{}^{1/2} + C_{D0}\,C_L{}^{-1/2}$$
$$\frac{\partial}{\partial C_L}\left(\frac{C_D}{\sqrt{C_L}}\right) = \frac{\partial}{\partial C_L}\left(K_1\,C_L{}^{3/2} + K_2\,C_L{}^{1/2} + C_{D0}\,C_L{}^{-1/2}\right)$$
$$\frac{\partial}{\partial C_L}\left(\frac{C_D}{\sqrt{C_L}}\right) = \frac{3}{2}\,K_1\,C_L{}^{1/2} + \frac{1}{2}\,K_2\,C_L{}^{-1/2} + \frac{-1}{2}\,C_{D0}\,C_L{}^{-3/2}$$
$$0 = \frac{3}{2}\,K_1\,C_L{}^{1/2} + \frac{1}{2}\,K_2\,C_L{}^{-1/2} + \frac{-1}{2}\,C_{D0}\,C_L{}^{-3/2}$$
$$0 = \frac{3}{2}\,K_1\,C_L{}^{2} + \frac{1}{2}\,K_2\,C_L{}^{1} + \frac{-1}{2}\,C_{D0}\,C_L{}^{0}$$
$$C_L = -\frac{\frac{1}{2}K_2}{2\left(\frac{3}{2}K_1\right)} \pm \frac{\sqrt{\left(\frac{1}{2}K_2\right)^2 - 4\left(\frac{3}{2}K_1\right)\left(\frac{-1}{2}C_{D0}\right)}}{2\left(\frac{3}{2}K_1\right)}$$
$$C_L = -\frac{K_2}{6\,K_1} + \frac{\sqrt{\frac{1}{4}K_2{}^2 + 3\,K_1\,C_{D0}}}{3\,K_1}$$
$$C_L = \frac{\sqrt{K_2{}^2 + 12\,K_1\,C_{D0}} - K_2}{6\,K_1} = \frac{\sqrt{(-0.008)^2 + 12\,(0.056)\,(0.014)} -- 0.008}{6\,(0.056)} = \boxed{0.3135}$$
$$C_D = (0.056)\,(0.3135)^2 + (-0.008)\,(0.3135) + 0.014 = \boxed{0.0170}$$

now solving for δ as before

$$\delta = \frac{2\,n\,W_{\text{initial}}}{\gamma\,P_{\text{ref}}\,M_0{}^2\,C_L\,S_w} = \frac{2\,(1)\left(100\,\frac{\text{lbf}}{\text{ft}^2}\right)}{1.4\left(2116.8\,\frac{\text{lbf}}{\text{ft}^2}\right)(0.8)^2\,(0.3135)} = 0.3365$$

this corresponds to a final altitude of about $\boxed{27,280\text{ ft.}}$

For initial velocity,

$$V = M\,a_{\text{std}}\,\sqrt{\theta} = 0.8\left(1116\,\frac{\text{ft}}{s}\right)\sqrt{0.8141} = \boxed{805.6\text{ ft}/s}$$

And final velocity is given by

$$V = V_i\,\sqrt{\frac{W_f}{W_i}} = 805.6\,\frac{\text{ft}}{s}\,\sqrt{0.9} = \boxed{764.2\text{ ft}/s}$$

Now, range factor

$$\text{RF} = \frac{C_L}{C_D}\,\frac{V}{\text{TSFC}}\,\frac{g_c}{g_0} = \frac{0.3135}{0.0170}\,\frac{1116\,\frac{\text{ft}}{s}\,(0.8)\left(\sqrt{0.8141}\right)\frac{3600\,s}{\text{hr}}}{0.8\,\frac{\text{lbm/hr}}{\text{lbf}}}\,\frac{32.174\,\frac{\text{lbm ft}}{\text{lbf }s^2}}{32.174\,\frac{\text{lbm ft}}{\text{lbf }s^2}}\,\frac{\text{nm}}{6080\text{ ft}} = 10,994\text{ nm}$$

And range comes from problem 1-16 since the altitude is too low for the Breguet range equation to apply

$$\frac{W_f}{W_i} = \left(1 - \frac{s}{2\,\text{RF}_i}\right)^2$$

$$s = 2\,\text{RF}_i\left(1 - \sqrt{\frac{W_f}{W_i}}\right) = 2\,(10\,994\text{ nm})\left(1 - \sqrt{0.9}\right)$$

$$\boxed{s = 1128.4\text{ nm}}$$

1.21 Rocket motor on static stand with exhaust of 100 lbm/s and exit velocity 2000 ft/s and pressure 50 psia. Exit area is 0.2 ft^2. For ambient pressure of 14.7 psia, determine effective exhaust velocity, thrust transmitted to test stand, and the specific impulse.

Effective exhaust velocity is defined in equation 1.53

$$C \equiv V_e + \frac{(P_e - P_a) A_e g_c}{\dot{m}_p}$$

$$C = 2000 \frac{\text{ft}}{s} + \frac{\left(50 \frac{\text{lbf}}{\text{in}^2} - 14.7 \frac{\text{lbf}}{\text{in}^2}\right) \frac{144 \text{ in}^2}{\text{ft}^2} \left(0.2 \text{ ft}^2\right) 32.174 \frac{\text{lbm ft}}{\text{lbf } s^2}}{100 \frac{\text{lbm}}{s}}$$

$$\boxed{C = 2327.1 \frac{\text{ft}}{s}}$$

Now static thrust is defined by equation 1.54

$$F = \frac{\dot{m}_p C}{g_c}$$

$$F = \frac{100 \frac{\text{lbm}}{s} 2327.1 \frac{\text{ft}}{s}}{32.174 \frac{\text{lbm ft}}{\text{lbf } s^2}}$$

$$\boxed{F = 7232.8 \text{ lbf}}$$

And I_{sp} is given by equation 1.56

$$I_{sp} = \frac{C}{g_0}$$

$$I_{sp} = \frac{2327.1 \frac{\text{ft}}{s}}{32.174 \frac{\text{ft}}{s^2}}$$

$$\boxed{I_{sp} = 72.3 \, s}$$

1.22 Rocket motor static testing with exhaust 50 kg/s at 800 m/s and 350 kPa. Exit area is 0.02 m^2. For ambient pressure 100 kPa, determine the effective exhaust velocity, thrust, and specific impulse.

We'll follow the same plan as 1.21

$$C \equiv V_e + \frac{(P_e - P_a) A_e g_c}{\dot{m}_p}$$

$$C = 800 \frac{m}{s} + \frac{(350\,000 \text{ Pa} - 100\,000 \text{ Pa}) (0.02\, m^2)}{50 \frac{\text{kg}}{s}}$$

$$\boxed{C = 900 \frac{m}{s}}$$

Now static thrust is defined by equation 1.54

$$F = \frac{\dot{m}_p C}{g_c}$$

$$F = \frac{50 \frac{\text{kg}}{s} 900 \frac{m}{s}}{1}$$

$$\boxed{F = 45\,000\, N}$$

And I_{sp} is given by equation 1.56

$$I_{\text{sp}} = \frac{C}{g_0}$$

$$I_{\text{sp}} = \frac{900 \frac{m}{s}}{9.8 \frac{m}{s^2}}$$

$$\boxed{I_{\text{sp}} = 91.84\, s}$$

2.12 Air at 1400K, 8 atm, and 0.3 Mach expands isotropically through a nozzle to 1 atm. Assuming a calorically perfect gas, find the exit temperature and the inlet and exit areas for a mass flow rate of 100 kg/s.

Let's start with equation 2.43

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{(\gamma-1)/\gamma}$$
$$T_2 = T_1\left(\frac{P_2}{P_1}\right)^{(\gamma-1)/\gamma} = 1400\,K\left(\frac{1\text{ atm}}{8\text{ atm}}\right)^{(1.4-1)/1.4} = \boxed{772.9\,K}$$

Now let's use equation 2.12b for the mass flow rate

$$\dot{m}_1 = \rho\, A_1\, V$$
$$A_1 = \frac{\dot{m}_1}{\rho\, V}$$

And equation 2.21 brings in pressure

$$P = \rho\, R\, T$$

Also, equation 2.36 helps us get V

$$V = a\, M = M\sqrt{\gamma\, R\, g_c\, T}$$

plugging these in for area,

$$A_1 = \frac{\dot{m}_1\, R\, T_1}{P_1\, M_1\sqrt{\gamma\, R\, g_c\, T_1}} = \frac{\dot{m}_1}{P_1\, M_1}\sqrt{\frac{R\, T_1}{\gamma\, g_c}} = \frac{100\text{ kg/}s}{8\text{ atm}\,\frac{101\,300\text{ Pa}}{\text{atm}}\,(0.3)}\sqrt{\frac{287\,\frac{J}{\text{kg}\,K}\,1400\,K}{1.4\left(1\,\frac{m^3}{\text{kg}\,s^2}\right)}}$$
$$\boxed{A_1 = 0.2204\, m^2}$$

Now use the equation in the middle of page 86, from the first law of thermodynamics

$$c_p\, T_1 + \frac{V_1^2}{2\, g_c} = c_p\, T_2 + \frac{V_2^2}{2\, g_c}$$

First, we apply equation 2.36

$$c_p\, T_1 + \frac{M_1^2\,\gamma\, R\, g_c\, T_1}{2\, g_c} = c_p\, T_2 + \frac{M_2^2\,\gamma\, R\, g_c\, T_2}{2\, g_c}$$
$$T_1\left(c_p + \frac{M_1^2\,\gamma\, R}{2}\right) = T_2\left(c_p + \frac{M_2^2\,\gamma\, R}{2}\right)$$
$$T_1\left(1 + \frac{M_1^2\,\gamma\, R}{2\, c_p}\right) = T_2\left(1 + \frac{M_2^2\,\gamma\, R}{2\, c_p}\right)$$

Next we apply equation 2.30

$$T_1\left(1+\frac{M_1^2\,\gamma}{2}\;\frac{\gamma-1}{\gamma}\right)=T_2\left(1+\frac{M_2^2\,\gamma}{2}\;\frac{\gamma-1}{\gamma}\right)$$
$$T_1\left(1+\frac{M_1^2}{2}\,(\gamma-1)\right)=T_2\left(1+\frac{M_2^2}{2}\,(\gamma-1)\right)$$

Solving for M_2

$$M_2=\sqrt{2\,\frac{T_1\left(1+\frac{M_1^2}{2}\,(\gamma-1)\right)-T_2}{T_2(\gamma-1)}}=\sqrt{2\,\frac{1400\,K\left(1+\frac{(0.3)^2}{2}\,(1.4-1)\right)-772.9}{772.9\,(1.4-1)}}=2.054$$

and, as before

$$A_2=\frac{\dot{m}_2}{P_2\,M_2}\sqrt{\frac{R\,T_2}{\gamma\,g_c}}=\frac{100\text{ kg}/s}{1\text{ atm}\,\frac{101\,300\text{ Pa}}{\text{atm}}\,(2.054)}\sqrt{\frac{287\,\frac{J}{\text{kg}\,K}\,772.9\,K}{1.4\,(1)}}$$
$$\boxed{A_2=0.1913\,m^2}$$

2.17 Air at 225K, 28 kPa, and M=2.0 enters an isentropic diffuser with an inlet area of $0.2 m^2$ and leaves at M=0.2. Assuming a calorically perfect gas, determine:

a) The velocity and mass flow rate of the entering air

We remember equation 2.36

$$V_1 = a\,M_1 = M_1 \sqrt{\gamma\, R\, g_c\, T_1} = 2.0 \sqrt{(1.4)\left(287\ \frac{J}{\text{kg}\,K}\right)(1)\,(225\ K)}$$

$$\boxed{V_1 = 601.3485\ m/s}$$

With the assistance of 2.12b and 2.21, we get

$$\dot{m}_1 = \frac{P\,A_1\,V}{\text{RT}} = \frac{28\,000\ \text{Pa}\left(0.2\ m^2\right)(601.3485\ m/s)}{287\ \frac{J}{\text{kg}\,K}\ 225\ K}$$

$$\boxed{\dot{m}_1 = 52.15\ \text{kg}/s}$$

b) The pressure and temperature of the leaving air

From problem 2.12, using Mach number and the first law of thermodynamics, we derived

$$T_1\left(1 + \frac{M_1^2}{2}(\gamma - 1)\right) = T_2\left(1 + \frac{M_2^2}{2}(\gamma - 1)\right)$$

$$T_2 = \frac{T_1\left(1+\frac{M_1^2}{2}(\gamma-1)\right)}{\left(1+\frac{M_2^2}{2}(\gamma-1)\right)} = \frac{225\ K\left(1+\frac{(2.0)^2}{2}(1.4-1)\right)}{\left(1+\frac{(0.2)^2}{2}(1.4-1)\right)}$$

$$\boxed{T_2 = 401.79\ K}$$

and now 2.43 lets us find pressure

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{(\gamma-1)/\gamma}$$

$$P_2 = P_1\left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} = 28\,000\ \text{Pa}\left(\frac{401.79\ K}{225\ K}\right)^{\frac{1.4}{1.4-1}}$$

$$\boxed{P_2 = 213.067\ \text{kPa}}$$

c) The exit area and magnitude and direction of the force on the diffuser (assume outside of diffuser is 28 kPa)

First we want exit area. Using the equation derived in problem 2.12

$$A_2 = \frac{\dot{m}_2}{P_2\,M_2}\sqrt{\frac{R\,T_2}{\gamma\,g_c}} = \frac{52.15\ \text{kg}/s}{213\,067\ \text{Pa}(0.2)}\sqrt{\frac{287\ \frac{J}{\text{kg}\,K}\ 401.79\ K}{1.4\,(1)}}$$

$$\boxed{A_2 = 0.3512\ m^2}$$

Now we can find V_2

$$V_2 = a\,M_2 = M_2\sqrt{\gamma\,R\,g_c\,T_2} = 0.2\sqrt{(1.4)\left(287\ \frac{J}{\text{kg}\,K}\right)(1)\,(401.79\ K)}$$

$$\boxed{V_2 = 80.3590\ m/s}$$

If the force F points from 1 to 2, then

$$\Sigma F_x = -(P_2 - P_a)\, A_2 + (P_1 - P_a)\, A_1 + F$$
$$\Sigma F_x = \frac{\dot{m}}{g_c}\,(V_2 - V_1)$$

so

$$F = \frac{\dot{m}}{g_c}\,(V_2 - V_1) + (P_2 - P_a)\, A_2 - (P_1 - P_a)\, A_1$$
$$F = \frac{52.15\ \frac{\text{kg}}{s}}{1}\left(80.3590\ \frac{m}{s} - 601.3485\ \frac{m}{s}\right) + (213\,067\ \text{Pa} - 28\,000\ \text{Pa})\ 0.3512\ m^2 - (28\,000\ \text{Pa} - 28\,000\ \text{Pa})\ 0.2\ m^2$$
$$\boxed{F = 37,\ 826\ N}$$

2.21 50 kg/s of air enters compressor at 1 atm and 20°C and leaves at 20 atm and 427°C. If the process is adiabatic, find the input power, specific volume at exit, and change in entropy. Is the process reversible. (Assume a calorically perfect gas.)

We can start with equation 2.13

$$\dot{Q} - \dot{W}_x = \dot{m}\left(h + \frac{V^2}{2 g_c} + \frac{g_z}{g_c}\right)_{\text{out}} - \dot{m}\left(h + \frac{V^2}{2 g_c} + \frac{g_z}{g_c}\right)_{\text{in}}$$

As shown on the middle of page 70, in this case

$$\dot{m}\left(h + \frac{V^2}{2 g_c} + \frac{g_z}{g_c}\right)_{\text{out}} - \dot{m}\left(h + \frac{V^2}{2 g_c} + \frac{g_z}{g_c}\right)_{\text{in}} = 0$$

and so

$$\dot{W}_x = \dot{Q} = \dot{m}\, q = \dot{m}\, c_p (T_2 - T_1) = 50\ \frac{\text{kg}}{s}\left(1003.5\ \frac{J}{\text{kg}\, K}\right)(700\ K - 293\ K)$$

$$\boxed{\dot{W}_x = 20.42\ \text{MW}}$$

the perfect gas equation PV=nRT gives specific volume as

$$V_2 = \frac{\text{RT}_2}{P_2} = \frac{\left(287\ \frac{J}{\text{kg}\, K}\right)(700\ K)}{20\ \text{atm}\ \frac{101\,300\ \text{Pa}}{\text{atm}}}$$

$$\boxed{V_2 = 0.0988\ \frac{m^3}{\text{kg}}}$$

To find change in entropy, we use equation 2.40

$$s_2 - s_1 = \Delta \text{s} = c_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{P_2}{P_1}\right) = 1003.5\ \frac{J}{\text{kg}\, K} \ln\left(\frac{700\ K}{293\ K}\right) - 287\ \frac{J}{\text{kg}\, K} \ln\left(\frac{20\ \text{atm}}{1\ \text{atm}}\right)$$

$$\boxed{\Delta \text{s} = 14.18\ \frac{J}{\text{kg}\, K}}$$

Since the entropy increases for an adiabatic process, the second law of thermodynamics tells us that this process is **not reversible.**

2.22 Given 200 lb/s of air enters a steady flow turbine at 20 atm and 3400°R. It leaves at 10 atm. For a turbine efficiency of 85%, determine the exit temperature, output power, and change in entropy. (Assume a calorically perfect gas.)

To find the exit temperature, let's use the definition of turbine efficiency. A modified version of equation 6.18 gives us

$$\eta_t = \frac{h_1 - h_2}{h_1 - h_{2s}} = \frac{T_1 - T_2}{T_1 - T_{2s}}$$
$$T_2 = T_1 - \eta_t (T_1 - T_{2s})$$

We also use

$$T_{2s} = T_1 \left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}} = 3400\ °\text{R} \left(\frac{10\ \text{atm}}{20\ \text{atm}}\right)^{\frac{1.4-1}{1.4}} = 2789.14\ °\text{R}$$
$$T_2 = 3400\ °\text{R} - 0.85\ (3400\ °\text{R} - 2789.14\ °\text{R})$$
$$\boxed{T_2 = 2880.77\ °\text{R}}$$

Now we use the equation from above, but switch T_1 and T_2 since we are using a compressor instead of a turbine.

$$\dot{W}_x = \dot{Q} = \dot{m}\, q = \dot{m}\, c_p (T_1 - T_2) = 200\ \frac{\text{lb}}{s} \left(0.24\ \frac{\text{Btu}}{\text{lb}\ °\text{R}}\right) (3400\ °\text{R} - 2880.77\ °\text{R})$$
$$\dot{W}_x = 24,926\ \frac{\text{Btu}}{s}$$
$$\dot{W}_x = 24,926\ \frac{\text{Btu}}{s} \frac{1055\ W}{\text{Btu}/s}$$
$$\boxed{\dot{W}_x = 26.298\ \text{MW}}$$

And, we can also find Δs as before

$$\Delta s = c_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{P_2}{P_1}\right) = 0.24\ \frac{\text{Btu}}{\text{lbm}\ °\text{R}} \ln\left(\frac{2880.77\ °\text{R}}{3400\ °\text{R}}\right) - 53.35\ \frac{\text{ft lbf}}{\text{lbm}\ °\text{R}} \frac{\text{Btu}}{778.16\ \text{lbf}} \ln\left(\frac{10\ \text{atm}}{20\ \text{atm}}\right)$$
$$\boxed{\Delta s = 0.0077\ \frac{\text{Btu}}{\text{lbm}\ °\text{R}}}$$