Lecture 10: exponentials and logarithms

Calculus I, section 10
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We now turn to our next (and more or less final) class of functions: exponential functions, as well as their inverses, logarithms.

Let b be any positive real number, and consider the function $f(x) = b^x$. We want to find $f'(x)$.

None of our rules are directly applicable, so let's just use the limit definition:

$$f'(x) = \frac{d}{dx} b^x = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h}.$$

Since $b^{x+h} = b^x \cdot b^h$, we can pull out a factor of b^x to get

$$\frac{d}{dx} b^x = b^x \cdot \lim_{h \to 0} \frac{b^h - 1}{h}.$$

This limit on the right is some unknown thing; we don't even know, a priori, that it exists. We could observe that it's equal to $f'(0)$, so this might remind you of the process for finding the derivatives of sine and cosine; for now let's just use the fact that it doesn't depend on x and call it some constant c. (Keep in mind that it *does* depend on b.)

Thus we've already figured out the main part of the claim:

$$\frac{d}{dx} b^x = cb^x$$

for some constant c, i.e. $f'(x)$ is proportionate to $f(x)$. Thus the larger $f(x)$ is, the faster it's increasing; conversely the smaller it is, the slower it changes. This is what's behind the rapidly increasing behavior of exponential functions. (Of course, c could be negative, in which case we replace "increasing" above with "decreasing.")

We'll come back to the question of what exactly this quantity c is, but first let's talk about logarithms. Logarithms are inverse functions of exponentials, i.e. with f as above,

$$f^{-1}(x) = \log_b(x).$$

Since we now know the derivative of $f(x)$ (at least up to identifying the constant c), we can apply the inverse rule:

$$\frac{d}{dx} \log_b(x) = \frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{cb^{\log_b(x)}} = \frac{1}{cx}.$$

This is already very interesting: the derivative takes $\log_b(x)$, which is a transcendental function defined as the inverse function of another transcendental function, into a simple rational function! We'll say more about this in a bit: one can even use it to define exponentials and logarithms.

We can now get a better handle on this number c. To simplify the situation, we plug in $x = 1$ to the above equation:

$$\left.\frac{d}{dx}\log_b(x)\right|_{x=1} = \frac{1}{c}.$$

On the other hand, by definition the derivative of $\log_b(x)$ at $x = 1$ is

$$\lim_{h\to 0}\frac{\log_b(1+h) - \log_b(1)}{h}.$$

By the properties of logarithms, $\log_b(1) = 0$ and this is

$$\lim_{h\to 0}\frac{1}{h}\log_b(1+h) = \lim_{h\to 0}\log_b\left((1+h)^{1/h}\right),$$

which by the composition limit law is

$$\log_b\left(\lim_{h\to 0}(1+h)^{1/h}\right)$$

so long as the inner limit exists and is in the domain of $\log_b(x)$.

What is this inner limit? Well, we can try plugging in some small numbers first to get some idea of what it looks like: if $h = 0.01$, then

$$(1+h)^{1/h} \approx 2.7048.$$

If $h = 0.001$, then

$$(1+h)^{1/h} \approx 2.7169;$$

if $h = 0.0001$, then

$$(1+h)^{1/h} \approx 2.7181.$$

At a first guess, then, this seems like it's approaching some constant of approximately 2.718.

This is indeed true: this number is called e, short for Euler's number, and is *defined* to be this limit:

$$e = \lim_{h\to 0}(1+h)^{1/h}.$$

An equivalent definition which you may have seen before is to think of, instead of h going to 0, taking $n = \frac{1}{h}$ so that $h \to 0$ is the same thing as $n \to \infty$. Then this definition is the same thing as

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.$$

Taking $n = 10^{100}$ gives the first hundred decimal digits:

$$e = 2.718281828459045235360287471352662497757247093699959$$
$$574966967627724076630353547594571382178525166427\ldots.$$

Now we can go back to our problem: combining the two expressions for the derivative of $\log_b(x)$ at $x = 1$, we get

$$\frac{1}{c} = \log_b(e),$$

and so

$$c = \frac{1}{\log_b(e)}.$$

Using the base change rule for logarithms,

$$\log_b(e) = \frac{\log_e(e)}{\log_e(b)} = \frac{1}{\log_e(b)},$$

and so this means

$$c = \log_e(b).$$

In particular, when $b = e$, $c = 1$, so we have

$$\frac{d}{dx}e^x = e^x$$

and

$$\frac{d}{dx}\log_e(x) = \frac{1}{x},$$

without any complicating scalar factor. This makes base e logarithms and exponentials more convenient to work with, despite the complicated definition of e, and suggests that e is in some sense the "natural" base for exponentials and logarithms. In particular, $\log_e(x)$ is sometimes called the "natural logarithm," and written $\ln(x)$ or just $\log(x)$, with no subscript. (Conventions vary, but in this class if you just see $\log(x)$ with no subscript it will always mean natural logarithm, not base 10 or base 2; I'll try and stick to $\ln(x)$ for natural logarithm to avoid confusion, but I may slip up.)

We can now go back and complete our computations of the derivatives of exponentials and logarithms. We have first

$$\frac{d}{dx}b^x = cb^x = b^x \ln(b),$$

and similarly

$$\frac{d}{dx}\log_b(x) = \frac{1}{x\ln(b)}.$$

Let's go back to the observation that differentiation turns logarithms into rational functions. In particular, the derivative of $\ln(x)$ is particularly simple:

$$\frac{d}{dx}\ln(x) = \frac{1}{x} = x^{-1}.$$

Recall from the power rule that usually we obtain powers of x as derivatives of other powers of x, up to a scalar:

$$\frac{d}{dx}x^n = nx^{n-1}$$

for constants n, and so

$$\frac{d}{dx}\frac{x^{n+1}}{n+1} = \frac{n+1}{n+1}x^{n+1-1} = x^n,$$

i.e. we obtain x^n as the derivative of a rescaled higher power. This works for every real number n except $n = -1$: in this case we have $n + 1 = 0$ in the denominator, so this doesn't exist. It makes sense that this shouldn't work for -1, since the power one up would be $x^0 = 1$, whose derivative is zero. Nevertheless the logarithm gives us a function whose derivative is $\frac{1}{x}$, which we're otherwise unable to get.

In fact, one could *define* $\ln(x)$ to be the function whose derivative is $\frac{1}{x}$; it turns out that, together with requiring $\ln(1) = 0$, this completely determines it. We'll come back to this perspective near the end of class when we talk about integrals; note for now that this gives a way of defining logarithms which has nothing to do with exponentials or inverse functions, and indeed we can then define e^x as the inverse function of the logarithm, and other exponential functions from there, without having to do all the step-by-step stuff we had to go through to define b^x for x not a positive integer. (The hard part is then showing that it has the right properties.)

Another approach along similar lines is via the observation that $\frac{d}{dx}e^x = e^x$. It turns out that, together with $e^0 = 1$, this completely determines e^x, i.e. it is the unique function $f(x)$ such that $f(0) = 1$ and $f'(x) = f(x)$. This is one of the most fundamental differential equations, which you might see in calculus 2 and will definitely see if you take more advanced math classes. We could also take this as the *definition* of e^x, instead of anything about exponentials, and then derive the properties of exponentials and logarithms from there.

Let's go back to the original problem of differentiating exponentials and give some examples. Let

$$f(x) = 2^{-x}.$$

There are two approaches we could take. One is to combine the above method with the chain rule: the derivative of $-x$ is -1 and the derivative of 2^x is $2^x \ln(2)$, so the total derivative is

$$f'(x) = -2^{-x}\ln(2).$$

The other is to observe that

$$2^{-x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x$$

and so

$$f'(x) = \left(\frac{1}{2}\right)^x \ln\left(\frac{1}{2}\right),$$

and since $\left(\frac{1}{2}\right)^x = 2^{-x}$ and $\ln\left(\frac{1}{2}\right) = -\ln(2)$ these are the same.

We can also combine functions using our various rules: for example,

$$\frac{d}{dx}xe^x = \left(\frac{d}{dx}x\right)e^x + x\left(\frac{d}{dx}e^x\right) = e^x + xe^x = (x+1)e^x.$$

Using our knowledge of derivatives of logarithms together with the chain rule, we can also get out another useful trick. Suppose we have some complicated function involving a lot of products or exponents which we'd like to differentiate. We don't necessarily know how to do this, but one thing we do know is that taking logarithms usually simplifies such functions. Then we compute by the chain rule

$$\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)},$$

and so if $\frac{d}{dx}\ln(f(x))$ is something we can compute, then we conclude that $f'(x) = f(x) \cdot \frac{d}{dx}\ln(f(x))$.

For example, consider the function

$$f(x) = x^x.$$

First, note that this only makes sense for $x > 0$; for example, if $x = -\frac{1}{2}$ then $f(-\frac{1}{2}) = (-1/2)^{-1/2} = \frac{1}{\sqrt{-1/2}}$ doesn't exist, so we restrict to $x > 0$. Other than that restriction, this is a reasonable if strange function; how do we differentiate it?

Recall a warning about this example from the power rule: we might be tempted to try and say something like $f'(x) = x \cdot x^{x-1} = x^x$. This is not correct!

Instead, we try taking logarithms. We have

$$\ln(f(x)) = \ln(x^x) = x\ln(x),$$

and so by the product rule

$$\frac{d}{dx}\ln(f(x)) = \frac{d}{dx}x\ln(x) = \left(\frac{d}{dx}x\right)\ln(x) + x\left(\frac{d}{dx}\ln(x)\right) = 1 \cdot \ln(x) + \frac{x}{x} = \ln(x) + 1.$$

On the other hand, as above by the chain rule this is equal to

$$\frac{f'(x)}{f(x)} = \frac{f'(x)}{x^x},$$

so

$$f'(x) = x^x(\ln(x) + 1).$$

Another example is, again, the power rule. We did a lot of piece-by-piece work to show that $\frac{d}{dx}x^n = nx^{n-1}$ not just for positive integers but also negative numbers and fractions; technically we never showed that it's true for all real numbers. With logarithmic differentiation, though, we can get a straightforward proof of all cases at once: for every real number n, we have

$$\ln(x^n) = n\ln(x),$$

and so

$$\frac{d}{dx}\ln(x^n) = \frac{d}{dx}n\ln(x) = \frac{n}{x}.$$

On the other hand as above this is equal to $\frac{\frac{d}{dx}x^n}{x^n}$, so we conclude

$$\frac{d}{dx}x^n = x^n \cdot \frac{d}{dx}\ln(x^n) = nx^{n-1}.$$