# 4. Central Forces

In this section we will study the three-dimensional motion of a particle in a central force potential. Such a system obeys the equation of motion

$$m\ddot{\mathbf{x}} = -\nabla V(r) \tag{4.1}$$

where the potential depends only on $r = |\mathbf{x}|$. Since both gravitational and electrostatic forces are of this form, solutions to this equation contain some of the most important results in classical physics.

Our first line of attack in solving (4.1) is to use angular momentum. Recall that this is defined as

$$\mathbf{L} = m\mathbf{x} \times \dot{\mathbf{x}}$$

We already saw in Section 2.2.2 that angular momentum is conserved in a central potential. The proof is straightforward:

$$\frac{d\mathbf{L}}{dt} = m\mathbf{x} \times \ddot{\mathbf{x}} = -\mathbf{x} \times \nabla V = 0$$

where the final equality follows because $\nabla V$ is parallel to $\mathbf{x}$.

The conservation of angular momentum has an important consequence: all motion takes place in a plane. This follows because $\mathbf{L}$ is a fixed, unchanging vector which, by construction, obeys

$$\mathbf{L} \cdot \mathbf{x} = 0$$

So the position of the particle always lies in a plane perpendicular to $\mathbf{L}$. By the same argument, $\mathbf{L} \cdot \dot{\mathbf{x}} = 0$ so the velocity of the particle also lies in the same plane. In this way the three-dimensional dynamics is reduced to dynamics on a plane.

## 4.1 Polar Coordinates in the Plane

We've learned that the motion lies in a plane. It will turn out to be much easier if we work with polar coordinates on the plane rather than Cartesian coordinates. For this reason, we take a brief detour to explain some relevant aspects of polar coordinates.

To start, we rotate our coordinate system so that the angular momentum points in the $z$-direction and all motion takes place in the $(x, y)$ plane. We then define the usual polar coordinates

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

Our goal is to express both the velocity and acceleration in polar coordinates. We introduce two unit vectors, $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ in the direction of increasing $r$ and $\theta$ respectively as shown in the diagram. Written in Cartesian form, these vectors are

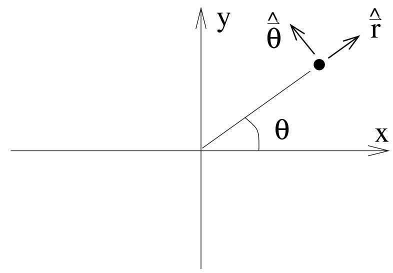


**Figure 11:**

$$\hat{\mathbf{r}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad , \quad \hat{\boldsymbol{\theta}} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

These vectors form an orthornormal basis at every point on the plane. But the basis itself depends on which angle $\theta$ we sit at. Moving in the radial direction doesn't change the basis, but moving in the angular direction we have

$$\frac{d\hat{\mathbf{r}}}{d\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \hat{\boldsymbol{\theta}} \quad , \quad \frac{d\hat{\boldsymbol{\theta}}}{d\theta} = \begin{pmatrix} -\cos\theta \\ -\sin\theta \end{pmatrix} = -\hat{\mathbf{r}}$$

This means that if the particle moves in a way such that $\theta$ changes with time, then the basis vectors themselves will also change with time. Let's see what this means for the velocity expressed in these polar coordinates. The position of a particle is written as the simple, if somewhat ugly, equation

$$\mathbf{x} = r\hat{\mathbf{r}}$$

From this we can compute the velocity, remembering that both $r$ and the basis vector $\hat{\mathbf{r}}$ can change with time. We get

$$\begin{aligned} \dot{\mathbf{x}} &= \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{d\theta}\dot{\theta} \\ &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} \end{aligned} \tag{4.2}$$

The second term in the above expression arises because the basis vectors change with time and is proportional to the *angular velocity*, $\dot{\theta}$. (Strictly speaking, this is the angular speed. In the next section, we will introduce a vector quantity which is the angular velocity).

Differentiating once more gives us the expression for acceleration in polar coordinates,

$$\begin{aligned} \ddot{\mathbf{x}} &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{r}}}{d\theta}\dot{\theta} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d\hat{\boldsymbol{\theta}}}{d\theta}\dot{\theta} \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} \end{aligned} \tag{4.3}$$

The two expressions (4.2) and (4.3) will be important in what follows.

### An Example: Circular Motion

Let's look at an example that we're already all familiar with. A particle moving in a circle has $\dot{r} = 0$. If the particle travels with constant angular velocity $\dot{\theta} = \omega$ then the velocity in the plane is

$$\dot{\mathbf{x}} = r\omega\hat{\boldsymbol{\theta}}$$

so the speed in the plane in $v = |\dot{\mathbf{x}}| = r\omega$. Similarly, the acceleration in the plane is

$$\ddot{\mathbf{x}} = -r\omega^2\hat{\mathbf{r}}$$

The magnitude of the acceleration is $a = |\ddot{\mathbf{x}}| = r\omega^2 = v^2/r$. From Newton's second law, if we want a particle to travel in a circle, we need to supply a force $F = mv^2/r$ towards the origin. This is known as a *centripetal force*.

## 4.2 Back to Central Forces

We've already seen that the three-dimensional motion in a central force potential actually takes place in a plane. Let's write the equation of motion (4.1) using the plane polar coordinates that we've just introduced. Since $V = V(r)$, the force itself can be written using

$$\nabla V = \frac{dV}{dr}\hat{\mathbf{r}}$$

and, from (4.3) the equation of motion becomes

$$m(\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} = -\frac{dV}{dr}\hat{\mathbf{r}} \tag{4.4}$$

The $\hat{\boldsymbol{\theta}}$ component of this is particularly simple. It is

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \qquad \Rightarrow \qquad \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$$

It looks as if we've found a new conserved quantity since we've learnt that

$$l = r^2\dot{\theta} \tag{4.5}$$

does not change with time. However, we shouldn't get too excited. This is something that we already know. To see this, let's look again at the angular momentum $\mathbf{L}$. We already used the fact that the direction of $\mathbf{L}$ is conserved when restricting motion to the plane. But what about the magnitude of $\mathbf{L}$? Using (4.2), we write

$$\mathbf{L} = m\mathbf{x} \times \dot{\mathbf{x}} = mr\hat{\mathbf{r}} \times \left(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}\right) = mr^2\dot{\theta}\,\left(\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}\right)$$

Since $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are orthogonal, unit vectors, $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}}$ is also a unit vector. The magnitude of the angular momentum vector is therefore

$$|\mathbf{L}| = ml$$

and $l$, given in (4.5), is identified as the angular momentum per unit mass, although we will often be lazy and refer to $l$ simply as the angular momentum.

Let's now look at the $\hat{\mathbf{r}}$ component of the equation of motion (4.4). It is

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{dV}{dr}$$

Using the fact that $l = r^2\dot{\theta}$ is conserved, we can write this as

$$m\ddot{r} = -\frac{dV}{dr} + \frac{ml^2}{r^3} \tag{4.6}$$

It's worth pausing to reflect on what's happened here. We started in (4.1) with a complicated, three dimensional problem. We used the direction of the angular momentum to reduce it to a two dimensional problem, and the magnitude of the angular momentum to reduce it to a one dimensional problem. This was all possible because angular momentum is conserved.

This should give you some idea of how important conserved quantities are when it comes to solving anything. Roughly speaking, this is also why it's not usually possible to solve the $N$-body problem with $N \geq 3$. In Section 5.1.5, we'll see that for the $N = 2$ mutually interacting particles, we can use the symmetry of translational invariance to solve the problem. But for $N \geq 3$, we don't have any more conserved quantities to come to our rescue.

Returning to our main storyline, we can write (4.6) in the suggestive form

$$m\ddot{r} = -\frac{dV_{\text{eff}}}{dr} \tag{4.7}$$

where $V_{\text{eff}}(r)$ is called the *effective potential* and is given by

$$V_{\text{eff}}(r) = V(r) + \frac{ml^2}{2r^2} \tag{4.8}$$

The extra term, $ml^2/2r^2$ is called the *angular momentum barrier* (also known as the centrifugal barrier). It stops the particle getting too close to the origin, since there is must pay a heavy price in"effective energy".

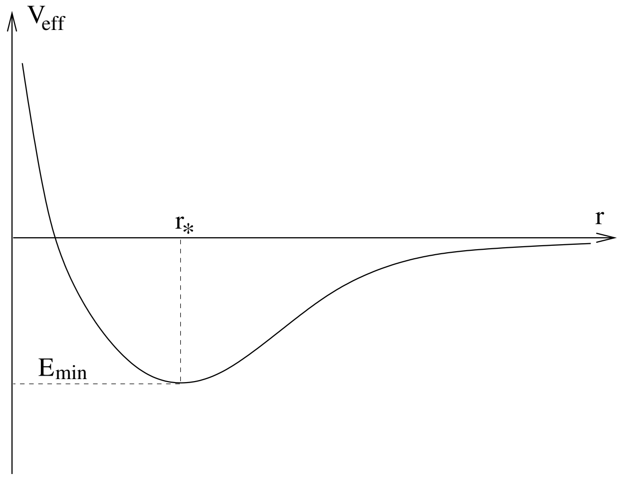


**Figure 12:** The effective potential arising from the inverse square force law.

### 4.2.1 The Effective Potential: Getting a Feel for Orbits

Let's just check that the effective potential can indeed be thought of as part of the energy of the full system. Using (4.2), we can write the energy of the full three dimensional problem as

$$\begin{aligned} E &= \frac{1}{2}m\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} + V(r) \\ &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r) \\ &= \frac{1}{2}m\dot{r}^2 + \frac{ml^2}{2r^2} + V(r) \\ &= \frac{1}{2}m\dot{r}^2 + V_{\rm eff}(r) \end{aligned}$$

This tells us that the energy $E$ of the three dimensional system does indeed coincide with the energy of the effective one dimensional system that we've reduced to. The effective potential energy is the real potential energy, together with a contribution from the angular kinetic energy.

We already saw in Section 2.1.1 how we can understand qualitative aspects of one dimensional motion simply by plotting the potential energy. Let's play the same game here. We start with the most useful example of a central potential: $V(r) = -k/r$, corresponding to an attractive inverse square law for $k > 0$. The effective potential is

$$V_{\rm eff} = -\frac{k}{r} + \frac{ml^2}{2r^2}$$

and is drawn in the figure.

The minimum of the effective potential occurs at $r_\star = ml^2/k$ and takes the value $V_{\text{eff}}(r_\star) = -k^2/2ml^2$. The possible forms of the motion can be characterised by their energy $E$.

- $E = E_{\text{min}} = -k^2/2ml^2$: Here the particle sits at the bottom of the well $r_\star$ and stays there for all time. However, remember that the particle also has angular velocity, given by $\dot\theta = l/r_\star^2$. So although the particle has fixed radial position, it is moving in the angular direction. In other words, the trajectory of the particle is a circular orbit about the origin.

  Notice that the radial position of the minimum depends on the angular momentum $l$. The higher the angular momentum, the further away the minimum. If there is no angular momentum, and $l = 0$, then $V_{\text{eff}} = V$ and the potential has no minimum. This is telling us the obvious fact that there is no way that $r$ can be constant unless the particle is moving in the $\theta$ direction. In a similar vein, notice that there is a relationship between the angular velocity $\dot\theta$ and the size of the orbit, $r_\star$, which we get by eliminating $l$: it is $\dot\theta^2 = k/mr_\star^3$. We'll come back to this relationship in Section 4.3.2 when we discuss Kepler's laws of planetary motion.

- $E_{\text{min}} < E < 0$: Here the 1d system sits in the dip, oscillating backwards and forwards between two points. Of course, since $l \neq 0$, the particle also has angular velocity in the plane. This describes an orbit in which the radial distance $r$ depends on time. Although it is not yet obvious, we will soon show that for $V = -k/r$, this orbit is an ellipse.

  The smallest value of $r$ that the particle reaches is called the *periapsis*. The furthest distance is called the *apoapsis*. Together, these two points are referred to as the *apsides*. In the case of motion around the Sun, the periapsis is called the *perihelion* and the apoapsis the *aphelion*.

- $E > 0$. Now the particle can sit above the horizontal axis. It comes in from infinity, reaches some minimum distance $r$, then rolls back out to infinity. We will see later that, for the $V = -k/r$ potential, this trajectory is hyperbola.

### 4.2.2 The Stability of Circular Orbits

Consider a general potential $V(r)$. We can ask: when do circular orbits exist? And when are they stable?

The first question is quite easy. Circular orbits exist whenever there exists a solution with $l \neq 0$ and $\dot{r} = 0$ for all time. The latter condition means that $\ddot{r} = 0$ which, in turn, requires

$$V'_{\rm eff}(r_\star) = 0$$

In other words, circular orbits correspond to critical points, $r_\star$, of $V_{\rm eff}$. The orbit is stable if small perturbations return us back to the critical point. This is the same kind of analysis that we did in Section 2.1.2: stability requires that we sit at the minimum of the effective potential. This usually translates to the requirement that

$$V''_{\rm eff}(r_\star) > 0$$

If this condition holds, small radial deviations from the circular orbit will oscillate about $r_\star$ with simple harmonic motion.

Although the criterion for circular orbits is most elegantly expressed in terms of the effective potential, sometimes it's necessary to go back to our original potential $V(r)$. In this language, circular orbits exist at points $r_\star$ obeying

$$V'(r_\star) = \frac{ml^2}{r_\star^3}$$

These orbits are stable if

$$V''(r_\star) + \frac{3ml^2}{r_\star^4} = V''(r_\star) + \frac{3}{r_\star}V'(r_\star) > 0 \tag{4.9}$$

We can even go right back to basics and express this in terms of the force (remember that?!), $F(r) = -V'(r)$. A circular orbit is stable if

$$F'(r_\star) + \frac{3}{r^\star}F(r_\star) < 0$$

**An Example**

Consider a central potential which takes the form

$$V(r) = -\frac{k}{r^n} \qquad n \geq 1$$

For what powers of $n$ are the circular orbits stable? By our criterion (4.9), stability requires

$$V'' + \frac{3}{r}V' = -\Big(n(n+1) - 3n\Big)\frac{k}{r^{n+2}} > 0$$

which holds only for $n < 2$. We can easily see this pictorially in the figures where we've plotted the effective potential for $n = 1$ and $n = 3$.

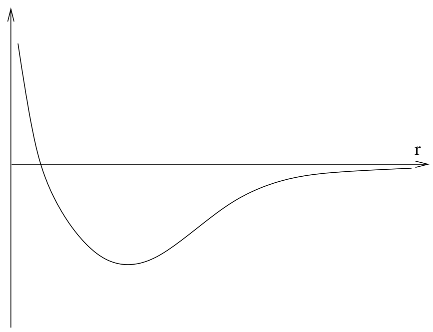


**Figure 13:** $V_{\rm eff}$ for $V = -1/r$

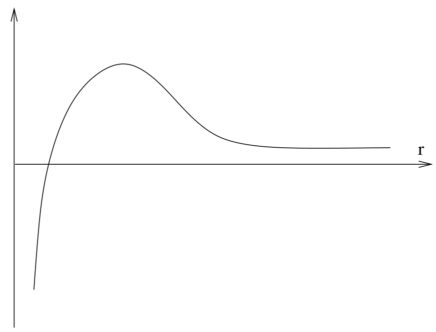


**Figure 14:** $V_{\rm eff}$ for $V = -1/r^3$

Curiously, in a Universe with $d$ spatial dimensions, the law of gravity would be $F \sim 1/r^{d-1}$ corresponding to a potential energy $V \sim -1/r^{d-2}$. We see that circular planetary orbits are only stable in $d < 4$ spatial dimensions. Fortunately, this includes our Universe. We should all be feeling very lucky right now.

### 4.3 The Orbit Equation

Let's return to the case of general $V_{\rm eff}$. If we want to understand how the radial position $r(t)$ changes with time, then the problem is essentially solved. Since the energy $E$ is conserved, we have

$$E = \frac{1}{2}m\dot{r}^2 + V_{\rm eff}(r)$$

which we can view as a first order differential equation for $dr/dt$. Integrating then gives

$$t = \pm\sqrt{\frac{m}{2}}\int \frac{dr}{\sqrt{E - V_{\rm eff}(r)}}$$

However, except for a few very special choices of $V_{\rm eff}(r)$, the integral is kind of a pain. What's more, often trying to figure out $r(t)$ is not necessarily the information that we're looking for. It's better to take a more global approach, and try to learn something about the whole trajectory of the particle, rather than its position at any given time. Mathematically, this means that we'll try to understand something about the shape of the orbit by computing $r(\theta)$.

In fact, to proceed, we'll also need a little trick. It's trivial, but it turns out to make the resulting equations much simpler. We introduce the new coordinate

$$u = \frac{1}{r}$$

I wish I had a reason to motivate this trick. Unfortunately, I don't. You'll just have to trust me and we'll see that it helps.

Let's put these things together. Firstly, we can rewrite the radial velocity as

$$\frac{dr}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\frac{l}{r^2} = -l\frac{du}{d\theta}$$

Meanwhile, the acceleration is

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(-l\frac{du}{d\theta}\right) = -l\frac{d^2u}{d\theta^2}\dot{\theta} = -l^2\frac{d^2u}{d\theta^2}\frac{1}{r^2} = -l^2u^2\frac{d^2u}{d\theta^2} \tag{4.10}$$

The equation of motion for the radial position, which we first derived back in (4.6), is

$$m\ddot{r} - \frac{ml^2}{r^3} = F(r)$$

where, we've reverted to expressing the right-hand side in terms of the force $F(r) = -dV/dr$. Using (4.10), and doing a little bit of algebra (basically dividing by $ml^2u^2$), we get the second order differential equation

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2u^2}F(1/u) \tag{4.11}$$

This is the *orbit equation*. Our goal is to solve this for $u(\theta)$. If we want to subsequently figure out the time dependence, we can always extract it from the equation $\dot{\theta} = lu^2$.

### 4.3.1 The Kepler Problem

The *Kepler problem* is the name given to understanding planetary orbits about a star. It is named after the astronomer Johannes Kepler – we'll see his contribution to the subject in the next section.

We saw in Section 2.3 that the inverse-square force law of gravitation is described by the central potential

$$V(r) = -\frac{km}{r} \tag{4.12}$$

where $k = GM$. However, the results that we will now derive will equally well apply to motion of a charged particle in a Coulomb potential if we instead use $k = -qQ/4\pi\epsilon_0 m$.

For the potential (4.12), the orbit equation (4.11) becomes very easy to solve. It is just

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{l^2}$$

But this is just the equation for a harmonic oscillator, albeit with its centre displaced by $k/l^2$. We can write the most general solution as

$$u = A\cos(\theta - \theta_0) + \frac{k}{l^2} \tag{4.13}$$

with $A$ and $\theta_0$ integration constants. (You might be tempted instead to write $u = A\cos\theta + B\sin\theta + k/l^2$ with $A$ and $B$ as integration constants. This is equivalent to our result above but, as we will now see, it's much more useful to use $\theta_0$ as the second integration constant).

At the point where the orbit is closest to the origin (the periapsis), $u$ is largest. From our solution, we have $u_{\text{max}} = A + k/l^2$. We will choose to orient our polar coordinates so that the periapsis occurs at $\theta = 0$. This choice means that set $\theta_0 = 0$. In terms of our original variable $r = 1/u$, we have the final expression for the orbit

$$r = \frac{r_0}{e\cos\theta + 1} \tag{4.14}$$

where

$$r_0 = \frac{l^2}{k} \quad \text{and} \quad e = \frac{Al^2}{k}$$

Notice that $r_0$ is fixed by the angular momentum, while the choice of $e$ is now effectively the integration constant in the problem.

You have seen equation (4.14) before (in the *Vectors and Matrices* course): it describes a *conic section*. If you don't remember this, don't worry! We'll derive all the necessary properties of this equation below. The integration constant $e$ is called the *eccentricity* and it determines the shape of the orbit.

**Ellipses:** $e < 1$

For $e < 1$, the radial position of the particle is bounded in the interval

$$\frac{r_0}{r} \in [1 - e, 1 + e]$$

We can convert (4.14) back to Cartesian coordinates $x = r\cos\theta$ and $y = r\sin\theta$, writing

$$r = r_0 - er\cos\theta \quad \Rightarrow \quad x^2 + y^2 = (r_0 - ex)^2$$

Multiplying out the square, collecting terms, and rearranging allow us to write this equation as

$$\frac{(x - x_c)^2}{a^2} + \frac{y^2}{b^2} = 1$$

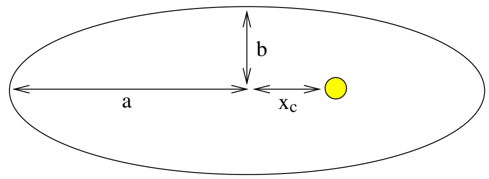


**Figure 15:** The elliptical orbit with the origin at a focus

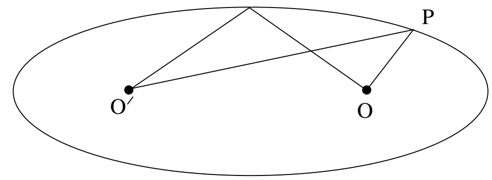


**Figure 16:** The distance from between the two foci and a point on the orbit is constant

with

$$x_c = -\frac{er_0}{1-e^2} \quad \text{and} \quad a^2 = \frac{r_0^2}{(1-e^2)^2} \quad \text{and} \quad b^2 = \frac{r_0^2}{1-e^2} < a^2 \qquad (4.15)$$

This is the formula for an ellipse, with its centre shifted to $x = x_c$. The orbit is drawn in the figure. The two semi-axes of the ellipse have lengths $a$ and $b$. The centre of attraction of the gravitational force (for example, the sun) sits at $r = 0$. This is marked by the yellow disc in the figure. Notice that it is not the centre of the ellipse: the two points differ by a distance

$$|x_c| = \frac{r_0 e}{1-e^2} = ea$$

The origin where the star sits has special geometric significance: it is called the *focus* of the ellipse. In fact, it is one of two foci: the other, shown as $O'$ in Figure above, sits at equal distance from the centre along the major axis. A rather nice geometric property of the ellipse is that the distance $OPO'$ shown in the second figure is the same for all points $P$ on the orbit. (You can easily prove this with some messy algebra).

When $e = 0$, the focus sits at the centre of the ellipse and lengths of the two axes coincide: $a = b$. This is a circular orbit.

In the Solar System, nearly all planets have $e < 0.1$. This means that the difference between the major and minor axes of their orbits is less than 1% and the orbits are very nearly circular. The only exception is Mercury, the closest planet to the Sun, which has $e \approx 0.2$. For very eccentric orbits, we need to look at comets. The most famous, Halley's comet, has $e \approx 0.97$, a fact which most scientists hold responsible for the Chas and Dave lyric "Halley's comet don't come round every year, the next time it comes into view will be the year 2062". However, according to astronomers, it will be the year 2061.

**Hyperbolae:** $e > 1$

For $e > 1$, there are two values of $\theta$ for which $r \to \infty$. They are $\cos\theta = -1/e$. Repeating the algebraic steps that lead to the ellipse equation, we instead find that the orbit is described by

$$\frac{1}{a^2}\left(x - \frac{r_0 e}{e^2-1}\right)^2 - \frac{y^2}{b^2} = 1$$

with $a^2 = r_0^2/(e^2-1)^2$ and $b^2 = r_0^2/(e^2-1)$. This is the equation for a hyperbola. It is plotted in the figure, where the dashed lines are the asymptotes. They meet at the point $x = r_0 e/(e^2-1)$. Again, the centre of the gravitational attraction sits at the origin denoted by the yellow disc. Notice that the orbit goes off to $r \to \infty$ when $\cos\theta = -1/e$. Since the right-hand side is negative, this must occur for some angle $\theta > \pi/2$. This is one way to see why the orbit sits in the left-hand quadrant as shown.

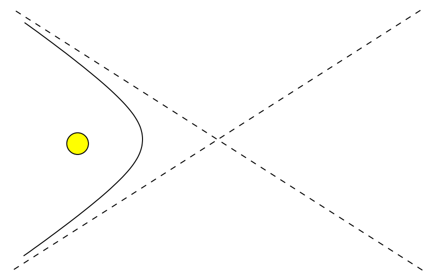

**Figure 17:** A hyperbola

**Parabolae:** $e = 1$

Finally, in the special case of $e = 1$, the algebra is particularly simple. The orbit is described by the equation for a parabola,

$$y^2 = r_0^2 - 2r_0 x$$

**The Energy of the Orbit Revisited**

We can tally our solutions with the general picture of orbits that we built in Section 4.2.1 by looking at the effective potential. The energy of a given orbit is

$$\begin{aligned} E &= \frac{1}{2}m\dot{r}^2 + \frac{ml^2}{2r^2} - \frac{km}{r} \\ &= \frac{1}{2}m\left(\frac{dr}{d\theta}\right)^2\dot{\theta}^2 + \frac{ml^2}{2r^2} - \frac{km}{r} \\ &= \frac{1}{2}m\left(\frac{dr}{d\theta}\right)^2\frac{l^2}{r^4} + \frac{ml^2}{2r^2} - \frac{km}{r} \end{aligned}$$

We can substitute in our solution (4.14) for the orbit to get

$$\frac{dr}{d\theta} = \frac{r_0 e \sin\theta}{(1+e\cos\theta)^2}$$

After a couple of lines of algebra, we find that all the $\theta$ dependence vanishes in the energy (as it must since the energy is a constant of the motion). We are left with the pleasingly simple result

$$E = \frac{mk^2}{2l^2}(e^2 - 1) \tag{4.16}$$

We can now compare this with the three cases we saw in Section 4.2.1:

- $e < 1 \;\Rightarrow\; E < 0$: These are the trapped, or bounded, orbits that we now know are ellipses.
- $e > 1 \;\Rightarrow\; E > 0$: These are the unbounded orbits that we now know are hyperbolae.
- $e = 0 \;\Rightarrow\; E = -mk^2/2l^2$. This coincides with the minimum of the effective potential $V_{\rm eff}$ which we previously understood corresponds to a circular orbit.

### A Repulsive Force

In the analysis above, we implicitly assumed that the force is attractive, so $k > 0$. This, in turn, ensures that $r_0 = l^2/k > 0$. For a repulsive interaction, we choose to write the solution (4.14) as

$$r = \frac{|r_0|}{e\cos\theta - 1} \tag{4.17}$$

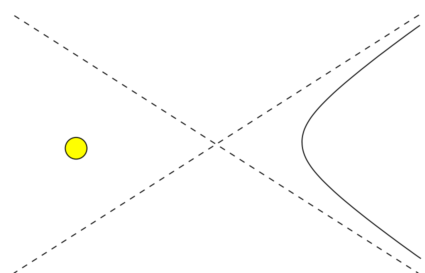

**Figure 18:**

where $|r_0| = l^2/|k|$ and $e = Al^2/|k|$. Note that with this choice of convention, $e > 0$. Since we must have $r > 0$, we only find solutions in the case $e > 1$. This is nice: we wouldn't expect to find bound orbits between two particles which repel each other. For $e > 1$, the unbounded hyperbolic orbits look like those shown in the figure. Notice that the orbits go off to $r \to \infty$ when $\cos\theta = 1/e$ which, since $e > 0$, must occur at an angle $\theta < \pi/2$. This is the reason that the orbit sits in the right-hand quadrant.

### 4.3.2 Kepler's Laws of Planetary Motion

In 1605, Kepler published three laws which are obeyed by the motion of all planets in the Solar System. These laws were the culmination of decades of careful, painstaking observations of the night sky, firstly by Tycho Brahe and later by Kepler himself. They are:

- **K1:** Each planet moves in an ellipse, with the Sun at one focus.

- **K2:** The line between the planet and the Sun sweeps out equal areas in equal times.

- **K3:** The period of the orbit is proportional to the radius$^{3/2}$.

Now that we understand orbits, let's see how Kepler's laws can be derived from Newton's inverse-square law of gravity.

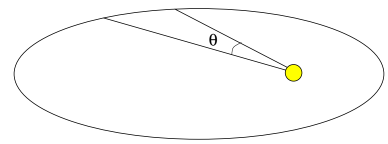


**Figure 19:**

We'll start with Kepler's second law. This is nothing more than the conservation of angular momentum. From the figure, we see that in time $\delta t$, the area swept out is

$$\delta A = \frac{1}{2}r^2\delta\theta \quad \Rightarrow \quad \frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{l}{2}$$

which we know is constant. This means that Kepler's second law would hold for *any* central force.

What about Kepler's third law? This time, we do need the inverse-square law itself. However, if we assume that the gravitational force takes the form $F = -GMm/r^2$, then Kepler's third law follows simply by dimensional analysis. The only parameter in the game is $GM$ which has dimensions

$$[GM] = L^3T^{-2}$$

So if we want to write down a formula relating the period of an orbit, $T$, with some average radius of the orbit $R$ (no matter how we define such a thing), the formula must take the form

$$T^2 \sim \frac{R^3}{GM}$$

We already saw a version of this in Section 4.2.1 where we noted that, for circular orbits, $\dot{\theta}^2 \sim 1/r^3$. For a general elliptical orbit, we can be more precise. The area of an ellipse is

$$A = \pi ab = \pi a^2\sqrt{1-e^2} = \frac{\pi r_0^2}{(1-e^2)^{3/2}}$$

Since area is swept out at a constant rate, $dA/dt = l/2$, the time for a single period is

$$T = \frac{2A}{l} = \frac{2\pi r_0^2}{l(1-e^2)^{3/2}} = \frac{2\pi}{\sqrt{GM}}\left(\frac{r_0}{1-e^2}\right)^{3/2}$$

The quantity in brackets indeed has the dimension of a length. But what length is it? In fact, it has a nice interpretation. Recall that the periapsis of the orbit occurs at $r_{\rm min} = r_0/(1+e)$ and the apoapsis at $r_{\rm max} = r_0/(1-e)$. It is then natural to define the average radius of the orbit to be $R = \frac{1}{2}(r_{\rm min} + r_{\rm max}) = r_0/(1-e^2)$. We have

$$T = \frac{2\pi}{\sqrt{GM}} R^{3/2}$$

The fact that the inverse-square law implies Kepler's third law was likely known to several of Newton's contemporaries, including Hooke, Wren and Halley. However, the proof that the inverse-square law also gives rise to Kepler's first law – a proof which we have spent much of this section deriving – was Newton's alone. This is one of the highlights of Newton's famous Principia.

### 4.3.3 Orbital Precession

For extremely massive objects, Newton's theory of gravity needs replacing. Its successor is Einstein's theory of general relativity which describes how gravity can be understood as the bending of space and time. You will have to be patient if you want to learn general relativity: it is offered as a course in Part II.

However, for certain problems, the full structure of general relativity reduces to something more familiar. It can be shown that for planets orbiting a star, much of the effect of the curvature of spacetime can be captured in a simple correction to the Newtonian force law, with the force now arising from the potential[3]

$$V(r) = -\frac{GMm}{r}\left(1 + \frac{3GM}{c^2 r}\right)$$

where $c$ is the speed of light. For $r \gg GM/c^2$, this extra term is negligible and we return to the Newtonian result. Here we will see the effect of keeping this extra term.

We again define $k = GM$. After a little bit of algebra, the orbit equation (4.11) can be shown to be

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{6k^2}{c^2 l^2}\right) u = \frac{k}{l^2}$$

[3]In the lecture notes on General Relativity we will actually derive a $1/r^3$ correction to Newton's law of gravity. But general relativity is subtle and there are different ways of parameterising the radial distance $r$. A different choice leads to the $1/r^2$ correction described above. Both approaches result in the same answer for the perihelion precession.

The solution to this equation is very similar to that of the Kepler problem (4.13). It is

$$u(\theta) = A\cos\left(\sqrt{1-\frac{6k^2}{c^2l^2}}\,\theta\right) + \frac{k}{l^2 - 6k^2/c^2}$$

where we have once again chosen our polar coordinates so that the integration constant is $\theta_0 = 0$.

This equation again describes an ellipse. But now the ellipse *precesses*, meaning that the periapsis (the point of closest approach to the origin) does not sit at the same angle on each orbit. This is simple to see. A periapsis occurs whenever the cos term is 1. This first happens at $\theta = 0$. But the next time round, it happens at

$$\theta = 2\pi\left(1 - \frac{6k^2}{c^2l^2}\right)^{-1/2} \approx\ 2\pi\left(1 + \frac{3k^2}{c^2l^2}\right)$$

We learn that the orbit does not close up. Instead the periapsis advances by an angle of $6\pi G^2M^2/c^2l^2$ each turn.

The general relativistic prediction of the perihelion advance of Mercury – the closest planet to the sun – was one of the first successes of Einstein's theory.

### 4.4 Scattering: Throwing Stuff at Other Stuff

In the past century, physicists have developed a foolproof and powerful method to understand everything and anything: you take the object that you're interested in and you throw something at it. Ideally, you throw something at it really hard. This technique was pioneered by Rutherford who used it to understand the structure of the atom. It was used by Franklin, Crick and Watson to understand the structure of DNA. And, more recently, it was used at the LHC to demonstrate the existence of the Higgs boson. In short, throwing stuff at other stuff is the single most important experimental method available to science. Because of this, it is given a respectable sounding name: it is called *scattering*.

Before we turn to any specific problem, there are a few aspects that apply equally well to particles scattering off any central potential $V(r)$. We will only need to assume $V(r) \to 0$ as $r \to \infty$. We do our experiment and throw the particle from a large distance which we will take to be $r \to \infty$. We want to throw the particle towards the origin, but our aim is not always spot on. If the interaction is repulsive, we expect the particle to be deflected and its trajectory will be something like that shown in the figure. (However, much of what we're about to say will hold whether the force is attractive or repulsive).

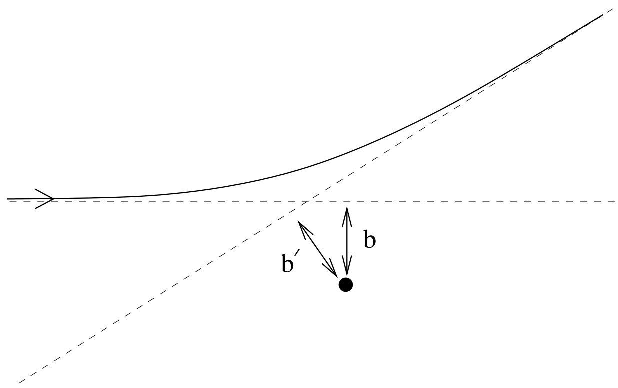


**Figure 20:**

Firstly, by energy conservation, the speed of the particle at the end of its trajectory must be the same as the initial speed. (This is true since at $r \to \infty$ at both the beginning and end and there is no contribution from the potential energy). Let's call this initial/final speed $v$.

But, in a central potential, we also have conservation of angular momentum, $\mathbf{L} = m\mathbf{r} \times \dot{\mathbf{r}}$. We can get an expression for $l = |\vec{L}|/m$ as follows: draw a straight line tangent to the initial velocity. The closest this line gets to the origin is distance $b$, known as the *impact parameter*. The modulus of the angular momentum is then

$$l = bv \tag{4.18}$$

If this equation isn't immediately obvious mathematically, the following words may convince you. Suppose that there was no force acting on the particle at all. In this case, the particle would indeed follow the straight line shown in the figure. When it's closest to the origin, its velocity $\dot{\mathbf{r}}$ is perpendicular to its position $\mathbf{r}$ and is its angular momentum is obviously $l = bv$. But angular momentum is conserved for a free particle, so this must also be its initial angular momentum. But, if this is the case, it is also the angular momentum of the particle moving in the potential $V(r)$ because there too the angular momentum is conserved and can't change from its initial value.

At the end of the trajectory, by the same kind of argument, the angular momentum $l$ is $l = b'v$ where $b'$ is the shortest distance from the origin to the exit asymptote as shown in the figure. But since the angular momentum is conserved, we must have

$$b = b'$$

### 4.4.1 Rutherford Scattering

*It was quite the most incredible event that ever happened to me in my life. It was almost as incredible as if you fired a 15-inch shell at a piece of tissue paper and it came back and hit you.*

Ernest Rutherford

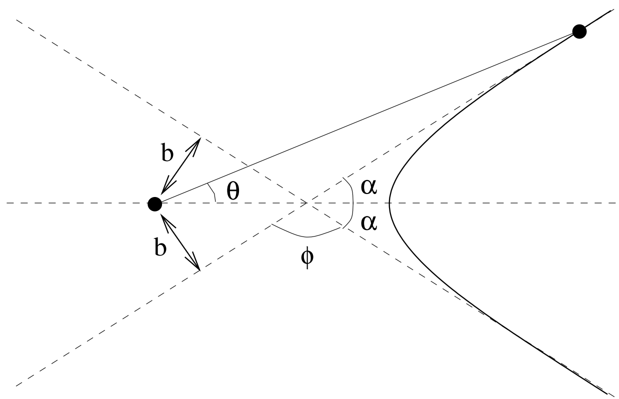


**Figure 21:**

Here we'll look at the granddaddy of all scattering experiments. We take a particle of charge $q$ and mass $m$ and throw it at a fixed particle of charge $Q$. We'll ignore the gravitational interaction and focus just on the repulsive Coulomb force. The potential is

$$V = \frac{qQ}{4\pi\epsilon_0 r}$$

This is mathematically identical to the gravitational force, so we can happily take all the results from the last section and replace $k = -qQ/4\pi\epsilon_0 m$ in our previous equations.

Using our knowledge that $b' = b$, we can draw another scatting event as shown. Here $\theta$ is the position of the particle. We will denote the total angle through which the particle is deflected as $\phi$. However, in the short term the angle $\alpha$, shown in the figure, will prove more useful. This is related to $\phi$ simply by

$$\phi = \pi - 2\alpha \tag{4.19}$$

Our goal is to understand how the scattering angle $\phi$ depends on the impact parameter $b$ and the initial velocity $v$. Using the expression (4.17) for the orbit that we derived earlier, we know that the particle asymptotes to $r \to \infty$ when the angle is at $\theta = \alpha$. This tells us that

$$\cos\alpha = \frac{1}{e}$$

As we mentioned previously, $e > 1$ which ensures that $\alpha < \pi/2$ as shown in the figure.

There are a number of ways to proceed from here. Probably the easiest is if we use the expression for energy. When the particle started its journey, it had $E = \frac{1}{2}mv^2$ (where $v$ is the initial velocity). We can equate this with (4.16) to get

$$E = \frac{1}{2}mv^2 = \frac{mk^2}{2l^2}(e^2 - 1) = \frac{mk^2}{2l^2}\tan^2\alpha$$

Finally, we replace $l = bv$ to get an the expression we wanted, relating the scattering angle $\phi$ to the impact parameter $b$,

$$\phi = 2\tan^{-1}\left(\frac{|k|}{bv^2}\right) \tag{4.20}$$

The result that we've derived here is for a potential with all the charge $Q$ sitting at the origin. We now know that this is a fairly good approximation to the nucleus of the atom. But, in 1909, when Rutherford, Geiger and Marsden, first did this experiment, firing alpha particles (Helium nuclei) at a thin film of gold, the standard lore was that the charge of the nucleus was smeared throughout the atom in the so-called "plum pudding model". In that case, the deflection of the particle at high velocities would be negligible. But, from (4.20), we see that, regardless of the initial velocity $v$, if you fire a particle directly at the nucleus, so that $b = 0$, the particle will always be deflected by a full $\phi = 180°$. This was the result that so surprised Rutherford.