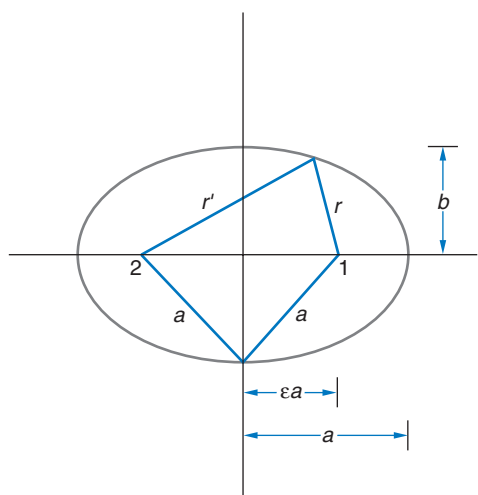


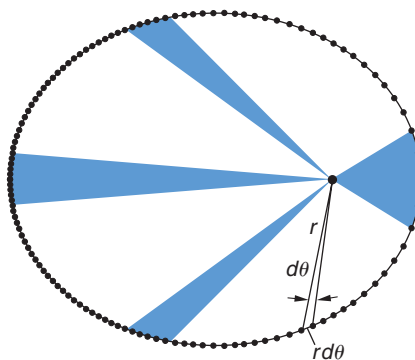
Energy of a Particle in an Elliptical Orbit

Two particles that attract each other by an inverse square law, such as an electron and an atomic nucleus (Coulomb's law) or Earth and the Sun (Newton's law of gravitation), move about their common center of mass (CM). Considering the CM as fixed, each particle can be considered as moving about the CM attracted by a force that varies like $1/r^2$, where r is the distance from the CM to the particle. As is illustrated in nearly every textbook on classical mechanics, the motion of each particle obeys Kepler's laws:

1. The particle's orbit or trajectory is a conic section, that is, an ellipse or a hyperbola (or one of their limiting cases, a circle or a parabola) with the CM at one focus (see Figure EO-1).
2. The radius vector from the focus to the particle sweeps out equal area in equal times (see Figure EO-2).
3. For elliptical (or circular) orbits, the square of the period is proportional to the cube of the major axis.



EO-1 The points 1 and 2 are the foci of the ellipse, a is the semimajor axis, b is the semiminor axis, and ϵ is the eccentricity. r and r' are the radius vectors from the foci to any particular point on the ellipse. $r + r' + 2a$, a constant.



EO-2 The times between adjacent dots around the ellipse are all equal. From Kepler's 2nd law, the four example shaded areas are then all equal. The area of the infinitesimal swept area is $r^2 d\theta / 2$.

Our interest here is a particle in an elliptical orbit. The geometrical definition of an ellipse is that the sum of the distances r and r' from the two foci to any point on the ellipse is a constant. If we let a be the length of the semimajor axis, it is apparent from Figure EO-1 that

$$r + r' = 2a \quad (\text{a constant}) \quad \text{EO-1}$$

The distance between the two foci is then $2a\varepsilon$, where ε is the *eccentricity* of the ellipse. You will note that, of necessity, $\varepsilon < 1$ and if $\varepsilon = 0$, that is, the foci coincide, the ellipse assumes its limiting form of a circle. The relation connecting r in Equation EO-1 with ε , a , and the polar coordinate θ can be found by first determining r' from Figure EO-1 and the law of cosines as

$$r' = [r^2 + (2a\varepsilon)^2 + 2r(2a\varepsilon) \cos \theta]^{1/2} \quad \text{EO-2}$$

and then substituting r' into Equation EO-1 written as $r' = 2a - r$. Squaring the resulting expression yields

$$(2a)^2 - 4ar + r^2 = r^2 + (2a\varepsilon)^2 + 2r(2a\varepsilon) \cos \theta \quad \text{EO-3}$$

Canceling the r^2 terms and then solving for r gives us

$$r = \frac{a(1 - \varepsilon^2)}{1 + \varepsilon \cos \theta} \quad \text{EO-4}$$

Using the elliptical orbit of a planet such as Earth as an example, where the Sun's mass M is so large that we can assume it to be fixed at one focus of the ellipse, the equation of motion of the planet of mass m is

$$m \frac{d^2 r}{dt^2} = -\frac{GmM}{r^2} + \frac{L^2}{mr^3} \quad \text{EO-5}$$

where G is the gravitational constant and L is the (constant) angular momentum. The general solution of Equation EO-5 (simplified by first making a variable substitution $u = 1/r$) is

$$\frac{1}{r} = \frac{Gm^2 M}{L^2} + A \cos \theta \quad \text{EO-6}$$

Rewriting Equation EO-4 as

$$\frac{1}{r} = \frac{1}{a(1 - \varepsilon^2)} + \frac{\varepsilon \cos \theta}{a(1 - \varepsilon^2)}$$

and comparing with Equation EO-6, we see that

$$A = \pm \varepsilon \frac{Gm^2 M}{L^2} \quad \text{EO-7}$$

and

$$a = \frac{L^2}{Gm^2 M(1 - \varepsilon^2)} \quad \text{EO-8}$$

Using Kepler's second law and referring to Figure EO-2, the area $d\alpha$ swept out by the radius r in time dt is given by

$$\frac{d\alpha}{dt} = \frac{1}{2} r \frac{rd\theta}{dt} = \frac{1}{2} r^2 \omega = \frac{L}{2m} \quad \text{EO-9}$$

Integrating Equation EO-9 over one complete orbit yields the period of the motion T :

$$T = \frac{2m}{L}\alpha \quad \text{EO-10}$$

where the area of the ellipse $\alpha = 2ab$ and, from Figure EO-1, $b = a\sqrt{1 - \varepsilon^2}$. Kepler's second law then gives the period of the orbital motion to be

$$T^2 = \frac{4\pi^2 a^3}{GM} \quad \text{EO-11}$$

Thus, the motion of the mass m in *all* elliptical orbits with the same major axis has the same period.

The total energy E of the mass m is given by

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2mr^2} - \frac{GmM}{r} \quad \text{EO-12}$$

where the second term on the right side of Equation EO-12 is the rotational kinetic energy and the third term is the gravitational potential energy. Since the energy of mass m is constant, Equation EO-12 can be evaluated at any time. An easy time to do so is when r is at either its maximum or minimum value; that is, at either end of the major axis where $dr/dt = 0$. Equations EO-6 and EO-7 then yield

$$\begin{aligned} \frac{1}{r_{\min}} &= \frac{Gm^2M}{L^2}(1 + \varepsilon) \\ \frac{1}{r_{\max}} &= \frac{Gm^2M}{L^2}(1 - \varepsilon) \end{aligned} \quad \text{EO-13}$$

Substituting into Equation EO-12 then yields

$$E = \frac{G^2m^3M^2}{2L^2}(\varepsilon^2 - 1) \quad \text{EO-14}$$

For elliptical orbits, where $\varepsilon < 1$, the total energy is negative. Writing E in terms of the length of the major axis with the aid of Equation EO-8 yields

$$E = -\frac{GmM}{2a} \quad \text{EO-15}$$

As was similarly the case with the period T , the mass m has the same total energy in *all* elliptical orbits that have the same major axis, regardless of the eccentricity of the ellipse.