Explanation of the Perihelion Motion of Mercury in Terms of a Velocity-Dependent Correction to Newton's Law of Gravitation

Randy Wayne*

Laboratory of Natural Philosophy, Section of Plant Biology, School of Integrative Plant Science, Cornell University, Ithaca, New York, USA

The first success of Einstein's General Theory of Relativity was to account for the anomalous precession of the perihelion of Mercury. This solution required time and space to be "*robbed of the last trace of objective reality*." Here I show that it is possible to interpret Einstein's relativistic correction for describing the precession of the perihelion of Mercury in terms of a gravitational force obeying Newton's law of gravitation corrected with a tangential-velocity-dependent term and operating through Euclidean space and Newtonian time.

Newton [1] explained the laws of planetary motion derived by Kepler in terms of a gravitational force that falls off with the square of the distance between the sun and each planet. Soon after, Halley [2] applied Newton's law of gravitation to predict the orbit and return of a comet, and Adams [3] and Le Verrier [4] independently used Newton's law of gravitation to predict the existence of Neptune as a result of an observed perturbation in the orbit of Uranus. Newton's law of gravitation appeared to be universal. However, in 1859, La Verrier [5], discovered a precession of the perihelion of Mercury of 38 arcseconds per century that could not be accounted for by Newton's law of gravitation. The precession of the perihelion of Mercury was more accurately determined in 1895 by Newcomb [6] to be 43 arcseconds per century. Many hypotheses were proffered to account for the perturbation that would give rise to the observed deviation from Newton's law of gravitation, but the true cause of the perturbation remained a mystery [7].

Then, in November 1915, a century ago, Einstein [8] proposed a correction to Newton's laws of motion that explained the anomalous precession of the perihelion of Mercury but it required time and space to be "*robbed of the last trace of objective reality*." When presented as an equation of motion, Einstein's relativistic correction to the equation of motion represents the effect of the mass of the sun on warping space-time as seen from locally measured proper time in Mercury's reference frame. Einstein's kinematic approach is not intuitive, and it is difficult to understand, especially when the constant (*A*) of the energy conservation law has dimensions of space squared over time squared and the constant of the angular momentum conservation law (*B*) has dimensions of space. *Mutatis mutandis*, Einstein's correction can be presented as an equation of motion using the familiar Hamiltonian [see Appendix]:

$$E = \frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} + \frac{L^2}{2mr^2}\left[1 - \frac{2GM}{rc^2}\right] \qquad (1)$$

Where, E is the total mechanical energy, G is the gravitational constant (6.67384×10^{-11} m^3 kg^{-1} s^{-2}), M is the mass of the sun (1.989×10^{30} kg), L is the orbital angular momentum of Mercury (9.1×10^{38} J s), m is the mass of Mercury (3.285×10^{23} kg), r is the distance between Mercury and the sun (the semi-major axis (a) is 5.791×10^{10} m) and c is the vacuum speed of light (2.99792458×10^8 m/s).

The success of Einstein's relativistic correction in explaining the anomalous precession of the perihelion of Mercury fulfilled Einstein's eight-year longing to explain the inadequacy of Newton's law of gravitation [9]. According to Einstein, the shortfall was due to Newton's conception of space and time as being absolute and independent of matter and of each other. Moreover, Einstein proposed that the geometry needed to describe the world would no longer be Euclidean but Riemannian, where time was the fourth dimension of space-time and space-time could be warped by matter. According to Einstein, the trajectory taken

*row1@cornell.edu

by a mass relative to a larger mass was not influenced directly by the force of gravity working through an independent space and time but was determined solely by the warping of space-time by the larger mass. Alexander Moszkowski [10] wrote, "*Whereas Leverrier in his time had pointed out a new planet, Einstein brought to view something far more important: a new truth.*"

I have previously shown that by taking the mechanical angular momentum and mechanical energy of the photon into consideration, it is possible to describe and explain the deflection of starlight and the gravitational redshift, which are typically explained by the General Theory of Relativity, in terms of Euclidean space and Newtonian time [11]. I have also shown that the relativity of simultaneity, the optics of moving bodies, the reason that moving bodies cannot exceed the speed of light, and the inertia of energy can be described and explained in terms of the second order Doppler effect taking place in Euclidean space and Newtonian time [12-15]. Here I show that it is also possible to interpret Einstein's relativistic correction for describing the precession of the perihelion of Mercury formally in terms of a gravitational force obeying Newton's law of gravitation corrected with a tangential velocity-dependent term operating through Euclidean space and Newtonian time. The tangential velocity-dependent correction has electrodynamic consequences that are capable of additionally increasing both the tangential kinetic energy and the gravitational potential energy as Mercury approaches the sun and additionally decreasing the tangential kinetic energy and gravitational potential energy as Mercury recedes from the sun. The predicted changes in two types of mechanical energy with opposite signs that make up the total energy are quantitatively equivalent to Einstein's predicted changes in space-time, where space and time have opposite signs in the equation of the space-time metric.

We can rewrite Einstein's correction that accounts for the anomalous precession of the perihelion of Mercury given in equation 1 in terms of mechanical energy:

$$E = KE_r + PE + KE_\theta\left[1-\frac{2GM}{rc^2}\right] \qquad (2)$$

by defining the orbital angular momentum ($L = mv_\theta r$), the tangential kinetic energy ($KE_\theta = \frac{mv_\theta^2}{2}$), the radial kinetic energy ($KE_r = \frac{1}{2}m\dot{r}^2$), the gravitational potential energy ($PE = -\frac{GMm}{r}$), and letting v_θ be the orbital velocity of Mercury (4.7360×10^4 m/s).

The tangential kinetic energy can be expressed in terms of a tangential velocity-dependent gravitational potential energy through the following algebraic transformation:

$$KE_\theta\left[1-\frac{2GM}{rc^2}\right] = KE_\theta - \frac{2GM\,KE_\theta}{rc^2} = KE_\theta - \frac{2GM\frac{mv_\theta^2}{2}}{rc^2}$$
$$= KE_\theta - \frac{GMm}{r}\frac{v_\theta^2}{c^2} = KE_\theta + PE\frac{v_\theta^2}{c^2} \qquad (3)$$

Thus Einstein's relativistic perturbation can be interpreted to change the gravitational potential energy in Euclidean space and Newtonian time in a periodic tangential velocity-dependent manner. Consequently, equation 2 can be written in an equivalent form:

$$E = KE_r + KE_\theta + PE\left[1+\frac{v_\theta^2}{c^2}\right] \qquad (4)$$

which shows that the perturbation that results in the precession of the perihelion of Mercury is formally a function of the ratio of the square of the tangential velocity of Mercury to the square of the speed of light. In the unperturbed orbit, the average Newtonian tangential kinetic energy term is approximately 3.6841×10^{32} J and the average Newtonian gravitational potential energy term is approximately -7.5300×10^{32} J. The average perturbation energy ($PE\frac{v_\theta^2}{c^2}$) is approximately 1.8792×10^{25} J, which is about five hundred million times smaller than the unperturbed gravitational potential energy term and formally vanishes as $v_\theta^2 \to 0$ as we return to the Newtonian condition. The proposed tangential velocity dependence of Newton's law of gravitation looks like so:

$$F_g = -\nabla PE = \frac{GMm}{r^2}\left[1+\frac{v_\theta^2}{c^2}\right] \qquad (5)$$

Where, F_g is the gravitational force. The simple appearance of the equation comes from the fact that it is the tangential velocity and not the radial velocity that modifies the potential and the force. Eqn. (5) is relativistic in that it describes the effect of relative motion on the gravitational force without the relativity of space and time. All of the equations presented here are relativistic in that the motions of Mercury and the sun are relative even though space and time are absolute. Equating equations 2 and 4, we get:

$$KE_\theta + PE\left[1+\frac{v_\theta^2}{c^2}\right] = PE + KE_\theta\left[1-\frac{2GM}{rc^2}\right] \quad (6)$$

Thus the energy integral can be written two equivalent ways:

$$\begin{aligned} E &= \frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} + \frac{L^2}{2mr^2}\left[1-\frac{2GM}{rc^2}\right] \\ &= \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}\left[1+\frac{v_\theta^2}{c^2}\right] \end{aligned} \quad (7)$$

In order to determine the shape of the orbital ($u(\theta)$) of Mercury, I define $u \equiv \frac{1}{r}$ and let $\frac{d\theta}{dt} = \frac{Lu^2}{m}$ and $\frac{dr}{du} = -\frac{1}{u^2}$. Then we can then use the chain rule to define $\dot{r}$ in terms of $\frac{du}{d\theta}$:

$$\dot{r} \equiv \frac{dr}{dt} = \frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{Lu^2}{m} = -\frac{du}{d\theta}\frac{L}{m} \quad (8)$$

I will obtain the equation for the precession of the perihelion of Mercury using the correction to the gravitational potential energy given in Eqn. (7). After substituting r with u in the R.H.S. of Eqn. (7), letting $L = mv_\theta r = \frac{mv_\theta}{u}$ in the last term, and simplifying, we get:

$$E\frac{2m}{L^2} = \left(\frac{du}{d\theta}\right)^2 + u^2 - \frac{2GMm^2u}{L^2} - \left[\frac{2GMu^3}{c^2}\right] \quad (9)$$

After differentiating u with respect to θ and simplifying we get:

$$0 = \frac{d^2u}{d\theta^2} + u - \frac{GMm^2}{L^2} - \left[\frac{3GMu^2}{c^2}\right] \quad (10)$$

And after rearranging, we get:

$$\frac{d^2u}{d\theta^2} + u = \frac{GMm^2}{L^2} + \left[\frac{3GMu^2}{c^2}\right] \quad (11)$$

Eqn. (11) differs from the unperturbed orbital equation $\left(\left(\frac{d^2u_o}{d\theta^2} + u_o\right) = \frac{GMm^2}{L^2}\right)$ by the last term on the right. The solution to the unperturbed orbital equation is: $u_o = A(1+\varepsilon\cos\theta)$ with $A \equiv \left(a(1-\varepsilon^2)\right)^{-1}$, where a is the semimajor axis and ε is the eccentricity of the orbit. Because the perturbation term is very small, we can use first-order perturbation theory and let $u = u_o + u_1$.

$$\begin{aligned} &\left(\frac{d^2u_o}{d\theta^2} + u_o\right) + \left(\frac{d^2u_1}{d\theta^2} + u_1\right) \\ &= \frac{GMm^2}{L^2} + \left[\frac{3GM(u_o+u_1)^2}{c^2}\right] \end{aligned} \quad (12)$$

Now we eliminate the part of Eqn. (12) that accounts for the unperturbed orbit:

$$\left(\frac{d^2u_1}{d\theta^2} + u_1\right) = \left[\frac{3GM(u_o+u_1)^2}{c^2}\right] \quad (13)$$

Since $\frac{3GM(u_o)^2}{c^2}$ is small, and $u_1 \ll u_0$ we can neglect u_1 on the right-hand side and substitute $u_o = A(1+\varepsilon\cos\theta)$:

$$\left(\frac{d^2u_1}{d\theta^2} + u_1\right) = \left[\frac{3GMA^2(1+\varepsilon\cos\theta)^2}{c^2}\right] \quad (14)$$

Expanding the right-hand side using the following trigonometric identity: $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, we get:

$$\begin{aligned} &\left(\frac{d^2u_1}{d\theta^2} + u_1\right) \\ &= \left[\frac{3GMA^2}{c^2}\left(1+\frac{\varepsilon^2}{2} + 2\varepsilon\cos\theta + \frac{\varepsilon^2}{2}\cos 2\theta\right)\right] \end{aligned} \quad (15)$$

Eqn. (15) is a linear equation that can be resolved into three separate linear equations:

$$\left(\frac{d^2u_1}{d\theta^2} + u_1\right) = \left[\frac{3GMA^2}{c^2}\left(1+\frac{\varepsilon^2}{2}\right)\right] \quad (16a)$$

whose solution after integration is: $u_1^1 = \left[\frac{3GMA^2}{c^2}\left(1+\frac{\varepsilon^2}{2}\right)\right]$

$$\left(\frac{d^2u_1}{d\theta^2}+u_1\right)=\left[\frac{3GMA^2}{c^2}\left(2\varepsilon\cos\theta\right)\right] \quad (16b)$$

whose solution after integration is: $u_1^2 = \left[\frac{3GMA^2}{c^2}\varepsilon\theta\sin\theta\right]$ and

$$\left(\frac{d^2u_1}{d\theta^2}+u_1\right)=\left[\frac{3GMA^2}{c^2}\left(\frac{\varepsilon^2}{2}\cos 2\theta\right)\right] \quad (16c)$$

whose solution after integration is: $u_1^3 = \left[-\frac{\varepsilon^2 3GMA^2}{6c^2}\cos 2\theta\right]$.

The complete solution for the perturbation is:

$$u_1 = u_1^1 + u_1^2 + u_1^3 = \left[\frac{3GMA^2}{c^2}\left(1+\varepsilon\theta\sin\theta-\frac{\varepsilon^2}{6}\cos 2\theta\right)\right] \quad (17)$$

The only sinusoidal term that can lead to an open orbit and the precession of the perihelion is: $\varepsilon\theta\sin\theta$. Therefore the equation for the elliptical orbit that includes the precession of the perihelion of Mercury is:

$$u = u_o + u_1 = A\left(1+\varepsilon\cos\theta+\frac{3GMA}{c^2}\varepsilon\theta\sin\theta\right) \quad (18)$$

Using the small angle approximations: $\cos\theta \cong 1$ and $\sin\theta \cong \theta$, and the trigonometric identity $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$, we get:

$$u = A\left(1+\varepsilon\cos\theta+\frac{3GMA}{c^2}\varepsilon\theta\sin\theta\right) \cong A\left[1+\varepsilon\cos\left(1-\frac{3GMA}{c^2}\right)\theta\right] \quad (19)$$

The radius ($r = 1/u$) traces out a precessing ellipse ($A[1+\varepsilon\cos(\theta)]$) with the following periodicity, which is greater than 2π and thus advances the perihelion:

$$\frac{2\pi}{\left(1-\frac{3GMA}{c^2}\right)} > 2\pi \quad (20)$$

The precession (δ) of the perihelion in radians per year is therefore:

$$\delta = \frac{2\pi}{\left(1-\frac{3GMA}{c^2}\right)} - 2\pi \cong 2\pi\left(1+\frac{3GMA}{c^2}\right)-2\pi = \left(2\pi+\frac{6\pi GMA}{c^2}\right)-2\pi = \frac{6\pi GMA}{c^2} \quad (21)$$

After substituting $A \equiv \left(a(1-\varepsilon^2)\right)^{-1}$ into Eqn. (21), we get:

$$\delta = \frac{6\pi GMA}{c^2} = \frac{6\pi GM}{a\left(1-\varepsilon^2\right)c^2} \quad (22)$$

By introducing the period ($P = \sqrt{\frac{(2\pi a)^2}{GM/a}}$) of the orbit, consistent with Kepler's third law, Eqn. (22) can be presented in the following form as given by Einstein [8]:

$$\delta = \frac{24\pi^3 a^2}{P^2c^2\left(1-\varepsilon^2\right)} \quad (23)$$

Returning to Eqn. (22) we substitute the velocity-induced perturbation of the average gravitational energy for the perturbation of the average tangential kinetic energy letting v_θ^2 represent the average tangential velocity and $r = a$ represents the semi-major axis. Using Eqn. (3) and letting $r = a$, we get:

$$\frac{PE}{KE_\theta}\frac{v_\theta^2}{c^2} = -\frac{2GM}{ac^2} \quad (24)$$

Thus the precession (δ) of the perihelion of a planet in radians per orbit is given by:

$$\delta = \frac{PE}{KE_\theta}\frac{3\pi v_\theta^2}{c^2\left(1-\varepsilon^2\right)} \quad (25)$$

And the precession of the perihelion of Mercury in arcsec per century (Δ) is given by:

$$\Delta = \delta \frac{360 \deg}{2\pi \ radians} \frac{3600 \ arc\sec}{\deg} \frac{1 \ orbit}{orbital \ period} \frac{365 \ days}{year} \frac{100 \ years}{century} \tag{26}$$

By inserting the orbital period for each planet, the above equation based on the velocity-dependent correction to Newton's law of gravitation predicts the anomalous precession of the perihelion of Mercury and the inner planets. The predicted and observed values for the anomalous precession of the perihelion of the inner planets are given in Table 1. The predicted values are identical to those given by General Relativity, but the interpretation is different. General Relativity posits that the precession of the perihelion of the planets occurs because the mass of the sun provides a perturbation that warps the space-time through which the planets move according to the principle of least time.

The velocity-dependent gravitational potential proposed here differs formally from other proposed velocity-dependent gravitational potentials [7] in that it is the tangential velocity ($\frac{rd\theta_\theta}{dt}$), perpendicular to a given radial distance and not the radial velocity ($\frac{dr}{dt}$), parallel to the radial distance that provides the feedback to moderate the gravitational potential.

Starting with the hypothesis that a tangential velocity-dependent gravitational potential can account for the anomalous precession of the perihelion of Mercury, it becomes important to find the cause of the velocity dependence. Let me say at the onset that I cannot identify the tangential velocity-dependent force with certainty. However, friction is velocity-dependent force that was prominent in Book II of Newton's *Principia* [1], but it is often an outsider in modern physical theory. I posit that the perturbation acts dynamically in a velocity-dependent manner producing a frictional force that is greatest at perihelion and least at aphelion. As Mercury approaches the sun, the increase in the frictional force would result in a simultaneous increase in the absolute value of the magnitudes of the tangential kinetic energy and gravitational potential energy and as Mercury recedes from the sun, the decrease in the frictional force would result in a simultaneous decrease in the absolute value of the magnitudes of the kinetic energy and gravitational potential energy. The tangential kinetic energy and gravitational potential energy can increase or decrease in parallel while conserving total energy because the kinetic energy is positive while the potential energy is negative. The net effect of this seemingly paradoxical behavior [16], known as the orbital paradox, qualitatively explains the observed precession of the perihelion of Mercury.

The idea that a perturbing resistance can result in an increase in the tangential kinetic energy of a comet passing close to the sun was first proposed by Encke who studied the orbital dynamics of the eponymous comet. Richardson [17] described Encke's proposal like so:

> *The idea that the velocity of a body can be increased by friction is so contrary to everyday experience as to seem ridiculous at first. It is true that a resisting medium in space by opposing the motion of a body does tend to make it move more slowly. But there is an important difference between the effect of friction upon the motion of a body revolving around the sun, and the effect of friction upon bodies moving at the surface of the earth. For in space the instant the speed of a body decreases it immediately starts to fall toward the sun thus diminishing the size of its orbit. We know that the closer a planet is to the sun the faster it moves…A complete mathematical discussion shows that the speed lost by friction is more than compensated by the speed gained from the shrinkage in the size of the orbit.*

Astronomers including Arago [18] and Airy [19] accepted the possibility of a resisting medium influencing the orbits of comets and Whewell [20] suggested that a resisting medium must have a similar although smaller effect on planets as it would on comets. Could the resisting medium also cause the precession of the perihelion of mercury? According to Lodge [21,22], as long as the resistance is greater at perihelion than at aphelion, "*the perturbation caused would be roughly parallel to the minor axis, so that it would give a large edω* [precession] *and a small de* [change in ellipticity]. *Which is what is wanted.*"

The cause of the velocity-dependent frictional force, while unknown, may be a result of Mercury moving through matter observed as zodiacal light, and/or through radiation. An optomechanical counterforce proffered as the cause of restraining moving objects to the speed of light depends on the square of the velocity and the square of the temperature [14]. The velocity-dependent optomechanical counterforce becomes more significant for planets closer to the sun since both the square of the velocity and the square of the temperature of space are inversely proportional to the distance from the sun (Fig. 1). This would result in a resistance that is greater at perihelion, where the velocity is maximal and the radius is minimal than at aphelion where the velocity is minimal and the radius is maximal.

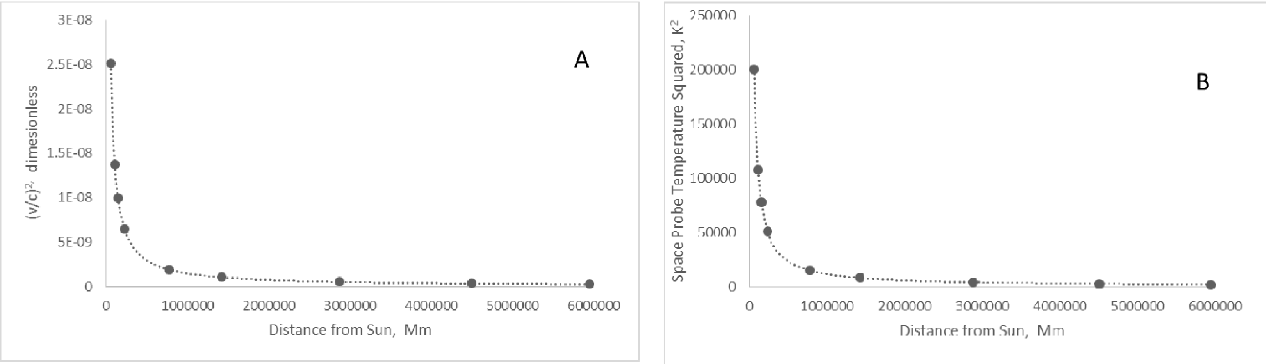

Fig.1: (A). The relationship between the square of the ratio of the velocity of the planets to the speed of light plotted as a function of distance from the sun [34]; (B). The relationship between the square of the temperature of space plotted as a function of distance from the sun [35].

The relative magnitude of the counterforce provided by radiation compared to the gravitational force depends on the dimensions of the body [23-25]. In general, the counterforce increases with the area of a body while the gravitational force increases with the volume. Consequently, radiation provides a dominant force on small orbiting particles or cosmic dust while it will only provide a perturbative force on comets and planets. The magnitude of the perturbative force necessary to cause the observed precession is approximately 10^{-7} times the gravitational force [26]. The perturbative effect of radiation alone is probably not enough to account for the precession of the perihelion of Mercury.

Poynting [23] considered the significant effect that the Doppler-shifted radiation would have in producing "*a force resisting the motion*" of small bodies orbiting the sun but considered the Doppler-shifted radiation to have a negligible effect on larger bodies such as planets. This may be true in part because Poynting and others considered the frictional force to be proportional to $\frac{v}{c^2}$ [27-29]. However, the overall effect of the Doppler-induced counterforce on the anomalous precession of the perihelion of Mercury would be greater if one took into consideration the second order Doppler effect, which makes the force proportional to $\frac{v^2}{c^2}$ [12-14]. As a consequence of the velocity dependence, the Doppler- induced counterforce increases as Mercury approaches the sun, reaches a maximum at perihelion, decreases as Mercury recedes from the sun, and reaches a minimum at aphelion.

Characterization of the perturbative effect of radiation in the solar system is difficult [30-33]. Further analysis of the effect of the resistance provided by the cumulative action of cosmic dust and the radiation of space on the observed dynamics of planets must await the development of orbital equations that take into consideration the square of the velocity, the square of the temperature of space [14] and the periodic nature of the Doppler-induced counterforce that increases as Mercury moves towards the sun and lessens as Mercury moves away. Complete orbital equations would also take the radiation pressure that acts on the radial velocity into consideration and the three dimensional temperature distribution of the planet [38]. Despite the current lack of equations, the above analysis shows that a tangential velocity-dependent gravitational potential can account for the anomalous precession of the perihelion of Mercury and that the relativity of space and time is sufficient but not necessary for explaining the anomalous precession of the perihelion of Mercury.

Appendix

In Eqn. (8) in his paper on the perihelion of Mercury, Einstein [8,39,40] introduced two equations of motion with the constants A and B.

$$\frac{1}{2}u^2 + \Phi = A \qquad \text{(A1a)}$$

$$r^2 \frac{d\phi}{ds} = B \qquad \text{(A1b)}$$

Where, Φ is the gravitational potential per unit mass, u is the velocity, r is the radius, and $\frac{d\phi}{ds}$ is the angular velocity where ds is an invariant space-time metric. $ds = cd\tau$ where $d\tau$ is the proper time of the planet orbiting in the coordinate system with the sun at its origin. The angular momentum (L) of a body if mass (m) in absolute Newtonian time (dt) is defined below:

$$mr^2 \frac{d\phi}{dt} = L \qquad \text{(A2a)}$$

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{L^2}{m^2 r^4} \qquad \text{(A2b)}$$

The following equations are useful combinations of Eqns. (A1b), (A2a) and (A2b):

$$r^2 \frac{d\phi}{dt} = Bc \qquad \text{(A3a)}$$

$$L = mr^2 \frac{d\phi}{dt} = Bmc \qquad \text{(A3b)}$$

$$B^2 = \left(\frac{L}{mc}\right)^2 \qquad \text{(S3c)}$$

In order to introduce a perturbation term into Newton's law of gravitation, Einstein defined the gravitational potential per unit mass in his Eqn. (7c), where $GM = \alpha$ like so:

$$\Phi = -\frac{\alpha}{2}\left[1 + \frac{B^2}{r^2}\right] \qquad \text{(A4)}$$

Mutatis mutandis, the following is what I think is the correct version of Einstein's Eqn. (7c):

$$\Phi = -\frac{\alpha}{r}\left[1 + \frac{B^2}{r^2}\right] = -\frac{GM}{r}\left[1 + \frac{B^2}{r^2}\right] \qquad \text{(A5)}$$

Earman and Janssen [41], the editors of the Collected Papers of Albert Einstein [42], and Vankov [43] detected the error in the original paper but gave a different correction—one that I cannot get to work ($\Phi = -\frac{\alpha}{2r}\left[1 + \frac{B^2}{r^2}\right]$). To obtain the Hamiltonian shown in Eqn. (1) of this paper, insert Eqn. (A5) into Eqn. (A1a):

$$\frac{1}{2}u^2 - \frac{GM}{r}\left[1 + \frac{B^2}{r^2}\right] = A \qquad \text{(A6)}$$

After expanding, we get:

$$\frac{1}{2}u^2 - \frac{GM}{r} - \frac{GM}{r}\frac{B^2}{r^2} = A \qquad \text{(A7)}$$

Resolve the square of the velocity (u^2) into its radial ($\left(\frac{dr}{dt}\right)^2$) and tangential ($r^2\left(\frac{d\phi}{dt}\right)^2$) components using Newtonian time:

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 - \frac{GM}{r} - \frac{GM}{r}\frac{B^2}{r^2} = A \qquad \text{(A8)}$$

Substitute $\left(\frac{d\phi}{dt}\right)^2 = \frac{L^2}{m^2 r^4}$ to remove the time dependence of the tangential component and substitute $B^2 = \left(\frac{L}{mc}\right)^2$ and then simplify:

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2m^2r^2} - \frac{GM}{r} - \frac{GM}{rc^2}\frac{L^2}{m^2r^2} = A \qquad \text{(A9)}$$

Multiply through by m to convert energy per unit mass (A) to the total energy (E):

$$\frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r} - \frac{GM}{rc^2}\frac{L^2}{mr^2} = mA = E \qquad \text{(A10)}$$

To get the final form of the Hamiltonian with a perturbation in Euclidean space and Newtonian time shown in Eqn. (1), rearrange Eqn. (A10) to get:

$$\frac{1}{2}m\left(\frac{dr}{dt}\right)^2 - \frac{GMm}{r} + \frac{L^2}{2mr^2}\left[1 - \frac{2GM}{rc^2}\right] = mA = E \qquad \text{(A11)}$$

Table 1: Physical characteristics of some planets, including the observed and predicted values of the anomalous precession of the perihelion based on the velocity-dependent correction to Newtons's law of gravitation. Tangential velocity, v_θ; mass, m; semimajor axis, a; orbital period, P; eccentricity, ε; average tangential kinetic energy, KE_θ; average gravitational potential energy, PE; precession in radians per year, δ; precession in arcseconds per century, Δ. The observed values come from references [36] and [37].

	Mercury	Venus	Earth	Mars	Jupiter
v_θ (m/s)	4.7360×10^4	3.5020×10^4	2.9780×10^4	2.4070×10^4	1.3060×10^4
m (kg)	3.301×10^{23}	4.8676×10^{24}	5.9726×10^{24}	6.4174×10^{23}	1.8983×10^{27}
a (m)	5.791×10^{10}	1.0821×10^{11}	1.4960×10^{11}	2.2792×10^{11}	7.7857×10^{11}
P (days)	88	224	365	686	4332
ε	0.2056	0.0067	0.0167	0.0935	0.0489
$KE_\theta = \frac{mv_\theta^2}{2}$ (J)	3.6841×10^{32}	2.9848×10^{33}	2.6484×10^{33}	1.8590×10^{32}	1.6189×10^{35}
$PE = -\frac{GMm}{a}$ (J)	-7.5300×10^{32}	-5.9712×10^{33}	-5.2996×10^{33}	-3.7376×10^{32}	-3.2365×10^{35}
$\frac{-PE}{KE_\theta}$	2.0439	2.0005	2.0011	2.0105	1.9992
$\delta = \frac{3\pi PE v_\theta^2}{KE_\theta c^2(1-\varepsilon^2)}$	5.0197×10^{-7}	2.5783×10^{-7}	1.8615×10^{-7}	1.2323×10^{-7}	0.3584×10^{-7}
$\Delta_{predicted}$	42.9447	8.6656	3.8396	1.3524	0.0623
$\Delta_{observed}$	42.98 ±0.04	8.6247 ±0.005	3.8387 ±0.004		

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