Lecture 26. KINETIC-ENERGY FOR PLANAR MOTION OF A RIGID BODY WITH APPLICATION EXAMPLES

Given: $T = m\frac{v^2}{2}$ for a particle

Find: Kinetic Energy for a rigid body

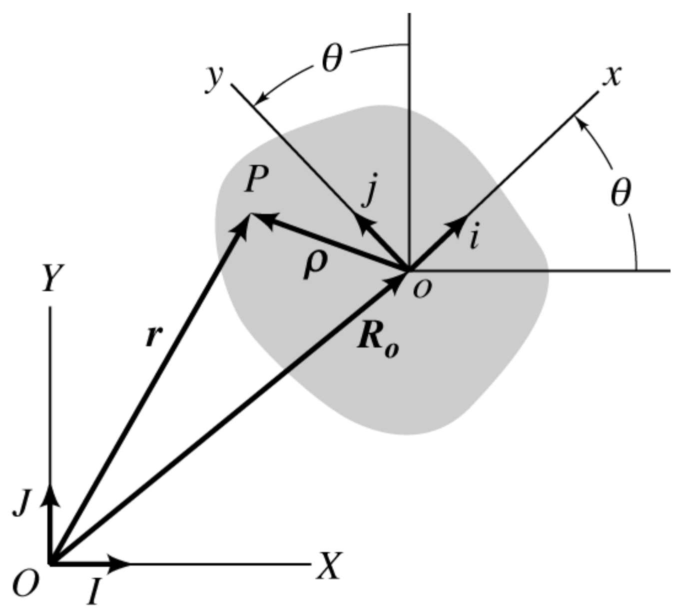


Figure 5.63 Rigid body with an imbedded *x, y, z* coordinate system. Point *o* , the origin of the *x,y,z* system, is located in the inertial *X, Y* system by the vector $\boldsymbol{R_o}$.

The mass center of the body is located in the *x, y, z* system by the position vector $\boldsymbol{b_{og}}$ defined earlier in section 5.2 as

$$m\boldsymbol{b_{og}} = m(\boldsymbol{i}b_{ogx} + \boldsymbol{j}b_{ogy} + \boldsymbol{k}b_{ogz}) = \int_V \boldsymbol{\rho}\gamma\, dx\, dy\, dz = \int_m \boldsymbol{\rho}\, dm \ , \quad (5.2)$$

where γ is the mass density of the body at point P. A point P in the body is located in the X, Y coordinate system by the position vector $\boldsymbol{r}$ and in the x, y, z system by the vector $\boldsymbol{\rho} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$.

The kinetic energy of the mass can be stated

$$T = \frac{1}{2}\int_m v^2 dm = \frac{1}{2}\int_m (\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}})dm \ , \qquad (5.178)$$

where $\dot{\boldsymbol{r}}$ is the velocity of a particle of mass dm at point P with respect to the X, Y coordinate system .

Since, points o and P are both fixed in the rigid body,

$$\boldsymbol{v}_{\boldsymbol{P}} = \boldsymbol{v}_{\boldsymbol{o}} + \boldsymbol{\omega} \times \boldsymbol{r}_{\boldsymbol{oP}} \ \Rightarrow \ \dot{\boldsymbol{r}} = \dot{\boldsymbol{R}}_{\boldsymbol{o}} + \boldsymbol{\omega} \times \boldsymbol{\rho}$$

Hence,

$$\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = |\dot{\boldsymbol{R}}_{\boldsymbol{o}}|^2 + 2\ \dot{\boldsymbol{R}}_{\boldsymbol{o}} \cdot \boldsymbol{\omega} \times \boldsymbol{\rho} + |\boldsymbol{\omega} \times \boldsymbol{\rho}|^2 \ . \qquad (5.179)$$

Substituting from Eq.(5.179) into the integral of Eq.(5.178) gives

$$T = \frac{m\,|\dot{\boldsymbol{R}}_{\boldsymbol{o}}|^2}{2} + \dot{\boldsymbol{R}}_{\boldsymbol{o}} \cdot \int_m \boldsymbol{\omega} \times \boldsymbol{\rho}\ d\ m + \frac{1}{2}\int_m |\boldsymbol{\omega} \times \boldsymbol{\rho}|^2\ dm \qquad (5.180)$$

where $\ddot{\boldsymbol{R}}_{\boldsymbol{o}}$ is not a function of the integration variables x, y, z and has been taken outside the integrals.

Continuing,

$$\boldsymbol{\omega} \times \boldsymbol{\rho} = \boldsymbol{k}\,\dot{\theta} \times (\boldsymbol{i}\,x + \boldsymbol{j}\,y + \boldsymbol{k}z) = \dot{\theta}\,(\boldsymbol{j}\,x - \boldsymbol{i}\,y)\ ,$$

and

$$|\boldsymbol{\omega}\times\boldsymbol{\rho}|^2 = (\boldsymbol{\omega} \times \boldsymbol{\rho}) \cdot (\boldsymbol{\omega} \times \boldsymbol{\rho}) = \dot{\theta}^2\,(x^2 + y^2)$$

$$\int (\boldsymbol{\omega} \times \boldsymbol{\rho})\,d\,m = \boldsymbol{\omega} \times \int \boldsymbol{\rho}\,d\,m = \boldsymbol{\omega} \times m\,\boldsymbol{b}_{og} \qquad (5.181)$$

Substituting from Eqs.(5.181) into (5.180) gives

$$T = \frac{m\,|\dot{\boldsymbol{R}}_o|^2}{2} + \dot{\boldsymbol{R}}_o \cdot (\boldsymbol{\omega} \times m\,\boldsymbol{b}_{og}) + \frac{I_o\,\dot{\theta}^2}{2}\ , \qquad (5.182)$$

where

$$I_{zzo} = I_o = \int (x^2 + y^2)\,dm$$

is the moment of inertia about a z axis through point o , the origin of the x, y, z system.

If o the origin of the x, y, z system coincides with g the body’s mass center, $\boldsymbol{b}_{og} = 0$, and Eq.(5.182) reduces to

$$T = \frac{m\,|\dot{\boldsymbol{R}}_{\boldsymbol{g}}|^2}{2} + \frac{I_g\,\dot{\theta}^2}{2} \,. \tag{5.183}$$

This equation states that the kinetic energy of a rigid body is the sum of the following terms:

a. The translational energy of the body assuming that all of its mass is concentrated at the mass center, and

b. The rotational energy of the rigid body from rotation about the mass center.

Rotation about a Fixed Axis

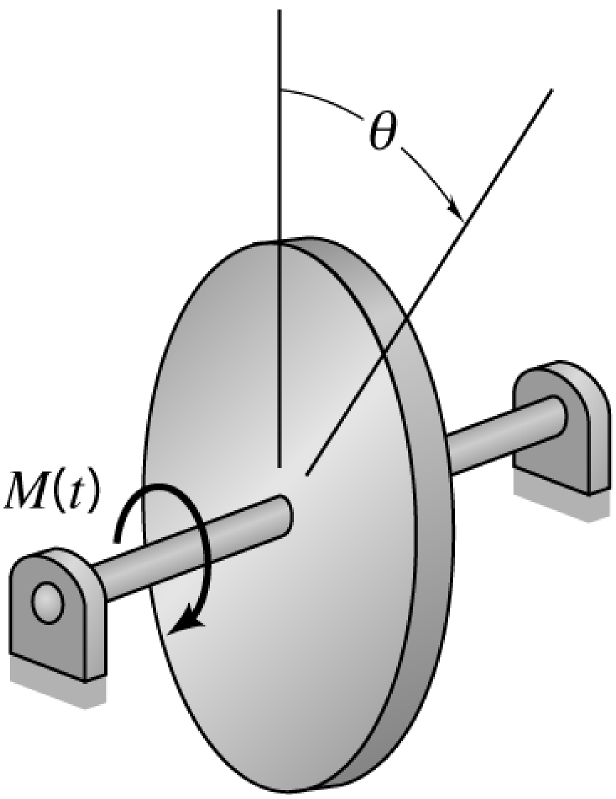
For pure rotation about *o*, $\dot{\boldsymbol{R}}_{\boldsymbol{o}} = 0$ in Eq.(5.182) and the following simplified definition applies

$$T = \frac{I_o\dot{\theta}^2}{2}. \tag{5.184}$$

Eq.(5.184) defines the kinetic energy of the body for pure rotation about an axis through a point *o* fixed in space.

Applications of the Energy Equation

Rotor in Bearings



Assume that the rotor has an initial angular velocity of $\dot{\theta}(0) = \omega_0$, *and is acted on by a constant drag moment* $\bar{M}$ *, how many revolutions will it take to come to rest?*

There is no change in potential energy, and the final kinetic energy is zero; hence, the energy equation
$Work_{n.c.} = \Delta(T + V)$ gives

$$Work_{n.c.} = (0 + 0) - (\frac{I_o \omega_o^2}{2} + 0) \quad . \qquad (5.207)$$

We need to calculate the work done by the resistance torque.

The differential work due to a applied force ***f*** acting through the differential distance ***ds*** is $dWk = \mathbf{f} \bullet \mathbf{ds}$. We can replace a moment M by a force acting at a fixed radius $\bar{r}$, such that $M = f\bar{r}$. When the moment M rotates through the differential angle $d\theta$, the force will act through the arc distance $ds = \bar{r}\,d\theta$, and the differential work will be

$$dWk = f\,ds = f\,\bar{r}\,d\theta = M d\theta \ . \tag{5.208}$$

Using Eq.(5.208), Eq.(5.207) becomes

$$\begin{aligned} \int_o^{\Delta\theta} -\bar{M} d\theta = -\bar{M}\Delta\theta = -(\frac{I_o\omega_o^2}{2}) \\ \therefore \ \Delta revolutions = \frac{\Delta\theta}{2\pi} = \frac{I_o\omega_o^2}{4\pi\bar{M}} \ . \end{aligned} \tag{5.209}$$

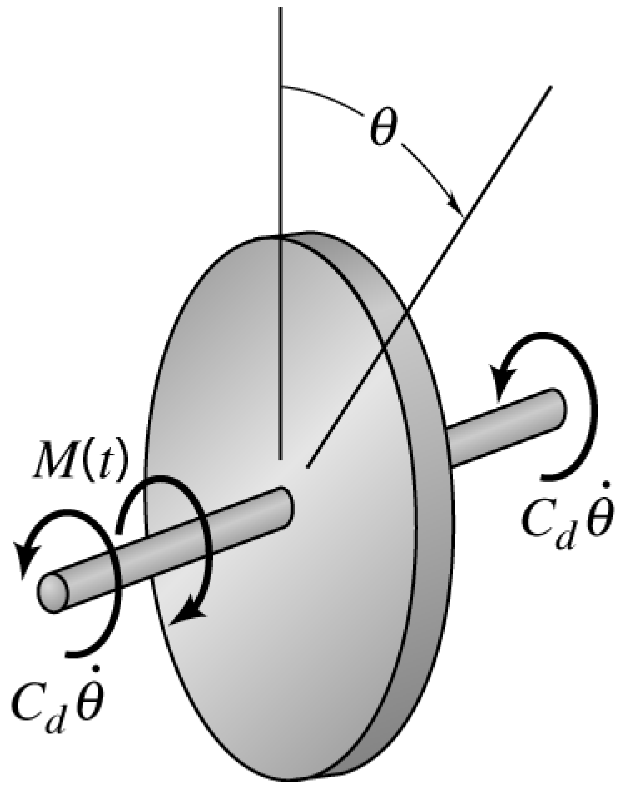
The work integral is negative because it decreases the energy of the system.

Assume that the rotor is acted on by the positive (in the direction of $+\theta$*) applied moment* $M(t)$, and derive the equation of motion. For this task, $Work_{n.c.} = \Delta(T+V)$ gives

$$\int_o^{\theta} M(t)\,dx = \frac{I_{oz}\dot{\theta}^2}{2} - T_o . \tag{5.210}$$

Differentiating Eq.(5.210) with respect to θ gives the differential equation of motion

$$I_{oz}\frac{d}{d\theta}\left(\frac{\dot{\theta}^2}{2}\right)=I_{oz}\ddot{\theta}=M(t) \quad .$$



Derive the governing equation of motion for the rotor including the applied moment M(t) and viscous drag moment $-C_d\dot{\theta}$.

For this task, $Work_{n.c.}=\Delta(T+V)$ becomes

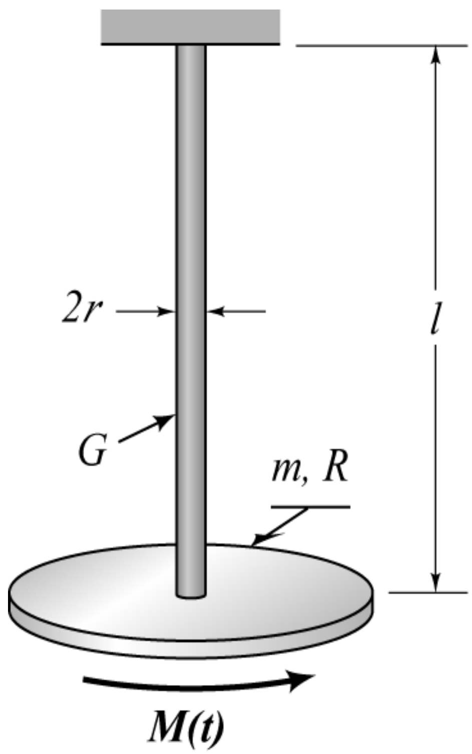
$$\int_o^{\theta}[M(t)-2C_d\dot{\theta}]\,dx=\frac{I_{oz}\dot{\theta}^2}{2}-T_o,$$

and differentiation with respect to θ gives

$$I_{oz}\frac{d}{dt}\left(\frac{\dot{\theta}^2}{2}\right)=I_{oz}\ddot{\theta}=M(t)-2C_d\dot{\theta}\;\Rightarrow\; I_{oz}\ddot{\theta}+2C_d\dot{\theta}=M(t) \quad .$$

Using the work-energy equation has no particular advantage in developing these last two equations of motion. As with the Newtonian approach, a free-body diagram is required to define the applied moment, and the nonconservative moments can not be integrated with respect to θ.

A Torsional-Vibration Example



(*a*)

Derive the governing equation of motion. The external moment $M(t)$ is adding energy to the system; hence, the work-energy equation is

$$\int_0^{\theta} M(t)\,dx = \left(\frac{I_o\dot{\theta}^2}{2} + V\right) - (T_0 + V_0)\ . \tag{5.211}$$

In this example, the potential energy of the system is stored in the shaft due to the torsional rotation θ. Recall that the reaction moment is defined from

$$M_\theta = -k_\theta \theta = -\frac{GJ}{l}\theta = -\frac{G}{l}\frac{\pi r^4}{2}\theta ,$$

where G is the shear modulus of the rod, and $J = \pi r^4/2$ is the rod's area polar moment of inertia. The requirement that a potential force (or moment) be derivable as the negative gradient of a potential function gives

$$M_\theta = -k_\theta \theta = -\frac{dV_\theta}{d\theta} \quad \Rightarrow \quad V_\theta = k_\theta(\frac{\theta^2}{2}) \ . \qquad (5.191)$$

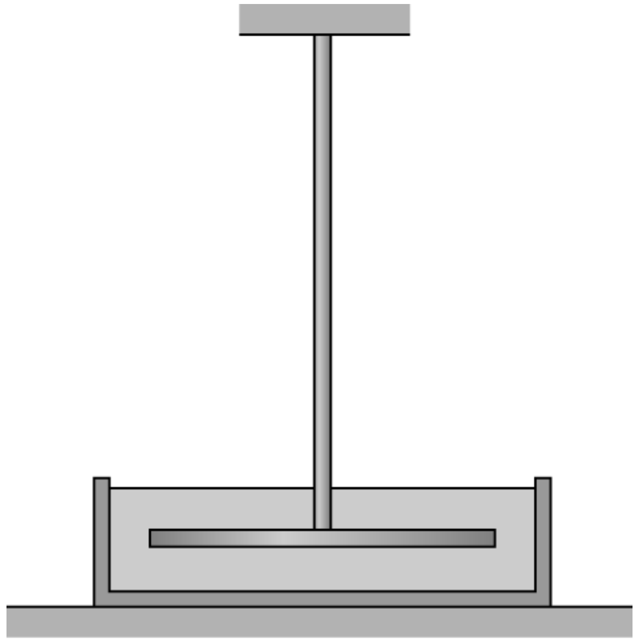
Substituting for V_θ into Eq.(5.211) gives

$$\int_0^\theta M(t)\, d\theta = (\frac{I_o \dot{\theta}^2}{2} + \frac{k_\theta \theta^2}{2}) - (T_0 + V_0) \ .$$

Differentiating with respect to θ gives the differential equation of motion

$$I_o \frac{d}{d\theta}(\frac{\dot{\theta}^2}{2}) + k_\theta \theta = M(t) \quad \Rightarrow \quad I_o \ddot{\theta} + k_\theta \theta = M(t) \ .$$

Torsional Vibration Example with viscous drag



For the viscous drag moment $-C_d\dot{\theta}$, the integrand in the work-energy Eq.(5.211) gives

$$\int_0^{\theta}[M(t)-C_d\dot{\theta}]\,dx=\frac{I_o\dot{\theta}^2}{2}+\frac{k_\theta\theta^2}{2}-(T_0+V_0)\ . \qquad (5.213)$$

Differentiating with respect to θ gives the final equation of motion

$$I_o\ddot{\theta}+C_d\dot{\theta}+k_\theta\theta=M(t)\,. \qquad (5.213a)$$

For this example, deriving the equation of motion using the moment or the work-energy equation requires about the same effort. The equation of motion can be derived by including the external driving moment $M(t)$ or viscous damping moment

$-C_d\dot{\theta}$ in the work integral, but the integral is now a function of t or $\dot{\theta}$, not θ, and can not be integrated.

Recall that the moment equation gave

$$I_o\ddot{\theta} = \Sigma M_o = M(t) - C_d\dot{\theta} - k_\theta\theta \ ,$$

where ΣM_o is the resultant moment about the vertical axis. Substituting the energy-integral substitution $\ddot{\theta} = d(\dot{\theta}^2/2)/d\theta$ yields

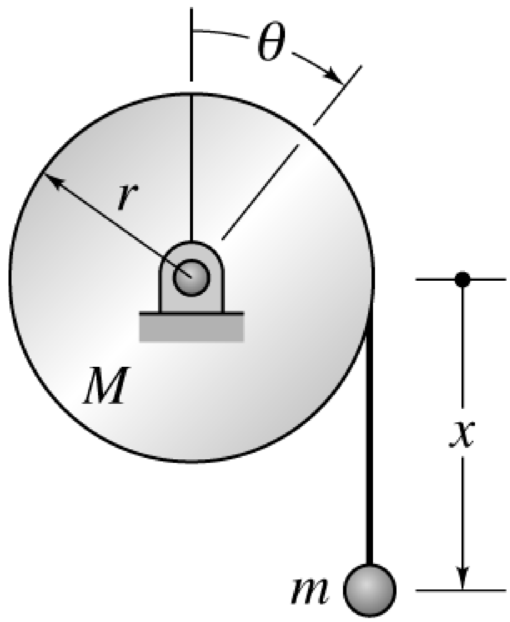
$$I_o\frac{d}{d\theta}(\frac{\dot{\theta}^2}{2}) = M(t) - C_d\dot{\theta} - k_\theta\theta \ . \qquad (5.214)$$

Multiplying through by $d\theta$ and integrating gives

$$I_o\frac{\dot{\theta}^2}{2} + k_\theta\frac{\theta^2}{2} = \int_{\theta_0}^{\theta}[M(t) - C_d\dot{\theta}]\,dx - (I_o\frac{\dot{\theta}_0^2}{2} + k_\theta\frac{\theta_0^2}{2}) \ . \qquad (5.215)$$

This result coincides with Eq.(5.213a), obtained from the work-energy-equation.

An Example Involving Connected Motion of a Disk and a Particle



Derive the governing equation of motion using conservation of energy.

There are no nonconservative forces ; hence, energy is conserved. Using a plane through the bearing as datum for potential energy due to gravity

$$T + V = T_0 + V_o \quad \Rightarrow \quad \frac{I_o \dot{\theta}^2}{2} + \frac{m\dot{x}^2}{2} - mgx = T_0 + V_0 \ .$$

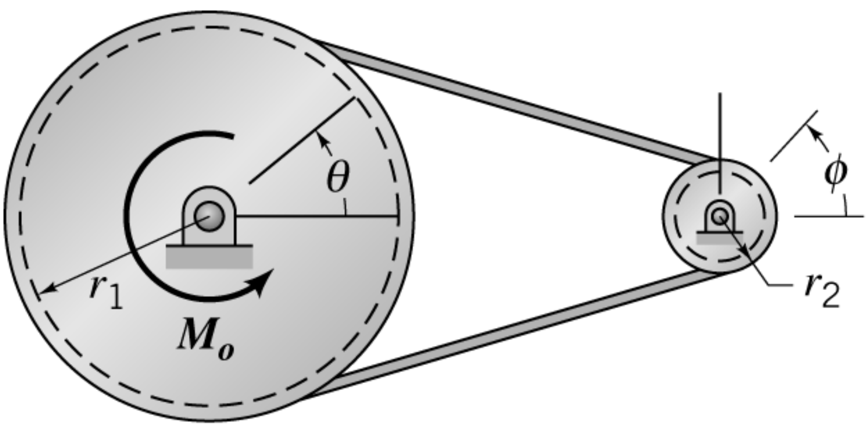
We need $I_o = Mr^2/2$, and the kinematics of Eq.(5.28), $x = r\theta, \dot{x} = r\dot{\theta}$, to obtain

$$\frac{\dot{\theta}^2 r^2}{2}\left(\frac{M}{2} + m\right) - mgr\theta = T_0 + V_0 \ .$$

Differentiating with respect to θ gives the differential equation of motion

$$(\frac{M}{2}+m)r^2\ddot{\theta}=wr \quad ,$$

Two Driven Pulleys Connected by a Belt



(*a*)

The left and right pulleys have radii and radii of gyrations (r_1, k_{g1}) and (r_2, k_{g2}), respectively. There is no energy dissipation (frictionless bearings), and the belt connecting the two pulleys does not slip.

a. If the system starts from rest, and the applied moment $M_o=\bar{M}$ is constant, find the angular velocity of both pulleys after 10 complete rotations of the left (driven)pulley.

b. Derive the differential equations of motion.

There is no change in the potential energy of this system; hence, $Work_{n.c.} = \Delta(T+V)$ yields

$$Work_{n.c.} = T - T_1 = \left(\frac{I_1\dot{\theta}^2}{2} + \frac{I_2\dot{\varphi}^2}{2}\right) - 0 = \frac{m_1 k_{g1}^2\dot{\theta}^2}{2} + \frac{m_2 k_{g2}^2\dot{\varphi}^2}{2} \quad (5.195)$$

Since the belt does not slip, the tangential velocities at the rims of the pulleys must equal, providing the kinematic condition, $r_1\dot{\theta} = r_2\dot{\varphi}$, which reduces Eq.(5.195) to

$$Work_{n.c.} = \frac{\dot{\theta}^2}{2}\left[m_1 k_{g1}^2 + \left(\frac{r_1}{r_2}\right)^2 m_2 k_{g2}^2\right] = \frac{I_{eff}\dot{\theta}^2}{2} \ . \quad (5.196)$$

Using $d\,Work = M\,d\theta$ to define the work integral on the left-hand side gives

$$\int_0^{\Delta\theta} \bar{M}\,d\theta = \bar{M}\Delta\theta = \frac{I_{eff}\dot{\theta}^2}{2} \ , \quad (5.197)$$

and the angular velocity after the moment has been applied for ten rotations $(\Delta\theta = 20\pi)$ is

$$\dot{\theta}(\Delta\theta = 20\pi) = \sqrt{\frac{2\bar{M}\cdot 20\pi}{I_{eff}}} \ ,$$

which concludes *Task a*. The work integral is positive in

Eq.(5.197) because it is increasing the mechanical energy of the system.

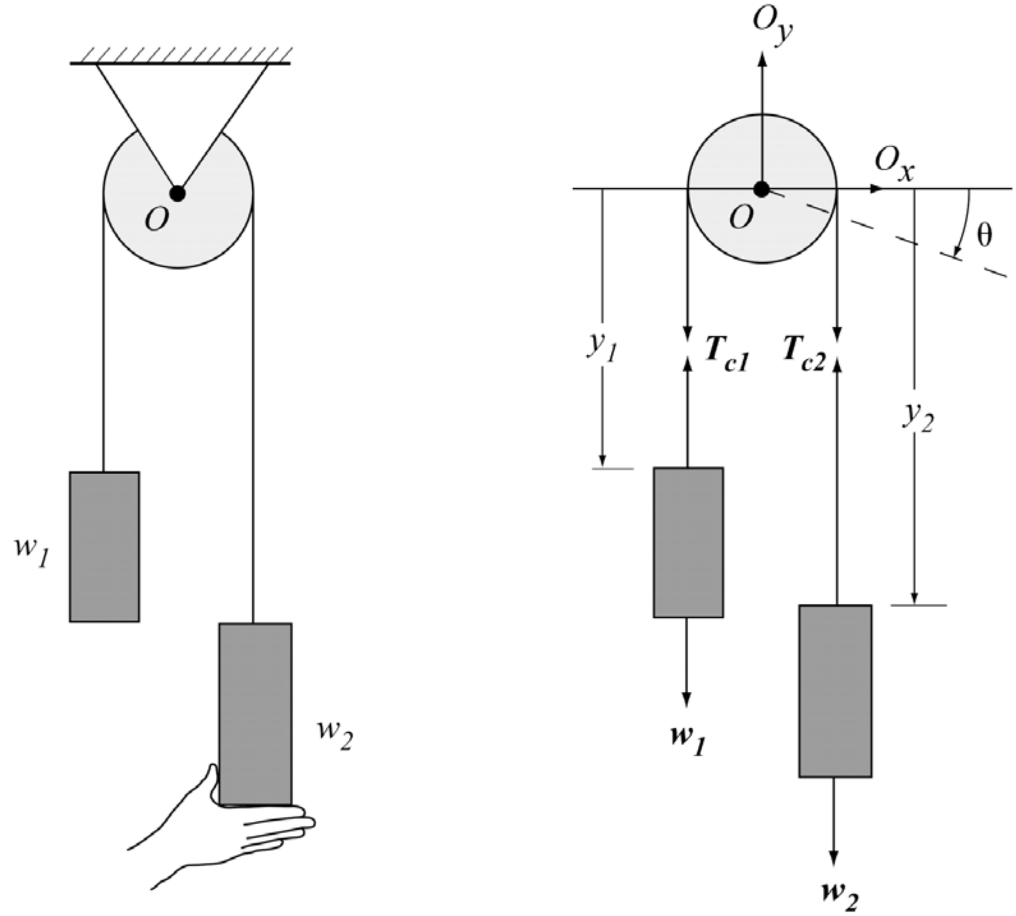
Differentiating Eq.(5.197) with respect to θ gives the governing differential equation of motion

$$I_{eff}\ddot{\theta} = M_o \ ,$$

which coincides with our earlier result and concludes *Task b*.

Particle Dynamics versus Rigid Body Dynamics

Derive the equation of motion for the system illustrated



Moment Equation

$$\sum M_0 = I_o \ddot{\theta} = T_{c2} r - T_{c1} r \tag{72}$$

Force Equations

$$\begin{aligned} &Body\ 1: \sum f_{y1} = w_1 - T_{c1} = m_1 \ddot{y}_1 \\ &Body\ 2: \sum f_{y2} = w_2 - T_{c2} = m_2 \ddot{y}_2 \end{aligned} \tag{73}$$

For negligible pulley inertia, $I_0 = 0 \Rightarrow T_{c1} = T_{c2} = T_c$, and Eq.(2) becomes

$$w_1 - T_c = m_1 \ddot{y}_1 \ , \ w_2 - T_c = m_2 \ddot{y}_2 \qquad \textbf{(3)}$$

Subtracting the 1st from the 2nd gives

$$m_2 \ddot{y}_2 - m_1 \ddot{y}_1 = w_2 - w_1 \qquad \textbf{(4)}$$

Kinematic Constraint

$$y_1 + y_2 + constant = l_c \Rightarrow \ddot{y}_1 = -\ddot{y}_2 \qquad \textbf{(5)}$$

Substitute from Eq.(6) into (5) gives

$$[m_2 \ddot{y}_2 - m_1(-\ddot{y}_2)] = (m_1 + m_2)\ddot{y}_2 = w_2 - w_1 \qquad \textbf{(6)}$$

This is a particle dynamics result.

For finite moment of inertia, $I_0 > 0$, solve for T_{1c} and T_{2c} from Eq.(2) and substitute into Eq.(1) to obtain

$$I_0 \ddot{\theta} = r(w_2 - m_2 \ddot{y}_2) - r(w_1 - m_1 \ddot{y}_1) \ ,$$

Substitute the kinematics relationships,

$$\ddot{y}_1 = -r\ddot{\theta} \ , \ \ddot{y}_2 = r\ddot{\theta} \ ,$$

to obtain

$$I_0 \ddot{\theta} = r w_2 - r m_2 (r\ddot{\theta}) - r w_1 + r m_1 (- r\ddot{\theta}_1) \ ,$$

or

$$(I_0 + m_1 r^2 + m_2 r^2)\ddot{\theta} = I_{eff}\ddot{\theta} = r(w_2 - w_1) \ . \qquad \textbf{(7)}$$

All of the contributions to I_{eff} should be positive.

Compare (7) and (6). Setting I_0 equal to zero in (7) and substituting $\ddot{\theta} = \ddot{y}_2 / r$ gives

$$(m_1 + m_2) r^2 (\frac{\ddot{y}_2}{r}) = r(w_2 - w_1) \ \Rightarrow \ (m_1 + m_2)\ddot{y}_2 = w_2 - w_1$$

Equation of Motion from conservation of energy

$$T = I_0 \frac{\dot{\theta}^2}{2} + m_1 \frac{\dot{y}_1^2}{2} + m_2 \frac{\dot{y}_2^2}{2} \;,\; V = -w_1 y_1 - w_2 y_2$$

The datum for V goes through the center of the pulley.

Kinematics:

$$y_1 = y_{10} - r\theta \;\Rightarrow\; \dot{y}_1 = -r\dot{\theta} \;,\; y_2 = y_{20} + r\theta \;\Rightarrow\; \dot{y}_2 = r\dot{\theta}$$

Substitution gives

$$T = I_0 \frac{\dot{\theta}^2}{2} + m_1 \frac{(-r\dot{\theta})^2}{2} + m \frac{(r\dot{\theta})^2}{2} = (I_0 + m_1 r^2 + m_2 r^2) \frac{\dot{\theta}^2}{2}$$

$$V = -w_1 (y_{10} - r\theta) - w_2 (y_{20} + r\theta) \;.$$

Hence, $T + V = T_0 + V_0$ gives

$$(I_0 + m_1 r^2 + m_2 r^2) \frac{\dot{\theta}^2}{2} - w_1 (y_{10} - r\theta) - w_2 (y_{20} + r\theta) = T_0 + V_0$$

Differentiate w.r.t. θ gives

$$(I_0 + m_1 r^2 + m_2 r^2) \frac{d}{d\theta} \frac{\dot{\theta}^2}{2} + w_1 r - w_2 r = 0 \;,$$

or

$$(I_0 + m_1 r^2 + m_2 r^2)\,\ddot{\theta} = w_2 r - w_1 r \ ,$$

which coincides with Eq.(7).