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16.07 Dynamics
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Lecture L20 - Energy Methods: Lagrange's Equations

The motion of particles and rigid bodies is governed by Newton's law. In this section, we will derive an alternate approach, placing Newton's law into a form particularly convenient for multiple degree of freedom systems or systems in complex coordinate systems. This approach results in a set of equations called Lagrange's equations. They are the beginning of a complex, more mathematical approach to mechanics called analytical dynamics. In this course we will only deal with this method at an elementary level. Even at this simplified level, it is clear that considerable simplification occurs in deriving the equations of motion for complex systems. These two approaches–Newton's Law and Lagrange's Equations–are totally compatible. No new physical laws result for one approach vs. the other. Many have argued that Lagrange's Equations, based upon conservation of energy, are a more fundamental statement of the laws governing the motion of particles and rigid bodies. We shall not enter into this debate.

Derivation of Lagrange's Equations in Cartesian Coordinates

We begin by considering the conservation equations for a large number (N) of particles in a conservative force field using cartesian coordinates of position x_i. For this system, we write the total kinetic energy as

$$T = \sum_{n=1}^{M} \frac{1}{2} m_i \dot{x}_i^2. \tag{1}$$

where M is the number of degrees of freedom of the system.

For particles traveling only in one direction, only one x_i is required to define the position of each particle, so that the number of degrees of freedom $M = N$. For particles traveling in three dimensions, each particle requires 3 x_i coordinates, so that $M = 3 * N$.

The momentum of a given particle in a given direction can be obtained by differentiating this expression with respect to the appropriate x_i coordinate. This gives the momentum p_i for this particular particle in this coordinate direction.

$$\frac{\partial T}{\partial \dot{x}_i} = p_i \tag{2}$$

The time derivative of the momentum is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) = m_i \ddot{x}_i \tag{3}$$

For a conservative force field, the force on a particle is given by the derivative of the potential at the particle position in the desired direction.

$$F_i = -\frac{\partial V}{\partial x_i} \tag{4}$$

From Newton's law we have

$$F_i = \frac{dp_i}{dt} \tag{5}$$

Equating like terms from our manipulations on kinetic energy and the potential of a conservative force field, we write

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) = -\frac{\partial V}{\partial x_i} \tag{6}$$

Now we make use of the fact that

$$\frac{\partial T}{\partial x_i} = 0 \tag{7}$$

and

$$\frac{\partial V}{\partial \dot{x}_i} = 0 \tag{8}$$

Using these results, we can rewrite Equation (6) as

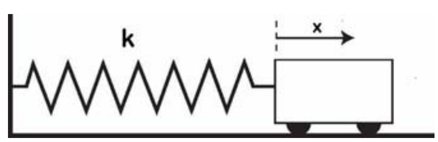
$$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{x}_i}\right) - \frac{\partial (T-V)}{\partial x_i} = 0 \tag{9}$$

We now define $L = T - V$: L is called the Lagrangian. Equation (9) takes the final form: Lagrange's equations in cartesian coordinates.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \tag{10}$$

where i is taken over all of the degrees of freedom of the system. Before moving on to more general coordinate systems, we will look at the application of Equation(10) to some simple systems.

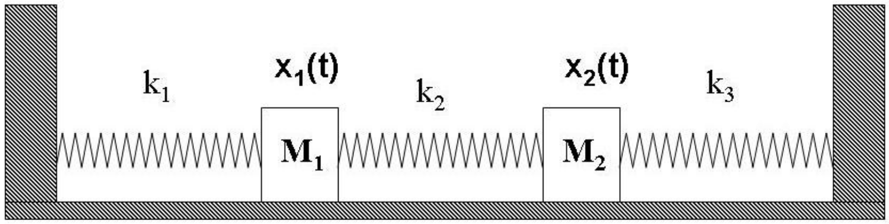
Mass-spring System



We first consider a simple mass spring system. This is a one degree of freedom system, with one x_i. Its kinetic energy is $T = 1/2m\dot{x}^2$; its potential is $V = 1/2kx^2$; its Lagrangian is $L = 1/2m\dot{x}^2 - 1/2kx^2$. Applying Equation (10) to the Lagrangian of this simple system, we obtain the familiar differential equation for the mass-spring oscillator.

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{11}$$

Clearly, we would not go through a process of such complexity to derive this simple equation. However, let's consider a more complex system, governed by the same laws.



Multi-degree of freedom systems

This is a 2 degree of freedom system, governed by 2 differential equations. The number of springs for this configuration is 3. These governing equations could be obtained by applying Newton's Law to the force balance that exists at each mass due to the deflection of the springs as was done in Lecture 19. The deflection of springs 1 and 3 are influenced by the boundary condition at either end of the slot; in this case the deflection is zero.

The governing equations can also be obtained by direct application of Lagrange's Equation. This approach is quite straightforward. The expression for kinetic energy is

$$T = \sum_{n=1}^{2} \frac{1}{2} m_i \dot{x}_i^2; \tag{12}$$

the expression for the potential is

$$V = 1/2 k_1 x_1^2 + 1/2 k_2 (x_2 - x_1)^2 + 1/2 k_3 x_2^2 \tag{13}$$

Applying Lagrange's equation to $T - V = L$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \tag{14}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \tag{15}$$

we obtain the governing equations as

$$m_1 \frac{d^2 x_1}{dt^2} = -k_1 x_1 + k_2 (x_2 - x_1) \tag{16}$$

$$m_2 \frac{d^2 x_2}{dt^2} = -k_2 (x_2 - x_1) - k_3 x_2 \tag{17}$$

Clearly, for multi degree of freedom systems, this approach has advantages over the force balancing approach using Newton's law.

Extension to General Coordinate Systems

A significant advantage of the Lagrangian approach to developing equations of motion for complex systems comes as we leave the cartesian x_i coordinate system and move into a general coordinate system. An example would be polar coordinates where for a two-dimensional position of a mass particle, x_1 and x_2 could be given by r and θ. A two-degree of freedom system remains two-degree so that the number of coordinate variables required remains two. r and θ and their counterparts in other coordinate systems will be referred to as ***generalized coordinates***. We introduce quite general notation for the relationship between the n cartesian variables of position x_i and their description in ***generalized coordinates***. (For some systems, the number of generalized coordinates is larger than the number of degrees of freedom and this is accounted for by introducing ***constraints*** on the system. This is an important part of the discussion of the Lagrange formulation. We shall not however develop these relations but will work directly with the number of variables equal to the number of degrees of freedom of the system.) We express the cartesian variable x_i using ***generalized coordinates*** q_j. (Polar coordinates r, θ would be an example.)

$$x_i = x_i(q_1, ..q_j, ...q_n). \tag{18}$$

In the general case, each x_i could be dependent upon every q_j.
What is remarkable about the Lagrange formulation, is that (10) holds in a general coordinate system with x_i replaced by q_i.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{19}$$

Before showing how this result can be derived from Newton's Law, we show two applications in polar coordinates to demonstrate the power of the approach.

Simple Pendulum by Lagrange's Equations

We first apply Lagrange's equation to derive the equations of motion of a simple pendulum in polar coordinates. This is a one degree of freedom system. However, it is convenient for later analysis of the double pendulum, to begin by describing the position of the mass point m_1 with cartesian coordinates x_1 and y_1 and then express the Lagrangian in the polar angle θ_1. Referring to a) in the figure below we have

$$x_1 = h_1 \sin\theta_1 \tag{20}$$

$$y_1 = -h_1 \cos\theta_1 \tag{21}$$

so that the kinetic energy is

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) = \frac{1}{2}m_1 h_1^2 \dot{\theta}_1^2 \tag{22}$$

The potential energy is

$$V = m_1 g y_1 = -m_1 g h_1 \cos\theta \tag{23}$$

The Lagrangian is

$$L = T - V = \frac{1}{2}m_1 h_1^2 \dot{\theta}_1^2 + m_1 g h_1 \cos\theta_1 \tag{24}$$

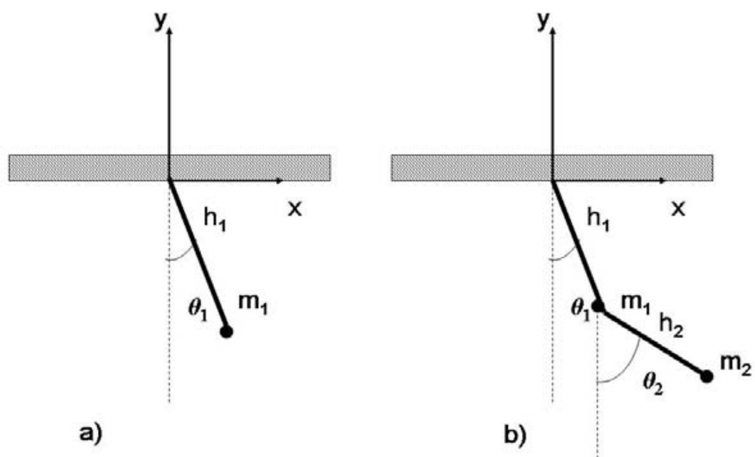
Applying (10) with $q_1 = \theta_1$, we obtain the differential equation governing the motion.

$$m_1 h_1^2 \ddot{\theta}_1 + m_1 g h_1 \sin\theta_1 = 0 \tag{25}$$

Again, for such a simple system, we would typically not go through this formalism to obtain this result. However, this framework will enable us to derive the equations of motion for the more complex systems such as the double pendulum shown in b).

Double Pendulum by Lagrange's Equations

Consider the double pendulum shown in b) consisting of two rods of length h_1 and h_2 with mass points m_1 and m_2 hung from a pivot. This systems has two degrees of freedom: θ_1 and θ_2.



To apply Lagrange's equations, we determine expressions for the kinetic energy and the potential as the system moves in angular displacement through the independent angles θ_1 and θ_2. From the geometry we have

$$x_1 = h_1 \sin\theta_1 \tag{26}$$

$$y_1 = -h_1 \cos\theta_1 \tag{27}$$

$$x_2 = h_1 \sin\theta_1 + h_2 \sin\theta_2 \tag{28}$$

$$y_2 = -h_1 \cos\theta_1 - h_2 \cos\theta_2 \tag{29}$$

The kinetic energy is $T = \frac{1}{2} m_1 \left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2\right)$. Expressed in variables θ_1 and θ_2, the kinetic energy of the system is

$$T = \frac{1}{2} m_1 h_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(h_1^2 \dot{\theta}_1^2 + h_2^2 \dot{\theta}_2^2 + 2 h_1 h_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right) \tag{30}$$

The potential energy of the system is

$$V = m_1 g y_1 + m_2 g y_2 = -(m_1 + m_2) g h_1 \cos\theta_1 - m_2 g h_2 \cos\theta_2. \tag{31}$$

The Lagrangian is then

$$L = T - V = \frac{1}{2}(m_1+m_2) h_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 h_2^2 \dot{\theta}_2^2 + m_2 h_1 h_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + (m_1+m_2) g h_1 \cos\theta_1 + m_2 g h_2 \cos\theta_2 \tag{32}$$

Since the generalized coordinates are now θ_1 and θ_2, Lagrange's equation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = 0 \tag{33}$$

for both $i = 1$ and $i = 2$.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0 \tag{34}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0 \tag{35}$$

Working out the details, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = (m_1 + m_2) h_1^2 \ddot{\theta}_1 + m_2 h_1 h_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 h_1 h_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \tag{36}$$

$$\frac{\partial L}{\partial \theta_1} = -h_1 g (m_1 + m_2) \sin(\theta_1) - m_2 h_1 h_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \tag{37}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 h_2^2 \ddot{\theta}_2 + m_2 h_1 h_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 h_1 h_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \tag{38}$$

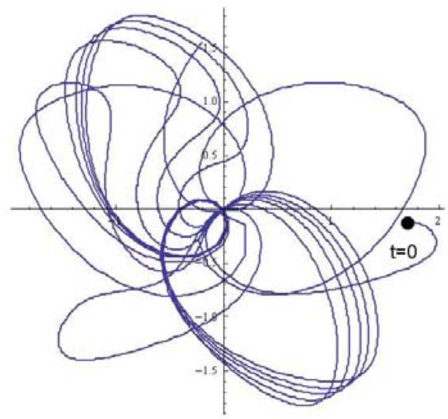
$$\frac{\partial L}{\partial \theta_2} = -h_2 g m_2 \sin(\theta_2) + m_2 h_1 h_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \tag{39}$$

From Equation(35-39), we obtain the final form of the governing equations for the double pendulum

$$(m_1 + m_2) h_1 \ddot{\theta}_1 + m_2 h_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 h_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + g(m_1 + m_2) \sin\theta_1 = 0 \tag{40}$$

$$m_2 h_2 \ddot{\theta}_2 + m_2 h_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 h_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g \sin\theta_2 = 0 \tag{41}$$

The double pendulum is a system of great interest, displaying conventional linear multi degree of freedom system behavior for small θ_1 and θ_2, but displaying chaotic behavior for large θ. A chaotic system is a deterministic system that exhibits great sensitivity to the initial conditions: the "butterfly" effect. A simulation of the motion of a double pendulum is available on **http://scienceworld.wolfram.com/physics/DoublePendulum.html**. For a particular choice of initial conditions, the position of m_2 with time is shown in the figure.



Double Pendulum Solution

Derivation of Lagrange's Equation for General Coordinate Systems

We now follow the earlier procedure we used to derive Lagrange's equation from Newton's law but using generalized coordinates instead of cartesian coordinates. (See additional reading: Slater and Frank and/or Marion and Thorton.) The cartesian variable are expressed in terms of the generalized coordinates as

$$x_i = x_i(q_1, ..q_j, ...q_n). \tag{42}$$

To obtain the velocity $\dot{x}_i$, we take the time derivative of (42) applying the chain rule.

$$\dot{x}_i = \sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j} \dot{q}_j. \tag{43}$$

Taking the partial derivative of (43) with respect to $\dot{q}_j$ we obtain a relation between these two derivatives,

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \frac{\partial x_i}{\partial q_j}, \tag{44}$$

a result which we shall need shortly. In deriving equation (44), we take advantage of the fact that since the generalized coordinates q_i are independent, $\frac{\partial q_i}{\partial q_j} = 0$ for $i \neq j$.

Following the approach leading to Equations (2-10), we ***define*** a ***generalized momentum*** as

$$p_i = \frac{\partial T}{\partial \dot{q}_i} \tag{45}$$

with $T = \sum_{n=1}^{M} \frac{1}{2} m_j \dot{x}_j^2$ as given by Equation (1).

We are now ready to express Newton's law in the ***generalized coordinates*** q_i that we have introduced.

For the ***generalized momentum*** we have

$$p_i = \frac{\partial T}{\partial \dot{q}_i} = \sum_{j=1}^{n} m_j \dot{x}_j \frac{\partial \dot{x}_j}{\partial \dot{q}_i} = \sum_{j=1}^{n} m_j \dot{x}_j \frac{\partial x_j}{\partial q_i}, \tag{46}$$

where we have made use of (44).

For a conservative force field, the work done during a displacement dx_i is given by

$$dW = \sum_{i=1}^{N} F_i dx_i \tag{47}$$

or expressed in the generalized coordinates q_j

$$dW = \sum_{i=1}^{N} F_i dx_i = \sum_{j=1}^{N} \sum_{i=1}^{N} F_i \frac{\partial x_i}{\partial q_j} dq_j \tag{48}$$

We identify

$$\sum_{i=1}^{N} F_i \frac{\partial x_i}{\partial q_j} dq_j \tag{49}$$

as the work done by a "displacement" through dq_j and define a generalized force Q_j as

$$Q_j = \sum_{i=1}^{N} F_i \frac{\partial x_i}{\partial q_j} \tag{50}$$

so that the work is expressed as.

$$dW = \sum_{j=1}^{N} Q_j dq_j \tag{51}$$

For a conservative system, the work done by a small displacement dq_j is

$$dW = -dV = -\frac{\partial V}{\partial q_j} dq_j, \tag{52}$$

where V is the potential function expressed in the coordinate system of the generalized coordinates and $\frac{\partial V}{\partial q_j}$ is the change in potential due to a change in the generalized coordinate q_j. So that the ***generalized force*** is

$$Q_j = -\frac{\partial V}{\partial q_j} \tag{53}$$

in analogy with (4). We now examine the time derivative of the ***generalized momentum***, p_i.

$$\frac{dp_i}{dt} = \frac{d}{dt}(\frac{\partial T}{\partial q_i}) = \sum_{j=1}^{N} (m_j \ddot{x}_j \frac{\partial x_j}{\partial q_i} + m_j \dot{x}_j \frac{d}{dt} \frac{\partial x_j}{\partial q_i}). \tag{54}$$

Since $\partial x_j / \partial q_i$ is a function of the $q's$ which are functions of time, we have by the chain rule

$$\frac{d}{dt}(\frac{\partial x_j}{\partial q_i}) = \sum_{k=1}^{N} \frac{\partial^2 x_j}{\partial q_i \partial q_k} \dot{q}_k \tag{55}$$

From Newton's law, $m_j \ddot{x}_j = F_j$, so that the first term in (54) is given by

$$\sum_{j=1}^{N} m_j \ddot{x}_j \frac{\partial x_j}{\partial q_i} = Q_i \tag{56}$$

Now let us consider the second term of (54). Consider the expression for $\partial T/\partial q_i$ and use (43, 44).

$$\frac{\partial T}{\partial q_i} = m_j \dot{x}_j \frac{\partial \dot{x}_j}{\partial q_i} = m_j \dot{x}_j \frac{\partial}{\partial q_i} \sum_{k=1}^{N} \frac{\partial x_i}{\partial q_k} \dot{q}_k \tag{57}$$

This is precisely the second term in equation (54). We have now identified simpler forms for the two expressions in the equation for the time derivative of the ***generalized momentum***. We may now write

$$\frac{dp_i}{dt} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = Q_i + \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i} + \frac{\partial T}{\partial q_i} \tag{58}$$

Since for a conservative system, the potential is independent of the velocities, we can place this equation into final form by defining the Lagrangian as $L = T - V$ to obtain the final form of Lagrange's equation as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{59}$$

in agreement with (19). Equation (59) is Lagrange's Equations in ***generalized coordinates***. In our previous example, we applied this equation to simple and double pendulums in polar coordinates using $q_1 = \theta_1$ and $q_2 = \theta_2$. What is significant about this equation, adding to its power, is that each i equation contains only derivatives with respect to that q_i and $\dot{q}_i$.

ADDITIONAL READING

Slater and Frank, *Mechanics*, Chapter IV

Marion and Thorton, Chapter 7

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