CHAPTER 10

ORDINARY DIFFERENTIAL EQUATIONS

10.1 Introduction

Relationship between rate of change of variables rather than variables themselves gives rise to differential equations. Mathematical formulation of most of the physical and engineering problems leads to differential equations. It is very important for engineers and scientists to know inception and solving of differential equations. These are of two types:

1) Ordinary Differential Equations (ODE)
2) Partial Differential Equations (PDE)

An ordinary differential equation (ODE) involves the derivatives of a dependent variable w.r.t. a single independent variable whereas a partial differential equation (PDE) contains the derivatives of a dependent variable w.r.t. two or more independent variables. In this chapter we will confine our studies to ordinary differential equations.

Prelims:

- $\boldsymbol{e^{\pm i\theta} = \cos\theta \pm i \sin\theta}$
- $\boldsymbol{\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})}$
- $\boldsymbol{\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})}$
- $\boldsymbol{\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})}$
- $\boldsymbol{\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})}$
- If u and v are functions of x and u vanishes after a finite number of differentiations

  $$\int u.v\, dx = uv_1 - u^{(1)}v_2 + u^{(2)}v_3 - u^{(3)}v_4 + \cdots$$

Here $\boldsymbol{u^{(n)}}$is derivative of $\boldsymbol{u^{(n-1)}}$ and $\boldsymbol{v_n}$ is integral of $\boldsymbol{v_{n-1}}$

For example

$$\int x^2.\sin x\ dx = (x^2)(-\cos x) - (2x)(-\sin x) + (2)(\cos x)$$

$$= -x^2\cos x + 2x\sin x + 2\cos x$$

Order and Degree of Ordinary Differential Equations (ODE)

A general ODE of n^{th} order can be represented in the form $F\left(x, y, \frac{\mathrm{dy}}{\mathrm{dx}}, \frac{\mathrm{d^2y}}{\mathrm{dx^2}}, \ldots, \frac{\mathrm{d^ny}}{\mathrm{dx^n}}\right)$=0 Order of an ordinary differential equation is that of the highest derivative occurring in it and the degree is the power of highest derivative after it has been freed from all radical signs.

The differential equation $\left(\frac{\mathrm{d^2y}}{\mathrm{dx^2}} + 2\mathrm{y}\right)^3 + \frac{\mathrm{d^3y}}{\mathrm{dx^3}} + y = 0$ is having order 3 and degree 1.

Whereas $\left(\frac{\mathrm{d^3y}}{\mathrm{dx^3}} + 2\mathrm{y}\right)^3 + \frac{\mathrm{d^2y}}{\mathrm{dx^2}} + y = 0$ is having order 3 and degree 3.

The differential equation $\sqrt{\frac{d^2y}{dx^2}} = \frac{d^3y}{dx^3} + y$ is of order 3 and degree 2.

10.2 First Order Linear Differential Equations (Leibnitz's Linear Equations)

A first order linear differential equation is of the form $\frac{dy}{dx} + Py = Q$, …….Ⓐ

where P and Q are functions of x alone or constants. To solve Ⓐ, multiplying throughout by $e^{\int P\,dx}$ (here $e^{\int P\,dx}$ is known as Integrating Factor (IF)), we get

$$\frac{dy}{dx} e^{\int P\,dx} + Py\, e^{\int P\,dx} = Q\, e^{\int P\,dx}$$

$$\Rightarrow \mathrm{d}\left(y\, e^{\int P\,dx}\right) = Q\, e^{\int P\,dx}$$

$$\Rightarrow y\, e^{\int P\,dx} = \int Q\, e^{\int P\,dx}\, dx + \mathrm{C}$$

Algorithm to solve a first order linear differential equation (Leibnitz's Equation)

1. Write the given equation in standard form i.e. $\frac{dy}{dx} + Py = Q$
2. Find the integrating factor (IF) = $e^{\int P\,dx}$
3. Solution is given by y. IF = $\int Q$ .IF dx + C , C is an arbitrary constant

Note: If the given equation is of the type $\frac{dx}{dy} + Px = Q$,

then IF = $e^{\int P\,dy}$ and the solution is given by x. IF = $\int Q$ .IF dy + C

Example 1 Solve the differential equation: $\frac{dy}{dx} = \frac{x + y \sin x}{1 + cosx}$

Solution: The given equation may be written as:

$\frac{dy}{dx} - \frac{\sin x}{1+cosx} y = \frac{x}{1+cosx}$ …… ①

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Where $P = -\frac{\sin x}{1+cosx}$ and $Q = \frac{x}{1+cosx}$

IF = $e^{\int P\,dx} = e^{\int -\frac{\sin x}{1+cosx}dx} = e^{log|1+\cos x|} = 1 + \cos x$

∴ Solution of ① is given by

$y.(1+\cos x) = \int \frac{x}{1+cosx}(1+\cos x)\,dx$ + C

⇒ $y\,(1+\cos x) = \frac{x^2}{2}$ + C

Example 2 Solve the differential equation: $\frac{dy}{dx} = (1+x) + (1-y)$

Solution: The given equation may be written as:

$\frac{dy}{dx} + y = 2 + x$ ……. ①

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Where $P = 1$ and $Q = 2 + x$

IF = $e^{\int P\,dx} = e^{\int dx} = e^x$

∴ Solution of ① is given by

$y.e^x = \int (2+x)e^x\,dx$ + C

⇒ $y = 1 + x$ + Ce^{-x}

Example 3 Solve the differential equation: $(x+y+1)\frac{dy}{dx} = 1$

Solution: The given equation may be written as:

$\frac{dx}{dy} = x + y + 1$ ⇒ $\frac{dx}{dy} - x = y + 1$ ......①

This is a linear differential equation of the form $\frac{dx}{dy} + Px = Q$

Where $P = -1$ and $Q = y + 1$

IF = $e^{\int P\,dy} = e^{\int -dy} = e^{-y}$

∴ Solution of ① is given by

$x.e^{-y} = \int (y+1)e^{-y}\,dy$ + C

⇒ $x\,e^{-y} = -(y+2)e^{-y}$ + C

⇒ $x = -(y+2)$ + C e^y

Example 4 Solve the differential equation: $x\log x\,\frac{dy}{dx} + y = 2\log x$

Solution: The given equation may be written as:

$\frac{dy}{dx} + \frac{1}{x\log x}y = \frac{2}{x}$ ……..①

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Where $P = \frac{1}{x\log x}$ and $Q = \frac{2}{x}$

IF = $e^{\int P\,dx} = e^{\int \frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x$

∴ Solution of ① is given by

$y.\log x = \int \frac{2}{x}\log x\;dx + C$

$\Rightarrow y\log x = (\log x)^2 + C$, C is an arbitrary constant

Example 5 Solve the differential equation: $\frac{dy}{dx} = \frac{e^{2\sqrt{x}+y}}{\sqrt{x}}$

Solution: The given equation may be written as:

$\frac{dy}{dx} - \frac{1}{\sqrt{x}}y = \frac{e^{2\sqrt{x}}}{\sqrt{x}}$ ……①

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Where $P = -\frac{1}{\sqrt{x}}$ and $Q = \frac{e^{2\sqrt{x}}}{\sqrt{x}}$

IF = $e^{\int P\,dx} = e^{\int \frac{-1}{\sqrt{x}}dx} = e^{-2\sqrt{x}}$

∴ Solution of ① is given by

$y.e^{-2\sqrt{x}} = \int \frac{e^{2\sqrt{x}}}{\sqrt{x}}e^{-2\sqrt{x}}\,dx + C$

$\Rightarrow y.e^{-2\sqrt{x}} = \int \frac{1}{\sqrt{x}}\,dx + C$

$\Rightarrow y.e^{-2\sqrt{x}} = 2\sqrt{x} + C$

$\Rightarrow y = 2\sqrt{x}\;e^{2\sqrt{x}} + C\,e^{2\sqrt{x}}$

10.3 Equations Reducible to Leibnitz's Equations (Bernoulli's Equations)

Differential equation of the form $\frac{dy}{dx} + Pf(y) = Qg(y)$, …….Ⓑ

where P and Q are functions of x alone or constant, is called Bernoulli's equation. Dividing both sides of Ⓑ by $g(y)$, we get $\frac{1}{g(y)}\frac{dy}{dx} + P\frac{f(y)}{g(y)} = Q$. Now putting $\frac{f(y)}{g(y)} = t$, Ⓑ reduces to Leibnitz's equation.

Example 6 Solve the differential equation: $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$ ……①

Solution: The given equation may be written as:

$e^{-y}\frac{dy}{dx} + \frac{1}{x}e^{-y} = \frac{1}{x^2}$ ……②

Putting $e^{-y} = t$, $-e^{-y}\frac{dy}{dx} = \frac{dt}{dx}$ ……③

Using ③ in ②, we get $\frac{dt}{dx} - \frac{1}{x}t = -\frac{1}{x^2}$ ……..④

④ is a linear differential equation of the form $\frac{dt}{dx} + Pt = Q$

Where $P = -\frac{1}{x}$ and $Q = -\frac{1}{x^2}$

IF = $e^{\int P\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

∴ Solution of ④ is given by

$t.\frac{1}{x} = \int -\frac{1}{x^2}.\frac{1}{x}\,dx + C$

$\Rightarrow t.\frac{1}{x} = \frac{1}{2x^2} + C$

Substituting $t = e^{-y}$

$\Rightarrow \frac{e^{-y}}{x} = \frac{1}{2x^2} + C$

$\Rightarrow 2x = e^y(2cx^2 + 1)$

Example 7 Solve the differential equation:

$\tan y\frac{dy}{dx} + \tan x = \cos y\, cos^3 x$ ……①

Solution: The given equation may be written as:

$\frac{\tan y}{\cos y}\frac{dy}{dx} + \frac{\tan x}{\cos y} = cos^3 x$

$\Rightarrow \sec y\ \tan y\frac{dy}{dx} + \sec y\ \tan x = cos^3 x$ ……②

Putting $\sec y = t$, $\sec y\ \tan y\frac{dy}{dx} = \frac{dt}{dx}$ ……③

Using ③ in ②, we get $\frac{dt}{dx} + (\tan x)\, t = cos^3 x$ ……..④

④ is a linear differential equation of the form $\frac{dt}{dx} + Pt = Q$

Where $P = \tan x$ and $Q = cos^3 x$

IF = $e^{\int P\,dx} = e^{\int \tan x\,dx} = e^{\log|\sec x|} = \sec x$

∴ Solution of ④ is given by

$t.\sec x = \int cos^3 x.\sec x \;dx + \text{C}$

$\Rightarrow \text{t}.\sec x = \int cos^2 x \;dx + \text{C}$

$\Rightarrow \text{t}.\sec x = \int \frac{1+\cos 2x}{2} \;dx + \text{C}$

$\Rightarrow \text{t}.\sec x = \frac{x}{2} + \frac{\sin 2x}{4} + \text{C}$

Substituting $\text{t} = \sec y$,

$\Rightarrow \sec x \sec y = \frac{x}{2} + \frac{\sin 2x}{4} + \text{C}$

Example 8 Solve the differential equation: $\frac{dy}{dx} = \frac{y}{x+\sqrt{xy}}$ ……①

Solution: The given equation may be written as:

$\frac{dx}{dy} = \frac{x+\sqrt{xy}}{y}$

$\Rightarrow \frac{dx}{dy} - \frac{1}{y}x = \sqrt{\frac{x}{y}}$

Dividing throughout by $\sqrt{x}$

$\Rightarrow \frac{1}{\sqrt{x}}\frac{dx}{dy} - \frac{1}{y}\sqrt{x} = \frac{1}{\sqrt{y}}$ ……②

Putting $\sqrt{x} = t$, $\frac{1}{2\sqrt{x}}\frac{dx}{dy} = \frac{dt}{dy}$ ……③

Using ③ in ②, we get $\frac{dt}{dy} - \frac{1}{2y}t = \frac{1}{2\sqrt{y}}$ ……..④

④ is a linear differential equation of the form $\frac{dt}{dy} + Pt = Q$

Where $P = -\frac{1}{2y}$ and $Q = \frac{1}{2\sqrt{y}}$

IF = $e^{\int P\,dx} = e^{\int \frac{-1}{2y}dy} = e^{\frac{-1}{2}\log y} = e^{\log \frac{1}{\sqrt{y}}} = \frac{1}{\sqrt{y}}$

∴ Solution of ④ is given by

$t.\frac{1}{\sqrt{y}} = \int \frac{1}{2\sqrt{y}}.\frac{1}{\sqrt{y}} \;dx + \text{C}$

$\Rightarrow t.\frac{1}{\sqrt{y}} = \int \frac{1}{2y} \;dx + \text{C}$

$\Rightarrow t.\frac{1}{\sqrt{y}} = \frac{1}{2}\,log\, y + \text{C}$

Substituting $t = \sqrt{x}$

$\sqrt{\frac{x}{y}} = log\sqrt{y} + \text{C}$

Example 9 Solve the differential equation: $x\frac{dy}{dx} + y = y^2 \log x$ .....①

Solution: The given equation may be written as:

$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x}\log x$

Dividing throughout by y^2

$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \frac{1}{x}\log x$......②

Putting $\frac{1}{y} = t$, $-\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$ ......③

Using ③ in ②, we get $\frac{dt}{dx} - \frac{1}{x}t = -\frac{1}{x}\log x$ ....④

④ is a linear differential equation of the form $\frac{dt}{dx} + Pt = Q$

Where $P = -\frac{1}{x}$ and $Q = -\frac{1}{x}\log x$

IF = $e^{\int P\,dx} = e^{\int \frac{-1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

∴ Solution of ④ is given by

$t.\frac{1}{x} = \int -\frac{1}{x}\log x\,.\frac{1}{x}\ dx + \text{C}$

$\Rightarrow t.\frac{1}{x} = \int -\frac{1}{x^2}\log x\ dx + \text{C}$

Putting $\log x = u, \frac{1}{x}dx = du$, also $x = e^u$

$\Rightarrow t.\frac{1}{x} = -\int ue^{-u}\ du + \text{C}$

$\Rightarrow t.\frac{1}{x} = -[u(-e^{-u}) - 1(e^{-u})] + \text{C}$

$\Rightarrow t.\frac{1}{x} = e^{-u}(u+1) + \text{C}$

$\Rightarrow t.\frac{1}{x} = \frac{1}{x}(log\,x + 1) + C$

Substituting $t = \frac{1}{y}$

$\Rightarrow \frac{1}{xy} = \frac{1}{x}(\log x + 1) + \text{C}$

$\Rightarrow \frac{1}{y} = (\log x + 1) + \text{C}\,x$, C is an arbitrary constant

Exercise 10A

Solve the following differential equations:

1. $e^{-y}sec^2y\, dy = dx + xdy$

   Ans. $\langle xe^y = C + \tan y \rangle$

2. $(x+1)\frac{dy}{dx} - 2y = (x+1)^4$

   Ans. $\langle y = \left(\frac{x^2}{2} + x + c\right)(x+1)^2 \rangle$

3. $\frac{dy}{dx} = \frac{e^{-2\sqrt{x}} - y}{\sqrt{x}}$ Ans. $y.\, e^{2\sqrt{x}} = 2\sqrt{x} + \mathrm{C}$
4. $\frac{dx}{dy} = \left(\frac{\sqrt{1+y^2}\sin y - xy}{1+y^2}\right)$ Ans. $x\sqrt{1+\ y^2} + \cos y = C$
5. $(x+2y^3)\frac{dy}{dx} = y$ Ans. $x = \ y^3 + Cy$
6. $\frac{dy}{dx}\cos x + ysin\ x = \sqrt{ysec\ x}$ Ans. $2\sqrt{ysec\ x} = \tan x + 2C$
7. $\frac{dy}{dx} - xy + \ y^3e^{-x^2} = 0$ Ans. $e^{x^2} = \ y^2(2x - C)$
8. $3x(1-x^2)y^2\frac{dy}{dx} + (2x^2-1)y^3 = x^3$

   Ans. $y^3 = x + Cx\sqrt{1-x^2}$

9. $\frac{dy}{dx} + \ ycos\ x = \ y^n sin\ 2x$ Ans. $(y^{-n+1} - 1)e^{\sin x} = \ C$
10. $\frac{dy}{dx} = \ e^{x-y}(e^x - e^y)$ Ans. $e^y = \ Ce^{-e^x} + \ e^x - 1$

10.4 Exact Differential Equations of First Order

A differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is said to be exact if it can be directly obtained from its primitive by differentiation.

Theorem: The necessary and sufficient condition for the equation $M(x,y)dx + N(x,y)dy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Working rule to solve an exact differential equation

1. For the equation $M(x,y)dx + N(x,y)dy = 0$, check the condition for exactness i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
2. Solution of the given differential equation is given by

$$\int M(taking\ y\ as\ constant)\ dx\ +\ \int N(terms\ not\ containg\ x)dy = C$$

Example 10 Solve the differential equation:

$(e^y + 1)\cos x\ dx + \ \ e^y \sin x\ dy = 0$ .....①

Solution: $M = (e^y + 1)\cos x$, $N = e^y \sin x$

$\frac{\partial M}{\partial y} = \ e^y \cos x$, $\frac{\partial N}{\partial x} = \ e^y \cos x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ given differential equation is exact.

Solution of ① is given by:

$$\int_{y\ constant} (e^y + 1)\cos x \ \ dx \ + \int 0\, dy = C$$

$\Rightarrow (e^y + 1)\sin x \ = C$

Example 11 Solve the differential equation:

$(\sec x \tan x \ \tan y - \ e^x)dx + (\sec x\, sec^2\, y)dy = 0$ .....①

Solution: $M = \sec x \tan x \ \tan y - \ e^x$, $N = \sec x\, sec^2\, y$

$\frac{\partial M}{\partial y} = \ \sec x \tan x\, sec^2\, y$, $\frac{\partial N}{\partial x} = \ \sec x \tan x\, sec^2\, y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ given differential equation is exact.

Solution of ① is given by:

$$\int_{y\ constant} (\sec x \tan x \ \tan y - \ e^x)\, dx \ + \int 0\, dy = C$$

$\Rightarrow \ \sec x \tan y - \ e^x = C$

Example 12 Solve the differential equation:

$\left[y\left(1 + \frac{1}{x}\right) + \cos y\right]dx + [x + \log x - x \sin y]dy = 0$ .....①

Solution: $M = y\left(1 + \frac{1}{x}\right) + \cos y$, $N = x + \log x - x\sin y$

$\frac{\partial M}{\partial y} = \ \left(1 + \frac{1}{x}\right) - \sin y$, $\frac{\partial N}{\partial x} = \left(1 + \frac{1}{x}\right) - \sin y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ given differential equation is exact.

Solution of ① is given by:

$$\int_{y\ constant} \left[y\left(1 + \frac{1}{x}\right) + \cos y\right] dx \ + \int 0\, dy = C$$

$\Rightarrow y\,(x + \log x) + x\cos y = C$

Example 13 Solve the differential equation:

$x\, dx + \ y\, dy = \frac{a^2(x\, dy - \ y\, dx)}{x^2+y^2}$ .....①

Solution: $\Rightarrow \left(x + \ \frac{a^2\, y}{x^2+y^2}\right)dx + \ \left(y - \ \frac{a^2\, x}{x^2+y^2}\right)dy = 0$

$M = x + \frac{a^2\, y}{x^2+y^2}$, $N = \ y - \ \frac{a^2\, x}{x^2+y^2}$

$\frac{\partial M}{\partial y} = \frac{a^2\,(x^2-y^2)}{(x^2+y^2)^2}$, $\frac{\partial N}{\partial x} = \frac{a^2\,(x^2-y^2)}{(x^2+y^2)^2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ given differential equation is exact.

Solution of ① is given by:

$$\int \left(x + \frac{a^2\,y}{x^2+y^2}\right) dx \;+\; \int y\,dy = C$$

y constant

$$\Rightarrow\ \frac{x^2}{2} + a^2 tan^{-1}\frac{x}{y} + \frac{y^2}{2} = \mathrm{C}$$

$$\Rightarrow\ x^2 + 2a^2 tan^{-1}\frac{x}{y} + y^2 = \mathrm{D}\ \ , D = 2C$$

10.5 Equations Reducible to Exact Differential Equations

Sometimes a differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is not exact i.e. $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. It can be made exact by multiplying the equation by some function of x and y known as integrating factor (IF).

10.5.1 Integrating Factor (IF) Found By Inspection

Some non-exact differential equations can be grouped or rearranged and solved directly by integration, after multiplying by an integrating factor (IF) which can be found just by inspection as shown below:

Term	IF	Result
$xdy + ydx$	1. $\frac{1}{\mathrm{xy}}$	$\frac{xdy + ydx}{xy} = \frac{1}{\mathrm{y}}\mathrm{dy} + \frac{1}{x}\,\mathrm{d}x = d[\log(xy)]$
	2. $\frac{1}{(\mathrm{xy})^{\mathrm{n}}}$, $n \neq 1$	$\frac{xdy + ydx}{(\mathrm{xy})^{\mathrm{n}}} = \frac{\mathrm{d}(xy)}{(\mathrm{xy})^{\mathrm{n}}} = -d\left[\frac{1}{(n-1)(\mathrm{xy})^{\mathrm{n}-1}}\right]$
$xdy - ydx$	1. $\frac{1}{\mathrm{x}^2}$	$\frac{xdy - ydx}{x^2} = d\left[\frac{y}{x}\right]$
	2. $\frac{1}{\mathrm{y}^2}$	$\frac{xdy - ydx}{y^2} = -d\left[\frac{x}{y}\right]$
	3. $\frac{1}{\mathrm{xy}}$	$\frac{xdy - ydx}{\mathrm{xy}} = d\left[log\,\frac{y}{x}\right]$
	4. $\frac{1}{\mathrm{x}^2+\mathrm{y}^2}$	

	5. $\frac{1}{x\sqrt{x^2-y^2}}$	$\frac{xdy - ydx}{x^2 + y^2} = d\left[tan^{-1}\frac{y}{x}\right]$ $\frac{xdy - ydx}{x\sqrt{x^2 - y^2}} = d\left[sin^{-1}\frac{y}{x}\right]$
$xdx + ydy$	1. $\frac{1}{x^2+y^2}$ 2. $\frac{1}{(x^2+y^2)^n}, n \neq 1$	$\frac{xdx + ydy}{x^2 + y^2} = \frac{1}{2}d[log(x^2 + y^2)]$ $\frac{xdx + ydy}{(x^2 + y^2)^n} = \frac{1}{2}d\left[\frac{(x^2 + y^2)^{-n+1}}{-n+1}\right]$

Example 14 Solve the differential equation:

$x\,dy - y\,dx + 2x^3dx = 0$ .....①

Solution: $\Rightarrow (-y + 2x^3)dx + xdy = 0$

$M = -y + 2x^3\ ,\ N = x$

$\frac{\partial M}{\partial y} = -1\ ,\ \frac{\partial N}{\partial x} = 1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Taking $\frac{1}{x^2}$ as integrating factor due to presence of the term $(x\,dy - y\,dx)$

① may be rewritten as : $\frac{xdy - ydx}{x^2} + 2x\,dx = 0$

$\Rightarrow d\left[\frac{y}{x}\right] + 2x\,dx = 0$ ......②

Integrating ②, solution is given by : $\frac{y}{x} + x^2 = C$

$\Rightarrow y + x^3 = Cx$

Example 15 Solve the differential equation:

$y\,dx - x\,dy + (1 + x^2)dx + x^2\cos y\,dy = 0$ .....①

Solution: $\Rightarrow (y + 1 + x^2)dx + (x^2\cos y - x)dy = 0$

$M = y + 1 + x^2,\ N = x^2\cos y - x$

$\frac{\partial M}{\partial y} = 1\ ,\ \frac{\partial N}{\partial x} = 2x\cos y - 1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Taking $\frac{1}{x^2}$ as integrating factor due to presence of the term $(y\,dx - x\,dy)$

① may be rewritten as : $\frac{ydy - xdx}{x^2} + \left(\frac{1}{x^2} + 1\right)dx + \cos y\,dy = 0$

$\Rightarrow -d\left[\frac{y}{x}\right] + \left(\frac{1}{x^2} + 1\right)dx + \cos y\,dy = 0$ ......②

Integrating ②, solution is given by : $-\frac{y}{x} + \left(-\frac{1}{x} + x\right) + \sin y = C$

$\Rightarrow x^2 - y - 1 + x\sin y = Cx$

Example 16 Solve the differential equation:

$x\,dx + y\,dy = a(x^2 + y^2)dy$ .....①

Solution: $\Rightarrow xdx + (y - a(x^2 + y^2))dy = 0$

$M = x$, $N = y - a(x^2 + y^2)$

$\frac{\partial M}{\partial y} = 0$, $\frac{\partial N}{\partial x} = -2ax$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Taking $\frac{1}{x^2 + y^2}$ as integrating factor due to presence of the term $(x\,dx + y\,dy)$

① may be rewritten as : $\frac{xdx + ydy}{x^2 + y^2} - a\,dy = 0$

$\Rightarrow \frac{1}{2}d[log(x^2 + y^2)] - a\,dy = 0$

$\Rightarrow d[log(x^2 + y^2)] - 2\,a\,dy = 0$ ......②

Integrating ②, solution is given by: $(x^2 + y^2) - 2ay = C$, C is an arbitrary constant

Example 17 Solve the differential equation:

$a(x\,dy + 2y\,dx) = xy\,dy$ .....①

Solution: $\Rightarrow 2aydx + (ax - xy)dy = 0$

$M = 2ay$, $N = ax - xy$

$\frac{\partial M}{\partial y} = 2a$, $\frac{\partial N}{\partial x} = a - y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Rewriting ① as $a(x\,dy + y\,dx) + ay\,dx = xy\,dy$ ......②

Taking $\frac{1}{\mathrm{xy}}$ as integrating factor due to presence of the term $(x\,dy + y\,dx)$

② may be rewritten as : $a\frac{xdy+ydx}{xy} + \frac{a}{\mathrm{x}}\,dx - dy = 0$

$\Rightarrow\ ad[\log(xy)] + \frac{a}{\mathrm{x}}\,dx - dy = 0$ ……③

Integrating ③ solution is given by: $a\log(xy) + a\log x - y = C$

$\Rightarrow a\log(x^2y) - y = C$, C is an arbitrary constant

Example 18 Solve the differential equation:

$$x^4\frac{dy}{dx} + x^3y + \sec(xy) = 0 \quad .....①$$

Solution: $\Rightarrow (x^3y + \sec(xy))dx + x^4dy = 0$

$M = x^3y + \sec(xy)\,,\ N = x^4$

$\frac{\partial M}{\partial y} = x^3 + x\sec(xy)\tan(xy)\,,\ \frac{\partial N}{\partial x} = 4x^3$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Rewriting ① as: $x^3(x\,dy + y\,dx) + \sec(xy)\,dx = 0$

$\Rightarrow \frac{(x\,dy + y\,dx)}{\sec(xy)} - x^{-3}dx = 0$

$\Rightarrow \cos(xy)\,(x\,dy + y\,dx) - x^{-3}dx = 0$

$\Rightarrow d\,[\sin(xy)] - \frac{1}{2}d(x^{-2})dx = 0$ ……②

Integrating ②, we get the required solution as:

$\sin(xy) - \frac{x^{-2}}{2} = C$

$\Rightarrow 2\,x^2\sin(xy) - 1 = Cx^2$

10.5.2 Integrating Factor (IF) of a Non-Exact Homogeneous Equation

If the equation $Mdx + Ndy = 0$ is a homogeneous equation, then the integrating factor (IF) will be $\frac{1}{Mx+Ny}$, provided $Mx + Ny \neq 0$

Example 19 Solve the differential equation:

$(x^3 + y^3)dx - xy^2\,dy = 0$ .....①

Solution: $M = x^3 + y^3$, $N = -\,xy^2$

$\frac{\partial M}{\partial y} = 3y^2$, $\frac{\partial N}{\partial x} = -y^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is a homogeneous equation , ∴ IF = $\frac{1}{Mx+Ny} = \frac{1}{x^4+xy^3-xy^3} = \frac{1}{x^4}$

∴ ① may be rewritten as : $\left(\frac{1}{x} + \frac{y^3}{x^4}\right) dx - \frac{y^2}{x^3} dy = 0$ ……②

New $M = \frac{1}{x} + \frac{y^3}{x^4}$, New $N = -\frac{y^2}{x^3}$

$\frac{\partial M}{\partial y} = \frac{3y^2}{x^4}$, $\frac{\partial N}{\partial x} = \frac{3y^2}{x^4}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ ② is an exact differential equation.

Solution of ② is given by:

$$\underset{y\ constant}{\int} \left(\frac{1}{x} + \frac{y^3}{x^4}\right) dx + \int 0\, dy$$

$\Rightarrow \log x - \frac{1}{3}\left(\frac{y}{x}\right)^3 = C$

Example 20 Solve the differential equation:

$(3y^4 + 3x^2y^2)dx + (x^3y - 3xy^3)\, dy = 0$ …..①

Solution: $M = 3y^4 + 3x^2y^2$, $N = x^3y - 3xy^3$

$\frac{\partial M}{\partial y} = 12y^3 + 6x^2y$, $\frac{\partial N}{\partial x} = 3x^2y - 3y^3$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is a homogeneous equation

∴ IF = $\frac{1}{Mx+Ny} = \frac{1}{3xy^4+3x^3y^2+x^3y^2-3xy^4} = \frac{1}{4x^3y^2}$

∴ ① may be rewritten after multiplying by IF as:

$\left(\frac{3y^2}{4x^3} + \frac{3}{4x}\right) dx + \left(\frac{1}{4y} - \frac{3y}{4x^2}\right) dy = 0$ ……②

New $M = \frac{3y^2}{4x^3} + \frac{3}{4x}$, New $N = \frac{1}{4y} - \frac{3y}{4x^2}$

$\frac{\partial M}{\partial y} = \frac{6y}{4x^3} = \frac{3y}{2x^3}$, $\frac{\partial N}{\partial x} = \frac{3y}{2x^3}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ ② is an exact differential equation.

Solution of ② is given by:

$$\underset{y\ constant}{\int} \left(\frac{3y^2}{4x^3} + \frac{3}{4x}\right) dx + \int \frac{1}{4y}\, dy$$

$\Rightarrow \frac{-3y^2}{8x^2} + \frac{3}{4}\log x + \frac{1}{4}\log y = \mathrm{C}$

$\Rightarrow \log x^3 y - \frac{3y^2}{2x^2} = \mathrm{D}\ ,\ \mathrm{D} = 4\mathrm{C}$

10.5.3 Integrating Factor of a Non-Exact Differential Equation of the Form

$\boldsymbol{y f_1(xy) dx + x f_2(xy)\, dy = 0}$ **:** If the equation $Mdx + Ndy = 0$ is of the given form**,** then the integrating factor (IF) will be $\frac{1}{Mx-Ny}$ provided $Mx - Ny \neq 0$

Example 21 Solve the differential equation:

$y(1+xy)dx + x(1-xy)\,dy = 0$ .....①

Solution: $M = y + xy^2,\ N = x - x^2y$

$\frac{\partial M}{\partial y} = 1 + 2xy\ ,\ \frac{\partial N}{\partial x} = 1 - 2xy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

∴ $\mathrm{IF} = \frac{1}{Mx-Ny} = \frac{1}{xy+x^2y^2-xy+x^2y^2} = \frac{1}{2x^2y^2}$

∴ ① may be rewritten after multiplying by IF as:

$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0$ ......②

New $M = \frac{1}{2x^2y} + \frac{1}{2x}$, New $N = \frac{1}{2xy^2} - \frac{1}{2y}$

$\frac{\partial M}{\partial y} = \frac{-1}{2x^2y^2}\ ,\ \frac{\partial N}{\partial x} = \frac{-1}{2x^2y^2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ ② is an exact differential equation.

Solution of ② is given by:

$$\int \left(\frac{1}{2x^2y} + \frac{1}{2x}\right) dx + \int -\frac{1}{2y} dy$$

y constant

$\Rightarrow \frac{-1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = \mathrm{C}$

$\Rightarrow \log\frac{\mathrm{x}}{\mathrm{y}} - \frac{1}{xy} = \mathrm{D},\ \mathrm{D} = 2\mathrm{C}$

Example 22 Solve the differential equation:

$y(xy + 2x^2y^2)dx + x(xy - x^2y^2)\,dy = 0$ .....①

Solution: $M = xy^2 + 2x^2y^3,\ N = x^2y - x^3y^2$

$\frac{\partial M}{\partial y} = 2xy + 6x^2y^2$, $\frac{\partial N}{\partial x} = 2xy - 3x^2y^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

∴ IF = $\frac{1}{Mx-Ny} = \frac{1}{x^2y^2+2x^3y^3-x^2y^2+x^3y^3} = \frac{1}{3x^3y^3}$

∴ ① may be rewritten after multiplying by IF as:

$\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$ ……②

New $M = \frac{1}{x^2y} + \frac{2}{x}$, New $N = \frac{1}{xy^2} - \frac{1}{y}$

$\frac{\partial M}{\partial y} = \frac{-1}{x^2y^2}$, $\frac{\partial N}{\partial x} = \frac{-1}{x^2y^2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,∴ ② is an exact differential equation.

Solution of ② is given by:

$\int \left(\frac{1}{x^2y} + \frac{2}{x}\right) dx + \int -\frac{1}{y}dy$

y constant

$\Rightarrow \frac{-1}{xy} + 2\log x - \log y = C$

$\Rightarrow \log\frac{x^2}{y} - \frac{1}{xy} = C$

10.5.4 Integrating Factor (IF) of a Non-Exact Differential Equation $\boldsymbol{Mdx + Ndx = 0}$ in which $\boldsymbol{\frac{\partial M}{\partial y}}$ and $\boldsymbol{\frac{\partial N}{\partial x}}$ are connected in a specific way as shown:

i. If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, a function of x alone, then IF = $e^{\int f(x)dx}$

ii. If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$, a function of y alone, then IF = $e^{\int -g(y)dy}$

Example 23 Solve the differential equation:

$(x^3 + y^2 + x)dx + xy\,dy = 0$.….①

Solution: $M = x^3 + y^2 + x$, $N = xy$

$\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is neither homogeneous nor of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

∴ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = y$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{y}{xy} = \frac{1}{x} = f(x)$ say

∴ IF = $e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

∴ ① may be rewritten after multiplying by IF as:

$(x^4 + xy^2 + x^2)dx + x^2y\, dy = 0$ ....②

New $M = x^4 + xy^2 + x^2$, New $N = x^2y$

$\frac{\partial M}{\partial y} = 2xy$, $\frac{\partial N}{\partial x} = 2xy$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,∴ ② is an exact differential equation.

Solution of ② is given by:

$\int (x^4 + xy^2 + x^2)\, dx + \int 0\, dy$
y constant

$\Rightarrow \frac{x^5}{5} + \frac{x^2y^2}{2} + \frac{x^3}{3} = C$

Example 24 Solve the differential equation:

$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)\, dy = 0$ .....①

Solution: $M = y^4 + 2y$, $N = xy^3 + 2y^4 - 4x$

$\frac{\partial M}{\partial y} = 4y^3 + 2$, $\frac{\partial N}{\partial x} = y^3 - 4$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is neither homogeneous nor of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

∴ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3y^3 + 6$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{3y^3+6}{y^4+2y} = \frac{3}{y} = g(y)$ say

∴ IF = $e^{\int -g(y)dy} = e^{\int -\frac{3}{y}dy} = e^{-3\log y} = \frac{1}{y^3}$

∴ ① may be rewritten after multiplying by IF as:

$\left(y + \frac{2}{y^2}\right) dx + \left(x + 2y - \frac{4x}{y^3}\right) dy = 0$ ......②

New $M = \mathrm{y} + \frac{2}{y^2}$, New $N = x + 2y - \frac{4x}{y^3}$

$\frac{\partial M}{\partial y} = 1 - \frac{4}{y^3}$, $\frac{\partial N}{\partial x} = 1 - \frac{4}{y^3}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,∴ ② is an exact differential equation.

Solution of ② is given by:

$$\int_{y\ constant} \left(\mathrm{y} + \frac{2}{y^2}\right) dx + \int 2\mathrm{y}\, dy$$

$\Rightarrow \left(\mathrm{y} + \frac{2}{y^2}\right)x + y^2 = \mathrm{C}$

Example 25 Solve the differential equation:

$(x^2 - y^2 + 2x)dx - 2y\,dy = 0$ .....①

Solution: $M = x^2 - y^2 + 2x,\ N = -2y$

$\frac{\partial M}{\partial y} = -2y$, $\frac{\partial N}{\partial x} = 0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is neither homogeneous nor of the form

$yf_1(xy)dx + xf_2(xy)dy = 0$,

∴ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2y$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-2y}{-2y} = 1 = f(x)$ say

∴ IF = $e^{\int f(x)dx} = e^{\int 1dx} = e^x$

∴ ① may be rewritten after multiplying by IF as:

$e^x(x^2 - y^2 + 2x)dx - 2e^x y\,dy = 0$.......②

New $M = e^x(x^2 - y^2 + 2x)$, New $N = -2e^x y$

$\frac{\partial M}{\partial y} = -2e^x y$, $\frac{\partial N}{\partial x} = -2e^x y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ ② is an exact differential equation.

Solution of ② is given by:

$$\int_{y\ constant} e^x(x^2 - y^2 + 2x)\, dx + \int 0\, dy$$

$\Rightarrow (x^2 - y^2 + 2x)e^x - (2x + 2)e^x + (2)e^x = \mathrm{C}$

$\Rightarrow (x^2 - y^2)e^x = \mathrm{C}$, C is an arbitrary constant

Example 26 Solve the differential equation:

$2ydx + (2x\log x - xy)\,dy = 0$ .....①

Solution: $M = 2y,\ N = 2x\log x - xy$

$\frac{\partial M}{\partial y} = 2\,,\ \frac{\partial N}{\partial x} = 2(1 + \log x) - y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is neither homogeneous nor of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

∴ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2\log x + y$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-2\log x + y}{x(2\log x - y)} = -\frac{1}{x} = f(x)$ say

∴ IF = $e^{\int f(x)dx} = e^{\int \frac{-1}{x}dx} = e^{\log x^{-1}} = \frac{1}{x}$

∴ ① may be rewritten after multiplying by IF as:

$\frac{2y}{x}dx + (2\log x - y)\,dy = 0$ ....②

New $M = \frac{2y}{x}$, New $N = 2\log x - y$

$\frac{\partial M}{\partial y} = \frac{2}{x} = \frac{\partial N}{\partial x} = \frac{2}{x}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ ② is an exact differential equation.

Solution of ② is given by:

$\int \frac{2y}{x}\,dx + \int -y\,dy$

y constant

$\Rightarrow\ 2y\log x - \frac{y^2}{2} = C$

10.5.4 Integrating Factor (IF) of a Non-Exact Differential Equation

$\boldsymbol{x^a y^b(m_1 ydx + n_1 xdy) + x^c y^d(m_2 ydx + n_2 xdy) = 0}$, where $a, b, c, d,$ m_1, n_1, m_2, n_2 are constants, is given by $x^{\alpha}y^{\beta}$**,** where α and β are connected by the relation $\frac{a+\alpha+1}{m_1} = \frac{b+\beta+1}{n_1}$ and $\frac{c+\alpha+1}{m_2} = \frac{d+\beta+1}{n_2}$

Example 27 Solve the differential equation:

$(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$ .....①

Solution: $M = y^2 + 2x^2y,\ N = 2x^3 - xy$

$\frac{\partial M}{\partial y} = 2y + 2x^2$, $\frac{\partial N}{\partial x} = 6x^2 - y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Rewriting ① as $x^2y^0(2ydx + 2xdy) + x^0y^1(ydx - xdy) = 0$ ....②

Comparing with standard form $a = 2,\ b = 0,\ c = 0,\ d = 1,$

$$m_1 = 2,\ n_1 = 2,\ m_2 = 1,\ n_2 = -1$$

∴ $\frac{2+\alpha+1}{2} = \frac{0+\beta+1}{2}$ and $\frac{0+\alpha+1}{1} = \frac{1+\beta+1}{-1}$

$\Rightarrow\ \alpha - \beta = -2$ and $\alpha + \beta = -3$

Solving we get $\alpha = \frac{-5}{2}$ and $\beta = \frac{-1}{2}$

∴ $\text{IF} = x^{\alpha}y^{\beta} = x^{\frac{-5}{2}}y^{\frac{-1}{2}}$

∴ ① may be rewritten after multiplying by IF as:

$x^{\frac{-5}{2}}y^{\frac{-1}{2}}(y^2 + 2x^2y)dx + x^{\frac{-5}{2}}y^{\frac{-1}{2}}(2x^3 - xy)dy = 0$.......②

$\Rightarrow \left(x^{\frac{-5}{2}}y^{\frac{3}{2}} + 2x^{\frac{-1}{2}}y^{\frac{1}{2}}\right)dx + \left(2x^{\frac{1}{2}}y^{\frac{-1}{2}} - x^{\frac{-3}{2}}y^{\frac{1}{2}}\right)dy = 0$

New $M = x^{\frac{-5}{2}}y^{\frac{3}{2}} + 2x^{\frac{-1}{2}}y^{\frac{1}{2}}$, New $N = 2x^{\frac{1}{2}}y^{\frac{-1}{2}} - x^{\frac{-3}{2}}y^{\frac{1}{2}}$

$\frac{\partial M}{\partial y} = \frac{3}{2}x^{\frac{-5}{2}}y^{\frac{1}{2}} + x^{\frac{-1}{2}}y^{\frac{-1}{2}}$, $\frac{\partial N}{\partial x} = \frac{3}{2}x^{\frac{-5}{2}}y^{\frac{1}{2}} + x^{\frac{-1}{2}}y^{\frac{-1}{2}}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,∴ ② is an exact differential equation.

Solution of ② is given by:

$$\int_{y\ constant} \left(x^{\frac{-5}{2}}y^{\frac{3}{2}} + 2x^{\frac{-1}{2}}y^{\frac{1}{2}}\right) dx + \int 0\, dy$$

$\Rightarrow\ 4(xy)^{\frac{1}{2}} - \frac{2}{3}\left(\frac{y}{x}\right)^{\frac{3}{2}} = C$, C is an arbitrary constant

Exercise 10B

Solve the following differential equations:

1. $\frac{dy}{dx} + \frac{ax+hy+g}{hx+by+f} = 0$

   Ans. $\langle ax^2 + 2hxy + by^2 + 2gx + 2fy = 0\rangle$
2. $\left(y^2e^{xy^2} + 4x^3\right)dx + \left(2xye^{xy^2} - 3y^2\right)dy = 0$

   Ans. $\langle e^{xy^2} + x^4 - y^3 = C\rangle$

3. $ydx - xdy + 3x^2y^2e^{x^3}dx = 0$

Ans. $\langle x + ye^{x^3} = Cy\rangle$

4. $y(2xy + e^x)dx = e^x dy$

Ans.$\langle \frac{e^x}{y} + x^2 = C\rangle$

5. $(y \, log \, x)dx + (x - log \, y)dy = 0$

Ans. $\langle (x \, log \, x) - \frac{1}{2}(\log y)^2 = C\rangle$

6. $(3xy - 2ay^2)dx + (x^2 - 2axy)dy = 0$

Ans. $\langle x^2(x - ay)y = C\rangle$

7. $y(x^2 y^2 + xy + 1)dx + x(x^2 y^2 - xy + 1)dy = 0$

Ans. $\langle 2x^2y^2 + xylog\frac{x^2}{y} - 2 = Cxy\rangle$

8. $(x^3 y^2 + x)dy + (x^2 y^3 - y)dx = 0$

Ans. $\langle log\frac{y}{x} + \frac{1}{2}x^2y^2 = C\rangle$

9. $(3 y^2 + 2xy)dx - (2xy + x^2)dy = 0$

Ans.$\langle x^3 = Cy(x + y)\rangle$

10. $(2 x^2y - xy^2 + y)dx + (x - y)dy = 0$

Ans. $\langle e^{x^2}(2xy - y^2) = C\rangle$

10.6 Previous Years Solved Questions

Q1. Solve $y(2xy + e^x)dx - e^x dy = 0$

$\langle \boldsymbol{Q1(g), GGSIPU, December\ 2012}\rangle$

Solution: $M = y(2xy + e^x)$, $N = -e^x$

$\frac{\partial M}{\partial y} = 4xy + e^x$, $\frac{\partial N}{\partial x} = -e^x$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Rearranging the equation as $(ye^x dx - e^x dy) + 2xy^2 dx = 0$ ....①

Taking $\frac{1}{y^2}$ as integrating factor, ① may be rewritten as:

$$\frac{ye^x dx - e^x dy}{y^2} + 2x\, dx = 0$$

$$\Rightarrow d\left[\frac{e^x}{y}\right] + 2x\, dx = 0 \ldots\ldots ②$$

Integrating ②, solution is given by : $\frac{e^x}{y} + x^2 = C$

$\Rightarrow e^x + yx^2 = Cy$

Q2. Solve the differential equation: $\langle \boldsymbol{Q8(a), GGSIPU, December\ 2012}\rangle$

$(x^2 + y^2 + 2x)dx + \; 2y\, dy = 0$ .....①

Solution: $M = \; x^2 + y^2 + 2x, \; N = 2y$

$\frac{\partial M}{\partial y} = \; 2y$, $\frac{\partial N}{\partial x} = \; 0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is neither homogeneous nor of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

∴ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \; = \; 2y$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \; \frac{2y}{2y} = \; 1 = \; f(x)$ say

∴ IF = $e^{\int f(x)dx} \; = \; e^{\int 1dx} \; = \; e^x$

∴ ① may be rewritten after multiplying by IF as:

$e^x(x^2 + y^2 + 2x)dx + \; 2e^x y\, dy = 0$.......②

New $M = \; e^x(x^2 + y^2 + 2x)$, New $N = \; 2e^x y$

$\frac{\partial M}{\partial y} = 2e^x y$, $\frac{\partial N}{\partial x} = \; 2e^x y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ ② is an exact differential equation.

Solution of ② is given by:

$\int e^x(x^2 + y^2 + 2x)\, dx \; + \; \int 0\, dy$

y constant

$\Rightarrow \; (x^2 + y^2 + 2x)e^x \; - \; (2x + 2)e^x \; + \; (2)e^x \; = \mathrm{C}$

$\Rightarrow \; (x^2 + y^2)e^x \; = \mathrm{C}$, C is an arbitrary constant.

Q3. Solve $(xy^2 + x)dx + \; (yx^2 + y)dy$

$\langle \boldsymbol{Q1(f)}, \; \boldsymbol{GGSIPU}, \; \boldsymbol{December\; 2013} \rangle$

Solution: $M = xy^2 + x$, $N = yx^2 + y$

$\frac{\partial M}{\partial y} = \; 2xy, \; \frac{\partial N}{\partial x} = \; 2xy$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ given differential equation is exact.

Solution is given by:

$\int (xy^2 + x)\, dx \; + \; \int y\, dy = C$

y constant

$\Rightarrow \frac{x^2y^2}{2} + \frac{x^2}{2} + \frac{y^2}{2} = C_1$

$\Rightarrow x^2y^2 + x^2 + y^2 = C$

Q4. Solve the differential equation: $\langle \boldsymbol{Q1(c)},\ \boldsymbol{GGSIPU, 2^{nd} term\ 2014} \rangle$

$(x^2y - 2xy^2)dx - (x^3 - 3x^2y\, dy = 0$ .....①

Solution: $M = x^2y - 2xy^2$, $N = -x^3 + 3x^2y$

$\frac{\partial M}{\partial y} = x^2 - 4xy$, $\frac{\partial N}{\partial x} = -3x^2 + 6xy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is a homogeneous equation

∴ $\text{IF} = \frac{1}{Mx+Ny} = \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2} = \frac{1}{x^2y^2}$

∴ ① may be rewritten as : $\left(\frac{1}{y} - \frac{2}{x}\right) dx - \left(\frac{x}{y^2} - \frac{3}{y}\right) dy = 0$ ......②

New $M = \frac{1}{y} - \frac{2}{x}$, New $N = -\frac{x}{y^2} + \frac{3}{y}$

$\frac{\partial M}{\partial y} = -\frac{1}{y^2}$, $\frac{\partial N}{\partial x} = -\frac{1}{y^2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ ② is an exact differential equation.

Solution of ② is given by:

$$\int\limits_{y\ constant} \left(\frac{1}{y} - \frac{2}{x}\right) dx + \int \left(\frac{3}{y}\right) dy$$

$\frac{x}{y} - 2\log|x| + 3\log|y| = C$

Q5. Solve the differential equation: $\langle \boldsymbol{Q3(a)},\ \boldsymbol{GGSIPU, 2^{nd} term\ 2014} \rangle$

$$\frac{dy}{dx} + \left(\frac{y}{x}\right) logy = \frac{y}{x}(logy)^2 \text{ .....①}$$

Solution: It is Bernoulli's equation which can be reduced to linear form

Dividing throughout by $y(logy)^2$

$\frac{1}{y(logy)^2}\frac{dy}{dx} + \frac{1}{x\, logy} = \frac{1}{x}$......②

Putting $\frac{1}{\log y} = t$, $-\frac{1}{y(logy)^2}\frac{dy}{dx} = \frac{dt}{dx}$ ......③

Using ③ in ②, we get $\frac{dt}{dx} - \frac{1}{x}t = -\frac{1}{x}$ ....④

④ is a linear differential equation of the form $\frac{dt}{dx} + Pt = Q$

Where $P = -\frac{1}{x}$ and $Q = -\frac{1}{x}$

IF = $e^{\int P\,dx} = e^{\int \frac{-1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

$\therefore$ Solution of ④ is given by

$t.\frac{1}{x} = \int -\frac{1}{x}.\frac{1}{x}\; dx + \text{C}$

$\Rightarrow t.\frac{1}{x} = \int -\frac{1}{x^2}\; dx + \text{C}$

$\Rightarrow t.\frac{1}{x} = \frac{1}{x} + \text{C}$

Substituting $t = \frac{1}{\log y}$

$\Rightarrow \frac{1}{x\log y} = \frac{1}{x} + \text{C}$

$\Rightarrow \frac{1}{\log y} = 1 + \text{C}x$, C is an arbitrary constant

Q6. Solve the differential equation: $\langle \boldsymbol{Q1(i), GGSIPU, December\ 2014}\rangle$

$(y^3 - 3xy^2)dx + (2x^2y - xy^2)\; dy = 0$ .....①

Solution: $M = y^3 - 3xy^2$, $N = 2x^2y - xy^2$

$\frac{\partial M}{\partial y} = 3y^2 - 6xy$, $\frac{\partial N}{\partial x} = 4xy - y^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, $\therefore$ given differential equation is not exact.

As ① is a homogeneous equation

$\therefore$ IF $= \frac{1}{Mx+Ny} = \frac{1}{xy^3-3x^2y^2+2x^2y^2-xy^3} = \frac{-1}{x^2y^2}$

$\therefore$ ① may be rewritten as : $\left(\frac{-y}{x^2} + \frac{3}{x}\right)dx + \left(-\frac{2}{y} + \frac{1}{x}\right)\; dy = 0$ ......②

New $M = \frac{-y}{x^2} + \frac{3}{x}$, New $N = -\frac{2}{y} + \frac{1}{x}$

$\frac{\partial M}{\partial y} = -\frac{1}{x^2}$, $\frac{\partial N}{\partial x} = -\frac{1}{x^2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\therefore$ ② is an exact differential equation.

Solution of ② is given by:

$$\int_{y\ constant} \left(\frac{-y}{x^2} + \frac{3}{x}\right)\, dx + \int -\frac{2}{y}\; dy$$

$\frac{y}{x} + 3\log|x| - 2\log|y| = \text{C}$