Ordinary Differential Equations: Graduate Level Problems and Solutions

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Contents

1 Preliminaries

Cauchy-Peano.

$$\begin{cases} \frac{du}{dt} = f(t,u) & t_0 \le t \le t_1 \\ u(t_0) = u_0 \end{cases} \tag{1.1}$$

$f(t,u)$ is continuous in the rectangle $R = \{(t,u):\ t_0 \le t \le t_0 + a,\ |u - u_0| \le b\}$. $M = \max_R |f(t,u)|$, and $\alpha = \min(a, \frac{b}{M})$. Then $\exists\ u(t)$ with continuous first derivative s.t. it satisfies (1.1) for $t_0 \le t \le t_0 + \alpha$.

Local Existence via Picard Iteration.
$f(t,u)$ is continuous in the rectangle $R = \{(t,u) : t_0 \le t \le t_0 + a, |u - u_0| \le b\}$.
Assume f is Lipschitz in u on R.
$|f(t,u) - f(t,v)| \le L|u - v|$
$M = \max_R |f(t,u)|$, and $\alpha = \min(a, \frac{b}{M})$. Then $\exists$ a unique $u(t)$, with $u, \frac{du}{dt}$ continuous on $[t_0, t_0 + \beta]$, $\beta \in (0, \alpha]$ s.t. it satisfies (1.1) for $t_0 \le t \le t_0 + \beta$.

Power Series.

$$\frac{du}{dt} = f(t,u)$$

$$u(0) = u_0$$

$$u(t) = \sum_{j=0}^{\infty} \frac{1}{j!}\frac{d^j u}{dt^j}(0)t^j \qquad \textit{i.e.}\ \ \frac{d^2u}{dt^2}(0) = (f_t + f_u f)|_0$$

Fixed Point Iteration.

$$|x_n - x^*| \le k^n |x_0 - x^*| \qquad k < 1$$

$$|x_{n+1} - x_n| \le k^n |x_1 - x_0| \qquad k < 1$$

$$\Rightarrow \quad |x^* - x_n| = \lim_{m\to\infty} |x_m - x_n| \le k^n(1 + k + k^2 + \cdots)|x_1 - x_0| = \frac{k^n}{1-k}|x_1 - x_0|$$

Picard Iteration. *Approximates (1.1). Initial guess: $u_0(t) = u_0$*

$$u_{n+1}(t) \ = \ Tu_n(t) \ = \ u_0 + \int_{t_0}^{t} f(s, u_n(s))ds.$$

Differential Inequality. *$v(t)$ piecewise continuous on $t_0 \le t \le t_0 + a$.*
$u(t)$ and $\frac{du}{dt}$ continuous on some interval. If

$$\frac{du}{dt} \le v(t)u(t)$$

$$\Rightarrow \quad u(t) \le u(t_0)e^{\int_{t_0}^{t} v(s)ds}$$

Proof. Multiply both sides by $e^{-\int_{t_0}^{t} v(s)ds}$. Then $\frac{d}{dt}[e^{-\int_{t_0}^{t} v(s)ds} u(t)] \le 0$. $\square$

1.1 Gronwall Inequality

Gronwall Inequality. *$u(t), v(t)$ continuous on $[t_0, t_0+a]$. $v(t) \geq 0$, $c \geq 0$.*

$$u(t) \leq c + \int_{t_0}^{t} v(s)u(s)ds$$

$$\Rightarrow \quad u(t) \leq c\, e^{\int_{t_0}^{t} v(s)ds} \qquad t_0 \leq t \leq t_0 + a$$

Proof. Multiply both sides by $v(t)$:

$$u(t)v(t) \leq v(t)\big\{c + \int_{t_0}^{t} v(s)u(s)ds\big\}$$

Denote $A(t) = c + \int_{t_0}^{t} v(s)u(s)ds \;\Rightarrow\; \frac{dA}{dt} \leq v(t)A(t)$. By differential inequality and hypothesis:

$$u(t) \leq A(t) \leq A(t_0)e^{\int_{t_0}^{t} v(s)ds} = ce^{\int_{t_0}^{t} v(s)ds}.$$

□

Error Estimates. *$f(t, u(t))$ continuous on $R = \{(t,u) : |t - t_0| \leq a, |u - u_0| \leq b\}$*
$f(t, u(t))$ Lipschitz in u: $|f(t,A) - f(t,B)| \leq L|A - B|$
$u_1(t), u_2(t)$ are ϵ_1, ϵ_2 approximate solutions

$$\frac{du_1}{dt} = f(t, u_1(t)) + R_1(t), \qquad |R_1(t)| \leq \epsilon_1$$

$$\frac{du_2}{dt} = f(t, u_2(t)) + R_2(t), \qquad |R_2(t)| \leq \epsilon_2$$

$$|u_1(t_0) - u_2(t_0)| \leq \delta$$

$$\Rightarrow \quad |u_1(t) - u_2(t)| \leq (\delta + a(\epsilon_1 + \epsilon_2))e^{a \cdot L} \qquad t_0 \leq t \leq t_0 + a$$

Generalized Gronwall Inequality. $w(s), u(s) \geq 0$

$$u(t) \leq w(t) + \int_{t_0}^{t} v(s)u(s)ds$$

$$\Rightarrow \quad u(t) \leq w(t) + \int_{t_0}^{t} v(s)w(s)\, e^{\int_{s}^{t} v(x)dx} ds$$

Improved Error Estimate (Fundamental Inequality).

$$|u_1(t) - u_2(t)| \leq \delta e^{L(t-t_0)} + \frac{(\epsilon_1 + \epsilon_2)}{L}(e^{L(t-t_0)} - 1)$$

1.2 Trajectories

Let $K \subset D$ compact. If for the trajectory $Z = \{(t, z(t)) : \alpha < t < \beta)\}$ we have that $\beta < \infty$, then Z lies outside of K for all t sufficiently close to β.

2 Linear Systems

2.1 Existence and Uniqueness

$A(t),\ g(t)$ continuous, then can solve

$$y' = A(t)y + g(t) \tag{2.1}$$

$$y(t_0) = y_0$$

For uniqueness, need RHS to satisfy Lipshitz condition.

2.2 Fundamental Matrix

A matrix whose columns are solutions of $y' = A(t)y$ is called a **solution matrix**. A solution matrix whose columns are linearly independent is called a **fundamental matrix**.
$F(t)$ is a fundamental matrix if:
1) $F(t)$ is a solution matrix;
2) $\det F(t) \neq 0$.
Either $\det M(t) \neq 0 \ \ \forall t \in \mathbb{R}$, or $\det M(t) = 0 \ \ \forall t \in \mathbb{R}$.
$F(t)c$ is a solution of (2.1), where c is a column vector.
If $F(t)$ is a fundamental matrix, can use it to solve:

$$y'(t) = A(t)y(t),\ y(t_0) = y_0$$

i.e. since $F(t)c|_{t_0} = F(t_0)c = y_0 \quad \Rightarrow \quad c = F^{-1}(t_0)y_0 \quad \Rightarrow$

$$\Rightarrow \ \ y(t) = F(t)F(t_0)^{-1}y_0$$

2.2.1 Distinct Eigenvalues or Diagonalizable

$$F(t) = [e^{\lambda_1 t}v_1, \ldots, e^{\lambda_n t}v_n] \qquad\qquad e^{At} = F(t)C$$

2.2.2 Arbitrary Matrix

i) Find generalized eigenspaces $X_j = \{x : (A - \lambda_j I)^{n_j} x = 0\}$;
ii) Decompose initial vector $\eta = v_1 + \cdots + v_k, \quad v_j \in X_j$,
solve for $v_1, \ldots, v_k$ in terms of components of η

$$y(t) = \sum_{j=1}^{k} e^{\lambda_j t}\Big[\sum_{i=0}^{n_j-1} \frac{t^i}{i!}(A - \lambda_j I)^i\Big]v_j \tag{2.2}$$

iii) Plug in $\eta = e_1, \ldots, e_n$ successively to get $y_1(t), \ldots, y_n(t)$ columns of $F(t)$.
Note: $y(0) = \eta,\ F(0) = I$.

2.2.3 Examples

Example 1. *Show that the solutions of the following system of differential equations remain bounded as $t \to \infty$:*

$$\begin{aligned} u' &= v - u \\ v' &= -u \end{aligned}$$

Proof. 1) $\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$. The eigenvalues of A are $\lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, so the eigenvalues are distinct $\Rightarrow$ diagonalizable. Thus, $F(t) = [e^{\lambda_1 t}v_1, e^{\lambda_2 t}v_2]$ is a fundamental matrix. Since $Re(\lambda_i) = -\frac{1}{2} < 0$, the solutions to $y' = Ay$ remain bounded as $t \to \infty$.

2) $u'' = v' - u' = -u - u'$,
$u'' + u' + u = 0$,
$u'u'' + (u')^2 + u'u = 0$,
$\frac{1}{2}\frac{d}{dt}((u')^2) + (u')^2 + \frac{1}{2}\frac{d}{dt}(u^2) = 0$,
$\frac{1}{2}((u')^2) + \frac{1}{2}(u^2) + \int_{t_0}^{t}(u')^2 dt = \text{const}$,
$\frac{1}{2}((u')^2) + \frac{1}{2}(u^2) \le \text{const}$,
$\Rightarrow \quad (u', u)$ is bounded. $\square$

Example 2. *Let A be the matrix given by: $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$. Find the eigenvalues, the generalized eigenspaces, and a fundamental matrix for the system $y'(t) = Ay$.*

Proof. • $\det(A - \lambda I) = (1-\lambda)^2(2-\lambda)$. The eigenvalues and their multiplicities:
$\lambda_1 = 1, n_1 = 2; \quad \lambda_2 = 2, n_2 = 1$.
• Determine subspaces X_1 and X_2, $\quad (A - \lambda_j I)^{n_j}x = 0$.
$(A - I)^2 x = 0 \qquad\qquad (A - 2I)x = 0$
To find X_1:

$$(A - I)^2 x = \begin{pmatrix} 0 & 0 & 3 \\ 2 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 3 \\ 2 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 8 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\Rightarrow \quad x_3 = 0,\ x_1, x_2 \text{ arbitrary} \quad \Rightarrow \quad X_1 = \left\{ \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}, \text{any } \alpha, \beta \in \mathbb{C} \right\}. \qquad \dim X_1 = 2.$$

To find X_2:

$$(A - 2I)x = \begin{pmatrix} -1 & 0 & 3 \\ 2 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} x = \begin{pmatrix} -1 & 0 & 3 \\ 0 & -1 & 8 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\Rightarrow \quad x_3 = \gamma,\ x_1 = 3\gamma,\ x_2 = 8\gamma \quad \Rightarrow \quad X_2 = \left\{ \gamma \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}, \text{any } \gamma \in \mathbb{C} \right\}. \qquad \dim X_2 = 1.$$

• Need to find $v_1 \in X_1,\ v_2 \in X_2$, such that initial vector η is decomposed as $\eta = v_1 + v_2$.

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} + \begin{pmatrix} 3\gamma \\ 8\gamma \\ \gamma \end{pmatrix}.$$

$$\Rightarrow \quad v_1 = \begin{pmatrix} \eta_1 - 3\eta_3 \\ \eta_2 - 8\eta_3 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 3\eta_3 \\ 8\eta_3 \\ \eta_3 \end{pmatrix}.$$

$$\bullet \quad y(t) = \sum_{j=1}^{k} e^{\lambda_j t}\Big[\sum_{i=0}^{n_j-1} \frac{t^i}{i!}(A-\lambda_j I)^i\Big]v_j = e^{\lambda_1 t}(I+t(A-I))v_1 + e^{\lambda_2 t}v_2$$

$$= e^t(I+t(A-I))v_1 + e^{2t}v_2 = e^t(I+t(A-I))\begin{pmatrix}\eta_1-3\eta_3\\ \eta_2-8\eta_3\\ 0\end{pmatrix} + e^{2t}\begin{pmatrix}3\eta_3\\ 8\eta_3\\ \eta_3\end{pmatrix}$$

$$= e^t\begin{pmatrix}1 & 0 & 3t\\ 2t & 1 & 2t\\ 0 & 0 & 1+t\end{pmatrix}\begin{pmatrix}\eta_1-3\eta_3\\ \eta_2-8\eta_3\\ 0\end{pmatrix} + e^{2t}\begin{pmatrix}3\eta_3\\ 8\eta_3\\ \eta_3\end{pmatrix}.$$

Note: $y(0) = \eta = \begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix}$.

• To find a fundamental matrix, putting η successively equal to $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ in this formula, we obtain the three linearly independent solutions that we use as columns of the matrix. If $\eta = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $y_1(t) = e^t\begin{pmatrix}1\\2t\\0\end{pmatrix}$. If $\eta = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $y_2(t) = e^t\begin{pmatrix}0\\1\\0\end{pmatrix}$.

If $\eta = \begin{pmatrix}0\\0\\1\end{pmatrix}$, $y_3(t) = e^t\begin{pmatrix}-3\\-6t-8\\0\end{pmatrix} + e^{2t}\begin{pmatrix}3\\8\\1\end{pmatrix}$. The fundamental matrix is

$$F(t) = e^{At} = \begin{pmatrix}e^t & 0 & -3e^t+3e^{2t}\\ 2te^t & e^t & (-6t-8)e^t+8e^{2t}\\ 0 & 0 & e^{2t}\end{pmatrix}$$

Note: At $t=0$, $F(t)$ reduces to I. $\square$

2.3 Asymptotic Behavior of Solutions of Linear Systems with Constant Coefficients

If all λ_j of A are such that $Re(\lambda_j) < 0$, then every solution $\phi(t)$ of the system $y' = Ay$ approaches zero as $t \to \infty$. $|\phi(t)| \leq \hat{K}e^{-\sigma t}$ or $|e^{At}| \leq Ke^{-\sigma t}$.
If, in addition, there are λ_j such that $Re(\lambda_j) = 0$ and are simple, then $|e^{At}| \leq K$, and hence every solution of $y' = Ay$ is bounded.
Also, see the section on Stability and Asymptotic Stability.

Proof. $\lambda_1, \lambda_2, \ldots, \lambda_k$ are eigenvalues and $n_1, n_2, \ldots, n_k$ are their corresponding multiplicities. Consider (2.2), i.e. the solution y satisfying $y(0) = \eta$ is

$$y(t) = e^{tA}\eta = \sum_{j=1}^{k} e^{\lambda_j t}\Big[\sum_{i=0}^{n_j-1} \frac{t^i}{i!}(A-\lambda_j I)^i\Big]v_j.$$

Subdivide the right hand side of equality above into two summations, i.e.:
1) λ_j, s.t. $n_j = 1$, $Re(\lambda_j) \leq 0$;
2) λ_j, s.t. $n_j \geq 2$, $Re(\lambda_j) < 0$.

$$\begin{aligned}
y(t) &= \underbrace{\sum_{j=1}^{k} e^{\lambda_j t} v_j}_{(n_j=1)\quad Re(\lambda_j)\leq 0} + \underbrace{\sum_{j=1}^{k} e^{\lambda_j t}\Big[I + t(A-\lambda_j I) + \cdots + \frac{t^{n_j-1}}{(n_j-1)!}(A-\lambda_j I)^{n_j-1}\Big]v_j}_{(n_j\geq 2)\qquad Re(\lambda_j)<0}. \\
|y(t)| &\leq \underbrace{\sum_{j=1}^{k} |e^{\lambda_j t} I||v_j|}_{Re(\lambda_j)\leq 0} + \underbrace{\widetilde{K}e^{-\sigma t}}_{-\sigma=\max(Re(\lambda_j),\ Re(\lambda_j)<0)} \leq c\sum_{j=1}^{k}|v_j| + \widetilde{K}e^{-\sigma t} \\
&\leq ck\max_j |v_j| + \widetilde{K}e^{-\sigma t} \leq \underbrace{\max(ck, \widetilde{K})}_{\text{const indep of t}} \Big[\underbrace{\max_j |v_j|}_{\text{indep of t}} + \underbrace{e^{-\sigma t}}_{\to 0 \text{ as } t\to\infty}\Big] \leq K.
\end{aligned}$$

□

2.4 Variation of Constants

Derivation: Variation of constants is a method to determine a solution of $y' = A(t)y + g(t)$, provided we know a fundamental matrix for the homogeneous system $y' = A(t)y$. Let F be a fundamental matrix. Look for solution of the form $\psi(t) = F(t)v(t)$, where v is a vector to be determined. (Note that if v is a constant vector, then ψ satisfies the homogeneous system and thus for the present purpose $v(t) \equiv c$ is ruled out.) Substituting $\psi(t) = F(t)v(t)$ into $y' = A(t)y + g(t)$, we get

$$\psi'(t) = F'(t)v(t) + F(t)v'(t) = A(t)F(t)v(t) + g(t)$$

Since F is a fundamental matrix of the homogeneous system, $F'(t) = A(t)F(t)$. Thus,

$$\begin{aligned}
F(t)v'(t) &= g(t), \\
v'(t) &= F^{-1}(t)g(t), \\
v(t) &= \int_{t_0}^{t} F^{-1}(s)g(s)ds. \\
\text{Therefore, } \psi(t) &= F(t)\int_{t_0}^{t} F^{-1}(s)g(s)ds.
\end{aligned}$$

Variation of Constants Formula: Every solution y of $y' = A(t)y + g(t)$ has the form:

$$\boxed{y(t) \;=\; \phi_h(t) \;+\; \psi_p(t) \;=\; F(t)\vec{c} \;+\; F(t)\int_{t_0}^{t} F^{-1}(s)g(s)ds}$$

where ψ_p is the solution satisfying initial condition $\psi_p(t_0) = 0$ and $\phi_h(t)$ is that solution of the homogeneous system satisfying the same initial condition at t_0 as y, $\phi_h(t_0) = y_0$.

$F(t) = e^{At}$ is the fundamental matrix of $y' = Ay$ with $F(0) = I$. Therefore, every solution of $y' = Ay$ has the form $y(t) = e^{At}c$ for a suitably chosen constant vector c.

$$\boxed{y(t) \;= e^{(t-t_0)A}y_0 \;+ \int_{t_0}^{t} e^{(t-s)A}g(s)ds}$$

That is, to find the general solution of (2.1), use (2.2) to get a fundamental matrix $F(t)$.

Then, add $\int_{t_0}^{t} e^{(t-s)A}g(s)ds \;=\; F(t)\int_{t_0}^{t} F^{-1}(s)g(s)ds$ to $F(t)\vec{c}$.

2.5 Classification of Critical Points

$y' = Ay$. Change of variable $y = Tz$, where T is nonsingular constant matrix (to be determined). $\Rightarrow$ $z' = T^{-1}ATz$ The solution is passing through (c_1, c_2) at $t = 0$.

1) λ_1, λ_2 are real. $z' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} z$

$\Rightarrow$ $z = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$

a) $\lambda_2 > \lambda_1 > 0 \Rightarrow z_2(t) = c(z_1(t))^p$, $p > 1$ **Improper Node** (tilted toward z_2-axis)
b) $\lambda_2 < \lambda_1 < 0 \Rightarrow z_2(t) = c(z_1(t))^p$, $p > 1$ **Improper Node** (tilted toward z_2-axis)
c) $\lambda_2 = \lambda_1$, A diagonalizable $\Rightarrow z_2 = cz_1$ **Proper Node**
d) $\lambda_2 < 0 < \lambda_1 \Rightarrow z_1(t) = c(z_2(t))^p$, $p < 0$ **Saddle Point**

2) $\lambda_2 = \lambda_1$, A non-diagonalizable, $z' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} z$

$\Rightarrow$ $z = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 t \\ c_2 \end{pmatrix} e^{\lambda t}$ **Improper Node**

3) $\lambda_{1,2} = \sigma \pm i\nu$. $z' = \begin{pmatrix} \sigma & \nu \\ -\nu & \sigma \end{pmatrix} z$

$\Rightarrow$ $z = e^{\sigma t} \begin{pmatrix} c_1 \cos(\nu t) + c_2 \sin(\nu t) \\ -c_1 \sin(\nu t) + c_2 \cos(\nu t) \end{pmatrix}$ **Spiral Point**

2.5.1 Phase Portrait

Locate stationary points by setting:
$\frac{du}{dt} = f(u, v) = 0$
$\frac{dv}{dt} = g(u, v) = 0$
(u_0, v_0) is a stationary point. In order to classify a stationary point, need to find eigenvalues of a linearized system at that point.

$$J(f(u, v), g(u, v)) = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{bmatrix}.$$

Find λ_j's such that $\det(J|_{(u_0,v_0)} - \lambda I) = 0$.

2.6 Problems

Problem (F'92, #4). *Consider the autonomous differential equation*

$$v_{xx} + v - v^3 - v_0 = 0$$

in which v_0 is a constant.
a) *Show that for $v_0^2 < \frac{4}{27}$, this equation has 3 stationary points and classify their type.*
b) *For $v_0 = 0$, draw the phase plane for this equation.*

Proof. **a)** We have

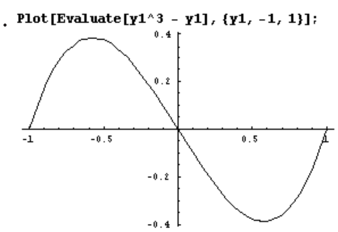
$$v'' + v - v^3 - v_0 = 0.$$

In order to find and analyze the stationary points of an ODE above, we write it as a first-order system.

$$y_1 = v,$$
$$y_2 = v'.$$

$$y_1' = v' = y_2 = 0,$$
$$y_2' = v'' = -v + v^3 + v_0 = y_1^3 - y_1 + v_0 = 0.$$

The function $f(y_1) = y_1^3 - y_1 = y_1(y_1^2 - 1)$ has zeros $y_1 = 0$, $y_1 = -1$, $y_1 = 1$.
See the figure.
It's derivative $f'(y_1) = 3y_1^2 - 1$ has zeros $y_1 = -\frac{1}{\sqrt{3}}$, $y_1 = \frac{1}{\sqrt{3}}$.
At these points, $f(-\frac{1}{\sqrt{3}}) = \frac{2}{3\sqrt{3}}$, $f(\frac{1}{\sqrt{3}}) = -\frac{2}{3\sqrt{3}}$.
If $v_0 = 0$, y_2' is exactly this function $f(y_1)$, with 3 zeros.
v_0 only raises or lowers this function. If $|v_0| < \frac{2}{3\sqrt{3}}$,
i.e. $v_0^2 < \frac{4}{27}$, the system would have 3 stationary points:



$$\boxed{\text{Stationary points: } (p_1, 0), (p_2, 0), (p_3, 0),}$$

with $p_1 < p_2 < p_3$.

$$y_1' = y_2 \qquad\qquad = f(y_1, y_2),$$
$$y_2' = y_1^3 - y_1 + v_0 = g(y_1, y_2).$$

In order to classify a stationary point, need to find eigenvalues of a linearized system at that point.

$$J(f(y_1,y_2), g(y_1,y_2)) = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2} \\ \frac{\partial g}{\partial y_1} & \frac{\partial g}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3y_1^2 - 1 & 0 \end{bmatrix}.$$

- For $(y_1, y_2) = (p_i, 0)$:

$$\det(J|_{(p_i,0)} - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 3p_i^2 - 1 & -\lambda \end{vmatrix} = \lambda^2 - 3p_i^2 + 1 = 0.$$

$$\lambda_\pm = \pm\sqrt{3p_i^2 - 1}.$$

At $y_1 = p_1 < -\frac{1}{\sqrt{3}}$, $\lambda_- < 0 < \lambda_+$. $\mathbf{(p_1,0)}$ is **Saddle Point**.
At $-\frac{1}{\sqrt{3}} < y_1 = p_2 < \frac{1}{\sqrt{3}}$, $\lambda_\pm \in \mathbb{C}$, $\mathbb{R}e(\lambda_\pm) = 0$. $\mathbf{(p_2,0)}$ is **Stable Concentric Circles**.
At $y_1 = p_3 > \frac{1}{\sqrt{3}}$, $\lambda_- < 0 < \lambda_+$. $\mathbf{(p_3,0)}$ is **Saddle Point**.

b) For $v_0 = 0$,

$$y_1' = y_2 = 0,$$
$$y_2' = y_1^3 - y_1 = 0.$$

$$\boxed{\text{Stationary points: } (-1,0),\ (0,0),\ (1,0).}$$

$$J(f(y_1,y_2), g(y_1,y_2)) = \begin{bmatrix} 0 & 1 \\ 3y_1^2 - 1 & 0 \end{bmatrix}.$$

• For $(y_1, y_2) = (0,0)$:

$$\det(J|_{(0,0)} - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0.$$

$$\lambda_\pm = \pm i.$$

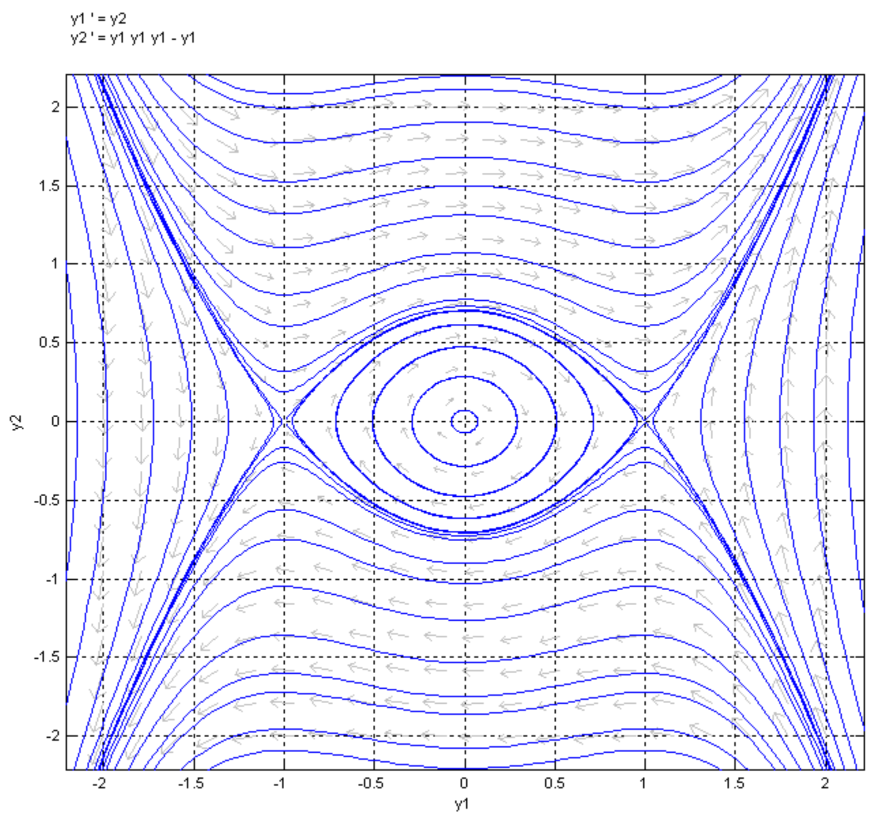
(0,0) is **Stable Concentric Circles (Center)**.

• For $(y_1, y_2) = (\pm 1, 0)$:

$$\det(J|_{(\pm 1,0)} - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - 2 = 0.$$

$$\lambda_\pm = \pm\sqrt{2}.$$

(-1,0) and **(1,0)** are **Saddle Points**.



□

Problem (F'89, #2). *Let $V(x,y) = x^2(x-1)^2 + y^2$. Consider the dynamical system*

$$\frac{dx}{dt} = -\frac{\partial V}{\partial x},$$
$$\frac{dy}{dt} = -\frac{\partial V}{\partial y}.$$

a) *Find the critical points of this system and determine their linear stability.*
b) *Show that V decreases along any solution of the system.*
c) *Use (b) to prove that if $z_0 = (x_0, y_0)$ is an isolated minimum of V then z_0 is an asymptotically stable equilibrium.*

Proof. **a)** We have

$$x' = -4x^3 + 6x^2 - 2x$$
$$y' = -2y.$$

$$\begin{cases} x' = -x(4x^2 - 6x + 2) = 0 \\ y' = -2y = 0. \end{cases}$$

$$\boxed{\text{Stationary points: } (0,0),\ \left(\frac{1}{2},0\right),\ (1,0).}$$

$$J(f(y_1,y_2), g(y_1,y_2)) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} -12x^2 + 12x - 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

- For $(x,y) = (0,0)$:

$$\det(J|_{(0,0)} - \lambda I) = \begin{vmatrix} -2-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix} = (-2-\lambda)(-2-\lambda) = 0.$$

$y' = Ay$, $\lambda_1 = \lambda_2 < 0$, A diagonalizable.
(0,0) is **Stable Proper Node**.

- For $(x,y) = \left(\frac{1}{2},0\right)$:

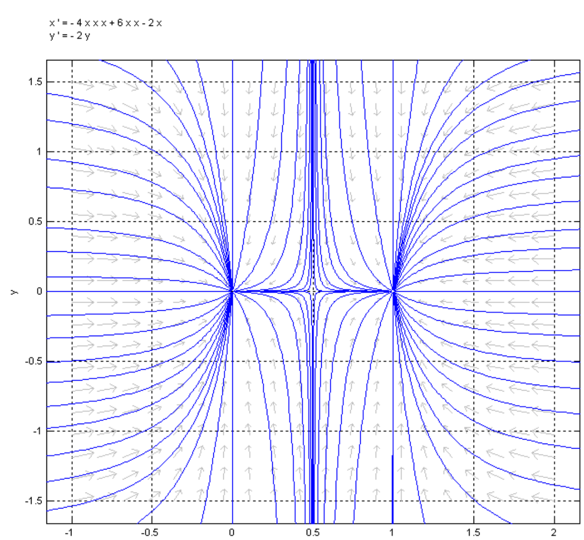
$$\det(J|_{(\frac{1}{2},0)} - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda) = 0.$$

$\lambda_1 = -2$, $\lambda_2 = 1$. $\lambda_1 < 0 < \lambda_2$.
$\left(\mathbf{\frac{1}{2},0}\right)$ is **Unstable Saddle Point**.

- For $(x,y) = (1,0)$:

$$\det(J|_{(1,0)} - \lambda I) = \begin{vmatrix} -2-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix} = (-2-\lambda)(-2-\lambda) = 0.$$

$y' = Ay$, $\lambda_1 = \lambda_2 < 0$, A diagonalizable.
(1,0) is **Stable Proper Node**.



b) *Show that V decreases along any solution of the system.*

$$\frac{dV}{dt} = V_x x_t + V_y y_t = V_x(-V_x) + V_y(-V_y) = -V_x^2 - V_y^2 < 0.$$

c) *Use (b) to prove that if $z_0 = (x_0, y_0)$ is an isolated minimum of V then z_0 is an asymptotically stable equilibrium.*

Lyapunov Theorem: If $\exists V(y)$ that is positive definite and for which $V^*(y)$ is negative definite in a neighborhood of 0, then the zero solution is **asymptotically stable**.
Let $W(x,y) = V(x,y) - V(x_0,y_0)$. Then, $W(x_0,y_0) = 0$.
$W(x,y) > 0$ in a neighborhood around (x_0,y_0), and $\frac{dW}{dt}(x,y) < 0$ by (b). ($\frac{dV}{dt}(x,y) < 0$ and $\frac{dV}{dt}(x_0,y_0) = 0$).
(x_0,y_0) is asymptotically stable. $\square$

Problem (S'98, #1). *Consider the undamped pendulum, whose equation is*

$$\frac{d^2p}{dt^2} + \frac{g}{l}\sin p = 0.$$

a) *Describe all possible motions using a phase plane analysis.*
b) *Derive an* ***integral expression for the period*** *of oscillation at a fixed energy E, and find the period at small E to first order.*
c) *Show that there exists a critical energy for which the motion is not periodic.*

Proof. **a)** We have

$$\begin{aligned} y_1 &= p \\ y_2 &= p'. \end{aligned}$$

$$\begin{aligned} y_1' &= p' = y_2 = 0 \\ y_2' &= p'' = -\frac{g}{l}\sin p = -\frac{g}{l}\sin y_1 = 0. \end{aligned}$$

$$\boxed{\text{Stationary points: } (n\pi, 0).}$$

$$\begin{aligned} y_1' &= y_2 &&= f(y_1, y_2), \\ y_2' &= -\frac{g}{l}\sin y_1 &&= g(y_1, y_2). \end{aligned}$$

$$J(f_1(y_1,y_2), f_2(y_1,y_2)) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos y_1 & 0 \end{bmatrix}.$$

- For $(y_1, y_2) = (n\pi, 0)$, n-even:

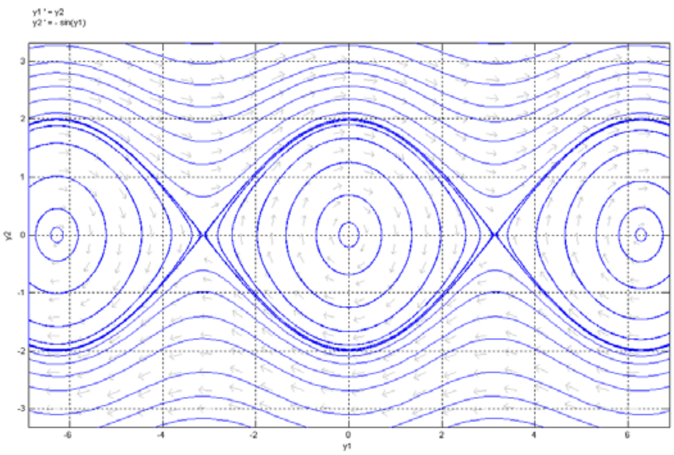
$$\det(J|_{(n\pi,0)} - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -\frac{g}{l} & -\lambda \end{vmatrix} = \lambda^2 + \frac{g}{l} = 0.$$

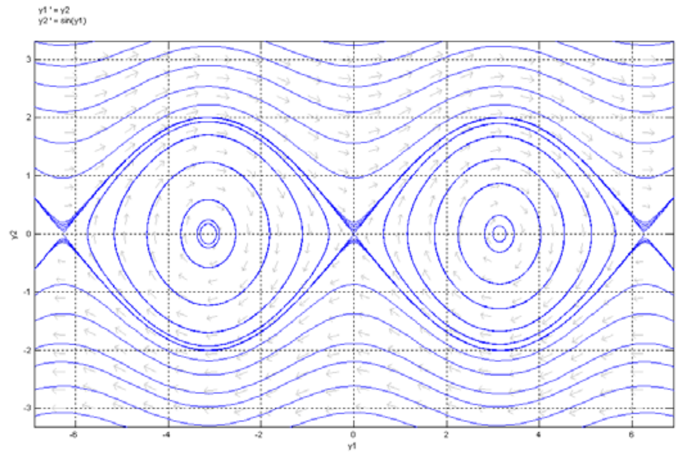
$$\lambda_\pm = \begin{cases} \pm i\sqrt{\frac{g}{l}} \in \mathbb{C}, & g > 0, \quad \Rightarrow \quad \textbf{(n}\boldsymbol{\pi}\textbf{,0), n-even, are Stable Centers}. \\ \pm\sqrt{-\frac{g}{l}} \in \mathbb{R}, & g < 0. \quad \Rightarrow \quad \textbf{(n}\boldsymbol{\pi}\textbf{,0), n-even, are Unstable Saddle Points}. \end{cases}$$

- For $(y_1, y_2) = (n\pi, 0)$, n-odd:

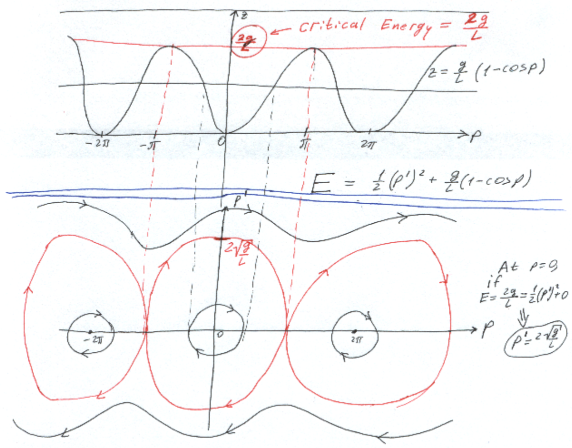
$$\det(J|_{(n\pi,0)} - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ \frac{g}{l} & -\lambda \end{vmatrix} = \lambda^2 - \frac{g}{l} = 0.$$

$$\lambda_\pm = \begin{cases} \pm\sqrt{\frac{g}{l}} \in \mathbb{R}, & g > 0, \quad \Rightarrow \quad \textbf{(n}\boldsymbol{\pi}\textbf{,0), n-odd, are Unstable Saddle Points}. \\ \pm i\sqrt{-\frac{g}{l}} \in \mathbb{C}, & g < 0, \quad \Rightarrow \quad \textbf{(n}\boldsymbol{\pi}\textbf{,0), n-odd, are Stable Centers}. \end{cases}$$


y1 ' = y2
y2 ' = - sin(y1)
y1
y2


y1 ' = y2
y2 ' = sin(y1)
y1
y2

b) We have



$$
\begin{aligned}
p'' + \frac{g}{l}\sin p &= 0, \\
p'p'' + \frac{g}{l}p'\sin p &= 0, \\
\frac{1}{2}\frac{d}{dt}(p')^2 - \frac{g}{l}\frac{d}{dt}(\cos p) &= 0, \\
\frac{1}{2}(p')^2 - \frac{g}{l}\cos p &= \tilde{E}.
\end{aligned}
$$

$$
\boxed{E = \frac{1}{2}(p')^2 + \frac{g}{l}(1 - \cos p).}
$$

Since we assume that $|p|$ is small, we could replace $\sin p$ by p, and perform similar calculations:

$$
\begin{aligned}
p'' + \frac{g}{l}p &= 0, \\
p'p'' + \frac{g}{l}p'p &= 0, \\
\frac{1}{2}\frac{d}{dt}(p')^2 + \frac{1}{2}\frac{g}{l}\frac{d}{dt}(p)^2 &= 0, \\
\frac{1}{2}(p')^2 + \frac{1}{2}\frac{g}{l}p^2 &= E_1, \\
(p')^2 + \frac{g}{l}p^2 &= E = \text{constant}.
\end{aligned}
$$

Thus,

$$
\frac{(p')^2}{E} + \frac{p^2}{\frac{lE}{g}} = 1,
$$

which is an ellipse with radii $\sqrt{E}$ on p'-axis, and $\sqrt{\frac{lE}{g}}$ on p-axis.

We derive an **Integral Expression for the Period** of oscillation at a fixed energy E.
Note that at maximum amplitude (maximum displacement), $p' = 0$.
Define $p = p_{\max}$ to be the maximum displacement:

$$
\begin{aligned}
E &= \frac{1}{2}(p')^2 + \frac{g}{l}(1 - \cos p), \\
p' &= \sqrt{2E - \frac{2g}{L}(1 - \cos p)},
\end{aligned}
$$

$$
\int_0^{\frac{T}{4}} \frac{p'}{\sqrt{2E - \frac{2g}{L}(1 - \cos p)}}\,dt = \int_0^{\frac{T}{4}} dt = \frac{T}{4},
$$

$$
T = 4\int_0^{\frac{T}{4}} \frac{p'}{\sqrt{2E - \frac{2g}{L}(1 - \cos p)}}\,dt. \qquad \left(T = 4\int_0^{p_{\max}} \frac{dp}{\sqrt{2E - \frac{2g}{L}(1 - \cos p)}}\right)
$$

Making change of variables: $\xi = p(t)$, $d\xi = p'(t)dt$, we obtain

$$
\boxed{T(p_{\max}) = 4\int_0^{p_{\max}} \frac{d\xi}{\sqrt{2E - \frac{2g}{L}(1 - \cos \xi)}}.}
$$

$\square$

Problem (F'94, #7).
The weakly nonlinear approximation to the pendulum equation ($\ddot{x} = -\sin x$) is

$$\ddot{x} = -x + \frac{1}{6}x^3. \tag{2.3}$$

a) *Draw the phase plane for (2.3).*
b) *Prove that (2.3) has periodic solutions $x(t)$ in the neighborhood of $x = 0$.*
c) *For such periodic solutions, define the amplitude as $a = \max_t x(t)$. Find an **integral formula for the period T** of a periodic solution as a function of the amplitude a.*
d) *Show that T is a non-decreasing function of a.*
Hint: Find a first integral of equation (2.3).

Proof. **a)**

$$\begin{aligned} y_1 &= x \\ y_2 &= x'. \end{aligned}$$

$$\begin{aligned} y_1' &= x' = y_2 = 0 \\ y_2' &= x'' = -x + \frac{1}{6}x^3 = -y_1 + \frac{1}{6}y_1^3 = 0. \end{aligned}$$

$$\boxed{\text{Stationary points: } (0,0),\ (-\sqrt{6},0),\ (\sqrt{6},0).}$$

$$\begin{aligned} y_1' &= y_2 &&= f(y_1,y_2), \\ y_2' &= -y_1 + \frac{1}{6}y_1^3 &&= g(y_1,y_2). \end{aligned}$$

$$J(f(y_1,y_2), g(y_1,y_2)) = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2} \\ \frac{\partial g}{\partial y_1} & \frac{\partial g}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 + \frac{1}{2}y_1^2 & 0 \end{bmatrix}.$$

- For $(y_1, y_2) = (0,0)$:

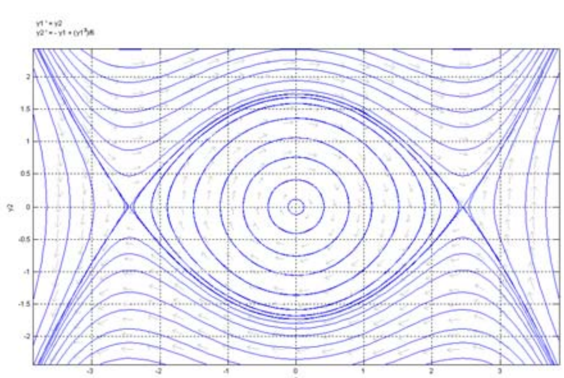
$$\det(J|_{(0,0)} - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0.$$

$\lambda_\pm = \pm i.$ **(0,0) is Stable Center**.

- For $(y_1, y_2) = (\pm\sqrt{6}, 0)$:

$$\det(J|_{(\pm\sqrt{6},0)} - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - 2 = 0.$$

$\lambda_\pm = \pm\sqrt{2}.$ **($\pm\sqrt{6}$,0) are Unstable Saddle Points**.



b) *Prove that* $\ddot{x} = -x + \frac{1}{6}x^3$ *has periodic solutions* $x(t)$ *in the neighborhood of* $x = 0$.
We have

$$\ddot{x} = -x + \frac{1}{6}x^3,$$

$$\dot{x}\ddot{x} = -\dot{x}x + \frac{1}{6}x^3\dot{x},$$

$$\frac{1}{2}\frac{d}{dt}(\dot{x}^2) = -\frac{1}{2}\frac{d}{dt}(x^2) + \frac{1}{24}\frac{d}{dt}(x^4),$$

$$\frac{d}{dt}\Big(\dot{x}^2 + x^2 - \frac{1}{12}x^4\Big) = 0.$$

$$\boxed{E \;=\; \dot{x}^2 + x^2 - \frac{1}{12}x^4.}$$

Thus the energy is conserved.
For $E > 0$ small enough, consider $\dot{x} = \pm\sqrt{E - x^2 + \frac{1}{12}x^4}$. For small E, $x \sim \sqrt{E}$.
Thus, there are periodic solutions in a neighborhood of 0.

c) *For such periodic solutions, define the amplitude as* $a = \max_t x(t)$. *Find an* ***Integral Formula for the Period T*** *of a periodic solution as a function of the amplitude* a.

Note that at maximum amplitude, $\dot{x} = 0$. We have

$$E \;=\; \dot{x}^2 + x^2 - \frac{1}{12}x^4,$$

$$\dot{x} \;=\; \sqrt{E - x^2 + \frac{1}{12}x^4},$$

$$\int_0^{\frac{T}{4}} \frac{\dot{x}}{\sqrt{E - x^2 + \frac{1}{12}x^4}}\,dt \;=\; \int_0^{\frac{T}{4}} dt \;=\; \frac{T}{4},$$

$$T \;=\; 4\int_0^{\frac{T}{4}} \frac{\dot{x}}{\sqrt{E - x^2 + \frac{1}{12}x^4}}\,dt.$$

Making change of variables: $\xi = x(t)$, $d\xi = \dot{x}(t)dt$, we obtain

$$\boxed{T(a) \;=\; 4\int_0^a \frac{d\xi}{\sqrt{E - \xi^2 + \frac{1}{12}\xi^4}}.}$$

d) *Show that* T *is a non-decreasing function of* a.

$$\frac{dT}{da} \;=\; 4\frac{d}{da}\int_0^a \frac{d\xi}{\sqrt{E - \xi^2 + \frac{1}{12}\xi^4}}.$$

□

Problem (S'91, #1). *Consider the autonomous ODE*

$$\frac{d^2x}{dt^2} + \sin x = 0.$$

a) *Find a nontrivial function $H(x, \frac{dx}{dt})$ that is* ***constant*** *along each solution.*[1]
b) *Write the equation as a system of 2 first order equations. Find all of the stationary points and analyze their type.*
c) *Draw a picture of the phase plane for this system.*

Proof. **a)** We have

$$\ddot{x} + \sin x = 0.$$

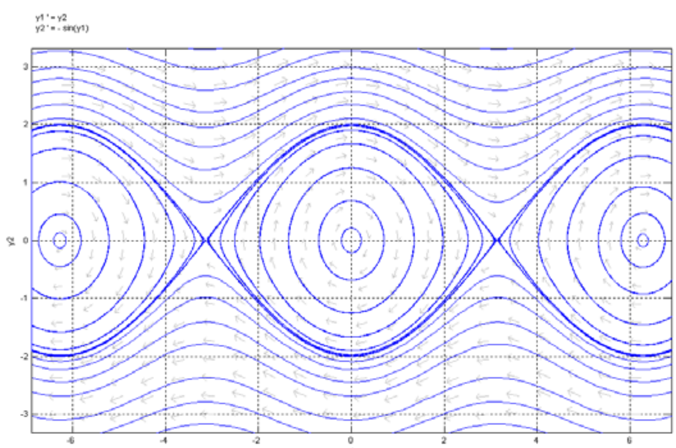
Multiply by $\dot{x}$ and integrate:

$$\begin{aligned} \dot{x}\ddot{x} + \dot{x}\sin x &= 0, \\ \frac{1}{2}\frac{d}{dt}(\dot{x}^2) + \frac{d}{dt}(-\cos x) &= 0, \\ \frac{\dot{x}^2}{2} - \cos x &= C, \\ H(x, \dot{x}) &= \frac{\dot{x}^2}{2} - \cos x. \end{aligned}$$

$H(x, \dot{x})$ is constant along each solution. Check:

$$\frac{d}{dt}H(x, \dot{x}) = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial \dot{x}}\ddot{x} = (\sin x)\dot{x} + \dot{x}(-\sin x) = 0.$$

b,c) [2]



$\square$

[1] Note that H does not necessarily mean that it is a Hamiltonian.
[2] See S'98 #1a.

2.7 Stability and Asymptotic Stability

$$y' = f(y) \tag{2.4}$$

An equilibrium solution y_0 of (2.4) is **stable** if $\forall\epsilon$, $\exists\delta(\epsilon)$ such that whenever any solution $\psi(t)$ of (2.4) satisfies $|\psi(t_0) - y_0| < \delta$, we have $|\psi(t) - y_0| < \epsilon$.

An equilibrium solution y_0 of (2.4) is **asymptotically stable** if it is stable, and $\exists\delta_0 > 0$, such that whenever any solution $\psi(t)$ of (2.4) satisfies $|\psi(t_0) - y_0| < \delta_0$, we have $\lim_{t\to\infty} |\psi(t) - y_0| = 0$.

$$y' = f(t, y) \tag{2.5}$$

A solution $\phi(t)$ of (2.5) is **stable** if $\forall\epsilon$, $\forall t_0 \geq 0$, $\exists\delta(\epsilon, t_0) > 0$ such that whenever any solution $\psi(t)$ of (2.5) satisfies $|\psi(t_0) - \phi(t_0)| < \delta$, we have $|\psi(t) - \phi(t)| < \epsilon$, $\forall t \geq t_0$.

A solution $\phi(t)$ of (2.5) is **asymptotically stable** if it is stable, and $\exists\delta_0 > 0$, such that whenever any solution $\psi(t)$ of (2.5) satisfies $|\psi(t_0) - \phi(t_0)| < \delta_0$, we have $\lim_{t\to\infty} |\psi(t) - \phi(t)| = 0$.

- $Re(\lambda_j) \leq 0$, and when $Re(\lambda_j) = 0$, λ_j is **simple** $\Rightarrow y \equiv 0$ is **stable**
- $Re(\lambda_j) < 0 \Rightarrow y \equiv 0$ is **asymptotically stable**
  $e^{A(t-t_0)}$ a fundamental matrix. $\exists K > 0, \sigma > 0$, s.t. $|e^{A(t-t_0)}| \leq Ke^{-\sigma(t-t_0)}$
- $Re(\lambda_0) > 0 \Rightarrow y \equiv 0$ is **unstable**.

$$y' = (A + B(t))y \tag{2.6}$$

Theorem. *$Re(\lambda_j) < 0$, $B(t)$ continuous for $0 \le t < \infty$ and such that $\int_0^\infty |B(s)|ds < \infty$. Then the zero solution of (2.6) is* ***asymptotically stable****.*

Proof. $y' = (A + B(t))y = Ay + \underbrace{B(t)y}_{g(t)}$, $\quad g(t)$ is an inhomogeneous term.

Let $\psi(t)$ be a solution to the ODE with $\psi(t_0) = y_0$.
By the variation of constants formula:

$$\begin{aligned}
\psi(t) &= e^{A(t-t_0)}y_0 + \int_{t_0}^t e^{A(t-s)}B(s)\psi(s)ds \\
&\quad \text{Note: } \psi(t_0) = y_0 \\
&\quad y_0 = e^{t_0A}\eta \quad \Rightarrow \quad \eta = e^{-t_0A}y_0 = e^{-t_0A}\psi(t_0). \\
|\psi(t)| &\le |e^{A(t-t_0)}||y_0| + \int_{t_0}^t |e^{A(t-s)}||\psi(s)||B(s)|ds \\
&\quad Re(\lambda_j) < 0 \quad \Rightarrow \quad \exists K, \sigma > 0, \text{ such that} \\
&\quad |e^{A(t-t_0)}| \le Ke^{-\sigma(t-t_0)}, \qquad t_0 \le t < \infty \\
&\quad |e^{A(t-s)}| \le Ke^{-\sigma(t-s)}, \qquad t_0 \le s < \infty \\
|\psi(t)| &\le Ke^{-\sigma(t-t_0)}|y_0| + K\int_{t_0}^t e^{-\sigma(t-s)}|\psi(s)||B(s)|ds \\
\underbrace{e^{\sigma t}|\psi(t)|}_{u(t)} &\le \underbrace{Ke^{\sigma t_0}|y_0|}_{c} + K\int_{t_0}^t \underbrace{e^{\sigma s}|\psi(s)|}_{u(s)}\underbrace{|B(s)|}_{v(s)}ds \\
&\quad \text{By Gronwall Inequality:} \\
e^{\sigma t}|\psi(t)| &\le Ke^{\sigma t_0}|y_0|e^{K\int_{t_0}^t |B(s)|ds} \\
|\psi(t)| &\le Ke^{-\sigma(t-t_0)}|y_0|e^{K\int_{t_0}^t |B(s)|ds} \\
&\quad \text{But } K\int_{t_0}^t |B(s)|ds \le M_0 < \infty \quad \Rightarrow \quad e^{K\int_{t_0}^t |B(s)|ds} \le e^{M_0} = M_1, \\
|\psi(t)| &\le KM_1e^{-\sigma(t-t_0)}|y_0| \to 0, \text{ as } t \to \infty.
\end{aligned}$$

Thus, the zero solution of $y' = (A + B(t))y$ is asymptotically stable. ☐

Theorem. *Suppose all solutions of $y' = Ay$ are bounded. Let $B(t)$ be continuous for $0 \le t < \infty$, and $\int_0^\infty |B(s)|ds < \infty$. Show all solutions of $y' = (A+B(t))y$ are* ***bounded*** *on $t_0 < t < \infty$.*

Proof.

$$y' = Ay \tag{2.7}$$

$$y' = (A + B(t))y \tag{2.8}$$

Solutions of (2.7) can be written as $e^{tA}c_0$, where e^{tA} is the fundamental matrix.
Since all solutions of (2.7) are bounded, $|e^{tA}c_0| \le c$, $0 \le t < \infty$.

Now look at the solutions of non-homogeneous equation (2.8). By the variation of constants formula and the previous exercise,

$$\begin{aligned}
\psi(t) &= e^{A(t-t_0)}y_0 + \int_{t_0}^{t} e^{A(t-s)}B(s)\psi(s)ds \\
|\psi(t)| &\le |e^{A(t-t_0)}||y_0| + \int_{t_0}^{t} |e^{A(t-s)}||\psi(s)||B(s)|ds \le c|y_0| + c\int_{t_0}^{t} |\psi(s)||B(s)|ds
\end{aligned}$$

By Gronwall Inequality,

$$|\psi(t)| \le c|y_0|e^{c\int_{t_0}^{t}|B(s)|ds}.$$

$$\text{But} \qquad \int_{t_0}^{t}|B(s)|ds < \infty \quad \Rightarrow \quad c\int_{t_0}^{t}|B(s)|ds < M_0, \ \Rightarrow \ e^{c\int_{t_0}^{t}|B(s)|ds} \le M_1.$$

$$\Rightarrow \qquad |\psi(t)| \le c|y_0|M_1 \le \widetilde{K}.$$

Thus, all solutions of (2.8) are bounded.
Claim: The zero solution of $y' = (A + B(t))y$ is **stable**.
An equilibrium solution y_0 is stable if $\forall\epsilon$, $\exists\delta(\epsilon)$ such that whenever any solution $\psi(t)$ satisfies $|\psi(t_0) - y_0| < \delta$, we have $|\psi(t) - y_0| < \epsilon$.
We had $|\psi(t)| \le c|\psi_0|M_1$. Choose $|\psi(t_0)|$ small enough such that $\forall\epsilon$, $\exists\delta(\epsilon)$ such that $|\psi(t_0)| < \delta < \frac{\epsilon}{CM_1}$
$\Rightarrow \quad |\psi(t) - 0| = |\psi(t)| \le c|\psi(t_0)|M_1 < c\delta M_1 < \epsilon$.
Thus, the zero solution of $y' = (A + B(t))y$ is stable. $\square$

$$y' = (A + B(t))y + f(t, y) \tag{2.9}$$

Theorem. *i)* $Re(\lambda_j) < 0$, $f(t, y)$ *and* $\frac{\partial f}{\partial y_j}(t, y)$ *are continuous in* (t, y).
ii) $\lim_{|y|\to 0} \frac{|f(t,y)|}{|y|} = 0$ *uniformly with respect to* t.
iii) $B(t)$ *continuous.* $\lim_{t\to\infty} B(t) = 0$.
Then the solution $y \equiv 0$ *of (2.9) is* ***asymptotically stable****.*

2.8 Conditional Stability

$$y' = Ay + g(y) \tag{2.10}$$

Theorem. $g, \frac{\partial g}{\partial y_j}$ *continuous,* $g(0) = 0$ *and* $\lim_{|y|\to 0} \frac{|g(y)|}{|y|} = 0$*. If the eigenvalues of* A *are* $\lambda, -\mu$ *with* $\lambda, \mu > 0$*, then* $\exists$ *a curve* C *in the phase plane of original equation passing through* 0 *such that if any solution* $\phi(t)$ *of (2.10) with* $|\phi(0)|$ *small enough starts on* C*, then* $\phi(t) \to 0$ *as* $t \to \infty$*. No solution* $\phi(t)$ *with* $|\phi(0)|$ *small enough that does not start on* C *can remain small. In particular,* $\phi \equiv 0$ *is unstable.*

2.9 Asymptotic Equivalence

$$x' = A(t)x \tag{2.11}$$

$$y' = A(t)y + f(t, y) \tag{2.12}$$

The two systems are **asymptotically equivalent** if to any solution $x(t)$ of (2.11) with $x(t_0)$ small enough there corresponds a solution $y(t)$ of (2.12) such that

$$\lim_{t\to\infty} |y(t) - x(t)| = 0$$

and if to any solution $\hat{y}(t)$ of (2.12) with $\hat{y}(t_0)$ small enough there corresponds a solution $\hat{x}(t)$ of (2.11) such that

$$\lim_{t\to\infty} |\hat{y}(t) - \hat{x}(t)| = 0$$

2.9.1 Levinson

Theorem. *A is a constant matrix such that all solutions of $x' = Ax$ are bounded on $0 \le t < \infty$. $B(t)$ is a continuous matrix such that $\int_0^\infty |B(s)|ds < \infty$. Then, the systems $x' = Ax$ and $y' = (A + B(t))y$ are **asymptotically equivalent**.*

3 Lyapunov's Second Method

Lagrange's Principle. *If the rest position of a conservative mechanical system has minimum potential energy, then this position corresponds to a stable equilibrium. If the rest position does not have minimum potential energy, then the equilibrium position is unstable.*

3.1 Hamiltonian Form

A system of 2 (or $2n$) equations determined by a single scalar function $H(y,z)$ (or $H(y_1,\ldots,y_n,z_1,\ldots,z_n)$) is called Hamiltonian if it is of the form

$$H(y,z) \qquad y' = \frac{\partial H}{\partial z} \qquad z' = -\frac{\partial H}{\partial y}$$

$$H(y_1,\ldots,y_n,z_1,\ldots,z_n) \qquad y_i' = \frac{\partial H}{\partial z_i} \qquad z_i' = -\frac{\partial H}{\partial y_i} \qquad (i=1,\ldots,n) \tag{3.1}$$

Problem. *If $\phi = (\phi_1,\ldots,\phi_{2n})$ is any solution of the Hamiltonian system (3.1), then $H(\phi_1,\ldots,\phi_{2n})$ is constant.*

Proof. Need to show $\frac{dH}{dt} = 0$.
Can relabel: $H(\phi_1,\ldots,\phi_n,\phi_{n+1},\ldots,\phi_{2n}) = H(y_1,\ldots,y_n,z_1,\ldots z_n)$.

$$\begin{aligned}
\frac{dH}{dt} &= \frac{d}{dt}H(\phi_1,\ldots,\phi_n,\phi_{n+1},\ldots,\phi_{2n}) \\
&= \frac{\partial H}{\partial \phi_1}\frac{d\phi_1}{dt} + \cdots + \frac{\partial H}{\partial \phi_n}\frac{d\phi_n}{dt} + \frac{\partial H}{\partial \phi_{n+1}}\frac{d\phi_{n+1}}{dt} + \cdots + \frac{\partial H}{\partial \phi_{2n}}\frac{d\phi_{2n}}{dt} \\
&= \sum_{i=1}^n \frac{\partial H}{\partial \phi_i}\frac{d\phi_i}{dt} + \sum_{i=1}^n \frac{\partial H}{\partial \phi_{n+i}}\frac{d\phi_{n+i}}{dt} = \sum_{i=1}^n \frac{\partial H}{\partial y_i}\frac{dy_i}{dt} + \sum_{i=1}^n \frac{\partial H}{\partial z_i}\frac{dz_i}{dt} \\
&= (\text{by } (3.1)) = \sum_{i=1}^n \frac{\partial H}{\partial y_i}\frac{\partial H}{\partial z_i} + \sum_{i=1}^n \frac{\partial H}{\partial z_i}\Big(-\frac{\partial H}{\partial y_i}\Big) = 0.
\end{aligned}$$

Thus, $H(\phi_1,\ldots,\phi_{2n})$ is constant. $\square$

Problem (F'92, #5). *Let $x = x(t)$, $p = p(t)$ be a solution of the* ***Hamiltonian system***

$$\frac{dx}{dt} = \frac{\partial}{\partial p} H(x,p), \qquad x(0) = y$$
$$\frac{dp}{dt} = \frac{\partial}{\partial x} H(x,p), \qquad p(0) = \xi.$$

Suppose that H is smooth and satisfies

$$\left|\frac{\partial H}{\partial x}(x,p)\right| \le C\sqrt{|p|^2+1}$$
$$\left|\frac{\partial H}{\partial p}(x,p)\right| \le C.$$

Prove that this system has a finite solution $x(t)$, $p(t)$ for $-\infty < t < \infty$.

Proof.

$$\begin{aligned} x(t) &= x(0) + \int_0^t \frac{dx}{ds}\, ds, \\ |x(t)| &\le |x(0)| + \int_0^t \left|\frac{dx}{ds}\right| ds \;=\; |x(0)| + \int_0^t \left|\frac{\partial H}{\partial p}\right| ds \;\le\; |x(0)| + C\int_0^t ds \;=\; |x(0)| + Ct. \end{aligned}$$

Thus, $x(t)$ is finite for finite t.

$$\begin{aligned} p(t) &= p(0) + \int_0^t \frac{dp}{ds}\, ds, \\ |p(t)| &\le |p(0)| + \int_0^t \left|\frac{dp}{ds}\right| ds \;=\; |p(0)| + \int_0^t \left|\frac{\partial H}{\partial x}\right| ds \;\le\; |p(0)| + C\int_0^t \sqrt{|p|^2+1}\, ds \\ &\le |p(0)| + C\int_0^t (1+|p|)\, ds = |p(0)| + Ct + C\int_0^t |p|\, ds \\ &\le (|p(0)| + Ct)e^{\int_0^t C\, ds} \;\le\; (|p(0)| + Ct)e^{Ct}, \end{aligned}$$

where we have used Gronwall (Integral) Inequality. [3] Thus, $p(t)$ is finite for finite t. $\square$

[3] **Gronwall (Differential) Inequality:** $v(t)$ piecewise continuous on $t_0 \le t \le t_0 + a$. $u(t)$ and $\frac{du}{dt}$ continuous on some interval. If

$$\frac{du}{dt} \le v(t)u(t)$$
$$\Rightarrow \quad u(t) \le u(t_0)e^{\int_{t_0}^t v(s)ds}$$

Gronwall (Integral) Inequality: $u(t), v(t)$ continuous on $[t_0, t_0 + a]$. $v(t) \ge 0$, $c \ge 0$.

$$u(t) \le c + \int_{t_0}^t v(s)u(s)ds$$
$$\Rightarrow \quad u(t) \le c\, e^{\int_{t_0}^t v(s)ds} \qquad t_0 \le t \le t_0 + a$$

3.2 Lyapunov's Theorems

Definitions: $y' = f(y)$
The scalar function $V(y)$ is said to be **positive definite** if $V(0) = 0$ and $V(y) > 0$ for all $y \neq 0$ in a small neighborhood of 0.
The scalar function $V(y)$ is **negative definite** if $-V(y)$ is positive definite.
The derivative of V with respect to the system $y' = f(y)$ is the scalar product

$$V^*(y) = \nabla V \cdot f(y)$$

$$\frac{d}{dt}V(y(t)) = \nabla V \cdot f(y) = V^*(y)$$

$\Rightarrow$ along a solution y the total derivative of $V(y(t))$ with respect to t coincides with the derivative of V with respect to the system evaluated at $y(t)$.

3.2.1 Stability (Autonomous Systems)

If $\exists V(y)$ that is positive definite and for which $V^*(y) \leq 0$ in a neighborhood of 0, then the zero solution is **stable**.
If $\exists V(y)$ that is positive definite and for which $V^*(y)$ is negative definite in a neighborhood of 0, then the zero solution is **asymptotically stable**.
If $\exists V(y),\ V(0) = 0$, such that $V^*(y)$ is either positive definite or negative definite, and every neighborhood of 0 contains a point $a \neq 0$ such that $V(a)V^*(a) > 0$, then the 0 solution is **unstable**.

Problem (S'00, #6).
a) *Consider the system of ODE's in $\mathbb{R}^{2n}$ given in vector notation by*

$$\frac{dx}{dt} = f(|x|^2)p \quad \textit{and} \quad \frac{dp}{dt} = -f'(|x|^2)|p|^2 x,$$

where $x = (x_1, \ldots, x_n)$, $p = (p_1, \ldots, p_n)$, and $f > 0$, smooth on $\mathbb{R}$. We use the notation $x \cdot p = x_1 p_1 + \cdots + x_n p_n$, $|x|^2 = x \cdot x$ and $|p|^2 = p \cdot p$.
Show that $|x|$ is increasing with t when $p \cdot x > 0$ and decreasing with t when $p \cdot x < 0$, and that $H(x,p) = f(|x|^2)|p|^2$ is constant on solutions of the system.
b) *Suppose $\frac{f(s)}{s}$ has a critical value at $s = r^2$. Show that solutions with $x(0)$ on the shpere $|x| = r$ and $p(0)$ perpendicular to $x(0)$ must remain on the sphere $|x| = r$ for all t. [Compute $\frac{d(p \cdot x)}{dt}$ and use part (a)].*

Proof. **a)**
- Consider $\boldsymbol{p \cdot x > 0}$:

Case ①: $p > 0$, $x > 0 \;\Rightarrow\; \frac{dx}{dt} > 0 \;\Rightarrow\; x = |x|$ is increasing.
Case ②: $p < 0$, $x < 0 \;\Rightarrow\; \frac{dx}{dt} < 0 \;\Rightarrow\; x = -|x|$ is decreasing $\;\Rightarrow\; |x|$ is increasing.
- Consider $\boldsymbol{p \cdot x < 0}$:

Case ③: $p > 0$, $x < 0 \;\Rightarrow\; \frac{dx}{dt} > 0 \;\Rightarrow\; x = -|x|$ is increasing $\;\Rightarrow\; |x|$ is decreasing.
Case ④: $p < 0$, $x > 0 \;\Rightarrow\; \frac{dx}{dt} < 0 \;\Rightarrow\; x = |x|$ is decreasing.
Thus, $|x|$ is increasing with t when $p \cdot x > 0$ and decreasing with t when $p \cdot x < 0$. ✓

To show $H(x,p) = f(|x|^2)|p|^2$ is constant on solutions of the system, consider

$$\begin{aligned}
\frac{dH}{dt} &= \frac{d}{dt}\Big[f(|x|^2)|p|^2\Big] = f'(|x|^2) \cdot 2x\dot{x}|p|^2 + f(|x|^2) \cdot 2p\dot{p} \\
&= f'(|x|^2) \cdot 2x f(|x|^2) p |p|^2 + f(|x|^2) \cdot 2p \cdot \big(-f'(|x|^2)|p|^2 x\big) = 0. \quad \checkmark
\end{aligned}$$

Thus, $H(x,p)$ is constant on solutions of the system.

b) $G(s) = \frac{f(s)}{s}$ has a critical value at $s = r^2$. Thus,

$$\begin{aligned}
G'(s) &= \frac{sf'(s) - f(s)}{s^2}, \\
G'(r^2) = 0 &= \frac{r^2 f'(r^2) - f(r^2)}{r^4}, \\
0 &= r^2 f'(r^2) - f(r^2).
\end{aligned}$$

Since $p(0)$ and $x(0)$ are perpendicular, $p(0) \cdot x(0) = 0$.

$$\begin{aligned}
\frac{d(p \cdot x)}{dt} &= x\frac{dp}{dt} + p\frac{dx}{dt} = -f'(|x|^2)|p|^2|x|^2 + f(|x|^2)|p|^2 = |p|^2\Big(f(|x|^2) - f'(|x|^2)|x|^2\Big), \\
\Rightarrow \qquad \frac{d(p \cdot x)}{dt}(t = 0) &= |p|^2\Big(f(r^2) - f'(r^2)r^2\Big) = |p|^2 \cdot 0 = 0.
\end{aligned}$$

Also, $\frac{d(p \cdot x)}{dt} = 0$ holds for all $|x| = r$. Thus, $p \cdot x = C$ for $|x| = r$. Since, $p(0) \cdot x(0) = 0$, $p \cdot x = 0$. Hence, p and x are always perpendicular, and solution never leaves the sphere.

Note: The system

$$\frac{dx}{dt} = f(|x|^2)p \quad \text{and} \quad \frac{dp}{dt} = -f'(|x|^2)|p|^2 x,$$

determined by $H(x,p) = f(|x|^2)|p|^2$ is Hamiltonian.

$$\dot{x} = \frac{\partial H}{\partial p} = 2f(|x|^2)|p|, \qquad \dot{p} = -\frac{\partial H}{\partial x} = -2xf'(|x|^2)|p|^2.$$

$\square$

Example 1. *Determine the stability property of the critical point at the origin for the following system.*

$$\begin{aligned} y_1' &= -y_1^3 + y_1 y_2^2 \\ y_2' &= -2y_1^2 y_2 - y_3^3 \end{aligned}$$

$$\begin{aligned} &Try \quad V(y_1, y_2) = y_1^2 + cy_2^2. \\ &\qquad V(0,0) = 0; \quad V(y_1, y_2) > 0, \ \forall y \neq 0 \quad \Rightarrow \quad V \ is \ positive \ definite. \end{aligned}$$

$$\begin{aligned} V^*(y_1, y_2) &= \frac{dV}{dt} = 2y_1 y_1' + 2cy_2 y_2' = 2y_1(-y_1^3 + y_1 y_2^2) + 2cy_2(-2y_1^2 y_2 - y_2^3) \\ &= -2y_1^4 - 2cy_2^4 + 2y_1^2 y_2^2 - 4cy_1^2 y_2^2. \\ If \ c = \frac{1}{2}, &\qquad V^*(y_1, y_2) = -2y_1^4 - y_2^4 < 0, \ \forall y \neq 0; \quad V^*(0,0) = 0 \\ \Rightarrow &\qquad V^* \ negative \ definite. \end{aligned}$$

*Since $V(y_1, y_2)$ is positive definite and $V^*(y_1, y_2)$ is negative definite, the critical point at the origin is **asymptotically stable**.*

Example 2. *Determine the stability property of the critical point at the origin for the following system.*

$$\begin{aligned} y_1' &= y_1^3 - y_2^3 \\ y_2' &= 2y_1 y_2^2 + 4y_1^2 y_2 + 2y_2^3 \end{aligned}$$

$$\begin{aligned} &Try \quad V(y_1, y_2) = y_1^2 + cy_2^2. \\ &\qquad V(0,0) = 0; \quad V(y_1, y_2) > 0, \ \forall y \neq 0 \quad \Rightarrow \quad V \ is \ positive \ definite. \end{aligned}$$

$$\begin{aligned} V^*(y_1, y_2) &= \frac{dV}{dt} = 2y_1 y_1' + 2cy_2 y_2' = 2y_1(y_1^3 - y_2^3) + 2cy_2(2y_1 y_2^2 + 4y_1^2 y_2 + 2y_2^3) \\ &= 2y_1^4 - 2y_1 y_2^3 + 4cy_1 y_2^3 + 8cy_1^2 y_2^2 + 4cy_2^4. \\ If \ c = \frac{1}{2}, &\qquad V^*(y_1, y_2) = 2y_1^4 + 4y_1^2 y_2^2 + 2y_2^4 > 0, \ \forall y \neq 0; \quad V^*(0,0) = 0 \\ \Rightarrow &\qquad V^* \ positive \ definite. \end{aligned}$$

*Since $V^*(y_1, y_2)$ is positive definite and $V(y)V^*(y) > 0$, $\forall y \neq 0$, the critical point at the origin is **unstable**.*

Example 3. *Determine the stability property of the critical point at the origin for the following system.*

$$\begin{aligned} y_1' &= -y_1^3 + 2y_2^3 \\ y_2' &= -2y_1 y_2^2 \end{aligned}$$

$$\begin{aligned} &Try \quad V(y_1, y_2) = y_1^2 + cy_2^2. \\ &\qquad V(0,0) = 0; \quad V(y_1, y_2) > 0, \ \forall y \neq 0 \quad \Rightarrow \quad V \ is \ positive \ definite. \end{aligned}$$

$$\begin{aligned} V^*(y_1, y_2) &= \frac{dV}{dt} = 2y_1 y_1' + 2cy_2 y_2' = 2y_1(-y_1^3 + 2y_2^3) + 2cy_2(-2y_1 y_2^2) \\ &= -2y_1^4 + 4y_1 y_2^3 - 4cy_1 y_2^3. \\ If \ c = 1, &\qquad V^*(y_1, y_2) = -2y_1^4 \leq 0, \ \forall y; \quad V^*(\vec{y}) = 0 \ for \ y = (0, y_2). \\ \Rightarrow &\qquad V^* \ is \ neither \ positive \ definite \ nor \ negative \ definite. \end{aligned}$$

Since V is positive definite and $V^*(y_1, y_2) \le 0$ in a neighborhood of 0, the critical point at the origin is ***at least stable****.*

V is positive definite, C^1, $V^*(y_1, y_2) \le 0$, $\forall y$. The origin is the only invariant subset of the set $E = \{y | V^*(y) = 0\} = \{(y_1, y_2) \mid y_1 = 0\}$. Thus, the critical point at the origin is ***asymptotically stable****.*

Problem (S'96, #1).
Construct a ***Liapunov function*** *of the form $ax^2 + cy^2$ for the system*

$$\begin{aligned}\dot{x} &= -x^3 + xy^2 \\ \dot{y} &= -2x^2y - y^3,\end{aligned}$$

and use it to show that the origin is a strictly stable critical point.

Proof. We let $V(x, y) = ax^2 + cy^2$.

$$\begin{aligned}V^*(x, y) &= \frac{dV}{dt} = 2ax\dot{x} + 2cy\dot{y} = 2ax(-x^3 + xy^2) + 2cy(-2x^2y - y^3) \\ &= -2ax^4 + 2ax^2y^2 - 4cx^2y^2 - 2cy^4 = -2ax^4 + (2a - 4c)x^2y^2 - 2cy^4.\end{aligned}$$

For $2a - 4c < 0$, i.e. $a < 2c$, we have $V^*(x, y) < 0$. For instance, $c = 1$, $a = 1$.
Then, $V(0, 0) = 0$; $V(x, y) > 0$, $\forall (x, y) \neq (0, 0) \quad \Rightarrow \quad V$ is positive definite.
Also, $V^*(0, 0) = 0$; $V^*(x, y) = -2ax^4 - 2x^2y^2 - 2cy^4 < 0$, $\forall (x, y) \neq (0, 0)$
$\Rightarrow \quad V^*$ is negative definite.
Since $V(x, y)$ is positive definite and $V^*(x, y)$ is negative definite, the critical point at the origin is **asymptotically stable**.

$\square$

Example 4. *Consider the equation $u'' + g(u) = 0$, where g is C^1 for $|u| < k$, $k > 0$, and $ug(u) > 0$ if $u \neq 0$. Thus, by continuity, $g(0) = 0$. Write the equation as a system*

$$\begin{aligned}y_1' &= y_2 \\ y_2' &= -g(y_1)\end{aligned}$$

and the origin is an isolated critical point. Set

$$V(y_1, y_2) = \frac{y_2^2}{2} + \int_0^{y_1} g(\sigma)d\sigma.$$

Thus, $V(0, 0) = 0$ and since $\sigma g(\sigma) > 0$, $\int_0^{y_1} g(\sigma)d\sigma > 0$ for $0 < |y_1| < k$.
Therefore, $V(y_1, y_2)$ is positive definite on $\Omega = \{(y_1, y_2) \mid |y_1| < k, |y_2| < \infty\}$.

$$V^*(y_1, y_2) = \frac{dV}{dt} = y_2y_2' + g(y_1)y_1' = y_2(-g(y_1)) + g(y_1)y_2 = 0.$$

Since V is positive definite and $V^*(y_1, y_2) \le 0$ in a neighborhood of 0, the critical point at the origin is ***stable****.*

Example 5. The Lienard Equation. *Consider the scalar equation*

$$u'' + u' + g(u) = 0$$

or, written as a system,

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -g(y_1) - y_2 \end{aligned}$$

where g is C^1, $ug(u) > 0$, $u \neq 0$. Try

$$V(y_1, y_2) = \frac{y_2^2}{2} + \int_0^{y_1} g(\sigma)d\sigma.$$

V is positive definite on $\Omega = \{(y_1, y_2) \mid |y_1| < k, |y_2| < \infty\}$.

$$V^*(y_1, y_2) = \frac{dV}{dt} = y_2 y_2' + g(y_1)y_1' = y_2(-g(y_1) - y_2) + g(y_1)y_2 = -y_2^2.$$

*Since $V^*(y_1, y_2) \leq 0$ in Ω, the solution is **stable**. But $V^*(y_1, y_2)$ is not negative definite on Ω ($V^*(y_1, y_2) = 0$ at all points $(y_1, 0)$). Even though the solution is asymptotically stable, we cannot infer this here by using Lyapunov's theorems.*[4]

[4] See the example in 'Invariant Sets and Stability' section.

3.3 Periodic Solutions

Problem. *Consider the 2-dimensional autonomous system $y' = f(y)$ where $f(y) \in C^1(\mathbb{R}^2)$. Let $\Omega \in \mathbb{R}^2$ be simply connected, such that $\forall y \in \Omega$, we have* $\mathbf{div\, f(y) \neq 0}$. *Show that the ODE system has* ***no periodic solutions*** *in Ω.*

Proof. Towards a contradiction, assume ODE system has a periodic solution in Ω. Let $\partial\Omega$ be a boundary on Ω.

$$y' = f(y) \quad \Rightarrow \quad \begin{cases} y_1' = f_1(y_1, y_2), \\ y_2' = f_2(y_1, y_2). \end{cases}$$

$n = (n_1, n_2) = (y_2', -y_1')$ is the normal to $\partial\Omega$. Recall Divergence Theorem:

$$\oint_{\partial\Omega} f \cdot n \; ds = \iint_{\Omega} div\, f \; dA.$$

Let y be a periodic solution with period T, i.e. $y(t+T) = y(t)$. Then, a path traversed by a solution starting from $t = a$ to $t = a+T$ is $\partial\Omega$. Then, $\partial\Omega$ is a closed curve.

$$\begin{aligned} \oint_{\partial\Omega} f \cdot n \; ds &= \int_{\partial\Omega} (f_1 n_1 + f_2 n_2) \; ds = \int_a^{a+T} (y_1' y_2' - y_2' y_1') \; dt = 0 \\ &\Rightarrow \quad \iint_{\Omega} div\, f \; dA = 0. \end{aligned}$$

However, by hypothesis, $div\, f(y) \neq 0$ and $f \in C^1$. Therefore, $div\, f \in C^0$, and either $div\, f > 0$ or $div\, f < 0$ on Ω. Thus, $\iint_\Omega div\, f \; dA > 0$ or $\iint_\Omega div\, f \; dA < 0$, a contradiction. □

Example. *Show that the given system has* ***no*** *non-trivial* ***periodic solutions:***

$$\begin{aligned} \frac{dx}{dt} &= x + y + x^3 - y^2, \\ \frac{dy}{dt} &= -x + 2y + x^2 y + \frac{y^3}{3}. \end{aligned}$$

Proof. $\frac{dx}{dt} = f_1(x, y)$, $\frac{dy}{dt} = f_2(x, y)$.

$$\mathrm{div}\, f(x, y) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = (1 + 2x^2) + (2 + x^2 + y^2) = 3 + 3x^2 + y^2 > 0.$$

By the problem above, the ODE system has no periodic solutions. □

Problem (F'04, #5).
Consider a generalized Volterra-Lotka system in the plane, given by

$$x'(t) = f(x(t)), \qquad x(t) \in \mathbb{R}^2, \tag{3.2}$$

where $f(x) = (f_1(x), f_2(x)) = (ax_1 - bx_1x_2 - ex_1^2,\ -cx_2 + dx_1x_2 - fx_2^2)$ *and* a, b, c, d, e, f *are positive constants. Show that*

$$\mathbf{div}(\varphi \mathbf{f}) \neq \mathbf{0} \qquad x_1 > 0,\ x_2 > 0,$$

where $\varphi(x_1, x_2) = 1/(x_1x_2)$*. Using this observation, prove that the autonomous system (3.2) has* ***no closed orbits*** *in the first quadrant.*

Proof.

$$\begin{aligned}
\varphi f &= \begin{pmatrix} \frac{ax_1 - bx_1x_2 - ex_1^2}{x_1x_2} \\ \frac{-cx_2 + dx_1x_2 - fx_2^2}{x_1x_2} \end{pmatrix} = \begin{pmatrix} ax_2^{-1} - b - ex_1x_2^{-1} \\ -cx_1^{-1} + d - fx_1^{-1}x_2 \end{pmatrix}, \\
div(\varphi f) &= \frac{\partial}{\partial x_1}(ax_2^{-1} - b - ex_1x_2^{-1}) + \frac{\partial}{\partial x_2}(-cx_1^{-1} + d - fx_1^{-1}x_2) = -ex_2^{-1} - fx_1^{-1} \neq 0,
\end{aligned}$$

for $x_1,\ x_2 > 0, \quad f,\ e > 0.$ ✓

Towards a contradiction, assume ODE system has a closed orbit in the first quadrant. Let Ω be a bounded domain with an orbit that is $\partial\Omega$.
Let x be a periodic solution with a period T, i.e. $x(t+T) = x(t)$.
$n = (n_1, n_2) = (x_2', -x_1')$ is the normal to $\partial\Omega$. By Divergence Theorem,

$$\begin{aligned}
\int_\Omega div(\varphi f)\, dx &= \int_{\partial\Omega} (\varphi f) \cdot n\, dS = \int_{\partial\Omega} \varphi(f_1n_1 + f_2n_2)\, dS \\
&= \int_a^{a+T} \varphi\, (x_1'x_2' - x_2'x_1')\, dt = 0.
\end{aligned}$$

Since $\varphi f \in C^1$ in Ω, then $div(\varphi f) \in C^0$ in Ω.
Thus, the above result implies $div(\varphi f) = 0$ for some $(x_1, x_2) \in \Omega$,
which contradicts the assumption. ∎

Problem (F'04, #4).
Prove that each solution (except $x_1 = x_2 = 0$) of the autonomous system

$$\begin{cases} x_1' = \quad x_2 + x_1(x_1^2 + x_2^2) \\ x_2' = -x_1 + x_2(x_1^2 + x_2^2) \end{cases}$$

blows up in finite time. What is the blow-up time for the solution which starts at the point $(1, 0)$ when $t = 0$?

Proof. We have $r^2 = x_1^2 + x_2^2$. Multiply the first equation by x_1 and the second by x_2:

$$\begin{aligned} x_1 x_1' &= x_1 x_2 + x_1^2(x_1^2 + x_2^2), \\ x_2 x_2' &= -x_1 x_2 + x_2^2(x_1^2 + x_2^2). \end{aligned}$$

Add equations:

$$\begin{aligned} x_1 x_1' + x_2 x_2' &= (x_1^2 + x_2^2)(x_1^2 + x_2^2), \\ \frac{1}{2}(x_1^2 + x_2^2)' &= (x_1^2 + x_2^2)(x_1^2 + x_2^2), \\ \frac{1}{2}(r^2)' &= r^4, \\ r r' &= r^4, \\ r' &= r^3, \\ \frac{dr}{dt} &= r^3, \\ \frac{dr}{r^3} &= dt, \\ -\frac{1}{2r^2} &= t + C, \\ r &= \sqrt{\frac{-1}{2(t+C)}}. \end{aligned}$$

Thus, solution blows up at $t = -C$. We determine C.
Initial conditions: $x_1(0) = 1$, $x_2(0) = 0 \quad \Rightarrow \quad r(0) = 1$.

$$\begin{aligned} 1 \;=\; r(0) &= \sqrt{\frac{-1}{2C}}, \\ C &= -\frac{1}{2}, \\ \Rightarrow \quad r &= \sqrt{\frac{-1}{2t-1}} = \sqrt{\frac{1}{1-2t}}. \end{aligned}$$

Thus, the blow-up time is $t = \frac{1}{2}$. $\square$

3.4 Invariant Sets and Stability

A set K of points in phase space is **invariant** with respect to the system $y' = f(y)$ if every solution of $y' = f(y)$ starting in K remains in K for all future time.
A point $p \in \mathbb{R}^n$ is said to lie in the **positive limit set** $L(C^+)$ (or is said to be a limit point of the orbit C^+) of the solution $\phi(t)$ iff for the solution $\phi(t)$ that gives C^+ for $t \geq 0$, $\exists$ a sequence $\{t_n\} \to +\infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \phi(t_n) = p$.
Remark: $V^* \leq 0$, $S_\lambda = \{y \in \mathbb{R}^n : V(y) \leq \lambda\}$.
For every λ the set S_λ, in fact, each of its components, is an **invariant** set with respect to $y' = f(y)$.
Reasoning: if $y_0 \in S_\lambda$ and $\phi(t, y_0)$ is solution $\Rightarrow$

$$
\begin{aligned}
&\Rightarrow \quad \frac{d}{dt} V(\phi(t, y_0)) = V^*(\phi(t, y_0)) \leq 0 \\
&\Rightarrow \quad V(\phi(t, y_0)) \leq V(\phi(0, y_0)), \quad \forall t \geq 0 \\
&\Rightarrow \quad \phi(t, y_0) \in S_\lambda, \quad \forall t \geq 0
\end{aligned}
$$

$\Rightarrow \quad S_\lambda$ invariant (as its components).
• If the solution $\phi(t, y_0)$ is **bounded** for $t \geq 0 \quad \Rightarrow \quad L(C^+)$ is a nonempty **closed, connected, invariant** set. Moreover, the solution $\phi(t, y_0) \to L(C^+)$ as $t \to \infty$.
• $V \in \Omega$ is C^1. $V^* \leq 0$ on Ω. Let $y_0 \in \Omega$ and $\phi(t, y_0)$ be bounded with $\phi(t, y_0) \in \Omega$, $\forall t \geq 0$. Assume that $L(C^+)$ lies in Ω. Then, $V^*(y) = 0$ at all points of $L(C^+)$.
• V positive definite, C^1, $V^* \leq 0$. Let the origin be the only invariant subset of the set $\{y | V^*(y) = 0\}$. Then the sero solution is **asymptotically stable**.
• V nonnegative, C^1, $V^* \leq 0$, $V(0) = 0$. Let M be the largest invariant subset of $\{y | V^*(y) = 0\}$. Then all bounded solutions approach the set M as $t \to \infty$.
• $L(C^+)$ contains a closed (periodic) orbit $\Rightarrow L(C^+)$ contains no other points.
• The limit set can not be a closed disk topologically.

Example. The Lienard Equation. *Consider the scalar equation*

$$u'' + f(u)u' + g(u) = 0$$

where $f(u) > 0$ *for* $u \neq 0$ *and* $ug(u) > 0$ *for* $u \neq 0$. *Written as a system,*

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -f(y_1)y_2 - g(y_1) \end{aligned}$$

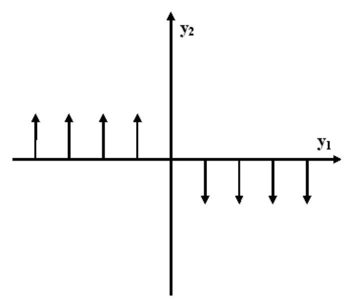
$$V(y_1, y_2) = \frac{y_2^2}{2} + \int_0^{y_1} g(\sigma)d\sigma.$$

$V(0,0) = 0$; $V(y_1, y_2) > 0$, $\forall y \neq 0$, *so* V *is positive definite.*

$$V^*(y_1, y_2) = \frac{dV}{dt} = y_2 y_2' + g(y_1)y_1' = y_2(-f(y_1)y_2 - g(y_1)) + g(y_1)y_2 = -\underbrace{f(y_1)}_{>0}\underbrace{y_2^2}_{\geq 0} \leq 0.$$

The zero solution is ***at least stable*** *by one of Lyapunov's theorems.*
$V^*(y_1, 0) = 0$ *on* y_1 *axis* $\Rightarrow$ $E = \{y \mid V^*(y) = 0\} = \{y \mid (y_1, 0)\}$ $\Rightarrow$ E *is* y_1*-axis.*
A set Γ *of points in phase space is invariant if every solution that starts in* Γ *remains in* Γ *for all time.*
On y_1*-axis* $(y_2 = 0)$:

$$\begin{aligned} \frac{dy_1}{dt} &= 0 \\ \frac{dy_2}{dt} &= -g(y_1) = -\begin{cases} > 0, y_1 > 0, \\ < 0, y_1 < 0. \end{cases} = \begin{cases} < 0, y_1 > 0, \\ > 0, y_1 < 0. \end{cases} \end{aligned}$$



The solution can remain on E $(y_2 = 0)$ *only if* $y_2' = -g(y_1) = 0$.
Thus, $(0,0)$ *is the largest (and only) invariant subset of* $E = \{y \mid V^*(y) = 0\}$.
Since V *is positive definite,* C^1 *on* $\mathbb{R}^2$, $V^* \leq 0$, $\forall y \in \mathbb{R}^2$, *and the origin is the only invariant subset of* E, *the zero solution is* ***asymptotically stable***.

Example. Van Der Pol Equation. Region of Asymptotic Stability.
Determine an estimate of the region of asymptotic stability in the phase plane for

$$u'' + \epsilon(1 - u^2)u' + u = 0, \qquad \epsilon > 0, \text{ a constant.}$$

Proof. Recall the Lienard equation: $u'' + f(u)u' + g(u) = 0$. In our case,
$f(u) = \epsilon(1 - u^2)$, $g(u) = u$.
Similar to assumptions made for the Lienard equation, we have
$g(0) = 0$, $ug(u) = u^2 > 0$, $u \neq 0$. Let $F(u) = \int_0^u f(\sigma)d\sigma$.

$$F(u) = \int_0^u f(\sigma)d\sigma = \int_0^u \epsilon(1 - \sigma^2)d\sigma = \epsilon u - \frac{\epsilon u^3}{3}.$$

Find $a > 0$ such that $uF(u) > 0$ for $0 < |u| < a$:

$$uF(u) = \epsilon u^2 - \frac{\epsilon u^4}{3} > 0 \quad \Rightarrow \quad 0 < |u| < \sqrt{3} = a. \tag{3.3}$$

Here, we employ a different equivalent system than we had done in previous examples,

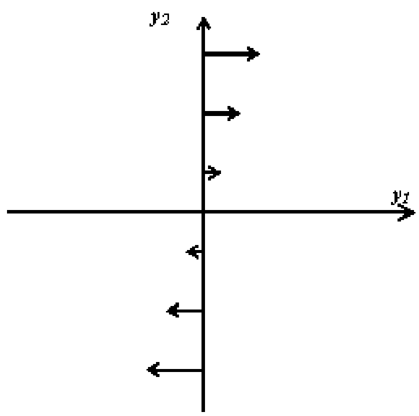
$$\begin{aligned} y_1 &= u, \\ y_2 &= u' + F(u), \qquad \text{which gives} \end{aligned}$$

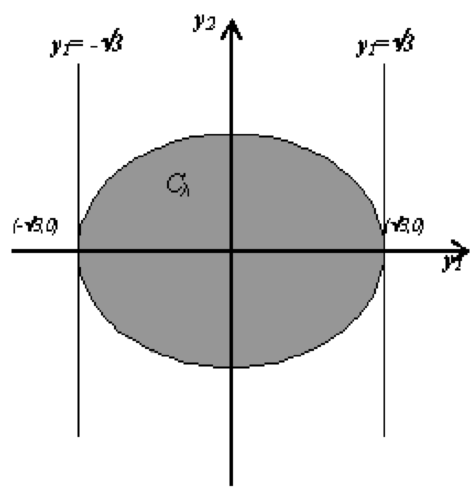
$$y_1' = y_2 - F(y_1),$$
$$y_2' = -y_1.$$

Define $G(y_1) = \int_0^{y_1} g(\sigma)\, d\sigma = \int_0^{y_1} \sigma\, d\sigma = \frac{y_1^2}{2}$.
Choose $V(y_1, y_2) = \frac{y_2^2}{2} + G(y_1) = \frac{y_2^2}{2} + \frac{y_1^2}{2} \quad \Rightarrow \quad V(y_1, y_2)$ is positive definite on $\mathbb{R}^2$.

$$V^*(y_1, y_2) = y_2 y_2' + y_1 y_1' = y_2(-y_1) + y_1(y_2 - F(y_1)) = -y_1 F(y_1) \le 0$$
on the strip $\quad \Omega = \{(y_1, y_2) \mid -\sqrt{3} < y_1 < \sqrt{3},\ -\infty < y_2 < \infty\}$, by (3.3).

Thus, the origin is stable.
$V^* = -y_1 F(y_1) = 0$ for $y_1 = 0$ (y_2−axis)
$\Rightarrow \quad E = \{y \mid V^*(y) = 0\} = \{(y_1, y_2) \mid y_1 = 0\}$. On $E: y_1' = y_2,\ y_2' = 0$.
Thus, 0 is the only invariant subset of E, and the zero solution is asymptotically stable.
Consider the curves $V(y_1, y_2) = \lambda$ $\left(\frac{y_1^2}{2} + \frac{y_2^2}{2} = \lambda\right)$ for $-\sqrt{3} < y_1 < \sqrt{3}$ with increasing values of λ, beginning with $\lambda = 0$. These are closed curves symmetric about the y_1-axis.





Since $V(y_1, y_2) = \frac{y_2^2}{2} + \frac{y_1^2}{2}$, $V(y_1, y_2)$ first makes contact with the boundary of Ω at one of the points $(-\sqrt{3}, 0)$ or $(\sqrt{3}, 0)$. The best value of $\hat{\lambda} = \min(G(\sqrt{3}), G(-\sqrt{3})) = \min(\frac{3}{2}, \frac{3}{2}) = \frac{3}{2}$ and $C_{\hat{\lambda}} = \{(y_1, y_2) \mid \frac{y_2^2}{2} + \frac{y_1^2}{2} < \hat{\lambda}\} = \{(y_1, y_2) \mid y_1^2 + y_2^2 < 3\}$.
$\Rightarrow$ Every solution that starts in C_λ approaches the origin.[5] $\square$

3.5 Global Asymptotic Stability

Theorem. *Let there exist a scalar function $V(y)$ such that:*

(i) $V(y)$ is positive definite on all $\mathbb{R}^n$;
(ii) $V(y) \to \infty$ as $|y| \to \infty$;
(iii) $V^*(y) \le 0$ on $\mathbb{R}^n$;
(iv) 0 it the onlty invariant subset of $E = \{y \mid V^*(y) = 0\}$.
*Then 0 is **globally asymptotically stable**.*

Corollary. *$V(y)$ satisfies (i) and (ii) above, and $V^*(y)$ is negative definite.*
*Then 0 is **globally asymptotically stable**.*

[5] Brauer, Nohel, Theorem 5.5, p. 214.

3.6 Stability (Non-autonomous Systems)

$$y' = f(t, y)$$

The scalar function $V(t, y)$ is **positive definite** if $V(t, 0) = 0$, $\forall t$ and $\exists W(y)$ positive definite, s.t. $V(t, y) \geq W(y)$ in $\Omega = \{(t, y) : t \geq 0, |y| \leq b, b > 0\}$.
The scalar function $V(t, y)$ is **negative definite** if $-V(t, y)$ is positive definite.

$$V^*(t, y) = \frac{d}{dt} V(t, y(t)) = \frac{\partial V}{\partial t} + \nabla V \cdot f(t, y)$$

If there exists a scalar function $V(t, y)$ that is positive definite and for which $V^*(t, y) \leq 0$ in Ω, then the zero solution is **stable**.
If there exists a scalar function $V(t, y)$ that is positive definite, satisfies an infinitesimal upper bound (i.e. $\lim_{\delta \to 0^+} \sup_{t \geq 0, |y| \leq \delta} |V(t, y)| = 0$), and for which $V^*(t, y)$ is negative definite, then the zero solution is **asymptotically stable**.

3.6.1 Examples

• $V(t, y) = y_1^2 + (1+t) y_2^2 \geq y_1^2 + y_2^2 = W(y) \;\Rightarrow\; V$ positive definite on $\Omega = \{(t, y) : t \geq 0)\}$
• $V(t, y) = y_1^2 + t y_2^2 \geq y_1^2 + a y_2^2 = W(y) \;\Rightarrow\; V$ positive definite on $\Omega = \{(t, y) : t \geq a, a > 0)\}$
• $V(t, y) = y_1^2 + \frac{y_2^2}{1+t}$. Since $V(t, 0, a_2) = \frac{a_2^2}{1+t} \to 0$ as $t \to \infty \;\Rightarrow\; V$ not positive definite even though $V(t, y) > 0$ for $y \neq 0$.

4 Poincare-Bendixson Theory

A **segment without contact** with respect to a vector field $V : \mathbb{R}^n \to \mathbb{R}^n$ is a finite, closed segment L of a straight line, s.t:

a) Every point of L is a regular point of V;
b) At no point of L the vector field V has the same direction as L.

Poincare-Bendixson Theorem. *Let C^+ be a positive semi-orbit contained in a closed and bounded set $K \subset \mathbb{R}^2$. If its limit set $L(C^+)$ contains no critical points of vector field $\vec{f}$, then $L(C^+)$ is a* ***periodic orbit****. Also, either:*
i) $C = L(C^+)$, or
ii) C approaches $L(C^+)$ spirally from either inside or outside.

Corollary. *If C^+ is a semiorbit contained in an* ***invariant compact*** *set K in which f has* ***no critical points****, then K contains a periodic orbit. Such a set cannot be equivalent to a disk.*

Example. *Prove that the second order differential equation*

$$z'' + (z^2 + 2(z')^2 - 1)z' + z = 0 \tag{4.1}$$

has a non-trivial ***periodic solution****.*

Proof. Write (4.1) as a first-order system:

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -y_1 - (y_1^2 + 2y_2^2 - 1)y_2. \end{aligned}$$

$$\text{Let } \ V(y_1, y_2) = \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2$$

$$\begin{aligned} V^*(y_1, y_2) &= y_1y_1' + y_2y_2' = y_1y_2 + y_2(-y_1 - (y_1^2 + 2y_2^2 - 1)y_2) \\ &= -y_2^2(y_1^2 + 2y_2^2 - 1) \end{aligned}$$

Use Poincare-Bendixson Theorem: If C^+ is a semiorbit contained in an **invariant compact** set K in which f has **no critical points**, then K contains a **periodic orbit**. Setting both equations of the system to 0, we see that $(0, 0)$ is the only critical point. Choose a compact set $K = \{(y_1, y_2) \mid \frac{1}{4} \le y_1^2 + y_2^2 \le 4\}$ and show that it is invariant.

$V^* = \nabla V \cdot \vec{f}$. Need $V^*|_{\Gamma_{out}} < 0$, $V^*|_{\Gamma_{in}} > 0$.

Check invariance of K:

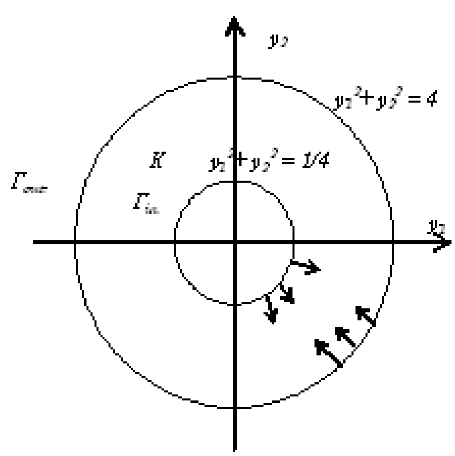
- $V^*|_{\Gamma_{out}} = -y_2^2(y_1^2 + 2y_2^2 - 1) \underbrace{<}_{need} 0,$

Need: $y_1^2 + 2y_2^2 - 1 > 0$,
$y_1^2 + 2y_2^2 - 1 \ge y_1^2 + y_2^2 - 1 = 4 - 1 = 3 > 0.$ ✓

- $V^*|_{\Gamma_{in}} = -y_2^2(y_1^2 + 2y_2^2 - 1) \underbrace{>}_{need} 0,$

Need: $y_1^2 + 2y_2^2 - 1 < 0$,
$y_1^2 + 2y_2^2 - 1 \le 2y_1^2 + 2y_2^2 - 1 = 2(\frac{1}{4}) - 1 = -\frac{1}{2} < 0.$ ✓

$\Rightarrow$ K is an invariant set. $(0, 0) \notin K$.
Thus K contains a periodic orbit.



□

Polar Coordinates. *Sometimes it is convenient to use* ***polar coordinates*** *when applying Poincare-Bendixson theorem.*
$y_1' = f_1(y_1, y_2)$
$y_2' = f_2(y_1, y_2)$
$V = \frac{y_1^2}{2} + \frac{y_2^2}{2}$
$V^* = \frac{dV}{dt} = y_1 y_1' + y_2 y_2' = r\,\cos\theta\; f_1(r,\theta) + r\,\sin\theta\; f_2(r,\theta).$

Example. Polar Coordinates. *Consider the system*

$$\begin{aligned} y_1' &= y_2 + y_1(1 - y_1^2 - y_2^2), \\ y_2' &= -y_1 + y_2(1 - y_1^2 - y_2^2). \end{aligned}$$

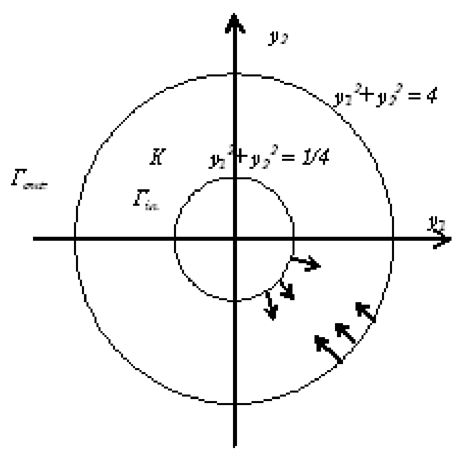
Proof. Let $V(y_1, y_2) = \frac{y_1^2}{2} + \frac{y_2^2}{2}$.

$$\begin{aligned} V^*(y_1, y_2) &= y_1 y_1' + y_2 y_2' = r\,\cos\theta\; f_1(r,\theta) + r\,\sin\theta\; f_2(r,\theta) \\ &= r\,\cos\theta\,(r\,\sin\theta + r\,\cos\theta(1-r^2)) + r\,\sin\theta\,(-r\,\cos\theta + r\,\sin\theta(1-r^2)) \\ &= r^2\cos\theta\sin\theta + r^2\cos^2\theta(1-r^2) - r^2\cos\theta\sin\theta + r^2\sin^2\theta(1-r^2) \\ &= r^2(1-r^2). \end{aligned}$$

Use Poincare-Bendixson Theorem: If C^+ is a semiorbit contained in an **invariant compact** set K in which f has **no critical points**, then K contains a **periodic orbit**.
Setting both equations of the system to 0,
we see that $(0,0)$ is the only critical point.
Choose a compact set $K = \{(y_1, y_2) \mid \frac{1}{4} \le y_1^2 + y_2^2 \le 4$
and show that it is invariant.
$V^* = \nabla V \cdot \vec{f}$. Need $V^*|_{\Gamma_{out}} < 0$, $V^*|_{\Gamma_{in}} > 0$.
Check invariance of K:

- $V^*|_{\Gamma_{out}} = r^2(1-r^2) = 4(1-4) < 0$. ✓
- $V^*|_{\Gamma_{in}} = r^2(1-r^2) = \frac{1}{4}(1-\frac{1}{4}) > 0$. ✓

$\Rightarrow$ K is an invariant set. $(0,0) \notin K$.
Thus K contains a periodic orbit.



□

Example. *Show that the autonomous system*

$$\begin{aligned}\frac{du}{dt} &= u - v - u^3 - uv^2\\ \frac{dv}{dt} &= u + v - v^3 - u^2 v\end{aligned}$$

has (a) a unique equilibrium point, (b) which is unstable, and (c) a unique closed solution curve.

Proof. **a)** Set above equations to 0 and multiply the first by v and the second by u:

$$\begin{aligned}&uv - v^2 - u^3 v - uv^3 = 0\\ &u^2 + uv - uv^3 - u^3 v = 0 \quad \Rightarrow \quad u^2 + v^2 = 0 \quad\Rightarrow\quad u^2 = -v^2 \quad\Rightarrow\quad u = 0,\ v = 0.\end{aligned}$$

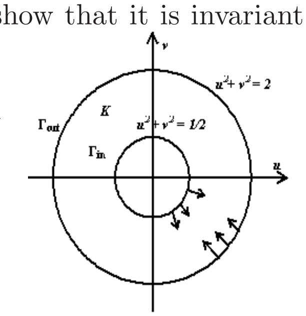
Thus, $(0,0)$ is a unique equilibrium point.
b) Let $V(u,v) = \frac{1}{2}u^2 + \frac{1}{2}v^2$, V is positive definite in $\mathbb{R}^2$.

$$\begin{aligned}V^*(u,v) &= uu' + vv' = u(u - v - u^3 - uv^2) + v(u + v - v^3 - u^2 v)\\ &= (u^2+v^2) - (u^2+v^2)^2 = (u^2+v^2)(1-(u^2+v^2)).\end{aligned}$$

$V^*(u,v)$ is positive definite in a small neighborhood of $(0,0)$, i.e. V^* is positive definite on $\Omega = \{(u,v) \mid u^2+v^2 = \frac{1}{2}\}$. Thus $(0,0)$ is unstable.
c) To show that the ODE system has a closed solution curve, use Poincare-Bendixson theorem: If C^+ is a semiorbit contained in an **invariant compact** set K in which f has **no critical points**, then K contains a **periodic orbit**.
Choose a compact set $K = \{(u,v) \mid \frac{1}{2} \le u^2+v^2 \le 2\}$ and show that it is invariant.
$V^* = \nabla V \cdot \vec{f}$. Need $V^*|_{\Gamma_{out}} < 0$, $V^*|_{\Gamma_{in}} > 0$.
Check invariance of K:

- $V^*|_{\Gamma_{out}} = (u^2+v^2)(1-(u^2+v^2)) = 2(1-2) = -2 < 0.$ ✓
- $V^*|_{\Gamma_{in}} = (u^2+v^2)(1-(u^2+v^2)) = \frac{1}{2}(1-\frac{1}{2}) = \frac{1}{4} > 0.$ ✓

$\Rightarrow$ K is an invariant set. $(0,0) \notin K$.
Thus K contains a periodic orbit.

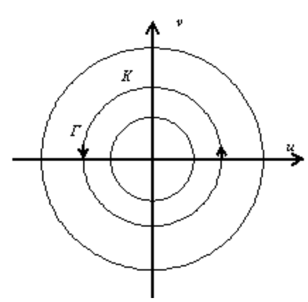


To show **uniqueness** of a periodic orbit, suppose Γ is the orbit of a periodic solution in K.

$$\begin{aligned}\int_\Gamma dV &= 0,\\ dV &= \frac{dV}{dt}dt = V^* dt\\ \Rightarrow \int_\Gamma V^* dt &= 0.\end{aligned}$$

$$V^*(u,v) = (u^2+v^2)(1-(u^2+v^2))$$
$$\Rightarrow \int_\Gamma (u^2+v^2)(1-(u^2+v^2))\,dt = 0.$$

$u^2+v^2 = 1$ is a periodic orbit.



Suppose there is another periodic orbit in K. We know that the following integral should be equal to 0 for a closed curve Γ:

$$\int_\Gamma \underbrace{(u^2+v^2)}_{\neq 0} \cdot \underbrace{(1-(u^2+v^2))}_{\textit{oscillates about 0 as going around}} \cdot dt = 0.$$

In order for integral above to be equal to 0, $(1-(u^2+v^2))$ should change sign as going around. At some point a, $\Gamma = \{(u,v) \mid u^2+v^2 = 1\}$ and Γ_2 defined by the second solution would intersect. But this is impossible, since at that point, there would be more than one possible solution. $\Rightarrow$ contradiction. Thus, the system has unique closed solution curve. $\square$

Problem (S'99, #8). *Consider the pair of ordinary differential equations*

$$\begin{aligned}\frac{dx_1}{dt} &= x_2\\ \frac{dx_2}{dt} &= -x_1 + (1-x_1^2-x_2^2)x_2\end{aligned}$$

a) Show any nontrivial solution has the property that $x_1^2+x_2^2$ decreases in time if its magnitude is greater than one and increases in time if its magnitude is less than one.
b) Use your work in (a) to show that on a periodic orbit, the integral

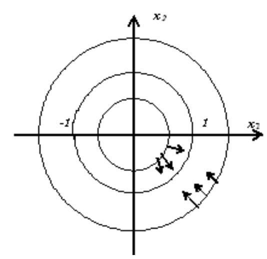
$$\int \big(1-x_1^2(t)-x_2^2(t)\big)\,x_2^2(t)\ dt \;=\; 0.$$

c) Consider the class of solutions $x_1=\sin(t+c)$, $x_2=\cos(t+c)$. Show that these are the only periodic orbits, for c any constant.
Hint: Use (b) to show that any periodic solution for which $1-x_1^2-x_2^2\neq 0$ must be such that $1-x_1^2-x_2^2$ changes sign on the orbit and use (a) to show this is impossible.

Proof. **a)** $(0,0)$ is the only equilibrium point.
Let $V(x_1,x_2)=\frac{1}{2}x_1^2+\frac{1}{2}x_2^2$; V is positive definite on $\mathbb{R}^2$.

$$V^*(x_1,x_2)=x_1x_1'+x_2x_2'=x_1x_2+x_2(-x_1+(1-x_1^2-x_2^2)x_2)=(1-x_1^2-x_2^2)x_2^2 \quad (4.2)$$

$V^*(x_1,x_2)\geq 0$ inside and $V^*(x_1,x_2)\leq 0$ outside the unit circle in the phase plane.
Since $V^*=0$ on $x_2=0$ (x_1-axis), it can not be concluded
that the statement to be proved is satisfied.
Let $r=\frac{1}{2}x_1^2+\frac{1}{2}x_2^2$ in (4.2), then



$$\begin{aligned}V^*(x_1,x_2) &= \frac{d}{dt}\Big(\frac{1}{2}x_1^2+\frac{1}{2}x_2^2\Big)=(1-x_1^2-x_2^2)x_2^2,\\ \frac{dr}{dt} &= (1-2r)x_2^2=\begin{cases}<0, & 2r>1\\ >0, & 2r<1\end{cases} = \begin{cases}<0, & x_1^2+x_2^2>\\ >0, & x_1^2+x_2^2<\end{cases}\end{aligned}$$

Thus, r (and thus, $x_1^2+x_2^2$) decreases if $x_1^2+x_2^2>1$ and increases if $x_1^2+x_2^2<1$.
If $r=\frac{1}{2}$, $\frac{dr}{dt}=0$, so $x_1^2+x_2^2=1$ is a circular orbit.
b) The only periodic orbit is $x_1^2+x_2^2=1$ where $V^*=0$:

$$\int_\Gamma dV=0,$$

$$dV=\frac{dV}{dt}dt=V^*dt$$

$$\Rightarrow\ \int_\Gamma V^*dt=0. \quad\Rightarrow\quad \int_\Gamma (1-x_1^2-x_2^2)x_2^2\ dt=0.$$

c) The class of solutions $x_1=\sin(t+c)$, $x_2=\cos(t+c)$ satisfy $x_1^2+x_2^2=1$, and therefore, are periodic orbits, for c any constant. Suppose there is another periodic orbit. We know that the following integral should be equal to 0 for a closed curve Γ:

$$\int_\Gamma \underbrace{(1-x_1^2-x_2^2)}_{\textit{oscillates about 0 as going around}}\cdot\underbrace{x_2^2}_{\neq 0}\cdot dt=0.$$

In order for integral above to be equal to 0, $1-x_1^2-x_2^2$ should change sign as going around.
At some point a, $\Gamma=\{(x_1,x_2)\mid x_1^2+x_2^2=1\}$ and Γ_2 defined by the second solution

would intersect. But this is impossible, since at that point, there would be more than one possible solution. $\Rightarrow$ contradiction. Thus, the system has a unique closed solution curve.
Also, by (a), we can conclude that solution curves either increase or decrease in time if the magnitude of $x_1^2 + x_2^2$ is not one. Thus, they approach the only periodic solution $x_1^2 + x_2^2 = 1$. $\square$

5 Sturm-Liouville Theory

Definition. *The differential equation*

$$(py')' + qy + r\lambda y = 0, \quad a \le x \le b \tag{5.1}$$

$$c_1 y(a) + c_2 y'(a) = 0, \quad c_3 y(b) + c_4 y'(b) = 0$$

is called a ***Sturm-Liouville equation****. A value of the parameter λ for which a* ***nontrivial solution*** *($y \ne 0$) exists is called an* ***eigenvalue*** *of the problem and corresponding nontrivial solution $y(x)$ of (5.1) is called an* ***eigenfunction*** *which is associated with that eigenvalue. Problem (5.1) is also called an* ***eigenvalue problem****.*

The coefficients p, q, and r must be real and continuous everywhere and $p > 0$ and $r > 0$ everywhere.

5.1 Sturm-Liouville Operator

Consider the Sturm-Liouville differential operator

$$Ly = (py')' + qy \qquad \left[L = \frac{d}{dx}\Big(p\frac{d}{dx}\Big) + q\right] \tag{5.2}$$

where $p > 0$, $r > 0$, and p', q and r are continuous on $[a, b]$. The differential equation (5.1) takes the operational form

$$Ly + \lambda r y = 0, \qquad a \le x \le b \tag{5.3}$$

$$c_1 y(a) + c_2 y'(a) = 0, \quad c_3 y(b) + c_4 y'(b) = 0.$$

5.2 Existence and Uniqueness for Initial-Value Problems

Theorem[6]**.** *Let $P(x)$, $Q(x)$ and $R(x)$ be continuous on $[a, b]$. If x_0 is a point in this interval and y_0 and y_1 are arbitrary numbers, then the initial-value problem*

$$y'' + P(x)y' + Q(x)y = R(x)$$
$$y(x_0) = y_0, \quad y'(x_0) = y_1$$

has a unique solution on $[a, b]$.

Note. *The unique solution of the initial-value problem with $R(x) = 0$, $y(x_0) = y'(x_0) = 0$, is the trivial solution.*

5.3 Existence of Eigenvalues

Theorem[7]**.** *The Sturm-Liouville problem (5.1) has an infinite number of eigenvalues, which can be written in increasing order as $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$, such that $\lim_{n\to\infty} \lambda_n = \infty$. The eigenfunctions $y_n(x)$ corresponding to λ_n has exactly $n-1$ zeros in (a, b).*

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[6] Bleecker and Csordas, Theorem 1, p. 260.
[7] Bleecker and Csordas, Theorem 2, p. 260.

5.4 Series of Eigenfunctions

Theorem[8]**.** *The eigenfunctions $\phi_n(x)$ form a "complete" set, meaning that any piece-wise smooth function $f(x)$ can be represented by a generalized Fourier series of eigen-functions:*

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x).$$

5.5 Lagrange's Identity

We calculate $uL(v) - vL(u)$, where u and v are any two functions. Recall that

$$L(u) = (pu')' + qu \quad \text{and} \quad L(v) = (pv')' + qv,$$

and hence

$$uL(v) - vL(u) = u(pv')' + quv - v(pu')' - quv = u(pv')' - v(pu')'.$$

The right hand side is manipulated to an exact differential:

$$\boxed{uL(v) - vL(u) = \big[p(uv' - vu')\big]'.}$$

5.6 Green's Formula

The integral form of the Lagrange's identity is known as Green's formula.

$$\boxed{\int_a^b \big[uL(v) - vL(u)\big]\,dx = p\big(uv' - vu'\big)\Big|_a^b}$$

for any functions u and v.

[8] Haberman, edition 4, Theorem 4, p. 163.

5.7 Self-Adjointness

With the additional restriction that the boundary terms vanish,

$$p\big(uv' - vu'\big)\Big|_a^b = 0,$$

we get

$$\int_a^b \big[uL(v) - vL(u)\big]\,dx = 0. \tag{5.4}$$

In fact, in the regular Sturm-Liouville eigenvalue problems, the boundary terms vanish.[9] When (5.4) is valid, we say that L is **self-adjoint**.

Definition[10]. *Let L and L^* denote the linear, second-order differential operators defined by*

$$\begin{aligned} Ly &= p_2(x)y'' + p_1(x)y' + p_0(x)y, \\ L^*y &= (yp_2(x))'' - (yp_1(x))' + yp_0(x). \end{aligned}$$

Then L^* is called the ***adjoint*** *of L and the differential equation $L^*y = 0$ is called the* ***adjoint equation****. The operator L is said to be* ***self-adjoint****, if $L = L^*$. A homogeneous, linear, second order ODE is said to be in* ***self-adjoint form*** *if the ODE has the form*

$$(p(x)y')' + q(x)y = 0.$$

Note: The linear, second-order differential operator

$$Ly = p_2(x)y'' + p_1(x)y' + p_0(x)y$$

is **self-adjoint** ($L = L^*$) if and only if $p_2'(x) = p_1(x)$, i.e.,

$$Ly = (p_2(x)y')' + p_0(x)y.$$

Proof. The adjoint L^* is given by

$$\begin{aligned} L^*y &= (yp_2(x))'' - (yp_1(x))' + yp_0(x) = y''p_2 + 2y'p_2' + yp_2'' - p_1'y - p_1y' + yp_0 \\ &= p_2y'' + (2p_2' - p_1)y' + (p_2'' - p_1' + p_0)y. \end{aligned}$$

Thus, $L = L^* \;\Rightarrow\; 2p_2' - p_1 = p_1, \;$ or $p_2' = p_1$. $\square$

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[9] Haberman, p. 176.
[10] Bleecker and Csordas, p. 264.

Problem (F'91, #6). *Consider the boundary value problem*

$$x\frac{d^2w}{dx^2} + (a-x)\frac{dw}{dx} = -\lambda w$$
$$w(L) = w(R) = 0,$$

where $a, L(> 0)$ and $R(> L)$ are real constants.

*By casting the problem in **self-adjoint form** shows that the eigenfunctions, w_1 and w_2, corresponding to different eigenvalues, λ_1 and λ_2, are **orthogonal** in the sense that*

$$\int_L^R e^{-x}x^{a-1}w_1w_2\,dx \;=\; \int_L^R e^{-x}x^a\frac{dw_1}{dx}\frac{dw_2}{dx}\,dx \;=\; 0.$$

Show also that

$$\lambda_i \;=\; \frac{\int_L^R e^{-x}x^a(\frac{dw_i}{dx})^2\,dx}{\int_L^R e^{-x}x^{a-1}w_i^2\,dx}$$

and hence that all eigenvalues are positive.

Proof. A homogeneous, linear, second order ODE is said to be in **self-adjoint form** if the ODE has the form

$$(p(x)u')' + q(x)u = 0.$$

We have

$$Lu \;=\; xu'' + (a-x)u'.$$

Multiply the equation by v so that it becomes of self-adjoint form:

$$vLu \;=\; xvu'' + (a-x)vu'.$$

Thus, we need

$$\begin{aligned}(pu')' &= xvu'' + (a-x)vu', \\ pu'' + p'u' &= xvu'' + (a-x)vu'.\end{aligned}$$

Thus, $p = xv$, and

$$\begin{aligned}(xv)' &= (a-x)v, \\ xv' + v &= av - xv, \\ \frac{v'}{v} &= \frac{a-x-1}{x}, \\ \frac{v'}{v} &= \frac{a-1}{x} - 1, \\ \ln v &= (a-1)\ln x - x, \\ \ln v &= \ln x^{a-1} - x, \\ v &= e^{ln\, x^{a-1}}e^{-x} = x^{a-1}e^{-x}.\end{aligned}$$

Thus, the self-adjoint form is

$$(xvu')' + \lambda uv = 0, \qquad \text{or}$$

$$\boxed{(x^a e^{-x}u')' + \lambda x^{a-1}e^{-x}u = 0. \qquad \circledast}$$

• Let λ_m, λ_n, be the eigenvalues and u_m, u_n be the corresponding eigenfunctions. We have

$$(x^a e^{-x} u_m')' + \lambda_m x^{a-1} e^{-x} u_m = 0, \tag{5.5}$$
$$(x^a e^{-x} u_n')' + \lambda_n x^{a-1} e^{-x} u_n = 0. \tag{5.6}$$

Multiply (5.5) by u_n and (5.6) by u_m and subtract equations from each other

$$\begin{aligned}
u_n(x^a e^{-x} u_m')' + \lambda_m x^{a-1} e^{-x} u_n u_m &= 0, \\
u_m(x^a e^{-x} u_n')' + \lambda_n x^{a-1} e^{-x} u_m u_n &= 0. \\
(\lambda_m - \lambda_n) x^{a-1} e^{-x} u_m u_n &= u_m(x^a e^{-x} u_n')' - u_n(x^a e^{-x} u_m')', \\
&= [x^a e^{-x}(u_m u_n' - u_n u_m')]'.
\end{aligned}$$

Integrating over (L, R) gives

$$(\lambda_m - \lambda_n) \int_L^R x^{a-1} e^{-x} u_m u_n \, dx = [x^a e^{-x}(u_m u_n' - u_n u_m')]_L^R = 0, \qquad \circledcirc$$

Since $\lambda_n \neq \lambda_m$, $u_n(x)$ and $u_m(x)$ are orthogonal on $[L, R]$.

• To show that u_m' and u_n' are **orthogonal** with respect to $x^{a-1}e^{-x}$, consider

$$\begin{aligned}
\int_L^R x^a e^{-x} u_m' u_n' \, dx &= x^a e^{-x} u_m' u_n |_L^R - \int_L^R (x^a e^{-x} u_m')' u_n \, dx \\
&= -\int_L^R (x^a e^{-x} u_m')' u_n \, dx = \circledast = \lambda_m \int_0^1 x^{a-1} e^{-x} u_m u_n \, dx = \circledcirc = 0.
\end{aligned}$$

• We now show that eigenvalues λ are **positive**. We have

$$(x^a e^{-x} u')' + \lambda x^{a-1} e^{-x} u = 0.$$

Multiplying by u and integrating, we get

$$\begin{aligned}
\int_L^R u(x^a e^{-x} u')' + \lambda x^{a-1} e^{-x} u^2 \, dx &= 0, \\
\underbrace{x^a e^{-x} u u' |_L^R}_{=0} - \int_L^R x^a e^{-x} u'^2 \, dx + \lambda \int_L^R x^{a-1} e^{-x} u^2 \, dx &= 0, \\
\lambda = \frac{\int_L^R x^a e^{-x} u'^2 \, dx}{\int_L^R x^{a-1} e^{-x} u^2 \, dx} &\geq 0.
\end{aligned}$$

The equality holds only if $u' \equiv 0$, which means $u = C$. Since $u(0) = u(1) = 0$, then $u \equiv 0$, which is not an eigenfunction. Thus, $\lambda > 0$. $\square$

Problem (F'01, #2). *Consider the differential operator*

$$L = \Big(\frac{d}{dx}\Big)^2 + 2\Big(\frac{d}{dx}\Big) + \alpha(x)$$

in which α is a real-valued function. The domain is $x \in [0,1]$, with Neumann boundary conditions

$$\frac{du}{dx}(0) = \frac{du}{dx}(1) = 0.$$

a) *Find a function $\phi = \phi(x)$ for which L is **self-adjoint** in the norm*

$$||u||^2 = \int_0^1 u^2\,\phi\,dx.$$

b) *Show that L must have a positive eigenvalue if α is not identically zero and*

$$\int_0^1 \alpha(x)\,dx \geq 0.$$

Proof. **a)** $Lu = u'' + 2u' + \alpha(x)u$. L is self-adjoint in the above norm, if

$$\begin{aligned}
\int_0^1 \big[uL(v) - vL(u)\big]\phi\,dx &= 0, \qquad \text{or} \\
\int_0^1 uL(v)\phi\,dx &= \int_0^1 vL(u)\phi\,dx, \\
\int_0^1 u(v'' + 2v' + \alpha(x)v)\phi\,dx &= \int_0^1 v(u'' + 2u' + \alpha(x)u)\phi\,dx, \\
\int_0^1 \underbrace{v''}_{g'}\,\underbrace{u\phi}_{f}\,dx + 2\int_0^1 uv'\phi\,dx + \int_0^1 \alpha(x)uv\,dx &= \int_0^1 \underbrace{u''}_{g'}\,\underbrace{v\phi}_{f}\,dx + 2\int_0^1 vu'\phi\,dx + \int_0^1 \alpha(x)uv\,dx, \\
v'u\phi|_0^1 - \int_0^1 v'(u'\phi + u\phi')\,dx + 2\int_0^1 uv'\phi\,dx &= u'v\phi|_0^1 - \int_0^1 u'(v'\phi + v\phi')\,dx + 2\int_0^1 vu'\phi\,dx.
\end{aligned}$$

Boundary terms are 0 due to boundary conditions. Cancelling out other terms, we get

$$\begin{aligned}
-\int_0^1 uv'\phi'\,dx \,+\, 2\int_0^1 uv'\phi\,dx &= -\int_0^1 vu'\phi'\,dx \,+\, 2\int_0^1 vu'\phi\,dx, \\
-uv'\phi' + 2uv'\phi &= -vu'\phi' + 2vu'\phi, \\
(vu' - uv')\phi' &= 2(vu' - uv')\phi \\
\phi' &= 2\phi. \qquad \text{Thus,}
\end{aligned}$$

$$\boxed{\phi = ae^{2x}.}$$

b) Divide by u and integrate:

$$\begin{aligned}
u'' + 2u' + \alpha(x)\, u &= \lambda u, \\
\int_0^1 \frac{u''}{u}\, dx + 2\int_0^1 \frac{u'}{u}\, dx + \int_0^1 \alpha(x)\, dx &= \int_0^1 \lambda\, dx, \\
\int_0^1 \underbrace{\frac{1}{u}}_{f} \underbrace{u''}_{g'}\, dx + 2\int_0^1 \frac{u'}{u}\, dx + \int_0^1 \alpha(x)\, dx &= \lambda, \\
\frac{1}{u}u'|_0^1 - \int_0^1 \underbrace{-\frac{1}{u^2}u'}_{f'} \underbrace{u'}_{g}\, dx + 2\int_0^1 \frac{u'}{u}\, dx + \int_0^1 \alpha(x)\, dx &= \lambda, \\
\int_0^1 \frac{u'^2}{u^2}\, dx + 2\int_0^1 \frac{u'}{u}\, dx + \int_0^1 \alpha(x)\, dx &= \lambda.
\end{aligned}$$

In order to have $\lambda > 0$, we must prove that there exists $u(x)$ such that

$$\int_0^1 \Big[\Big(\frac{u'}{u}\Big)^2 + 2\frac{u'}{u}\Big]\, dx \;>\; 0.$$

We can choose to have

$$\Big(\frac{u'}{u}\Big)^2 + 2\frac{u'}{u} \;>\; 0,$$

which means that $\frac{u'}{u} > 0$ or $\frac{u'}{u} < -2$. For example, if $u(x) = e^{cx}$ with $c > 0$, we have

$$\frac{u'}{u} \;=\; c \;>\; 0.$$

For such $u(x)$, $\lambda > 0$.

$\square$

Problem (F'99, #7). *Consider the differential operator*

$$L = \Big(\frac{d}{dx}\Big)^2 + 2\Big(\frac{d}{dx}\Big).$$

The domain is $x \in [0,1]$, with boundary conditions $u(0) = u(1) = 0$.
a) *Find a function $\phi = \phi(x)$ for which L is **self-adjoint** in the norm*

$$||u||^2 = \int_0^1 u^2\, \phi\, dx.$$

b) *If $a < 0$ show that $L + aI$ is **invertible**.*
c) *Find a value of a, so that $(L + aI)u = 0$ has a **nontrivial** solution.*

Proof. **a)** $Ly = y'' + 2y'$. L is self-adjoint in the above norm, if

$$
\begin{aligned}
\int_0^1 \big[uL(v) - vL(u)\big]\phi\,dx &= 0, \qquad \text{or} \\
\int_0^1 uL(v)\phi\,dx &= \int_0^1 vL(u)\phi\,dx, \\
\int_0^1 u(v'' + 2v')\phi\,dx &= \int_0^1 v(u'' + 2u')\phi\,dx, \\
\int_0^1 \underbrace{v''}_{g'}\underbrace{u\phi}_{f}\,dx + 2\int_0^1 uv'\phi\,dx &= \int_0^1 \underbrace{u''}_{g'}\underbrace{v\phi}_{f}\,dx + 2\int_0^1 vu'\phi\,dx, \\
v'u\phi|_0^1 - \int_0^1 v'(u'\phi + u\phi')\,dx + 2\int_0^1 uv'\phi\,dx &= u'v\phi|_0^1 - \int_0^1 u'(v'\phi + v\phi')\,dx + 2\int_0^1 vu'\phi\,dx.
\end{aligned}
$$

Boundary terms are 0 due to boundary conditions. Cancelling out other terms, we get

$$
\begin{aligned}
-\int_0^1 uv'\phi'\,dx \,+\, 2\int_0^1 uv'\phi\,dx &= -\int_0^1 vu'\phi'\,dx \,+\, 2\int_0^1 vu'\phi\,dx, \\
-uv'\phi' + 2uv'\phi &= -vu'\phi' + 2vu'\phi, \\
(vu' - uv')\phi' &= 2(vu' - uv')\phi, \\
\phi' &= 2\phi,
\end{aligned}
$$

Thus,

$$\boxed{\phi = ae^{2x}.}$$

b) $L + aI$ is **invertible** if the following holds:

$$(L + aI)u = 0 \qquad \Leftrightarrow \qquad u \equiv 0.$$

- $\Leftarrow$ | $u = 0 \quad \Rightarrow \quad (L + aI)u = 0.$
- $\Rightarrow$ | We have

$$\begin{aligned}(L + aI)u &= 0, \\ Lu + au &= 0, \\ u'' + 2u' + au &= 0.\end{aligned}$$

Multiply by u and integrate:

$$\begin{aligned} \int_0^1 uu''\,dx + \int_0^1 2uu'\,dx + \int_0^1 au^2\,dx &= 0, \\ \underbrace{uu'|_0^1}_{=0} - \int_0^1 (u')^2\,dx + \underbrace{2uu|_0^1}_{=0} \underbrace{- \int_0^1 2u'u\,dx}_{=0, \text{ since } \int_0^1 2u'u = -\int_0^1 2u'u} + \int_0^1 au^2\,dx &= 0, \end{aligned}$$

$$\begin{aligned} -\int_0^1 (u')^2\,dx + \int_0^1 au^2\,dx &= 0, \\ \int_0^1 \underbrace{\big(-(u')^2 + au^2\big)}_{\leq 0,\ (a<0)}\,dx &= 0. \end{aligned}$$

Thus, $u \equiv 0$.

- $\Rightarrow$ | We could also solve the equation directly and show $u \equiv 0$.

$$\begin{aligned} (L + aI)u &= 0, \\ Lu + au &= 0, \\ u'' + 2u' + au &= 0, \\ u &= ce^{sx}, \qquad \text{(anzats)} \\ u(x) &= c_1 e^{(-1+\sqrt{1-a})x} + c_2 e^{(-1-\sqrt{1-a})x}, \\ u(0) &= 0 = c_1 + c_2 \quad \Rightarrow \quad c_1 = -c_2. \\ u(1) &= 0 = c_1 e^{-1+\sqrt{1-a}} - c_1 e^{-1-\sqrt{1-a}}, \\ 0 &= c_1 e^{-1}(e^{\sqrt{1-a}} - e^{-\sqrt{1-a}}), \\ &\Rightarrow \; c_1 = 0 \quad \Rightarrow \; c_2 = 0 \quad \Rightarrow u \equiv 0. \end{aligned}$$

c) We want to find a value of a, so that $(L + aI)u = 0$ has a **nontrivial** solution.

$$\begin{aligned} &u'' + 2u' + au = 0, \\ &u(x) = c_1 e^{(-1+\sqrt{1-a})x} + c_2 e^{(-1-\sqrt{1-a})x}. \end{aligned}$$

Let $a = 1 + \pi^2$. Then

$$\begin{aligned} u(x) &= c_1 e^{(-1+\sqrt{-\pi^2})x} + c_2 e^{(-1-\sqrt{-\pi^2})x} = c_1 e^{(-1+i\pi)x} + c_2 e^{(-1-i\pi)x} \\ &= c_1 e^{-x} e^{i\pi x} + c_2 e^{-x} e^{-i\pi x} = c_1 e^{-x}(\cos \pi x + i \sin \pi x) + c_2 e^{-x}(\cos \pi x - i \sin \pi x), \\ u(0) &= 0 = c_1 + c_2 \quad \Rightarrow \quad c_1 = -c_2. \\ u(x) &= c_1 e^{-x}(\cos \pi x + i \sin \pi x) - c_1 e^{-x}(\cos \pi x - i \sin \pi x) = 2ic_1 e^{-x} \sin \pi x. \\ u(1) &= 0. \;\checkmark \end{aligned}$$

Let $c_1 = -i$. Then, $u(x) = 2e^{-x}\sin \pi x$, is a nontrivial solution. $\square$

Problem (F'90, #6). *Consider the differential-difference operator*

$$Lu \;=\; u''(x) + u'(x-1) + 3u(x)$$

defined on $0 < x < 3/2$ *along with the boundary conditions* $u(x) \equiv 0$ *on* $-1 \le x \le 0$ *and* $u(3/2) = 0$. *Determine the* ***adjoint*** *operator and the adjoint boundary conditions. Hint: Take the inner product to be* $(u,v) \equiv \int_0^{3/2} u(x)v(x)\,dx$.

Proof. The adjoint operator of L is L^*, such that

$$\int_0^{\frac{3}{2}} \big[uLv - vL^*u\big]\,dx \;=\; H(x)\Big|_0^{\frac{3}{2}}.$$

$$\begin{aligned}
\int_0^{\frac{3}{2}} uLv\,dx \;&=\; \int_0^{\frac{3}{2}} u\big(v''(x) + v'(x-1) + 3v(x)\big)\,dx \\
&=\; \int_0^{\frac{3}{2}} u(x)v''(x) + \int_0^{\frac{3}{2}} u(x)v'(x-1) + 3\int_0^{\frac{3}{2}} u(x)v(x) \;=\; \circledast
\end{aligned}$$

Change of variables: $y = x-1$, $dy = dx$, then

$$\int_0^{\frac{3}{2}} u(x)v'(x-1)\,dx = \int_{-1}^{\frac{1}{2}} u(y+1)v'(y)\,dy = \int_{-1}^{\frac{1}{2}} u(x+1)v'(x)\,dx.$$

$$\begin{aligned}
\circledast \;&=\; \int_0^{\frac{3}{2}} u(x)v''(x) + \int_{-1}^{\frac{1}{2}} u(x+1)v'(x)\,dx + 3\int_0^{\frac{3}{2}} u(x)v(x) \\
&=\; \underbrace{u(x)v'(x)\Big|_0^{\frac{3}{2}}}_{=0} - \int_0^{\frac{3}{2}} u'(x)v'(x) + \underbrace{u(x+1)v(x)\Big|_{-1}^{\frac{1}{2}}}_{=0} - \int_{-1}^{\frac{1}{2}} u'(x+1)v(x) + 3\int_0^{\frac{3}{2}} u(x)v(x) \\
&=\; \underbrace{-u'(x)v(x)\Big|_0^{\frac{3}{2}}}_{=0} + \int_0^{\frac{3}{2}} u''(x)v(x) - \int_{-1}^{\frac{1}{2}} u'(x+1)v(x) + 3\int_0^{\frac{3}{2}} u(x)v(x) \\
&=\; \int_0^{\frac{3}{2}} u''(x)v(x) - \int_{-1}^{\frac{1}{2}} u'(x+1)v(x) + 3\int_0^{\frac{3}{2}} u(x)v(x) \\
&=\; \int_0^{\frac{3}{2}} u''(x)v(x) - \int_0^{\frac{3}{2}} u'(x+1)v(x) + 3\int_0^{\frac{3}{2}} u(x)v(x) \qquad (\text{if } u \equiv 0 \text{ for } x \in [-1,0], [\tfrac{1}{2}, \tfrac{3}{2}]) \\
&=\; \int_0^{\frac{3}{2}} \big(u''(x)v(x) - u'(x+1)v(x) + 3u(x)v(x)\big)\,dx \\
&=\; \int_0^{\frac{3}{2}} v\big(u''(x) - u'(x+1) + 3u(x)\big)\,dx \;=\; \int_0^{\frac{3}{2}} vL^*u\,dx.
\end{aligned}$$

Thus, the adjoint boundary conditions are $u \equiv 0$ for $-1 \le x \le 0$, $\frac{1}{2} \le x \le \frac{3}{2}$, and

$$\boxed{L^*u = u''(x) - u'(x+1) + 3u(x).}$$

∎

Problem (S'92, #2). *Consider the two point boundary value problem*

$$y'''' + a(x)y''' + b(x)y' + c(x)y = F \qquad 0 \le x \le 1$$

with boundary conditions

$$y(0) = 0, \qquad y''(0) = \alpha y'''(0), \qquad y(1) = 0, \qquad y''(1) = \beta y'''(1).$$

Here a, b, c are real C^∞-smooth functions and α, β are real constants.
a) *Derive necessary and sufficient conditions for a, b, c, α, β such that the problem is* ***self-adjoint****.*

Proof. **a) METHOD I:** L is self-adjoint if

$$\begin{aligned}
L &= L^*, \\
y'''' + ay''' + by' + cy &= y'''' - (ay)''' - (by)' + cy, \\
ay''' + by' &= -(ay)''' - (by)', \\
ay''' + by' &= -a'''y - 3a''y' - 3a'y'' - ay''' - b'y - by', \\
2ay''' + 3a'y'' &+ (3a'' + 2b)y' + (a''' + b')y = 0, \\
\Rightarrow & \quad a = 0,\ b = 0, \quad c \text{ arbitrary.}
\end{aligned}$$

METHOD II: L is self-adjoint if

$$(Lu|v) = (u|Lv), \qquad \text{or}$$
$$\int_0^1 uL(v)\,dx = \int_0^1 vL(u)\,dx.$$

In the procedure below, we integrate each term of $uL(v)$ by parts at most 4 times to get

$$\int_0^1 uL(v)\,dx = \int_0^1 vL(u)\,dx + F(x),$$

and set $F(x) = 0$, which determines the conditions on a, b and c.

$$\begin{aligned}
&\int_0^1 uL(v)\,dx \;=\; \int_0^1 u(v'''' + av''' + bv' + cv)\,dx \\
=\;& \int_0^1 uv'''' + \int_0^1 auv''' + \int_0^1 buv' + \int_0^1 cuv \\
=\;& \underbrace{uv'''|_0^1}_{=0} - \int_0^1 u'v''' + \underbrace{auv''|_0^1}_{=0} - \int_0^1 (a'uv'' + au'v'') + \underbrace{buv|_0^1}_{=0} - \int_0^1 (b'uv + bu'v) + \int_0^1 cuv \\
=\;& -u'v''|_0^1 + \int_0^1 u''v'' - \underbrace{a'uv'|_0^1}_{=0} + \int_0^1 (a''uv' + a'u'v') - au'v'|_0^1 + \int_0^1 (a'u'v' + au''v') - \int_0^1 (b'uv + bu'v) + \int_0 \\
=\;& -u'v''|_0^1 + u''v'|_0^1 - \int_0^1 u'''v' + \underbrace{a''uv|_0^1}_{=0} - \int_0^1 (a'''uv + a''u'v) + \underbrace{a'u'v|_0^1}_{=0} - \int_0^1 (a''u'v + a'u''v) \\
-\;& au'v'|_0^1 + \underbrace{a'u'v|_0^1}_{=0} - \int_0^1 (a''u'v + a'u''v) + \underbrace{au''v|_0^1}_{=0} - \int_0^1 (a'u''v + au'''v) - \int_0^1 (b'uv + bu'v) + \int_0^1 cuv \\
=\;& -u'v''|_0^1 + u''v'|_0^1 - \underbrace{u'''v|_0^1}_{=0} + \int_0^1 u''''v - \int_0^1 (a'''uv + a''u'v) - \int_0^1 (a''u'v + a'u''v) \\
-\;& au'v'|_0^1 - \int_0^1 (a''u'v + a'u''v) - \int_0^1 (a'u''v + au'''v) - \int_0^1 (b'uv + bu'v) + \int_0^1 cuv \\
=\;& -u'v''|_0^1 + u''v'|_0^1 - au'v'|_0^1 \\
+\;& \int_0^1 (u'''' - a'''u - a''u' - a''u' - a'u'' - a''u' - a'u'' - a'u'' - au''' - b'u - bu' + cu)v \\
=\;& -u'v''|_0^1 + u''v'|_0^1 - au'v'|_0^1 + \int_0^1 (u'''' - a'''u - 3a''u' - 3a'u'' - au''' - b'u - bu' + cu)v \\
=\;& \int_0^1 v(u'''' + au''' + bu' + cu) \\
-\;& u'v''|_0^1 + u''v'|_0^1 - au'v'|_0^1 + \int_0^1 (-a'''u - 3a''u' - 3a'u'' - 2au''' - b'u - 2bu')v \\
=\;& \int_0^1 vL(u)\,dx - u'v''|_0^1 + u''v'|_0^1 - au'v'|_0^1 + \int_0^1 \big((-a''' - b')u - (3a'' + 2b)u' - 3a'u'' - 2au'''\big)v.
\end{aligned}$$

Thus, L is self-adjoint if $\int_0^1 \big((-a''' - b')u - (3a'' + 2b)u' - 3a'u'' - 2au'''\big)v = 0$, or $a = 0$, $b = 0$, c arbitrary. Also, need

$$\begin{aligned}
-u'(1)v''(1) + u'(0)v''(0) + u''(1)v'(1) - u''(0)v'(0) - \underbrace{au'v'|_0^1}_{=0,\ (a=0)} &= 0, \\
-\beta u'(1)v'''(1) + \alpha u'(0)v'''(0) + \beta u'''(1)v'(1) - \alpha u'''(0)v'(0) &= 0.
\end{aligned}$$

Thus, α, $\beta = 0$.
Note that both Methods I and II give the same answers. However, we need to use Method II in order to obtain information about boundary conditions.

b) *Assume that $c(x) = c_0$ is constant and that the problem is **self-adjoint**. Determinte the **eigenvalues and eigenfunctions** and show that they form a complete*

orthonormal set.
From part (a), we have

$$y'''' + c_0 y = F \qquad 0 \le x \le 1$$

$$y(0) = 0, \qquad y''(0) = 0, \qquad y(1) = 0, \qquad y''(1) = 0.$$

The eigenvalue problem is

$$\begin{aligned} y'''' + c_0 y &= \lambda y, \\ \Rightarrow \quad y'''' - (\lambda - c_0) y &= 0. \end{aligned}$$

To determine eigenfunctions, try $y = a\cos(\lambda - c_0)^{\frac{1}{4}}x + b\sin(\lambda - c_0)^{\frac{1}{4}}x$. Initial conditions give

$$\begin{aligned} y(0) &= a = 0 \qquad \Rightarrow \quad y = b\sin(\lambda - c_0)^{\frac{1}{4}}x, \\ y(1) &= b\sin(\lambda - c_0)^{\frac{1}{4}} = 0 \quad \Rightarrow \quad (\lambda - c_0)^{\frac{1}{4}} = n\pi \quad \Rightarrow \quad \lambda_n = n^4\pi^4 + c_0. \end{aligned}$$

Thus, the eigenvalues and eigenfunctions are

$$\boxed{\lambda_n = n^4\pi^4 + c_0, \qquad y_n = \sin(\lambda_n - c_0)^{\frac{1}{4}}x = \sin n\pi x, \qquad n = 1, 2, \ldots.}$$

$\lceil$ We could also use the table to find out that the eigenfunctions are $y = \sin\frac{n\pi x}{L} = \sin n\pi x$. We have

$$\begin{aligned} y'''' + c_0 y &= \lambda y, \\ (\sin n\pi x)'''' + c_0 \sin n\pi x &= \lambda \sin n\pi x, \\ n^4\pi^4 \sin n\pi x + c_0 \sin n\pi x &= \lambda \sin n\pi x, \\ n^4\pi^4 + c_0 &= \lambda_n. \quad \rfloor \end{aligned}$$

The normalized eigenfunctions form an orthonormal set

$$\int_0^1 (\sqrt{2}\sin n\pi x)\,(\sqrt{2}\sin m\pi x)\,dx = \begin{cases} 0 & n \ne m \\ 1 & n = m \end{cases}$$

Any smooth function f can be written in terms of eigenfunctions $f(x) = \sum_{n=1}^{\infty} a_n \sqrt{2}\sin n\pi x$.

c) *Use the eigenfunctions to construct the* ***Green's function****.*

We have

$$y'''' + c_0 y = F(x), \tag{5.7}$$

$$y(0) = 0, \qquad y''(0) = 0, \qquad y(1) = 0, \qquad y''(1) = 0. \tag{5.8}$$

The related eigenvalue problem is

$$\begin{aligned} &y'''' + c_0 y = \lambda y \\ &y(0) = 0, \qquad y''(0) = 0, \qquad y(1) = 0, \qquad y''(1) = 0. \end{aligned}$$

The eigenvalues are $\lambda_n = n^4\pi^4 + c_0$, and the corresponding eigenfunctions are $\sin n\pi x$, $n = 1, 2, \ldots$.
Writing $y = \sum a_n \phi_n = \sum a_n \sin n\pi x$ and inserting into (5.7), we get

$$\begin{aligned} \sum_{n=1}^{\infty} \big(a_n n^4 \pi^4 \sin n\pi x + c_0 a_n \sin n\pi x\big) &= F(x), \\ \sum_{n=1}^{\infty} a_n (n^4\pi^4 + c_0) \sin n\pi x &= F(x), \\ \int_0^1 \sum_{n=1}^{\infty} a_n (n^4\pi^4 + c_0)\, \sin n\pi x \, \sin m\pi x \, dx &= \int_0^1 F(x)\, \sin m\pi x \, dx, \\ a_n (n^4\pi^4 + c_0) \frac{1}{2} &= \int_0^1 F(x)\, \sin n\pi x \, dx, \\ a_n &= \frac{2\int_0^1 F(x)\, \sin n\pi x\, dx}{n^4\pi^4 + c_0}. \end{aligned}$$

$$y(x) = \sum a_n \sin n\pi x = \sum_{n=1}^{\infty} \frac{2\int_0^1 F(\xi)\, \sin n\pi x\, \sin n\pi\xi\, d\xi}{n^4\pi^4 + c_0},$$

$$\boxed{y = \int_0^1 F(\xi)\, \underbrace{\left[2\sum_{n=1}^{\infty} \frac{\sin n\pi x\, \sin n\pi\xi}{n^4\pi^4 + c_0}\right]}_{=\,G(x,\xi)}\, d\xi.}$$

See a less complicated problem, $y'' = f$, in Poisson Equation subsection of Eigenvalues of the Laplacian section (PDEs). $\square$

Problem (S'91, #5). *Define the operator*

$$Lu = u_{xxxx} + a(x)u_{xx} + b(x)u_x + c(x)u$$

for $0 \leq x \leq 2\pi$ *with boundary conditions*

$$u = u_{xx} = 0 \qquad on \quad x = 0,\, 2\pi.$$

a) *Find conditions on the functions* a, b *and* c *so that* L *is* ***self-adjoint****.*
b) *For* $a = b = 0$ *and* $c =$ *constant, find the* ***fundamental solution*** *for the PDE*

$$u_t = -Lu$$

as a ***Fourier series*** *in* x.

Proof. **a) METHOD I:** L is self-adjoint if

$$\begin{aligned}
L &= L^*, \\
u'''' + au'' + bu' + cu &= u'''' + (au)'' - (bu)' + cu, \\
au'' + bu' &= (au)'' - (bu)', \\
au'' + bu' &= a''u + 2a'u' + au'' - b'u - bu', \\
0 &= a''u + 2a'u' - b'u - 2bu', \\
0 &= (a'' - b')u + 2(a' - b)u', \\
\Rightarrow &\qquad a' = b, \quad c \text{ arbitrary.}
\end{aligned}$$

METHOD II: L is self-adjoint if

$$\begin{aligned}
(Lu|v) = (u|Lv), &\qquad \text{or} \\
\int_0^{2\pi} uL(v)\,dx = \int_0^{2\pi} vL(u)\,dx.&
\end{aligned}$$

In the procedure below, we integrate each term of $uL(v)$ by parts at most 4 times to get

$$\int_0^{2\pi} uL(v)\,dx = \int_0^{2\pi} vL(u)\,dx + F(x),$$

and set $F(x) = 0$, which determines the conditions on a, b and c.

$$\begin{aligned}
&\int_0^{2\pi} uL(v)\,dx \;=\; \int_0^{2\pi} u(v'''' + av'' + bv' + cv)\,dx \\
= \;& \int_0^{2\pi} uv'''' + \int_0^{2\pi} auv'' + \int_0^{2\pi} buv' + \int_0^{2\pi} cuv \\
= \;& \underbrace{uv'''|_0^{2\pi}}_{=0} - \int_0^{2\pi} u'v''' + \underbrace{auv'|_0^{2\pi}}_{=0} - \int_0^{2\pi} (a'uv' + au'v') + \underbrace{buv|_0^{2\pi}}_{=0} - \int_0^{2\pi} (b'uv + bu'v) + \int_0^{2\pi} cuv \\
= \;& \underbrace{-u'v''|_0^{2\pi}}_{=0} + \int_0^{2\pi} u''v'' - \underbrace{a'uv|_0^{2\pi}}_{=0} + \int_0^{2\pi} (a''uv + a'u'v) - \underbrace{au'v|_0^{2\pi}}_{=0} + \int_0^{2\pi} (a'u'v + au''v) - \int_0^{2\pi} (b'uv + bu'v) + \\
= \;& \underbrace{u''v'|_0^{2\pi}}_{=0} - \int_0^{2\pi} u'''v' + \int_0^{2\pi} (a''uv + a'u'v) + \int_0^{2\pi} (a'u'v + au''v) - \int_0^{2\pi} (b'uv + bu'v) + \int_0^{2\pi} cuv \\
= \;& \underbrace{-u'''v|_0^{2\pi}}_{=0} + \int_0^{2\pi} (u''''v + a''uv + a'u'v + a'u'v + au''v - b'uv - bu'v + cuv) \\
= \;& \int_0^{2\pi} v(u'''' + au'' + bu' + cu) + \int_0^{2\pi} (a''uv + 2a'u'v - b'uv - 2bu'v) \\
= \;& \int_0^{2\pi} vL(u)\,dx \;+\; \int_0^{2\pi} (a''uv + 2a'u'v - b'uv - 2bu'v).
\end{aligned}$$

Thus, L is self-adjoint if $\int_0^{2\pi}(a''u + 2a'u' - b'u - 2bu')v = 0$, or $a' = b$, c arbitrary.

b) *For $a = b = 0$ and $c =$ constant, find the* ***fundamental solution*** *for the PDE*

$$u_t = -Lu$$

as a ***Fourier series*** *in x.*

We have $u_t = -Lu = -u'''' - cu$. We first need to find eigenfunctions and eigenvalues. The eigenvalue problem is

$$\begin{aligned} & u'''' + cu = \lambda u, \\ \Rightarrow\ & u'''' - (\lambda - c)u = 0, \\ & u = u_{xx} = 0 \quad \text{on} \quad x = 0,\, 2\pi. \end{aligned}$$

To determine eigenfunctions, try $u = a\cos(\lambda - c)^{\frac{1}{4}}x + b\sin(\lambda - c)^{\frac{1}{4}}x$. Initial conditions:

$$\begin{aligned} u(0) &= a = 0 \qquad \Rightarrow \quad u = b\sin(\lambda - c)^{\frac{1}{4}}x, \\ u(2\pi) &= 0 = b\sin(\lambda - c)^{\frac{1}{4}}2\pi = 0 \quad \Rightarrow \quad (\lambda - c)^{\frac{1}{4}}2\pi = n\pi \quad \Rightarrow \quad \lambda_n = \frac{n^4}{16} + c. \end{aligned}$$

Thus, the eigenvalues and eigenfunctions are

$$\boxed{\lambda_n = \frac{n^4}{16} + c, \qquad u_n = \sin(\lambda_n - c)^{\frac{1}{4}}x = \sin\frac{nx}{2}, \qquad n = 1, 2, \ldots.}$$

Let $\quad u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\frac{nx}{2}$.

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty}\Big(u_n'(t)\sin\frac{nx}{2} + u_n(t)\frac{n^4}{16}\sin\frac{nx}{2} + cu_n(t)\sin\frac{nx}{2}\Big) = 0, \\ u_n'(t) &+ u_n(t)\frac{n^4}{16} + cu_n(t) = 0, \\ u_n'(t) &+ \Big(\frac{n^4}{16} + c\Big)u_n(t) = 0, \\ u_n(t) &= c_n e^{-(\frac{n^4}{16}+c)t}. \end{aligned}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(\frac{n^4}{16}+c)t}\sin\frac{nx}{2}.}$$

In order to determine c_n we need initial conditions $u(x,0) = f(x)$. Then [11]

$$\begin{aligned} u(x,0) &= \sum_{n=1}^{\infty} c_n \sin\frac{nx}{2}\,dx = f(x). \\ \pi c_n &= \int_0^{2\pi} f(x)\sin\frac{nx}{2}\,dx, \\ c_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin\frac{nx}{2}\,dx. \\ \Rightarrow \qquad & u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(\frac{n^4}{16}+c)t}\sin\frac{nx}{2} = \sum_{n=1}^{\infty}\frac{1}{\pi}\int_0^{2\pi} f(\xi)\sin\frac{n\xi}{2}e^{-(\frac{n^4}{16}+c)t}\sin\frac{nx}{2}\,d\xi. \end{aligned}$$

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[11] ChiuYen's solutions list $G(x,t;x_0,t_0) = \sum_{n=1}^{\infty}\frac{1}{\pi}\sin\frac{nx_0}{2}e^{-(\frac{n^4}{16}+c)(t-\mathbf{t_0})}\sin\frac{nx}{2}$. Similar result may be found in Haberman, p. 383.

$$\boxed{u(x,t) = \int_0^{2\pi} f(\xi) \underbrace{\sum_{n=1}^{\infty} \frac{1}{\pi} \sin\frac{n\xi}{2} \sin\frac{nx}{2}\, e^{-(\frac{n^4}{16}+c)t}}_{=\,G(x,t;x_0,t_0)} d\xi.}$$

5.8 Orthogonality of Eigenfunctions

□

Definition[12]**.** *A positive, continuous function $r(x)$ defined on $[a,b]$ is called a weight function. Two continuous functions $f(x)$ and $h(x)$ defined on $[a,b]$ are said to be orthogonal on $[a,b]$ with respect to the weight function $r(x)$, if*

$$\int_a^b f(x)h(x)r(x)dx = 0.$$

Theorem[13]**.** *Let λ_m and λ_n be two* ***distinct*** *eigenvalues of the Sturm-Liouville problem (5.3). Then the corresponding eigenfunctions $y_m(x)$ and $y_n(x)$ are* ***orthogonal*** *on $[a,b]$ with respect to the weight function $r(x)$.*

$$\int_a^b y_m(x)y_n(x)r(x)dx = 0.$$

Proof. We have the relations

$$(py_m')' + qy_m + \lambda_m r y_m = 0, \tag{5.9}$$

$$(py_n')' + qy_n + \lambda_n r y_n = 0. \tag{5.10}$$

Multiply (5.9) by y_n and (5.10) by y_m and subtract equations from each other [14]

$$(\lambda_n - \lambda_m) r y_m y_n = y_n(py_m')' - y_m(py_n')' = [p(y_n y_m' - y_m y_n')]'. \tag{5.11}$$

Integrating both sides of (5.11) over (a,b) gives

$$(\lambda_n - \lambda_m)\int_a^b y_m y_n r = [p(y_n y_m' - y_m y_n')]_a^b.$$

The boundary conditions in (5.3) ensure that the right side vanishes (e.g. if $c_2 \neq 0$, then $y'(a) = -\frac{c_1}{c_2}y(a)$, and $y_n(a)y_m'(a) - y_m(a)y_n'(a) = -y_n(a)\frac{c_1}{c_2}y_m(a) + y_m(a)\frac{c_1}{c_2}y_n(a) = 0$). Thus,

$$(\lambda_n - \lambda_m)\int_a^b y_m y_n r = 0.$$

Since $\lambda_n \neq \lambda_m$, $y_n(x)$ and $y_m(x)$ are orthogonal on $[a,b]$ with respect to the weight function $r(x)$. □

[12] Bleecker and Csordas, p. 266.
[13] Bleecker and Csordas, Theorem 5, p. 267.
[14] Note an important identity:

$$\boxed{y_n(py_m')' - y_m(py_n')' = [p(y_n y_m' - y_m y_n')]'.}$$

Problem (S'90, #3). *Consider the eigenvalue problem*

$$a(x)\frac{d^2u(x)}{dx^2} = \lambda u(x), \qquad 0 < x < 1,$$

with the boundary conditions $u(0) = 0$, $u'(1) = 0$. *Here* $0 < c_1 \le a(x) \le c_2$ *is a smooth function on* $[0, 1]$. *Let* λ_n, $n = 1, \ldots$, *be the eigenvalues and* $\varphi_n(x)$ *be the corresponding eigenfunctions. Prove that there is a weight* $\rho(x)$ *such that*

$$\int_0^1 \varphi_m(x)\varphi_n(x)\rho(x)\,dx = 0 \qquad \text{for} \;\; m \ne n.$$

Proof. Rewrite the equation as

$$u'' - \lambda\frac{1}{a(x)}u = 0.$$

Let λ_m, λ_n, be the eigenvalues and u_m, u_n be the corresponding eigenfunctions. We have

$$u_m'' - \lambda_m\frac{1}{a(x)}u_m = 0, \tag{5.12}$$

$$u_n'' - \lambda_n\frac{1}{a(x)}u_n = 0. \tag{5.13}$$

Multiply (5.12) by u_n and (5.13) by u_m and subtract equations from each other

$$\begin{aligned} u_n u_m'' &= \lambda_m\frac{1}{a(x)}u_m u_n, \\ u_m u_n'' &= \lambda_n\frac{1}{a(x)}u_n u_m. \\ (\lambda_m - \lambda_n)\frac{1}{a(x)}u_m u_n &= u_n u_m'' - u_m u_n'' = (u_n u_m' - u_m u_n')'. \end{aligned}$$

Integrating over $(0, 1)$ gives

$$(\lambda_m - \lambda_n)\int_0^1 \frac{1}{a(x)}u_m u_n\,dx \;=\; [u_n u_m' - u_m u_n']_0^1 = 0.$$

Since $\lambda_n \ne \lambda_m$, $u_n(x)$ and $u_m(x)$ are orthogonal on $[0, 1]$ with respect to the weight function $\rho(x) = \frac{1}{a(x)}$. ◻

5.9 Real Eigenvalues

Theorem[15]**.** *For any regular Sturm-Liouville problem, all the eigenvalues* λ *are real.*

Proof. We can use orthogonality of eigenfunctions to prove that the eigenvalues are real. Suppose that λ is a complex eigenvalue and $\phi(x)$ the corresponding eigenfunction (also allowed to be complex since the differential equation defining the eigenfunction would be complex):

$$L(\phi) + \lambda r\phi = 0. \tag{5.14}$$

Thus, the complex conjugate of (5.14) is also valid:

$$\overline{L(\phi)} + \overline{\lambda}r\overline{\phi} = 0, \tag{5.15}$$

[15] Haberman, edition 4, p. 178.

assuming that r is real. Since the coefficients of a linear operator $L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$ are real, $\overline{L(\phi)} = L(\overline{\phi})$. Thus,

$$L(\overline{\phi}) + \overline{\lambda} r \overline{\phi} = 0.$$

If ϕ satisfies boundary conditions with real coefficients, for example $c_1\phi(a)+c_2\phi'(a) = 0$, then $\overline{\phi}$ satisfies the same boundary conditions, $c_1\overline{\phi}(a) + c_2\overline{\phi}'(a) = 0$. Equation (5.14) and the boundary conditions show that $\overline{\phi}$ satisfies the Sturm-Liouville problem, but with eigenvalue being $\overline{\lambda}$. Thus, **if λ is a complex eigenvalue with corresponding eigenfunction ϕ, then $\overline{\lambda}$ is also an eigenvalue with corresponding eigenfunction $\overline{\phi}$.**
Using orthogonality of eigenfunctions, ϕ and $\overline{\phi}$ are orthogonal (with weight r). Thus,

$$\int_a^b \phi\,\overline{\phi}\,r\,dx = 0.$$

Since $\phi\overline{\phi} = |\phi|^2 \geq 0$ and $r > 0$, the integral above is ≥ 0. In fact, the integral can equal 0 only if $\phi \equiv 0$, which is prohibited since ϕ is an eigenfunction. Thus, $\lambda = \overline{\lambda}$, and hence λ is real. $\square$

5.10 Unique Eigenfunctions

Theorem. *Consider the Sturm-Liouville problem (5.3). If $y_1(x)$ and $y_2(x)$ are two eigenfunctions corresponding to the same eigenvalue λ, then $y_1(x) = \alpha y_2(x)$, $a \le x \le b$, for some nonzero constant α, (i.e., $y_1(x)$ and $y_2(x)$ are linearly dependent).*

Proof. [16] **Method 1**: Suppose that there are two different eigenfunctions y_1 and y_2 corresponding to the same eigenvalue λ. In this case,

$$\begin{aligned} L(y_1) + \lambda r y_1 &= 0, \\ L(y_2) + \lambda r y_2 &= 0. \end{aligned}$$

$$0 = y_2\big(L(y_1) + \lambda r y_1\big) - y_1\big(L(y_2) + \lambda r y_2\big) = y_2 L(y_1) - y_1 L(y_2) = \Big[p\big(y_2 y_1' - y_1 y_2'\big)\Big]',$$

where the Lagrange's identity was used in the last equality. It follows that

$$p\big(y_2 y_1' - y_1 y_2'\big) = \text{constant}.$$

This constant is evaluated from the boundary conditions and is equal to 0 if the boundary conditions are of the Sturm-Liouville type. Thus,

$$y_2 y_1' - y_1 y_2' = 0.$$

This is equivalent to $\frac{d}{dx}(\frac{y_1}{y_2}) = 0$, and hence for these boundary conditions

$$y_2 = c y_1.$$

Thus, the two eigenfunctions are dependent; the eigenfunction is unique. $\square$

Proof. [17] **Method 2**: Consider the function

$$w(x) = y_2'(a) y_1(x) - y_1'(a) y_2(x),$$

and suppose that

$$y_1'(a)^2 + y_2'(a)^2 \neq 0. \tag{5.16}$$

Then $w(x)$ satisfies the following initial-value problem

$$\begin{aligned} &Lw + \lambda r w = 0 \qquad a \le x \le b \qquad \big[\ (pw')' + qw + \lambda r w = 0\ \big] \\ &w(a) = w'(a) = 0. \end{aligned}$$

⌈ Check that $w(x)$ indeed satisfies the initial-value problem:

$$\begin{aligned} (pw')' + qw + \lambda r w &= \Big[p\big(y_2'(a) y_1(x) - y_1'(a) y_2(x)\big)'\Big]' + q\big(y_2'(a) y_1(x) - y_1'(a) y_2(x)\big) \\ &\qquad + \lambda r\big(y_2'(a) y_1(x) - y_1'(a) y_2(x)\big) \\ &= y_2'(a)\Big[(p y_1'(x))' + q y_1(x) + \lambda r y_1(x)\Big] - y_1'(a)\Big[(p y_2'(x))' + q y_2(x) + \lambda r y_2(x)\Big] = 0, \end{aligned}$$

since y_1 and y_2 are eigenfunctions. Also,

$$\begin{aligned} w(a) &= y_2'(a) y_1(a) - y_1'(a) y_2(a) = -\frac{c_1}{c_2} y_2(a) y_1(a) + \frac{c_1}{c_2} y_1(a) y_2(a) = 0, \\ w'(a) &= y_2'(a) y_1'(a) - y_1'(a) y_2'(a) = 0. \quad ⌋ \end{aligned}$$

[16] Haberman, edition 4, p. 179.
[17] Bleecker and Csordas, Theorem 3, p. 265.

By the uniqueness theorem for initial-value problems, $w(x) \equiv 0$. Therefore,

$$y_2'(a)y_1(x) - y_1'(a)y_2(x) \equiv 0, \qquad a \le x \le b. \tag{5.17}$$

Since $y_1(x)$ and $y_2(x)$ are eigenfunctions, $y_1(x)$ and $y_2(x)$ are not identically 0. Hence, (5.16) and (5.17) imply that $y_1'(a)y_2'(a) \neq 0$. Thus, by (5.17), $y_1(x) = \alpha y_2(x)$, where $\alpha = y_1'(a)/y_2'(a)$. $\square$

Remark: In the theorem above, we showed that, for the Sturm-Liouville problem (5.3), there is only one linearly independent eigenfunction associated with each eigenvalue λ. For this reason, λ is said to be **simple**.

5.11 Rayleigh Quotient

Theorem[18]**.** *Any eigenvalue can be related to its eigenfunction by the* ***Rayleigh quotient****:*

$$\lambda = \frac{-p\phi\phi'|_a^b + \int_a^b \left[p(\phi')^2 - q\phi^2\right] dx}{\int_a^b \phi^2 r\, dx}.$$

Proof. The Rayleigh quotient can be derived from the Sturm-Liouville differential equation,

$$(p\phi')' + q\phi + \lambda r\phi = 0, \tag{5.18}$$

by multiplying (5.18) by ϕ and integrating:

$$\int_a^b \Big[\phi(p\phi')' + q\phi^2\Big]\, dx + \lambda \int_a^b r\phi^2\, dx = 0.$$

Since $\int_a^b r\phi^2 > 0$, we can solve for λ:

$$\lambda = \frac{\int_a^b \left[-\phi(p\phi')' - q\phi^2\right] dx}{\int_a^b r\phi^2\, dx}.$$

Integrating by parts gives

$$\lambda = \frac{-p\phi\phi'|_a^b + \int_a^b \left[p(\phi')^2 - q\phi^2\right] dx}{\int_a^b r\phi^2\, dx}.$$

$\square$

Note: Given the equation:

$$\frac{1}{x}(xf')' + \lambda f = 0,$$

we can obtain

$$\lambda = \frac{\int_0^1 xf'^2\, dx}{\int_0^1 xf^2\, dx} \quad \ge \quad 0. \qquad \circledast$$

[18]Haberman, edition 4, Theorem 6, p. 189.

We can establish the **Rayleigh-Ritz principle**, namely that

$$F(f) = \frac{\int_0^1 x(f')^2\,dx}{\int_0^1 xf^2\,dx}$$

is an **upper bound on the smallest eigenvalue**.
Let $f(x) = \sum a_n f_n$, where f_n's are eigenfunctions. Then,

$$\begin{aligned} F(f) &= \frac{\int_0^1 x(f')^2\,dx}{\int_0^1 xf^2\,dx} = \frac{\int_0^1 x(\sum a_n f_n')^2\,dx}{\int_0^1 x(\sum a_n f_n)^2\,dx} \qquad \text{(by orthogonality)} \\ &= \frac{\sum a_n^2 \int_0^1 x f_n'^2\,dx}{\sum a_n^2 \int_0^1 x f_n^2\,dx} = \circledast = \frac{\sum a_n^2 \lambda_n \int_0^1 x f_n^2\,dx}{\sum a_n^2 \int_0^1 x f_n^2\,dx} > \lambda_{\min} \frac{\sum a_n^2 \int_0^1 x f_n^2\,dx}{\sum a_n^2 \int_0^1 x f_n^2\,dx} = \lambda_{\min}. \end{aligned}$$

5.12 More Problems

Example. *Determine the eigenvalues and eigenfunctions of the Sturm-Liouville problem*

$$y'' + \lambda y = 0, \quad 0 \le x \le L$$
$$y(0) = 0, \quad y(L) = 0.$$

Proof. Note that we get this equation from (5.1) with $p \equiv 1$, $q \equiv 0$, $r \equiv 1$, $a = 0$, $b = L$. We consider the three cases $\lambda > 0$, $\lambda = 0$, $\lambda < 0$.
• If $\lambda = 0$, the ODE reduces to $y'' = 0$. Try $y(x) = Ax + B$.
Applying the first boundary condition gives $y(0) = 0 = B$. The second boundary condition gives $y(L) = 0 = AL$, or $A = 0$. Therefore, the only solution for this case is the trivial solution, $y(x) \equiv 0$, which is not an eigenfunction, and therefore, 0 is not an eigenvalue.
• If $\lambda < 0$, or $\lambda = -\beta^2$, the ODE becomes

$$y'' - \beta^2 y = 0.$$

The anzats $y = e^{sx}$ gives $s^2 - \beta^2 = 0$, or $s = \pm\beta$. Thus the general solution is

$$y(x) = Ae^{\beta x} + Be^{-\beta x}.$$

Applying the first boundary condition gives

$$y(0) = 0 = A + B, \quad \text{or} \quad B = -A.$$

The second boundary condition gives

$$y(L) = 0 = A(e^{\beta L} - e^{-\beta L}) = 2A \sinh \beta L, \quad \text{or} \quad A = 0.$$

Thus, the only solution is the trivial solution, $y(x) \equiv 0$, which is not an eigenfunction, and therefore, there are no negative eigenvalues.
• If $\lambda > 0$, try $\lambda = +\beta^2$

$$y'' + \beta^2 y = 0,$$

with the anzats $y = e^{sx}$, which gives $s = \pm i\beta$ with the family of solutions

$$y(x) = A \sin \beta x + B \cos \beta x.$$

Applying the first boundary condition gives

$$y(0) = 0 = B.$$

The second boundary condition gives

$$y(L) = 0 = A \sin \beta L.$$

Since we want nontrivial solutions, $A \neq 0$, and we set $A \sin \beta L = 0$, obtaining $\beta L = n\pi$. Thus the eigenvalues and the corresponding eigenfunctions are

$$\lambda = \lambda_n = \Big(\frac{n\pi}{L}\Big)^2, \qquad y_n(x) = A_n \sin\Big(\frac{n\pi x}{L}\Big).$$

⌈ Also, the eigenfunctions can always be used to represent any piecewise smooth function $f(x)$,

$$f(x) \sim \sum_{n=1}^{\infty} a_n y_n(x).$$

Thus, for our example,

$$f(x) \sim \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}. \qquad \rfloor$$

$\square$

Problem (F'98, #3). *Consider the eigenvalue problem*

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0,$$
$$\phi(0) - \frac{d\phi}{dx}(0) = 0, \quad \phi(1) + \frac{d\phi}{dx}(1) = 0.$$

a) *Show that all eigenvalues are* ***positive****.*
b) *Show that there exist a sequence of eigenvalues $\lambda = \lambda_n$, each of which satisfies*

$$\tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}.$$

Proof. **a) Method ①.** • If $\lambda = 0$, the ODE reduces to $\phi'' = 0$. Try $\phi(x) = Ax + B$. From the first boundary condition,

$$\phi(0) - \phi'(0) = 0 = B - A \quad \Rightarrow \quad B = A.$$

Thus, the solution takes the form $\phi(x) = Ax + A$. The second boundary condition gives

$$\phi(1) + \phi'(1) = 0 = 3A \quad \Rightarrow \quad A = B = 0.$$

Thus the only solution is $\phi \equiv 0$, which is not an eigenfunction, and 0 not an eigenvalue. ✓
• If $\lambda < 0$, try $\phi(x) = e^{sx}$, which gives $s = \pm\sqrt{-\lambda} = \pm\beta \in \mathbb{R}$.
Hence, the family of solutions is $\phi(x) = Ae^{\beta x} + Be^{-\beta x}$. Also, $\phi'(x) = \beta Ae^{\beta x} - \beta Be^{-\beta x}$. The boundary conditions give

$$\phi(0) - \phi'(0) = 0 = A + B - \beta A + \beta B = A(1-\beta) + B(1+\beta), \tag{5.19}$$

$$\phi(1)+\phi'(1) = 0 = Ae^{\beta} + Be^{-\beta} + \beta Ae^{\beta} - \beta Be^{-\beta} = Ae^{\beta}(1+\beta) + Be^{-\beta}(1-\beta). \tag{5.20}$$

From (5.19) and (5.20) we get

$$\frac{1+\beta}{1-\beta} = -\frac{A}{B} \quad \text{and} \quad \frac{1+\beta}{1-\beta} = -\frac{B}{A}e^{-2\beta}, \qquad \text{or} \qquad \frac{A}{B} = e^{-\beta}.$$

From (5.19), $\beta = \dfrac{A+B}{A-B}$ and thus, $\dfrac{A}{B} = e^{\frac{A+B}{B-A}}$, which has no solutions. ✓

Method ②. Multiply the equation by ϕ and integrate from 0 to 1.

$$\int_0^1 \phi\phi''\,dx + \lambda\int_0^1 \phi^2\,dx = 0,$$
$$\phi\phi'|_0^1 - \int_0^1 (\phi')^2\,dx + \lambda\int_0^1 \phi^2\,dx = 0,$$
$$\lambda = \frac{-\phi(1)\phi'(1) + \phi(0)\phi'(0) + \int_0^1(\phi')^2\,dx}{\int_0^1 \phi^2\,dx} = \frac{\phi(1)^2 + \phi(0)^2 + \int_0^1(\phi')^2\,dx}{\int_0^1 \phi^2\,dx}.$$

Thus, $\lambda > 0$ for ϕ not identically 0.
b) Since $\lambda > 0$, the anzats $\phi = e^{sx}$ gives $s = \pm i\sqrt{\lambda}$ and the family of solutions takes the form

$$\phi(x) = A\sin(x\sqrt{\lambda}) + B\cos(x\sqrt{\lambda}).$$

Then, $\phi'(x) = A\sqrt{\lambda}\cos(x\sqrt{\lambda}) - B\sqrt{\lambda}\sin(x\sqrt{\lambda})$. The first boundary condition gives

$$\phi(0) - \phi'(0) \;=\; 0 = B - A\sqrt{\lambda} \qquad \Rightarrow \qquad B = A\sqrt{\lambda}.$$

Hence, $\phi(x) = A\sin(x\sqrt{\lambda}) + A\sqrt{\lambda}\cos(x\sqrt{\lambda})$. The second boundary condition gives

$$\begin{aligned}\phi(1)+\phi'(1)=0 &= A\sin(\sqrt{\lambda}) + A\sqrt{\lambda}\cos(\sqrt{\lambda}) + A\sqrt{\lambda}\cos(\sqrt{\lambda}) - A\lambda\sin(\sqrt{\lambda}) \\ &= A\big[(1-\lambda)\sin(\sqrt{\lambda}) + 2\sqrt{\lambda}\cos(\sqrt{\lambda})\big]\end{aligned}$$

$A \neq 0$ (since $A = 0$ implies $B = 0$ and $\phi = 0$, which is not an eigenfunction). Therefore, $-(1-\lambda)\sin(\sqrt{\lambda}) = 2\sqrt{\lambda}\cos(\sqrt{\lambda})$, and thus $\tan(\sqrt{\lambda}) = \frac{2\sqrt{\lambda}}{\lambda-1}$. $\square$

Problem (F'02, #2). *Consider the second order differential operator L defined by*

$$Lu = -u'' + \epsilon x u$$

for $0 < x < \pi$ with boundary conditions

$$u(0) = u(\pi) = 0.$$

a) *For $\epsilon = 0$ find the leading (i.e. smallest) eigenvalue λ_0 and the corresponding eigenfunction ϕ_0 for L.*
b) *For $\epsilon > 0$ look for the eigenvalues and eigenfunctions to have an expansion of the form*

$$\begin{aligned}\lambda &= \lambda_0 + \epsilon\lambda_1 + O(\epsilon^2),\\ \phi &= \phi_0 + \epsilon\phi_1 + O(\epsilon^2).\end{aligned}$$

Find formulas for λ_1 and ϕ_1 (your formulas will contain definite integrals which you do not need to evaluate).

Proof. **a)** Since $\epsilon = 0$, the eigenvalue problem for $\lambda = \nu^2$ becomes

$$u'' + \nu^2 u = 0.$$

The equation has solutions in the form

$$u(x) = A\sin\nu x + B\cos\nu x.$$

The first boundary condition gives $u(0) = 0 = B$, and the second gives $u(\pi) = 0 = A\sin\nu\pi$. Since we are looking for nontrivial solutions, $A \neq 0$ and $\sin\nu\pi = 0$, which gives $\nu = 1, 2, 3, \ldots$. Thus, the smallest eigenvalue and the corresponding eigenfunction are

$$\lambda_0 = 1, \qquad \phi_0 = \sin x.$$

b) For $\epsilon > 0$, we have

$$\begin{aligned}&-u'' + \epsilon x u - \lambda u = 0,\\ &-(\phi_0 + \epsilon\phi_1)'' + \epsilon x(\phi_0 + \epsilon\phi_1) - (\lambda_0 + \epsilon\lambda_1)(\phi_0 + \epsilon\phi_1) = 0,\\ &-\phi_0'' - \epsilon\phi_1'' + \epsilon x\phi_0 + \epsilon^2 x\phi_1 - \lambda_0\phi_0 - \epsilon\lambda_0\phi_1 - \epsilon\lambda_1\phi_0 - \epsilon^2\lambda_1\phi_1 = 0.\end{aligned}$$

Drop $O(\epsilon^2)$ terms. Since $\phi_0'' + \lambda_0\phi_0 = 0$,

$$\begin{aligned}&-\epsilon\phi_1'' + \epsilon x\phi_0 - \epsilon\lambda_0\phi_1 - \epsilon\lambda_1\phi_0 = 0,\\ &-\phi_1'' + x\phi_0 - \lambda_0\phi_1 - \lambda_1\phi_0 = 0,\\ &-\phi_1'' + x\sin x - \phi_1 - \lambda_1\sin x = 0,\\ &\ \ \phi_1'' + \phi_1 = x\sin x - \lambda_1\sin x.\end{aligned}$$

Multiplying by ϕ_0 and using orthogonality of the eigenfunctions[19], we get

$$\begin{aligned}\int_0^\pi \phi_0\phi_1''dx + \underbrace{\int_0^\pi \phi_0\phi_1 dx}_{=0} &= \int_0^\pi (x\sin^2 x - \lambda_1\sin^2 x)dx,\\ \phi_0\phi_1'|_0^\pi - \int_0^\pi \phi_0'\phi_1' dx &= \int_0^\pi (x\sin^2 x - \lambda_1\sin^2 x)dx, \qquad \text{(integration by parts)}\\ 0 &= \int_0^\pi (x\sin^2 x - \lambda_1\sin^2 x)dx,\\ \lambda_1\int_0^\pi \sin^2 x\, dx &= \int_0^\pi x\sin^2 x\, dx,\end{aligned}$$

[19]Bleecker and Csordas, p. 267, p. 274.

$$\boxed{\lambda_1 = \frac{\int_0^\pi x \sin^2 x \, dx}{\int_0^\pi \sin^2 x \, dx}}$$

Since λ_1 is known, we should be able to solve the ODE $\phi_1'' + \phi_1 = x \sin x - \lambda_1 \sin x$ by the variation of parameters. $\square$

Problem (F'00, #5). *Consider the eigenvalue problem on the interval* $[0, 1]$,

$$
\begin{aligned}
-y''(t) + p(t)y(t) &= \lambda y(t), \\
y(0) = y(1) &= 0.
\end{aligned}
$$

a) *Prove that all eigenvalues* λ *are* ***simple***.
b) *Prove that there is at most a* ***finite number of negative eigenvalues***.

a) In order to show that λ is simple, need to show that there is only one linearly independent eigenfunction associated with each eigenvalue λ.

Proof. **Method 1**: Let $y_1(x)$ and $y_2(x)$ be two eigenfunctions corresponding to the same eigenvalue λ. We will show that y_1 and y_2 are linearly dependent. We have

$$
\begin{aligned}
-y_1'' + py_1 - \lambda y_1 &= 0, \\
-y_2'' + py_2 - \lambda y_2 &= 0.
\end{aligned}
$$

$$0 = y_2\big(-y_1'' + py_1 - \lambda y_1\big) - y_1\big(-y_2'' + py_2 - \lambda y_2\big) = y_1y_2'' - y_2y_1'' = [y_1y_2' - y_2y_1']',$$

where Lagrange's identity was used in the last equality. It follows that

$$y_1y_2' - y_2y_1' = \text{constant}.$$

Using boundary conditions,

$$\big(y_1y_2' - y_2y_1'\big)(0) = 0.$$

Therefore, $y_1y_2' - y_2y_1' \equiv 0$. This is equivalent to $\big(\frac{y_2}{y_1}\big)' = 0$, and hence

$$y_2 = cy_1.$$

Thus the two eigenfunctions are dependent; the eigenfunction is unique, and λ simple. $\square$

Proof. **Method 2**: Let $y_1(x)$ and $y_2(x)$ be two eigenfunctions corresponding to the same eigenvalue λ. We will show that y_1 and y_2 are linearly dependent. We only consider the case with

$$y_1'(0)^2 + y_2'(0)^2 \neq 0. \tag{5.21}$$

Consider the function

$$w(x) = y_2'(0)y_1(x) - y_1'(0)y_2(x),$$

Then $w(x)$ satisfies the following initial-value problem

$$
\begin{aligned}
-w'' + pw - \lambda w = 0 \qquad & 0 \leq x \leq 1 \\
w(0) = w'(0) = 0. &
\end{aligned}
$$

$\lceil$ Check that $w(x)$ indeed satisfies the initial-value problem:

$$
\begin{aligned}
-w'' + pw - \lambda w &= -\big[y_2'(0)y_1(x) - y_1'(0)y_2(x)\big]'' + p\big[y_2'(0)y_1(x) - y_1'(0)y_2(x)\big] \\
&\qquad -\lambda\big[y_2'(0)y_1(x) - y_1'(0)y_2(x)\big] \\
&= y_2'(0)\big[-y_1''(x) + py_1(x) - \lambda y_1(x)\big] - y_1'(0)\big[-y_2''(x) + py_2(x) - \lambda y_2(x)\big] = 0,
\end{aligned}
$$

since y_1 and y_2 are eigenfunctions. Also,

$$
\begin{aligned}
w(0) &= y_2'(0)y_1(0) - y_1'(0)y_2(0) = y_2'(0)\cdot 0 - y_1'(0)\cdot 0 = 0, \\
w'(0) &= y_2'(0)y_1'(0) - y_1'(0)y_2'(0) = 0. \qquad \rfloor
\end{aligned}
$$

Then, by the uniqueness theorem for initial-value problems, $w(x) \equiv 0$. Therefore,

$$y_2'(0)y_1(x) - y_1'(0)y_2(x) \equiv 0, \qquad 0 \le x \le 1. \tag{5.22}$$

Since $y_1(x)$ and $y_2(x)$ are eigenfunctions, $y_1(x)$ and $y_2(x)$ are not identically 0. Hence, (5.21) and (5.22) imply that $y_1'(0)y_2'(0) \neq 0$. Thus, by (5.22), $y_1(x) = \alpha y_2(x)$, where $\alpha = y_1'(0)/y_2'(0)$.

$$\begin{aligned} -y''(t) + p(t)y(t) &= \lambda y(t), \\ y(0) = y(1) &= 0. \end{aligned}$$

b) *Prove that there is at most a* ***finite number of negative eigenvalues****.*

We need to show that the eigenvalues are bounded below

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \ldots; \quad \text{with } \lambda_n \to \infty \ \text{ as } n \to \infty.$$

Multiply the equation by y and integrate:

$$\begin{aligned} -yy'' + py^2 &= \lambda y^2, \\ -\int_0^1 yy''\,dt + \int_0^1 py^2\,dt &= \lambda \int_0^1 y^2\,dt, \\ \underbrace{-yy'\big|_0^1}_{=0} + \int_0^1 (y')^2\,dt + \int_0^1 py^2\,dt &= \lambda \int_0^1 y^2\,dt, \\ \lambda &= \frac{\int_0^1 (y')^2\,dt + \int_0^1 py^2\,dt}{\int_0^1 y^2\,dt}. \end{aligned}$$

The Poincare inequality gives:

$$\begin{aligned} \int_0^1 y^2\,dt &\leq C\int_0^1 (y')^2\,dt, \qquad \text{or} \\ -\int_0^1 y^2\,dt &\geq -C\int_0^1 (y')^2\,dt. \end{aligned}$$

Thus, we have

$$\begin{aligned} \lambda &= \frac{\int_0^1 (y')^2\,dt + \int_0^1 py^2\,dt}{\int_0^1 y^2\,dt} \geq \frac{\int_0^1 (y')^2\,dt - \max_{0\leq x\leq 1}|p| \int_0^1 y^2\,dt}{\int_0^1 y^2\,dt} \\ &\geq \frac{\frac{1}{C}\int_0^1 y^2\,dt - \max_{0\leq x\leq 1}|p| \int_0^1 y^2\,dt}{\int_0^1 y^2\,dt} = \frac{\left(\frac{1}{C} - \max|p|\right)\int_0^1 y^2\,dt}{\int_0^1 y^2\,dt} \\ &= \frac{1}{C} - \max|p|. \end{aligned}$$

$\square$

Problem (S'94, #6). *Consider the eigenvalue problem*

$$-\frac{d^2u}{dx^2} + v(x)u = \lambda u \qquad on \quad [0,1]$$

with the boundary conditions $\frac{du}{dx}(0) = \frac{du}{dx}(1) = 0$. *Show that if* $\int_0^1 v(x)\,dx = 0$ *then there is a negative eigenvalue, unless* $v(x) \equiv 0$.

Proof. Divide by u and integrate:

$$\begin{aligned}
-u'' + v(x)u &= \lambda u, \\
-\int_0^1 \frac{u''}{u}\,dx + \underbrace{\int_0^1 v(x)\,dx}_{=0} &= \int_0^1 \lambda\,dx, \\
-\int_0^1 \underbrace{\frac{1}{u}}_{f} \underbrace{u''}_{g'}\,dx &= \lambda, \\
-\frac{1}{u}u'|_0^1 + \int_0^1 \underbrace{-\frac{1}{u^2}u'}_{f'} \underbrace{u'}_{g}\,dx &= \lambda, \\
0 > -\int_0^1 \frac{u'^2}{u^2}\,dx &= \lambda.
\end{aligned}$$

Thus, $\lambda < 0$. ☐

Problem (S'95, #1). *Find the* ***eigenfunctions/eigenvalues*** *for the following operator*

$$\begin{aligned} Lf &= \frac{d^2}{dx^2}f + 4f \qquad -\pi < x < \pi \\ f & \quad 2\pi - \textit{periodic}. \end{aligned}$$

Find all solutions (periodic or non-periodic) for the problems
a) $Lf = \cos x$,
b) $Lf = \cos 2x$.

Proof. To find eigenfunctions and eigenvalues for L, consider

$$f'' + 4f + \lambda f = 0,$$
$$f'' + (\lambda + 4)f = 0.$$

The anzats $f = e^{sx}$ gives $s^2 + (\lambda + 4) = 0$, or $s = \pm\sqrt{-\lambda - 4}$.

Case 1: $-\lambda - 4 < 0 \;\Rightarrow\; s = \pm i \underbrace{\sqrt{\lambda + 4}}_{\in\mathbb{R}}$.

Thus, eigenfunctions are $\cos\sqrt{\lambda + 4}\,x$, $\sin\sqrt{\lambda + 4}\,x$. To make these 2π periodic, need

$$n = \sqrt{\lambda_n + 4} \;\Rightarrow\; \lambda_n + 4 = n^2 \;\Rightarrow\; \lambda_n = -4 + n^2, \;\; n = 0, 1, 2, \ldots \quad (\text{note: } -\lambda - 4 < 0).$$

Thus, the eigenvalues and eigenfunctions are

$$\boxed{\lambda_n = -4 + n^2, \qquad \cos nx, \; n = 0, 1, 2, \ldots, \quad \sin nx, \; n = 1, 2, \ldots.}$$

For example, with $n = 1$, eigenvalues and eigenfunctions are:

$$\lambda_1 = -3, \qquad \cos x, \quad \sin x.$$

Note that $-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \ldots;$ with $\lambda_n \to \infty$ as $n \to \infty$.

Case 2: $-\lambda - 4 = 0, \quad (\lambda + 4 = 0)$

$\Rightarrow \quad f'' = 0 \quad f = ax + b$. Since $a \neq 0$ does not satisfy periodicity (being a linear function), $a = 0$. Since an eigenfunction can not be 0 everywhere $b \neq 0$. Thus,

$$\boxed{\lambda = -4, \qquad f = b \neq 0}$$

is 2π periodic.

Case 3: $-\lambda - 4 > 0 \;\Rightarrow\; s = \pm\sqrt{-\lambda - 4}$

Eigenfunctions $e^{-\sqrt{-\lambda-4}x}$, $e^{\sqrt{-\lambda-4}x}$ are **not** 2π-periodic.

- As in F'92 #3, could take $f(x) = \sum a_n e^{inx}$, $2\pi-$periodic. Then

$$f'' + 4f + \lambda f = 0,$$
$$-n^2 + 4 + \lambda = 0,$$
$$\lambda_n = -4 + n^2.$$

e^{inx}, $n = 0, 1, 2, \ldots$, are eigenfunctions.

a) $f'' + 4f = \cos x$. We first solve the homogeneous equation $f'' + 4f = 0$. Substitution $f = e^{sx}$ gives $s^2 + 4 = 0$. Hence, $s_{1,2} = \pm 2i$ and the superposition principle gives:

$$f_h(x) = A\cos 2x + B\sin 2x.$$

Find a particular solution of the inhomogeneous equation $f'' + 4f = \cos x$.
Try $f(x) = C\cos x + D\sin x$. Then,

$$\begin{aligned} -C\cos x - D\sin x + 4C\cos x + 4D\sin x &= \cos x, \\ 3C\cos x + 3D\sin x &= \cos x, \\ C = \frac{1}{3}, \quad D &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} f_p(x) &= \frac{1}{3}\cos x. \\ f(x) &= f_h(x) + f_p(x) = A\cos 2x + B\sin 2x + \frac{1}{3}\cos x. \end{aligned}$$

b) $f'' + 4f = \cos 2x$. In part (a), we already found

$$f_h(x) = A\cos 2x + B\sin 2x.$$

to be a homogeneous equation. To find a particular solution of the inhomogeneous equation, we try

$$\begin{aligned} f_p(x) &= Cx\cos 2x + Dx\sin 2x, \\ f_p'(x) &= -2Cx\sin 2x + C\cos 2x + 2Dx\cos 2x + D\sin 2x, \\ f_p''(x) &= -4Cx\cos 2x - 2C\sin 2x - 2C\sin 2x - 4Dx\sin 2x + 2D\cos 2x + 2D\cos 2x \\ &= -4Cx\cos 2x - 4C\sin 2x - 4Dx\sin 2x + 4D\cos 2x. \end{aligned}$$

Substitution into $f'' + 4f = \cos 2x$ gives:

$$-4Cx\cos 2x - 4C\sin 2x - 4Dx\sin 2x + 4D\cos 2x \quad + \quad 4Cx\cos 2x + 4Dx\sin 2x = \cos 2x,$$

which gives $\quad -4C\sin 2x + 4D\cos 2x = \cos 2x, \quad$ or $\quad C = 0,\ D = \frac{1}{4}$.
Thus,

$$\begin{aligned} f_p(x) &= \frac{1}{4}x\sin 2x, \\ f(x) &= f_h(x) + f_p(x) = A\cos 2x + B\sin 2x + \frac{1}{4}x\sin 2x. \end{aligned}$$

$\square$

Problem (F'92, #3). *Denote*

$$Lf = \frac{\partial^4 f}{\partial x^4} + 3\frac{\partial^2 f}{\partial x^2} + f \qquad \textit{for} \ \ 0 < x < \pi$$

for f satisfying

$$f = \frac{\partial^2}{\partial x^2} f = 0 \qquad \textit{for} \ \ x = 0 \ \textit{and} \ x = \pi.$$

a) *Find the* ***eigenfunctions and eigenvalues*** *for* L.
b) *Solve the problem*

$$\frac{\partial}{\partial t} f = Lf$$
$$f(x, t=0) = \frac{e^{ix}}{1-\frac{1}{2}e^{ix}} - \frac{e^{-ix}}{1-\frac{1}{2}e^{-ix}}$$

with the boundary conditions above.

Proof. **a)** In order to find eigenfunctions and eigenvalues for L, consider

$$f'''' + 3f'' + f = \lambda f.$$

$$\text{Let} \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

$f(0) = f(\pi) = 0 \quad \Rightarrow \quad a_n = 0, \quad n = 0, 1, 2, \ldots.$
$f''(0) = f''(\pi) = 0 \quad \Rightarrow \quad a_n = 0, \quad n = 0, 1, 2, \ldots.$

$$\Rightarrow \quad f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

Thus, the eigenfunctions are $\sin nx$, $n = 1, 2, \ldots.$ We have

$$\begin{aligned}
(\sin nx)'''' + 3(\sin nx)'' + \sin nx &= \lambda \sin nx, \\
(n^4 - 3n^2 + 1) \sin nx &= \lambda \sin nx, \\
n^4 - 3n^2 + 1 &= \lambda_n.
\end{aligned}$$

Thus, the eigenvalues and eigenfunctions are

$$\boxed{\lambda_n = n^4 - 3n^2 + 1, \quad f_n(x) = \sin nx, \quad n = 1, 2, \ldots.}$$

b) We have

$$\begin{aligned}
f_t &= f_{xxxx} + 3f_{xx} + f, \\
f(x, 0) &= \frac{e^{ix}}{1-\frac{1}{2}e^{ix}} - \frac{e^{-ix}}{1-\frac{1}{2}e^{-ix}}
\end{aligned}$$

Let $f(x,t) = \sum f_n(t) \sin nx$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} f_n'(t) \sin nx &= \sum_{n=1}^{\infty} f_n(t) n^4 \sin nx - 3f_n(t) n^2 \sin nx + f_n(t) \sin nx, \\
f_n'(t) &= (n^4 - 3n^2 + 1) f_n(t), \\
f_n'(t) - (n^4 - 3n^2 + 1) f_n(t) &= 0, \\
f_n(t) &= c_n e^{(n^4 - 3n^2 + 1)t}, \\
f(x,t) &= \sum_{n=1}^{\infty} c_n e^{(n^4 - 3n^2 + 1)t} \sin nx.
\end{aligned}$$

Using initial conditions, we have

$$\begin{aligned}
f(x,0)=\sum_{n=1}^{\infty} c_n \sin nx &= \frac{e^{ix}}{1-\frac{1}{2}e^{ix}} - \frac{e^{-ix}}{1-\frac{1}{2}e^{-ix}} = \sum_{n=0}^{\infty} e^{ix}\big(\frac{1}{2}e^{ix}\big)^n - \sum_{n=0}^{\infty} e^{-ix}\big(\frac{1}{2}e^{-ix}\big)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{2^n} e^{ix(n+1)} - \sum_{n=0}^{\infty} \frac{1}{2^n} e^{-ix(n+1)} = \sum_{n=0}^{\infty} \frac{1}{2^n}\big(e^{ix(n+1)} - e^{-ix(n+1)}\big) \\
&= \sum_{n=0}^{\infty} \frac{1}{2^{n-1}} \frac{i}{2i}\big(e^{ix(n+1)} - e^{-ix(n+1)}\big) = \sum_{n=0}^{\infty} \frac{i}{2^{n-1}} \sin((n+1)x) \\
&= \sum_{n=1}^{\infty} \frac{i}{2^{n-2}} \sin nx.
\end{aligned}$$

Thus, $c_n = i/2^{n-2}, \quad n = 1, 2, \ldots$, and

$$\boxed{f(x,t) = \sum_{n=1}^{\infty} \frac{i}{2^{n-2}}\, e^{(n^4-3n^2+1)t}\, \sin nx.}$$

$\square$

Problem (W'02, #2). ***a)*** *Prove that*

$$\int_0^\pi |u(x)|^2 dx \le \int_0^\pi \Big|\frac{du}{dx}\Big|^2 dx$$

for all continuously differentiable functions u satisfying $u(0) = u(\pi) = 0$.
b) *Consider the differential operator*

$$Lu = -\frac{d^2u}{dx^2} + q(x)u, \quad 0 < x < \pi$$

with the boundary conditions $u(0) = u(\pi) = 0$. Suppose q is continuous on $[0, \pi]$ and $q(x) > -1$ on $[0, \pi]$. Prove that all ***eigenvalues of*** L ***are positive.***

Proof. **a)** Use eigenvalues of the Laplacian for $u'' + \lambda u = 0$, $u(0) = u(\pi) = 0$.
Then $\phi_n = \sin nx$, $\lambda_n = n^2$, $n = 1, 2, \ldots$.
Then

$$\begin{aligned}
\int_0^\pi u^2\,dx &= \int_0^\pi \big(\sum_m a_m\phi_m\big)\big(\sum_n a_n\phi_n\big)\,dx = \sum_n a_n^2 \int_0^\pi \sin^2 nx\,dx \\
&= \sum_n a_n^2 \int_0^\pi \frac{1-\cos 2nx}{2}\,dx = \sum_n a_n^2 \Big[\frac{x}{2} - \frac{1}{4n}\sin 2nx\Big]_0^\pi = \frac{\pi}{2}\sum_n a_n^2, \\
\int_0^\pi (u')^2\,dx &= u(\pi)u'(\pi) - u(0)u'(0) - \int_0^\pi uu''\,dx = -\int_0^\pi uu''\,dx \\
&= -\int_0^\pi \big(\sum_m a_m\phi_m\big)\big(\sum_n -\lambda_n a_n\phi_n\big)\,dx \\
&= \sum_n \lambda_n a_n^2 \int_0^\pi \sin^2 nx\,dx = \frac{\pi}{2}\sum_n \lambda_n a_n^2.
\end{aligned}$$

Since $\lambda_n = n^2$, $n = 1, 2, \ldots \;\Rightarrow\; \lambda_n \ge 1$, so [20]

$$\int_0^\pi u^2\,dx \;=\; \frac{\pi}{2}\sum_n a_n^2 \;\le\; \frac{\pi}{2}\sum_n \lambda_n a_n^2 \;=\; \int_0^\pi (u')^2\,dx.$$

b) We have

$$\begin{aligned}
-u'' + q(x)u - \lambda u &= 0, \\
-uu'' + q(x)u^2 - \lambda u^2 &= 0, \\
\int_0^\pi -uu''\,dx + \int_0^\pi q(x)u^2\,dx - \int_0^\pi \lambda u^2\,dx &= 0, \\
-uu'|_0^\pi + \int_0^\pi (u')^2\,dx + \int_0^\pi q(x)u^2\,dx - \int_0^\pi \lambda u^2\,dx &= 0, \\
\int_0^\pi (u')^2\,dx + \int_0^\pi q(x)u^2\,dx = \lambda \int_0^\pi u^2\,dx.
\end{aligned}$$

Since $q(x) > -1$, and using result from part (a),

$$0 \underbrace{\le}_{(a)} \int_0^\pi (u')^2\,dx - \int_0^\pi u^2\,dx \underbrace{<}_{q>-1} \int_0^\pi (u')^2\,dx + \int_0^\pi q(x)u^2\,dx = \lambda \int_0^\pi u^2\,dx.$$

Since $\int_0^\pi u^2\,dx \ge 0$, we have $\lambda > 0$. $\square$

[20] See similar Poincare Inequality PDE problem.

Problem (F'02, #5; F'89, #6). ***a)*** *Suppose that u is a continuously differentiable function on $[0,1]$ with $u(0)=0$. Starting with $u(x)=\int_0^x u'(t)\,dt$, prove the (sharp) estimate*

$$\max_{[0,1]} |u(x)|^2 \leq \int_0^1 |u'(t)|^2\,dt. \tag{5.23}$$

b) *For any function p define $p_-(x) = -\min\{p(x),0\}$.*[21] *Using the inequality (5.23), if p is continuous on $[0,2]$, show that all eigenvalues of*

$$Lu = -u'' + pu \quad \text{on } [0,2]$$

with $u(0)=u(2)=0$ are ***strictly positive*** *if $\int_0^2 p_-(t)\,dt < 1$.*

Proof. **a)** By the Fundamental Theorem of Calculus,

$$\begin{aligned}
\int_0^x u'(t)\,dt &= u(x)-u(0) = u(x),\\
\max_{[0,1]}|u(x)| &= \Big|\int_0^1 u'(t)\,dt\Big| \leq \int_0^1 |u'(t)|\,dt \leq ||1||_{L^2}\Big(\int_0^1 |u'(t)|^2\,dt\Big)^{\frac{1}{2}} = \Big(\int_0^1 |u'(t)|^2\,dt\Big)^{\frac{1}{2}},\\
\max_{[0,1]}|u(x)|^2 &\leq \int_0^1 |u'(t)|^2\,dt. \ \checkmark
\end{aligned}$$

b) We have

$$\begin{aligned}
-u''+pu &= \lambda u,\\
\int_0^2 -uu''\,dt + \int_0^2 pu^2\,dt &= \int_0^2 \lambda u^2\,dt,\\
\underbrace{-uu'\big|_0^2}_{=0} + \int_0^2 |u'|^2\,dt + \int_0^2 pu^2\,dt &= \int_0^2 \lambda u^2\,dt.
\end{aligned}$$

If we define $p_+(x)=\max\{p(x),0\}$ and $p_-(x) = -\min\{p(x),0\}$, then $p = p_+ - p_-$.

[21] Note that p_+ and p_- are defined by

$$p_+(x) = \begin{cases} p(x) & \text{for } p(x)\geq 0\\ 0 & \text{for } p(x)<0, \end{cases} \qquad p_-(x) = \begin{cases} 0 & \text{for } p(x)\geq 0\\ |p(x)| & \text{for } p(x)<0. \end{cases}$$

$$
\begin{aligned}
\int_0^2 |u'|^2\,dt + \int_0^2 p_+ u^2\,dt - \int_0^2 p_- u^2\,dt &= \lambda \int_0^2 u^2\,dt, \\
\int_0^2 |u'|^2\,dt - \int_0^2 p_- u^2\,dt &\le \lambda \int_0^2 u^2\,dt, \\
\max_{[0,2]} |u|^2 - \int_0^2 p_- u^2\,dt &\le \lambda \int_0^2 u^2\,dt, \\
\max_{[0,2]} |u|^2 - \max_{[0,2]} |u|^2 \int_0^2 p_-\,dt &\le \lambda \int_0^2 u^2\,dt, \\
\max_{[0,2]} |u|^2 \Big(1 - \underbrace{\int_0^2 p_-\,dt}_{<1}\Big) &\le \lambda \int_0^2 u^2\,dt, \\
c^2 \max_{[0,2]} |u|^2 &\le \lambda \int_0^2 u^2\,dt.
\end{aligned}
$$

Thus, $\lambda > 0$. $\square$

Problem (F'95, #6). *Define*

$$Ly(x) = -y''(x) + q(x)y(x) \qquad on \quad (0,a).$$

Denote $q_-(x) = \min(q(x), 0)$. *We seek conditions on* $q_-(x)$ *so that* L *will be nonnegative definite on* $C_0^\infty(0,a)$, *i.e.,*

$$(L\phi, \phi) = \int_0^a \phi(x) \cdot L\phi(x)\, dx \geq 0 \qquad \forall \phi \in C_0^\infty(0,a). \tag{5.24}$$

Find optimal conditions on $q_-(x)$ *so that (5.24) holds.*
Can $q_-(x)$ *be unbounded and (5.24) still hold?*

Proof. Define $q_+ = \max(q(x), 0)$. We have

$$\begin{aligned}
\int_0^a \phi(x) \cdot L\phi(x)\, dx &= \int_0^a \phi \cdot (-\phi'' + q\phi)\, dx = \int_0^a (-\phi\phi'' + q\phi^2)\, dx \\
&= \underbrace{-\phi\phi'|_0^a}_{=0} + \int_0^a (\phi')^2 + q\phi^2\, dx = \int_0^a (\phi')^2\, dx + \int_0^a q\phi^2\, dx \\
\circledast \quad &\geq \left(\frac{\pi}{a}\right)^2 \int_0^a \phi^2\, dx + \int_0^a q\phi^2\, dx \geq \left(\frac{\pi}{a}\right)^2 \int_0^a \phi^2\, dx + \int_0^a q_-\phi^2\, dx \\
&= \int_0^a \left(\left(\frac{\pi}{a}\right)^2 + q_-\right)\phi^2\, dx \underbrace{\geq}_{need} 0.
\end{aligned}$$

Thus, if $(\frac{\pi}{a})^2 + q_- \geq 0$, L will be nonnegative definite on $C_0^\infty(0,a)$. ✓

Proof of $\circledast$:
Use eigenvalues of the Laplacian for $\phi'' + \lambda\phi = 0$, $\phi(0) = \phi(a) = 0$.
Then $\phi_n = \sin(\frac{n\pi}{a})x$, $\lambda_n = (\frac{n\pi}{a})^2$, $n = 1, 2, \ldots$. We have

$$\begin{aligned}
\int_0^a \phi^2\, dx &= \int_0^a \big(\sum_m a_m\phi_m\big)\big(\sum_n a_n\phi_n\big)\, dx = \sum_n a_n^2 \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx, \\
\int_0^a (\phi')^2\, dx &= \phi\phi'|_0^a - \int_0^a \phi\phi''\, dx = -\int_0^a \phi\phi''\, dx \\
&= -\int_0^a \big(\sum_m a_m\phi_m\big)\big(\sum_n -\lambda_n a_n\phi_n\big)\, dx \\
&= \sum_n \lambda_n a_n^2 \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx.
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\pi}{a}\right)^2 \int_0^a \phi^2\, dx &= \left(\frac{\pi}{a}\right)^2 \sum_n a_n^2 \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx \leq \sum_n \lambda_n a_n^2 \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx \\
&= \int_0^a (\phi')^2\, dx. \\
\Rightarrow \qquad & \int_0^a (\phi')^2\, dx \geq \left(\frac{\pi}{a}\right)^2 \int_0^a \phi^2\, dx. \quad ✓
\end{aligned}$$

□

Problem (W'04, #4). *Consider boundary value problem on* $[0, \pi]$*:*

$$
\begin{aligned}
-y''(x) + p(x)y(x) = f(x), \qquad 0 < x < \pi, \\
y(0) = 0, \quad y'(\pi) = 0.
\end{aligned}
$$

Find the smallest λ_0 *such that the boundary value problem has a* ***unique*** *solution whenever* $p(x) > \lambda_0$ *for all* x*. Justify your answer.*

Proof. Suppose y_1 and y_2 are two solutions of the problem. Let $w = y_1 - y_2$. Then

$$
\begin{aligned}
-w'' + pw = 0, \qquad 0 < x < \pi, \\
w(0) = 0, \quad w'(\pi) = 0.
\end{aligned}
$$

Multiply by w and integrate

$$
\begin{aligned}
-\int_0^1 ww''\,dx + \int_0^\pi pw^2\,dx = 0, \\
\underbrace{-ww'|_0^\pi}_{=0} + \int_0^\pi (w')^2\,dx + \int_0^\pi pw^2\,dx = 0, \\
\int_0^\pi (w')^2\,dx + \int_0^\pi pw^2\,dx = 0. \qquad \circledast
\end{aligned}
$$

We will derive the Poincare inequality for this boundary value problem.
Use eigenvalues of the Laplacian for $w'' + \lambda w = 0$, $w(0) = w'(\pi) = 0$.
Expand w in eigenfunctions: $w = \sum_n a_n \phi_n$. Then $\phi_n(x) = a_n \cos\sqrt{\lambda_n}x + b_n \sin\sqrt{\lambda_n}x$.
Boundary conditions give:

$$
\lambda_n = \Big(n + \frac{1}{2}\Big)^2, \qquad \phi_n(x) = \sin\Big(n + \frac{1}{2}\Big)x, \qquad n = 0, 1, 2, \ldots. \qquad \text{Then,}
$$

$$
\begin{aligned}
\int_0^\pi w^2\,dx &= \int_0^\pi \big(\sum_m a_m\phi_m\big)\big(\sum_n a_n\phi_n\big)\,dx = \sum_n a_n^2 \int_0^\pi \phi_n^2(x)\,dx, \\
\int_0^\pi (w')^2\,dx &= ww'|_0^\pi - \int_0^\pi ww''\,dx = -\int_0^\pi ww''\,dx \\
&= -\int_0^\pi \big(\sum_m a_m\phi_m\big)\big(\sum_n -\lambda_n a_n\phi_n\big)\,dx \;=\; \sum_n \lambda_n a_n^2 \int_0^\pi \phi_n^2\,dx.
\end{aligned}
$$

Thus, the Poincare inequality is:

$$
\frac{1}{4}\int_0^\pi w^2\,dx \;=\; \frac{1}{4}\sum_n a_n^2 \int_0^\pi \phi_n^2\,dx \;\le\; \sum_n \lambda_n a_n^2 \int_0^\pi \phi_n^2\,dx \;=\; \int_0^\pi (w')^2\,dx.
$$

Thus, from $\circledast$:

$$
0 \;=\; \int_0^\pi (w')^2\,dx + \int_0^\pi pw^2\,dx \;\ge\; \frac{1}{4}\int_0^\pi w^2\,dx + \int_0^\pi pw^2\,dx \;=\; \int_0^\pi \Big(\frac{1}{4} + p\Big)w^2\,dx.
$$

If $\frac{1}{4} + p(x) > 0$, $\quad (p(x) > -\frac{1}{4})$, $\quad \forall x$, then $w \equiv 0$, and we obtain uniqueness. $\square$

Problem (F'97, #5). ***a)*** *Prove that all eigenvalues of the Sturm-Liouville problem*

$$\frac{d}{dx}\Big(p(x)\frac{du}{dx}\Big) + \lambda u(x) = 0, \qquad 0 < x < a,$$
$$u(0) = 0, \qquad \frac{du(a)}{dx} + hu(a) = 0,$$

are ***positive****. Here $h > 0$, $p(x) > 0$ and continuous on $[0, a]$.*
b) *Show that the same is true when $h < 0$ and $|h|$ is sufficiently small.*

Proof. **a)** Let ϕ be an eigenfunction. We have

$$(p\phi')' + \lambda\phi = 0. \tag{5.25}$$

Multiply (5.25) by ϕ and integrate from 0 to a,

$$\int_0^a \big((p\phi')'\phi + \lambda\phi^2\big)\,dx \;\;=\;\; 0.$$

Since $\int_0^a \phi^2\,dx > 0$, we can solve for λ:

$$\lambda = \frac{-\int_0^a (p\phi')'\phi\,dx}{\int_0^a \phi^2\,dx}.$$

Integrating by parts and plugging in the boundary conditions give

$$\lambda = \frac{-p\phi\phi'|_0^a + \int_0^a p(\phi')^2\,dx}{\int_0^a \phi^2\,dx} = \frac{hp(a)\phi^2(a) + \int_0^a p(x)(\phi'(x))^2\,dx}{\int_0^a \phi^2(x)\,dx} \;\;\geq\;\; 0.$$

To show that $\lambda > 0$, assume $\lambda = 0$. Then the ODE becomes

$$(pu')' = 0 \;\;\Rightarrow\;\; p(x)\,u'(x) = C, \text{ a constant.}$$

Then

$$p(a)\,u'(a) = -h\,p(a)\,u(a) = C$$

Wrong assumption follows: $u = 0$.

b) $h < 0$. For $|h|$ is sufficiently small, i.e.

$$|hp(a)\phi^2(a)| < \int_0^a p(x)(\phi'(x))^2\,dx,$$

we have

$$\lambda = \frac{hp(a)\phi^2(a) + \int_0^a p(x)(\phi'(x))^2\,dx}{\int_0^a \phi^2(x)\,dx} > 0.$$

□

Problem (S'93, #7). ***a)*** *Show that the general solution of*

$$\frac{1}{x}\frac{d}{dx}\left[x\frac{df}{dx}\right] = -\lambda f, \tag{5.26}$$

where λ is a constant, is a linear combination of f_1 and f_2, where

$$f_1 = O(1), \qquad f_2 = O(\ln x), \qquad x \to 0.$$

Proof. **a)** We use the **method of dominant balance**. We have

$$\begin{aligned}
\frac{1}{x}(xf')' &= -\lambda f, \\
\frac{1}{x}(xf'' + f') &= -\lambda f, \\
f'' + \frac{1}{x}f' &= -\lambda f, \\
xf'' + f' &= -\lambda x f.
\end{aligned}$$

As $x \to 0$, $f'(x) \to 0$, i.e. $f(x) \to C$. (Incomplete) $\square$

b) *Consider the eigenvalue problem posed by (5.26) and the conditions*

$$f(0) = O(1), \qquad f(1) = 0. \tag{5.27}$$

Assuming that the spectrum of λ is discrete, show that the eigenfunctions belonging to different λ are ***orthogonal****:*

$$\int_0^1 x f_i f_j\, dx \;=\; \int_0^1 x\frac{df_i}{dx}\frac{df_j}{dx}\, dx \;=\; 0, \qquad \lambda_i \neq \lambda_j,$$

and that all eigenvalues are ***positive****.*

Proof. Rewrite the equation as

$$\frac{1}{x}(xf')' + \lambda f = 0. \qquad \circledast$$

Let λ_m, λ_n, be the eigenvalues and f_m, f_n be the corresponding eigenfunctions. We have

$$\frac{1}{x}(xf_m')' + \lambda_m f_m = 0, \tag{5.28}$$

$$\frac{1}{x}(xf_n')' + \lambda_n f_n = 0. \tag{5.29}$$

Multiply (5.28) by f_n and (5.29) by f_m and subtract equations from each other

$$\begin{aligned}
f_n\frac{1}{x}(xf_m')' + \lambda_m f_n f_m &= 0, \\
f_m\frac{1}{x}(xf_n')' + \lambda_n f_m f_n &= 0. \\
(\lambda_m - \lambda_n) f_m f_n &= f_m\frac{1}{x}(xf_n')' - f_n\frac{1}{x}(xf_m')', \\
(\lambda_m - \lambda_n) x f_m f_n &= f_m(xf_n')' - f_n(xf_m')' = [x(f_m f_n' - f_n f_m')]'.
\end{aligned}$$

Integrating over $(0,1)$ gives

$$\begin{aligned}(\lambda_m - \lambda_n)\int_0^1 x f_m f_n \, dx &= [x(f_m f_n' - f_n f_m')]_0^1 \\ &= 1\cdot(f_m f_n' - f_n f_m')(1) - 0\cdot(f_m f_n' - f_n f_m')(0) = 0, \qquad \circledcirc\end{aligned}$$

since $f_m(1) = f_n(1) = 0$. Since $\lambda_n \neq \lambda_m$, $f_n(x)$ and $f_m(x)$ are orthogonal on $[0,1]$.

- To show that f'_m and f'_n are **orthogonal** with respect to x, consider

$$\begin{aligned}
\int_0^1 x f'_m f'_n\,dx &= x f'_m f_n|_0^1 - \int_0^1 (x f'_m)' f_n\,dx \\
&= 1\cdot f'_m(1) f_n(1) - 0\cdot f'_m(0) f_n(0) - \int_0^1 (x f'_m)' f_n\,dx \\
&= -\int_0^1 (x f'_m)' f_n\,dx = \circledast = \lambda_m \int_0^1 x f_m f_n\,dx = \circledcirc = 0.
\end{aligned}$$

- We now show that eigenvalues λ are **positive**. We have

$$\frac{1}{x}(xf')' + \lambda f = 0,$$
$$(xf')' + \lambda x f = 0.$$

Multiplying by f and integrating, we get

$$\begin{aligned}
\int_0^1 f(xf')'\,dx + \lambda \int_0^1 x f^2\,dx &= 0, \\
\underbrace{f x f'|_0^1}_{=0} - \int_0^1 x f'^2\,dx + \lambda \int_0^1 x f^2\,dx &= 0, \\
\lambda = \frac{\int_0^1 x f'^2\,dx}{\int_0^1 x f^2\,dx} &\geq 0. \qquad \circledast\circledast
\end{aligned}$$

The equality holds only if $f' \equiv 0$, which means $f = C$. Since $f(1) = 0$, then $f \equiv 0$, which is not an eigenfunction. Thus, $\lambda > 0$. □

c) *Let $f(x)$ be any function that can be expanded as a linear combination of eigenfunctions of (5.26) and (5.27). Establish the* ***Rayleigh-Ritz principle****, namely that*

$$F(f) = \frac{\int_0^1 x(f')^2\,dx}{\int_0^1 x f^2\,dx}$$

is an ***upper bound on the smallest eigenvalue****.*

Proof. Let $f(x) = \sum a_n f_n$, where f_n's are eigenfunctions. Then,

$$\begin{aligned}
F(f) &= \frac{\int_0^1 x(f')^2\,dx}{\int_0^1 x f^2\,dx} = \frac{\int_0^1 x(\sum a_n f'_n)^2\,dx}{\int_0^1 x(\sum a_n f_n)^2\,dx} && \text{(by orthogonality)} \\
&= \frac{\int_0^1 x \sum a_n^2 f_n'^2\,dx}{\int_0^1 x \sum a_n^2 f_n^2\,dx} = \frac{\sum a_n^2 \int_0^1 x f_n'^2\,dx}{\sum a_n^2 \int_0^1 x f_n^2\,dx} && (\text{by } \circledast\circledast) \\
&= \frac{\sum a_n^2 \lambda_n \int_0^1 x f_n^2\,dx}{\sum a_n^2 \int_0^1 x f_n^2\,dx} > \lambda_{\min} \frac{\sum a_n^2 \int_0^1 x f_n^2\,dx}{\sum a_n^2 \int_0^1 x f_n^2\,dx} = \lambda_{\min}.
\end{aligned}$$

Thus, $\lambda_{\min} < F(f)$, i.e. $F(f)$ is an upper bound on $\lambda_{\min}$. □

d) *The Bessel function* $J_0(r)$ *is* $O(1)$ *at* $r = 0$ *and obeys*

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dJ_0}{dr}\right] = -J_0.$$

Obtain an upper bound for the smallest positive zero of J_0.

Problem (F'90, #8). *Consider the differential equation*

$$\begin{aligned}
&-u_{xx} + (1+x^2)u = \lambda u, \\
&u(0) = u(a) = 0.
\end{aligned}$$

a) *Find a* ***variational characterization*** *for the eigenvalues* λ_i, $i = 1, 2, \ldots$.
b) *Show that the eigenvalues are all* ***positive****, i.e.* $\lambda_i > 0$.
c) *Consider the problem for two different values of* a*:* $a = a_1$ *and* $a = a_2$ *with* $a_1 < a_2$*. Show that the eigenvalues* $\lambda_1(a_1)$ *for* $a = a_1$ *is larger than (or equal to) the first eigenvalues* $\lambda_1(a_2)$ *for* a_2*, i.e.*

$$\lambda_1(a_1) \geq \lambda_1(a_2) \qquad \textit{for} \;\; a_1 < a_2.$$

d) *Is this still true for* $i > 1$*, i.e. is*

$$\lambda_i(a_1) \geq \lambda_i(a_2) \qquad \textit{for} \;\; a_1 < a_2?$$

Proof. **a)** We have

$$\begin{aligned}
-u'' + (1+x^2)u &= \lambda u, \\
-\int_0^a uu''\,dx + \int_0^a (1+x^2)u^2\,dx &= \lambda \int_0^a u^2\,dx, \\
\underbrace{-uu'|_0^a}_{=0} + \int_0^a (u')^2\,dx + \int_0^a (1+x^2)u^2\,dx &= \lambda \int_0^a u^2\,dx,
\end{aligned}$$

$$\boxed{\lambda = \frac{\int_0^a \left((u')^2 + (1+x^2)u^2\right)dx}{\int_0^a u^2\,dx}.}$$

b) $$\lambda = \frac{\int_0^a \left((u')^2 + (1+x^2)u^2\right)dx}{\int_0^a u^2\,dx} > 0, \qquad \text{if } u \text{ not identically } 0.$$

c) $$\begin{aligned}
\lambda_1(a_1) &= \min_{[0,a_1]} \frac{\int_0^{a_1} \left((u')^2 + (1+x^2)u^2\right)dx}{\int_0^{a_1} u^2\,dx}, \\
\lambda_1(a_2) &= \min_{[0,a_2]} \frac{\int_0^{a_2} \left((u')^2 + (1+x^2)u^2\right)dx}{\int_0^{a_2} u^2\,dx}.
\end{aligned}$$

The minimum value in a small interval is greater then or equal to the minimum value in the larger interval. Thus, $\lambda_1(a_1) \geq \lambda_1(a_2)$ for $a_1 < a_2$.
We may also think of this as follows: We can always make a 0 extension of u from a_1 to a_2. Then, we can observe that the minimum of λ for such extended functions would be greater.

d) □

6 Variational (V) and Minimization (M) Formulations

Consider

$$(\mathbf{D}) \qquad \begin{cases} -u''(x) = f(x) & \text{for } \ 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases}$$

$$(\mathbf{V}) \qquad \text{Find } \ u \in V, \ \text{s.t.} \ \ a(u,v) = L(v) \quad \forall v \in V,$$

$$(\mathbf{M}) \qquad \text{Find } \ u \in V, \ \text{s.t.} \ \ F(u) \le F(v) \quad \forall v \in V, \qquad (F(u) = \min_{v \in V} F(v)).$$

$V = \{v : \ v \in C^0[0,1],\ v'$ piecewise continous and bounded on $[0,1]$, and $v(0) = v(1) = 0\}$.

$$F(v) = \frac{1}{2} a(v,v) - L(v)$$

$$(\mathbf{D}) \Leftrightarrow (\mathbf{V}) \Leftrightarrow (\mathbf{M})$$

(D) ⇒ (V)
Multiply the equation by $v \in V$, and integrate over $(0,1)$:

$$\begin{aligned} -u'' &= f(x), \\ \int_0^1 -u''v\,dx &= \int_0^1 fv\,dx, \\ \underbrace{-u'v|_0^1}_{=0} + \int_0^1 u'v'\,dx &= \int_0^1 fv\,dx, \\ \int_0^1 u'v'\,dx &= \int_0^1 fv\,dx, \\ a(u,v) &= L(v) \qquad \forall v \in V. \end{aligned}$$

(V) ⇒ (M)
We have $a(u,v) = L(v), \ \ \forall v \in V \ \ \circledast$. Suppose $v = u + w, \ \ w \in V$. We have

$$\begin{aligned} F(v) &= F(u+w) = \frac{1}{2} a(u+w, u+w) - L(u+w) \\ &= \frac{1}{2} a(u,u) + a(u,w) + \frac{1}{2} a(w,w) - L(u) - L(w) \\ &= \underbrace{\frac{1}{2} a(u,u) - L(u)}_{=F(u)} + \frac{1}{2} a(w,w) + \underbrace{a(u,w) - L(w)}_{=0, \ \ by \ \circledast} \\ &\ge F(u). \end{aligned}$$

(M) ⇒ (V)
We have $F(u) \le F(u + \varepsilon v)$, for any $v \in V$, since $u + \varepsilon v \in V$. Thus, the function

$$\begin{aligned} g(\varepsilon) \ \equiv \ F(u+\varepsilon v) &= \frac{1}{2} a(u+\varepsilon v, u+\varepsilon v) - L(u+\varepsilon v) \\ &= \frac{1}{2} a(u,u) + \varepsilon a(u,v) + \frac{\varepsilon^2}{2} a(v,v) - L(u) - \varepsilon L(v), \end{aligned}$$

has a minimum at $\varepsilon = 0$ and hence $g'(0) = 0$. We have

$$\begin{aligned} g'(\varepsilon) &= a(u,v) + \varepsilon a(v,v) - L(v), \\ 0 = g'(0) &= a(u,v) - L(v), \\ a(u,v) &= L(v). \end{aligned}$$

(V) $\Rightarrow$ (D)
We have

$$\int_0^1 u'v'\,dx - \int_0^1 fv\,dx = 0 \qquad \forall v \in V.$$

Assume u'' exists and is continuous, then

$$\underbrace{u'v|_0^1}_{=0} - \int_0^1 u''v\,dx - \int_0^1 fv\,dx = 0,$$
$$-\int_0^1 (u''+f)v\,dx = 0 \qquad \forall v \in V.$$

Since $u''+f$ is continuous, then

$$(u''+f)(x) = 0 \qquad 0 < x < 1.$$

We can show that **(V)** is uniquely determined if $a(u,v) = (u',v') = \int_0^1 u'v'\,dx$.
Suppose $u_1,\, u_2 \in V$ and

$$(u_1', v') = L(v) \qquad \forall v \in V,$$
$$(u_2', v') = L(v) \qquad \forall v \in V.$$

Subtracting these equations gives

$$(u_1' - u_2', v') = 0 \qquad \forall v \in V.$$

Choose $v = u_1 - u_2 \in V$. We get

$$\begin{aligned} (u_1' - u_2', u_1' - u_2') &= 0, \\ \int_0^1 (u_1' - u_2')^2\,dx &= \int_0^1 (u_1 - u_2)'^{\,2}\,dx = 0, \end{aligned}$$

which shows that

$$(u_1 - u_2)'(x) = 0 \qquad \Rightarrow \qquad u_1 - u_2 = \text{constant}.$$

The boundary conditions $u_1(0) = u_2(0) = 0$ give $u_1(x) = u_2(x)$, $x \in [0,1]$.

Problem (F'91, #4). *Consider a boundary value problem in a bounded plane domain Ω:*

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + a(s)u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

where $a(s)$ is a smooth function on $\partial\Omega$.
a) *Find the variational formulation of this problem, i.e. find a functional $F(v)$ defined on smooth functions in the $\overline{\Omega}$ such that the **Euler-Lagrange equation** for this functional is equivalent to (6.1).*

Proof. **a) (D) $\Rightarrow$ (M)**
We will proceed as follows: **(D) $\Rightarrow$ (V) $\Rightarrow$ (M)**. We have

$$\begin{cases} \triangle u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + a(s)u = 0 & \text{on } \partial\Omega. \end{cases}$$

- **(D) $\Rightarrow$ (V)**

Multiply the equation by $v \in V$, and integrate over Ω:

$$\begin{aligned} \triangle u &= f, \\ \int_\Omega \triangle u v \, dx &= \int_\Omega f v \, dx, \\ \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds - \int_\Omega \nabla u \cdot \nabla v \, dx &= \int_\Omega f v \, dx, \\ -\int_{\partial\Omega} a(s) u v \, ds - \int_\Omega \nabla u \cdot \nabla v \, dx &= \int_\Omega f v \, dx, \\ \underbrace{\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} a(s) u v \, ds}_{a(u,v)} &= \underbrace{-\int_\Omega f v \, dx}_{L(v)}. \end{aligned}$$

- **(V) $\Rightarrow$ (M)**

$$\begin{aligned} a(u,v) &= L(v), \\ a(u,v) &= \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} a(s) u v \, ds, \\ L(v) &= -\int_\Omega f v \, dx, \\ F(v) &= \frac{1}{2} a(v,v) - L(v). \qquad \circledast \end{aligned}$$

$$\boxed{F(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 \, dx + \frac{1}{2}\int_{\partial\Omega} a(s) v^2 \, ds + \int_\Omega f v \, dx.}$$

We show that $F(v)$, defined as $\circledast$, minimizes the functional.
We have $a(u,v) = L(v), \quad \forall v \in V \quad \circledast$. Suppose $v = u + w, \quad w \in V$. We have

$$\begin{aligned} F(v) &= F(u+w) = \frac{1}{2} a(u+w, u+w) - L(u+w) \\ &= \frac{1}{2} a(u,u) + a(u,w) + \frac{1}{2} a(w,w) - L(u) - L(w) \\ &= \underbrace{\frac{1}{2} a(u,u) - L(u)}_{=F(u)} + \frac{1}{2} a(w,w) + \underbrace{a(u,w) - L(w)}_{=0, \ by \ \circledast} \ \geq \ F(u). \end{aligned}$$

$\square$

b) *Prove that if $a(s) > 0$, then the solution of (6.1) is* ***unique*** *in the class of smooth functions in $\overline{\Omega}$.*

Proof. • Let u_1, u_2 be two solutions of (6.1), and set $w = u_1 - u_2$. Then

$$\begin{aligned} a(u_1, v) &= L(v), \\ a(u_2, v) &= L(v), \\ a(w, v) &= 0. \end{aligned}$$

Let $v = w \in V$. Then,

$$a(w, w) = \int_\Omega |\nabla w|^2 \, dx + \int_{\partial\Omega} a(s) w^2 \, ds = 0.$$

Since $a(s) > 0$, $w \equiv 0$.

• We can also begin from considering

$$\begin{cases} \triangle w = 0 & \text{in } \Omega, \\ \frac{\partial w}{\partial n} + a(s) w = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiplying the equation by w and integrating, we obtain

$$\begin{aligned} \int_\Omega w \triangle w \, dx &= 0, \\ \int_{\partial\Omega} w \frac{\partial w}{\partial n} \, ds - \int_\Omega |\nabla w|^2 \, dx &= 0, \\ -\int_{\partial\Omega} a(s) w^2 \, ds - \int_\Omega |\nabla w|^2 \, dx &= 0, \\ \int_{\partial\Omega} a(s) w^2 \, ds + \int_\Omega |\nabla w|^2 \, dx &= 0. \end{aligned}$$

Since $a(s) > 0$, $w \equiv 0$. $\square$

Problem (W'04, #2). *Let $C^2(\overline{\Omega})$ be the space of all twice continuously differentiable functions in the bounded smooth closed domain $\overline{\Omega} \subset \mathbb{R}^2$. Let $u_0(x,y)$ be the function that minimizes the functional*

$$D(u) = \int\int_\Omega \Big[\Big(\frac{\partial u(x,y)}{\partial x}\Big)^2 + \Big(\frac{\partial u(x,y)}{\partial y}\Big)^2 + f(x,y)u(x,y) \Big] \, dxdy + \int_{\partial\Omega} a(s)\, u^2(x(s), y(s))\, ds,$$

where $f(x,y)$ and $a(s)$ are given continuous functions.
Find the differential equation and the boundary condition that u_0 satisfies.

Proof. **(M) ⇒ (D)**
We will proceed as follows: **(M) ⇒ (V) ⇒ (D)**. We have

$$F(v) = \int_\Omega (|\nabla v|^2 + fv)\, dx + \int_{\partial\Omega} a(s)\, v^2\, ds,$$

$$\begin{aligned} F(v) &= \frac{1}{2} a(v,v) - L(v), \\ a(u,v) &= 2\int_\Omega \nabla u \cdot \nabla v \, dx + 2\int_{\partial\Omega} a(s) uv \, ds, \\ L(v) &= -\int_\Omega fv\, dx. \end{aligned}$$

- **(M) ⇒ (V)** Since u_0 minimizes $F(v)$ we have

$$F(u_0) \le F(v), \quad \forall v \in V.$$

Thus, the function

$$\begin{aligned} g(\varepsilon) \equiv F(u_0 + \varepsilon v) &= \frac{1}{2} a(u_0 + \varepsilon v, u_0 + \varepsilon v) - L(u_0 + \varepsilon v) \\ &= \frac{1}{2} a(u_0, u_0) + \varepsilon a(u_0, v) + \frac{\varepsilon^2}{2} a(v,v) - L(u_0) - \varepsilon L(v), \end{aligned}$$

has a minimum at $\varepsilon = 0$ and hence $g'(0) = 0$. We have

$$\begin{aligned} g'(\varepsilon) &= a(u_0, v) + \varepsilon a(v,v) - L(v), \\ 0 = g'(0) &= a(u_0, v) - L(v), \\ a(u_0, v) &= L(v). \end{aligned}$$

$$2\int_\Omega \nabla u_0 \cdot \nabla v \, dx + 2\int_{\partial\Omega} a(s) u_0 v\, ds = -\int_\Omega fv\, dx.$$

- **(V) ⇒ (D)**

$$\begin{aligned} 2\int_\Omega \nabla u_0 \cdot \nabla v\, dx + 2\int_{\partial\Omega} a(s) u_0 v\, ds &= -\int_\Omega fv\, dx, \\ 2\int_{\partial\Omega} \frac{\partial u_0}{\partial n} v\, ds - 2\int_\Omega \triangle u_0 v\, dx + 2\int_{\partial\Omega} a(s) u_0 v\, ds &= -\int_\Omega fv\, dx, \\ \int_\Omega (-2\triangle u_0 + f) v\, dx + 2\int_{\partial\Omega} \Big(\frac{\partial u_0}{\partial n} + a(s) u_0\Big) v\, ds &= 0. \end{aligned}$$

If $\frac{\partial u_0}{\partial n} + a(s) u_0 = 0$, we have

$$\int_\Omega (-2\triangle u_0 + f) v\, dx = 0 \qquad \forall v \in V.$$

Since $-2\triangle u_0 + f$ is continuous, then $-2\triangle u_0 + f = 0$.

$$\boxed{\begin{cases} -2\triangle u_0 + f = 0, & x \in \Omega, \\ \frac{\partial u_0}{\partial n} + a(s)u_0 = 0, & x \in \partial\Omega. \end{cases}}$$

See the preferred solution in the Euler-Lagrange Equations section. $\square$

7 Euler-Lagrange Equations

Consider the problem of determining a C^1 function $u(x)$ for which the integral

$$E = \int_\Omega J(\vec{x}, u, \nabla u)\, d\vec{x}$$

takes on a *minimum* value.
Suppose $u(x)$ is the actual minimizing function, and choose any C^1 function $\eta(x)$.
Since u is the minimizer

$$E(u + \varepsilon\eta) \geq E(u), \qquad \forall\varepsilon.$$

$E(u+\varepsilon\eta)$ has a minimum at $\varepsilon = 0$. Thus,

$$\frac{dE}{d\varepsilon}(u + \varepsilon\eta)\big|_{\varepsilon=0} = 0.$$

7.1 Rudin-Osher-Fatemi

$$E \;=\; \int_\Omega |\nabla u| + \lambda(u-f)^2\, dx.$$

$$\begin{aligned}
\frac{dE}{d\varepsilon}(u+\varepsilon\eta)\big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\int_\Omega |\nabla(u+\varepsilon\eta)| \,+\, \lambda(u+\varepsilon\eta-f)^2\ dx\ \big|_{\varepsilon=0} \\
&= \int_\Omega \frac{\nabla(u+\varepsilon\eta)}{|\nabla(u+\varepsilon\eta)|}\cdot\nabla\eta \,+\, 2\lambda(u+\varepsilon\eta-f)\eta\ dx\ \big|_{\varepsilon=0} \\
&= \int_\Omega \frac{\nabla u}{|\nabla u|}\cdot\nabla\eta \,+\, 2\lambda(u-f)\eta\ dx \\
&= \int_{\partial\Omega} \eta\frac{\nabla u}{|\nabla u|}\cdot n\, ds \,-\, \int_\Omega \nabla\cdot\Big(\frac{\nabla u}{|\nabla u|}\Big)\eta\, dx \,+\, \int_\Omega 2\lambda(u-f)\eta\, dx \\
&= \int_{\partial\Omega} \eta\frac{\nabla u}{|\nabla u|}\cdot n\, ds \,-\, \int_\Omega \Big[\nabla\cdot\Big(\frac{\nabla u}{|\nabla u|}\Big) \,-\, 2\lambda(u-f)\Big]\eta\ dx \;=\; 0.
\end{aligned}$$

Choose $\eta \in C_c^1(\Omega)$. The Euler-Lagrange equations [22] are

$$\boxed{\nabla\cdot\Big(\frac{\nabla u}{|\nabla u|}\Big) \,-\, 2\lambda(u-f) = 0 \qquad \text{on } \ \Omega,}$$

---

[22] Hildebrand's (p.124-128) definition of Euler-Lagrange equations in one dimension:

$$\int_{x_1}^{x_2}\Big[\frac{\partial J}{\partial y}\eta - \frac{d}{dx}\Big(\frac{\partial J}{\partial y'}\Big)\eta\Big]\, dx + \Big[\frac{\partial J}{\partial y'}\eta(x)\Big]_{x_1}^{x_2} \;=\; 0.$$

$$\boxed{\frac{d}{dx}\Big(\frac{\partial J}{\partial y'}\Big) = \frac{\partial J}{\partial y}.}$$

$$\boxed{\Big[\frac{\partial J}{\partial y'}\Big]_{x=x_1} = 0, \qquad \Big[\frac{\partial J}{\partial y'}\Big]_{x=x_2} = 0.}$$

In n dimensions:

$$\boxed{\nabla_x\cdot(\nabla_p J) = \nabla_u J \qquad \text{on } \ \Omega,}$$

$$\boxed{\nabla_p J\cdot n = 0 \qquad \text{on } \ \partial\Omega,}$$

where $p = \nabla u = (u_x, u_y)$.

$$\boxed{\nabla u \cdot n = 0 \qquad \text{on } \partial\Omega.}$$

7.1.1 Gradient Descent

If we want to find a local minimum of a function f in $\mathbb{R}^1$, we have

$$\frac{dx}{dt} = -\frac{df}{dx}.$$

To minimize the energy E (in $\mathbb{R}^2$), we would have

$$\frac{du}{dt} = -\frac{dE(u)}{du}.$$

Also, consider

$$E = \int_\Omega |\nabla u| + \lambda(u-f)^2 \, dx.$$

We want $E(u(x,t))$ to decrease, that is,

$$\frac{d}{dt}E(u(x,t)) \le 0, \qquad \text{for all } t.$$

Assume $\nabla u \cdot n = 0$ on $\partial\Omega$. We have

$$\begin{aligned}
\frac{d}{dt}E(u(x,t)) &= \frac{d}{dt}\int_\Omega |\nabla u| + \lambda(u-f)^2 \, dx \\
&= \int_\Omega \frac{\nabla u \cdot \nabla u_t}{|\nabla u|} + 2\lambda(u-f)\, u_t \, dx \\
&= \int_\Omega -\nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) u_t + 2\lambda(u-f)\, u_t \, dx \\
&= \int_\Omega u_t \Big[\underbrace{-\nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) + 2\lambda(u-f)}_{①}\Big] \, dx \le \circledast \le 0.
\end{aligned}$$

To ensure that $\circledast$ holds, we need to choose u_t to be negative of ①, or

$$\boxed{u_t = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) - 2\lambda(u-f).}$$

7.2 Chan-Vese

$$\begin{aligned}
F^{CV} &= \mu \int_\Omega \delta(\phi)|\nabla\phi|\,dx \;+\; \nu\int_\Omega (1-H(\phi))\,dx \\
&+\; \lambda_1\int_\Omega |u_0-c_1|^2(1-H(\phi))\,dx \;+\; \lambda_2\int_\Omega |u_0-c_2|^2 H(\phi)\,dx.
\end{aligned}$$

$$\begin{aligned}
\frac{dF^{CV}}{d\varepsilon}(\phi+\varepsilon\eta)\Big|_{\varepsilon=0} &= \mu\frac{d}{d\varepsilon}\int_\Omega \delta(\phi+\varepsilon\eta)|\nabla(\phi+\varepsilon\eta)|\,dx \;+\; \nu\frac{d}{d\varepsilon}\int_\Omega(1-H(\phi+\varepsilon\eta))\,dx \\
&\quad + \lambda_1\frac{d}{d\varepsilon}\int_\Omega (u_0-c_1)^2(1-H(\phi+\varepsilon\eta))\,dx \;+\; \lambda_2\frac{d}{d\varepsilon}\int_\Omega(u_0-c_2)^2 H(\phi+\varepsilon\eta)\,dx\,\Big|_{\varepsilon=0} \\
&= \mu\int_\Omega\Big[\delta'(\phi+\varepsilon\eta)\,\eta\,|\nabla(\phi+\varepsilon\eta)| + \delta(\phi+\varepsilon\eta)\frac{\nabla(\phi+\varepsilon\eta)}{|\nabla(\phi+\varepsilon\eta)|}\cdot\nabla\eta\Big]\,dx \\
&\quad + \nu\int_\Omega -H'(\phi+\varepsilon\eta)\,\eta\,dx \\
&\quad + \lambda_1\int_\Omega (u_0-c_1)^2(-H'(\phi+\varepsilon\eta))\,\eta\,dx \\
&\quad + \lambda_2\int_\Omega (u_0-c_2)^2 H'(\phi+\varepsilon\eta)\,\eta\,dx\,\Big|_{\varepsilon=0} \\
&= \mu\int_\Omega\Big[\delta'(\phi)\,\eta\,|\nabla\phi| + \delta(\phi)\frac{\nabla\phi}{|\nabla\phi|}\cdot\nabla\eta\Big]\,dx \\
&\quad - \nu\int_\Omega H'(\phi)\,\eta\,dx \\
&\quad - \lambda_1\int_\Omega (u_0-c_1)^2 H'(\phi)\,\eta\,dx \\
&\quad + \lambda_2\int_\Omega (u_0-c_2)^2 H'(\phi)\,\eta\,dx \\
&= \mu\int_\Omega \delta'(\phi)\,|\nabla\phi|\,\eta\,dx + \mu\int_{\partial\Omega}\frac{\delta(\phi)}{|\nabla\phi|}\frac{\partial\phi}{\partial n}\,\eta\,ds \\
&\quad - \mu\int_\Omega \delta'(\phi)\nabla\phi\frac{\nabla\phi}{|\nabla\phi|}\,\eta\,dx - \mu\int_\Omega \delta(\phi)\,\nabla_x\cdot\Big(\frac{\nabla\phi}{|\nabla\phi|}\Big)\,\eta\,dx \\
&\quad - \nu\int_\Omega \delta(\phi)\,\eta\,dx \\
&\quad - \lambda_1\int_\Omega (u_0-c_1)^2\delta(\phi)\,\eta\,dx \\
&\quad + \lambda_2\int_\Omega (u_0-c_2)^2\delta(\phi)\,\eta\,dx \\
&= \mu\int_{\partial\Omega}\frac{\delta(\phi)}{|\nabla\phi|}\frac{\partial\phi}{\partial n}\,\eta\,ds \\
&\quad + \int_\Omega \delta(\phi)\Big[-\mu\nabla\cdot\Big(\frac{\nabla\phi}{|\nabla\phi|}\Big) - \nu - \lambda_1(u_0-c_1)^2 + \lambda_2(u_0-c_2)^2\Big]\eta\,dx \;=\; 0.
\end{aligned}$$

Choose $\eta \in C_c^1(\Omega)$. The Euler-Lagrange equations are

$$\boxed{\delta(\phi)\Big[\mu\,\nabla\cdot\Big(\frac{\nabla\phi}{|\nabla\phi|}\Big) + \nu + \lambda_1(u_0-c_1)^2 - \lambda_2(u_0-c_2)^2\Big] \;=\; 0 \qquad \text{on } \Omega,}$$

$$\boxed{\frac{\delta(\phi)}{|\nabla\phi|}\frac{\partial\phi}{\partial n} \;=\; 0 \qquad \text{on } \partial\Omega.}$$

7.3 Problems

The problem below was solved in the previous section. However, the approach below is preferable.

Problem (W'04, #2). *Let $C^2(\overline{\Omega})$ be the space of all twice continuously differentiable functions in the bounded smooth closed domain $\overline{\Omega} \subset \mathbb{R}^2$. Let $u_0(x, y)$ be the function that minimizes the functional*

$$D(u) = \int\int_\Omega \Big[\Big(\frac{\partial u(x,y)}{\partial x}\Big)^2 + \Big(\frac{\partial u(x,y)}{\partial y}\Big)^2 + f(x,y)u(x,y)\Big]\,dxdy + \int_{\partial\Omega} a(s)\,u^2(x(s), y(s))\,ds,$$

where $f(x, y)$ and $a(s)$ are given continuous functions.
Find the differential equation and the boundary condition that u_0 satisfies.

Proof. Suppose $u(x)$ is the actual minimizing function, and choose any C^1 function $\eta(x)$.
Since u is the minimizer

$$F(u + \varepsilon\eta) \geq F(u), \qquad \forall\varepsilon.$$

$F(u + \varepsilon\eta)$ has a minimum at $\varepsilon = 0$. Thus,

$$\frac{dF}{d\varepsilon}(u + \varepsilon\eta)\big|_{\varepsilon=0} = 0.$$

$$F(u) = \int_\Omega (|\nabla u|^2 + fu)\,dx + \int_{\partial\Omega} a(s)\,u^2\,ds,$$

$$\begin{aligned}
\frac{dF}{d\varepsilon}(u + \varepsilon\eta)\big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\int_\Omega \big(|\nabla(u+\varepsilon\eta)|^2 + f\cdot(u+\varepsilon\eta)\big)\,dx + \frac{d}{d\varepsilon}\int_{\partial\Omega} a(s)\,(u+\varepsilon\eta)^2\,ds\,\big|_{\varepsilon=0} \\
&= \int_\Omega \big(2\,\nabla(u+\varepsilon\eta)\cdot\nabla\eta + f\eta\big)\,dx + \int_{\partial\Omega} 2\,a(s)\,(u+\varepsilon\eta)\,\eta\,ds\,\big|_{\varepsilon=0} \\
&= \int_\Omega \big(2\,\nabla u\cdot\nabla\eta + f\eta\big)\,dx + \int_{\partial\Omega} 2\,a(s)\,u\,\eta\,ds \\
&= \int_{\partial\Omega} 2\frac{\partial u}{\partial n}\eta\,ds - \int_\Omega \big(2\,\triangle u\,\eta - f\eta\big)\,dx + \int_{\partial\Omega} 2\,a(s)\,u\,\eta\,ds \\
&= 2\int_{\partial\Omega}\Big(\frac{\partial u}{\partial n} + a(s)u\Big)\,\eta\,ds - \int_\Omega \big(2\,\triangle u - f\big)\,\eta\,dx = 0.
\end{aligned}$$

The Euler-Lagrange equations are

$$\boxed{\begin{cases} 2\triangle u = f, & x \in \Omega, \\ \frac{\partial u}{\partial n} + a(s)u = 0, & x \in \partial\Omega. \end{cases}}$$

$\square$

Problem (F'92, #7). *Let a_1 and a_2 be positive constants with $a_1 \neq a_2$ and define*

$$a(x) = \begin{cases} a_1 & \textit{for} \;\; 0 < x < \frac{1}{2} \\ a_2 & \textit{for} \;\; \frac{1}{2} < x < 1 \end{cases}$$

and let $f(x)$ be a smooth function. Consider the functional

$$F(u) \, = \, \int_0^1 a(x) u_x^2 \, dx \, - \, \int_0^1 f(x) u(x) \, dx$$

in which u is continuous on $[0,1]$, twice differentiable on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, and has a possible jump discontinuity in u_x at $x = \frac{1}{2}$. Find the ***Euler-Lagrange equation*** *for $u(x)$ that minimizes the functional $F(u)$. In addition find the boundary conditions on u at $x = 0$, $x = \frac{1}{2}$ and $x = 1$.*

Proof. Suppose $u(x)$ is the actual minimizing function, and choose any C^1 function $\eta(x)$.
Since u is the minimizer

$$F(u + \varepsilon\eta) \geq F(u), \qquad \forall \varepsilon.$$

$F(u + \varepsilon\eta)$ has a minimum at $\varepsilon = 0$. Thus,

$$\frac{dF}{d\varepsilon}(u + \varepsilon\eta)\big|_{\varepsilon=0} = 0.$$

$$F(u) \, = \, \int_0^{\frac{1}{2}} a_1 u_x^2 \, dx \, + \, \int_{\frac{1}{2}}^1 a_2 u_x^2 \, dx \, - \, \int_0^1 f(x) u(x) \, dx$$

$$\begin{aligned} \frac{dF}{d\varepsilon}(u + \varepsilon\eta)\big|_{\varepsilon=0} \;&=\; \frac{d}{d\varepsilon} \int_0^{\frac{1}{2}} a_1 (u_x + \varepsilon\eta_x)^2 \, dx \, + \, \frac{d}{d\varepsilon} \int_{\frac{1}{2}}^1 a_2 (u_x + \varepsilon\eta_x)^2 \, dx \, - \, \frac{d}{d\varepsilon} \int_0^1 f(x)(u + \varepsilon\eta) \, dx \, \bigg|_{\varepsilon=0} \\ &=\; \int_0^{\frac{1}{2}} 2a_1 (u_x + \varepsilon\eta_x)\eta_x \, dx \, + \, \int_{\frac{1}{2}}^1 2a_2 (u_x + \varepsilon\eta_x)\eta_x \, dx \, - \, \int_0^1 f(x)\eta \, dx \, \bigg|_{\varepsilon=0} \\ &=\; \int_0^{\frac{1}{2}} 2a_1 u_x \eta_x \, dx \, + \, \int_{\frac{1}{2}}^1 2a_2 u_x \eta_x \, dx \, - \, \int_0^1 f(x)\eta \, dx \\ &=\; 2a_1 u_x \eta \Big|_0^{\frac{1}{2}} \, - \, \int_0^{\frac{1}{2}} 2a_1 u_{xx} \eta \, dx \, + \, 2a_2 u_x \eta \Big|_{\frac{1}{2}}^1 \, - \, \int_{\frac{1}{2}}^1 2a_2 u_{xx} \eta \, dx \, - \, \int_0^1 f(x)\eta \, dx \\ &=\; 2a_1 u_x \eta \Big|_0^{\frac{1}{2}} \, + \, 2a_2 u_x \eta \Big|_{\frac{1}{2}}^1 \, - \, \int_0^1 2a(x) u_{xx} \eta \, dx \, - \, \int_0^1 f(x)\eta \, dx. \end{aligned}$$

Thus,

$$a_1 u_x\big(\frac{1}{2}\big)\eta\big(\frac{1}{2}\big) \, - \, a_1 u_x(0)\eta(0) \, + \, a_2 u_x(1)\eta(1) \, - \, a_2 u_x\big(\frac{1}{2}\big)\eta\big(\frac{1}{2}\big) = 0.$$

$$\int_0^1 \Big[2a(x) u_{xx} \, + \, f(x) \Big] \eta \, dx = 0.$$

$$\boxed{\begin{cases} 2a(x)u_{xx} + f(x) = 0, \\ u_x(0) = 0, \\ u_x(1) = 0, \\ a_1 u_x\left(\frac{1}{2}-\right) = a_2 u_x(\frac{1}{2}+). \end{cases}}$$

The process of finding Euler-Lagrange equations (given the minimization functional) is equivalent to **(D)** $\Leftarrow$ **(V)** $\Leftarrow$ **(M)**. $\square$

Problem (F'00, #4). *Consider the following functional*

$$F(v) = \int\int\int_\Omega \Big[\sum_{j,k=1}^{3} \Big(\frac{\partial v_j}{\partial x_k}\Big)^2 + \alpha\Big(\sum_{j=1}^{3} v_j^2(x) - 1\Big)^2\Big]\, dx,$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $v(x) = (v_1(x), v_2(x), v_3(x))$, $\Omega \in \mathbb{R}^3$ bounded, and $\alpha > 0$ is a constant. Let $u(x) = (u_1(x), u_2(x), u_3(x))$ be the minimizer of $F(v)$ among all smooth functions satisfying the Dirichlet condition, $u_k(x) = \varphi_k(x)$, $k = 1, 2, 3$. Derive the system of differential equations that $u(x)$ satisfies.

Proof. **(M) ⇒ (D)**
Suppose $u(x)$ is the actual minimizing function, and choose any C^1 function $\eta(x) = (\eta_1(x), \eta_2(x), \eta_3(x))$.
Since u is the minimizer

$$F(u + \varepsilon\eta) \geq F(u), \qquad \forall\varepsilon.$$

$F(u + \varepsilon\eta)$ has a minimum at $\varepsilon = 0$. Thus,

$$\frac{dF}{d\varepsilon}(u + \varepsilon\eta)\big|_{\varepsilon=0} = 0.$$

$$\begin{aligned}
\frac{dF}{d\varepsilon}(u + \varepsilon\eta)\big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\int_\Omega \Big[\sum_{j,k=1}^{3} \Big(\frac{\partial u_j}{\partial x_k} + \varepsilon\frac{\partial \eta_j}{\partial x_k}\Big)^2 + \alpha\Big(\sum_{j=1}^{3}(u_j + \varepsilon\eta_j)^2 - 1\Big)^2\Big]\, dx\,\Big|_{\varepsilon=0} \\
&= \int_\Omega \Big[2\sum_{j,k=1}^{3} \Big(\frac{\partial u_j}{\partial x_k} + \varepsilon\frac{\partial \eta_j}{\partial x_k}\Big)\frac{\partial \eta_j}{\partial x_k} + 2\alpha\Big(\sum_{j=1}^{3}(u_j + \varepsilon\eta_j)^2 - 1\Big)2\sum_{j=1}^{3}(u_j + \varepsilon\eta_j)\eta_j\Big]\, dx\,\Big|_{\varepsilon=0} \\
&= \int_\Omega \Big[2\sum_{j,k=1}^{3} \Big(\frac{\partial u_j}{\partial x_k}\Big)\frac{\partial \eta_j}{\partial x_k} + 4\alpha\Big(\sum_{j=1}^{3} u_j^2 - 1\Big)\sum_{j=1}^{3} u_j\eta_j\Big]\, dx \\
&= \int_\Omega \Big[2\big(\nabla u_1 \cdot \nabla\eta_1 + \nabla u_2 \cdot \nabla\eta_2 + \nabla u_3 \cdot \nabla\eta_3\big) \\
&\qquad + 4\alpha\big(u_1^2 + u_2^2 + u_3^2 - 1\big)\big(u_1\eta_1 + u_2\eta_2 + u_3\eta_3\big)\Big]\, dx \\
&= \int_{\partial\Omega} \Big[2\Big(\frac{\partial u_1}{\partial n}\eta_1 + \frac{\partial u_2}{\partial n}\eta_2 + \frac{\partial u_3}{\partial n}\eta_3\Big)\, ds + \int_\Omega \Big[2\big(\triangle u_1\eta_1 + \triangle u_2\eta_2 + \triangle u_2\eta_3\big)\, dx \\
&\qquad + 4\alpha\big(u_1^2 + u_2^2 + u_3^2 - 1\big)\big(u_1\eta_1 + u_2\eta_2 + u_3\eta_3\big)\Big]\, dx \;=\; 0.
\end{aligned}$$

If we assume that $u_1^2 + u_2^2 + u_3^2 - 1 = 1$, we have

$$\begin{aligned}
\frac{dF}{d\varepsilon}(u + \varepsilon\eta)\big|_{\varepsilon=0} &= \int_{\partial\Omega} \Big[2\Big(\frac{\partial u_1}{\partial n}\eta_1 + \frac{\partial u_2}{\partial n}\eta_2 + \frac{\partial u_3}{\partial n}\eta_3\Big)\, ds + \int_\Omega \Big[2\big(\triangle u_1\eta_1 + \triangle u_2\eta_2 + \triangle u_2\eta_3\big)\, dx \\
&\qquad + 4\alpha\big(u_1\eta_1 + u_2\eta_2 + u_3\eta_3\big)\Big]\, dx \;=\; 0.
\end{aligned}$$

$$\boxed{\triangle u_i + 2\alpha u_i = 0, \qquad \text{in } \Omega}$$

$$\boxed{\frac{\partial u_i}{\partial n} = 0, \quad i = 1, 2, 3, \qquad \text{on } \partial\Omega.}$$

□

8 Integral Equations

Fredholm Equation: $\alpha(x)y(x) = F(x) + \lambda \int_a^b K(x,\xi)y(\xi)d\xi$
Volterra Equation: $\alpha(x)y(x) = F(x) + \lambda \int_a^x K(x,\xi)y(\xi)d\xi$

When $\alpha \equiv 0$, the equation is said to be an **integral equation of the first kind**.
When $\alpha \equiv 1$, the equation is said to be an **integral equation of the second kind**.

$$\frac{d}{dx}\int_{A(x)}^{B(x)} F(x,\xi)d\xi = \int_A^B \frac{\partial F(x,\xi)}{\partial x}d\xi + F(x,B(x))\frac{dB}{dx} - F(x,A(x))\frac{dA}{dx}.$$

8.1 Relations Between Differential and Integral Equations

Example 1. *Consider the boundary-value problem*

$$y'' + \lambda y = 0,$$
$$y(0) = y(L) = 0.$$

After the first integration over $(0,x)$, we obtain

$$y'(x) = -\lambda \int_0^x y(\xi)\,d\xi + C,$$

where C represents the unknown value of $y'(0)$. A second integration over $(0,x)$ gives

$$\begin{aligned} y(x) &= -\lambda \int_0^x \underbrace{ds}_{v'} \underbrace{\int_0^s y(\xi)\,d\xi}_{u} + Cx + D = -\lambda\Big[\Big[s\int_0^s y(\xi)\,d\xi\Big]_0^x - \int_0^x sy(s)\,ds\Big] + Cx + D \\ &= -\lambda\Big[x\int_0^x y(\xi)\,d\xi - \int_0^x \xi y(\xi)\,d\xi\Big] + Cx + D = -\lambda\int_0^x (x-\xi)y(\xi)\,d\xi + Cx + D. \qquad \circledast \end{aligned}$$

$y(0) = 0$ gives $D = 0$. Since $y(L) = 0$, then

$$\begin{aligned} y(L) = 0 &= -\lambda \int_0^L (L-\xi)y(\xi)\,d\xi + CL, \\ C &= \frac{\lambda}{L}\int_0^L (L-\xi)y(\xi)\,d\xi. \end{aligned}$$

If the values of C and D are introduced into $\circledast$, this relation takes the form

$$\begin{aligned} y(x) &= -\lambda\int_0^x (x-\xi)y(\xi)\,d\xi + \lambda\frac{x}{L}\int_0^L (L-\xi)y(\xi)\,d\xi \\ &= -\lambda\int_0^x (x-\xi)y(\xi)\,d\xi + \lambda\int_0^x \frac{x}{L}(L-\xi)y(\xi)\,d\xi + \lambda\int_x^L \frac{x}{L}(L-\xi)y(\xi)\,d\xi \\ &= \lambda\int_0^x \frac{\xi}{L}(L-\xi)y(\xi)\,d\xi + \lambda\int_x^L \frac{x}{L}(L-\xi)y(\xi)\,d\xi. \end{aligned}$$

Thus,

$$\boxed{y(x) = \lambda\int_0^L K(x,\xi)y(\xi)\,d\xi}$$

where

$$\boxed{K(x,\xi) = \begin{cases} \frac{\xi}{L}(L-\xi), & \xi < x \\ \frac{x}{L}(L-\xi), & \xi > x \end{cases}}$$

Note, $K(x,\xi)$ *is symmetric:* $K(x,\xi) = K(\xi,x)$.
The kernel K *is continuous at* $x = \xi$.

Example 2. *Consider the boundary-value problem*

$$y'' + Ay' + By = 0,$$
$$y(0) = y(1) = 0.$$

Integrating over $(0, x)$ *twice, we obtain*

$$\begin{aligned}
y'(x) &= -Ay(x) - B\int_0^x y(\xi)\,d\xi + C, \\
y(x) &= -A\int_0^x y(\xi)\,d\xi - B\int_0^x \underbrace{ds}_{v'} \underbrace{\int_0^s y(\xi)\,d\xi}_{u} + Cx + D \\
&= -A\int_0^x y(\xi)\,d\xi - B\bigg[\Big[s\int_0^s y(\xi)\,d\xi\Big]_0^x - \int_0^x sy(s)\,ds\bigg] + Cx + D \\
&= -A\int_0^x y(\xi)\,d\xi - B\bigg[x\int_0^x y(\xi)\,d\xi - \int_0^x \xi y(\xi)\,d\xi\bigg] + Cx + D \\
&= \int_0^x \Big[-A - B(x-\xi)\Big]y(\xi)\,d\xi + Cx + D. \quad \circledast
\end{aligned}$$

$y(0) = 0$ *gives* $D = 0$*. Since* $y(1) = 0$*, then*

$$\begin{aligned}
y(1) = 0 &= \int_0^1 \Big[-A - B(1-\xi)\Big]y(\xi)\,d\xi + C, \\
C &= \int_0^1 \Big[A + B(1-\xi)\Big]y(\xi)\,d\xi.
\end{aligned}$$

If the values of C *and* D *are introduced into* $\circledast$*, this relation takes the form*

$$\begin{aligned}
y(x) &= \int_0^x \Big[-A - B(x-\xi)\Big]y(\xi)\,d\xi + x\int_0^1 \Big[A + B(1-\xi)\Big]y(\xi)\,d\xi \\
&= \int_0^x \Big[-A - B(x-\xi)\Big]y(\xi)\,d\xi + \int_0^x \Big[Ax + Bx(1-\xi)\Big]y(\xi)\,d\xi + \int_x^1 \Big[Ax + Bx(1-\xi)\Big]y(\xi)\,d\xi \\
&= \int_0^x \Big[A(x-1) + B\xi(1-x)\Big]y(\xi)\,d\xi + \int_x^1 \Big[Ax + Bx(1-\xi)\Big]y(\xi)\,d\xi
\end{aligned}$$

Thus,

$$\boxed{y(x) = \int_0^1 K(x,\xi)y(\xi)\,d\xi}$$

where

$$\boxed{K(x,\xi) = \begin{cases} A(x-1) + B\xi(1-x), & \xi < x \\ Ax + Bx(1-\xi), & \xi > x \end{cases}}$$

Note, $K(x,\xi)$ *is **not** symmetric:* $K(x,\xi) \neq K(\xi,x)$*, unless* $A = 0$*.*
The kernel K *is **not** continuous at* $x = \xi$*, since*

$$\lim_{x\to\xi^+} A(x-1)+B\xi(1-x) \;=\; A(\xi-1)+B\xi(1-\xi) \;\neq\; A\xi+B\xi(1-\xi) \;=\; \lim_{x\to\xi^-} Ax+Bx(1-\xi).$$

8.2 Green's Function

Given the differential operator

$$L = \frac{d}{dx}\big(p\frac{d}{dx}\big) + q,$$

consider the differential equation

$$\begin{aligned} &Ly + F(x) = 0, \qquad a \le x \le b \\ &c_1 y(a) + c_2 y'(a) = 0, \quad c_3 y(b) + c_4 y'(b) = 0 \end{aligned}$$

where F may also depend upon x indirectly through $y(x)$, $F(x) = F(x, y(x))$.

We construct a **Green's function** G which, for a given number ξ, is given by $u(x)$ when $x < \xi$ and by $v(x)$ when $x > \xi$, and which has the following four properties:
① The functions u and v satisfy the equation $LG = 0$ in their intervals of definition; that is $Lu = 0$ when $x < \xi$, and $Lv = 0$ when $x > \xi$.
② u satisfies the boundary condition at $x = a$, and v that at $x = b$.
③ G is continuous at $x = \xi$; that is $u(\xi) = v(\xi)$.
④ $v'(\xi) - u'(\xi) = -1/p(\xi)$.
⌈ When $G(x, \xi)$ exists, the original formulation of the problem can be transformed to

$$y(x) = \int_a^b G(x, \xi) F(\xi) d\xi. \ ⌋$$

Thus, conditions ① and ② imply

$$G = \begin{cases} u(x), & x < \xi, \\ v(x), & x > \xi. \end{cases} \tag{8.1}$$

where u and v satisfty respective boundary conditions, and conditions ③ and ④ determine additional properties of u and v (i.e. constants in terms of ξ):

$$c_2 v(\xi) - c_1 u(\xi) \ = \ 0, \tag{8.2}$$

$$c_2 v'(\xi) - c_1 u'(\xi) = -\frac{1}{p(\xi)}. \tag{8.3}$$

Example. *Transform the problem*

$$\frac{d^2y}{dx^2} + y + \epsilon y^2 = f(x),$$
$$y(0) = 0, \quad y(1) = 0$$

to a nonlinear Fredholm integral equation in each of the two following ways. Use
a) $Ly = y''$.
b) $Ly = y'' + y$.

Proof. **a)** We have

$$\underbrace{y''}_{Ly} + \underbrace{y + \epsilon y^2 - f(x)}_{F(x)} = 0$$

① $Ly = y'' = 0 \quad \Rightarrow \quad y = ax + b$
$u(x) = ax + b, \;\; v(x) = cx + d.$
② $u(0) = 0 = b \quad \Rightarrow \quad \underline{u(x) = ax}.$
$v(1) = 0 = c + d \quad \Rightarrow \quad \underline{v(x) = c(x-1)}.$
Determine a and c in terms of ξ:
③ $u(\xi) = v(\xi),$
$a\xi = c(\xi - 1),$
$\xi = \frac{c}{c-a}.$
④ $v'(\xi) - u'(\xi) = c - a = -\frac{1}{p(\xi)} = -1,$
$\Rightarrow \quad c = -\xi, \;\; a = 1 - \xi.$ Thus,

$$\boxed{G = \begin{cases} u(x), & x < \xi, \\ v(x), & x > \xi. \end{cases} = \begin{cases} x(1-\xi), & x < \xi, \\ \xi(1-\xi), & x > \xi. \end{cases}}$$

$$\boxed{y(x) = \int_0^1 G(x,\xi) F(\xi)\, d\xi = -\int_0^1 G(x,\xi) f(\xi)\, d\xi + \int_0^1 G(x,\xi)\big[y(\xi) + \epsilon y^2(\xi)\big]\, d\xi}$$

b) We have

$$\underbrace{y'' + y}_{Ly} + \underbrace{\epsilon y^2 - f(x)}_{F(x)} = 0$$

① $Ly = y'' + y = 0 \quad \Rightarrow \quad y = A\cos x + B\sin x$
$u(x) = a\cos x + b\sin x, \;\; v(x) = c\cos x + d\sin x.$
② $u(0) = 0 = a \quad \Rightarrow \quad \underline{u(x) = b\sin x}.$
$v(1) = 0 = c\cos 1 + d\sin 1 \quad \Rightarrow \quad \underline{v(x) = d(\sin x - \frac{\sin 1}{\cos 1}\cos x)}.$
Determine b and d in terms of ξ:
③ $u(\xi) = v(\xi),$
$b\sin\xi = d(\sin\xi - \frac{\sin 1}{\cos 1}\cos\xi),$
$b = d(1 - \frac{\sin 1}{\cos 1}\frac{\cos\xi}{\sin\xi}).$
④ $v'(\xi) - u'(\xi) = d(\frac{\sin 1}{\cos 1}\sin\xi + \cos\xi - b\cos\xi) = -\frac{1}{p(\xi)} = -1.$
After some algebra,
$u(x) = \frac{\sin(1-\xi)\sin x}{\sin 1},$
$v(x) = \frac{\sin(1-x)\sin\xi}{\sin 1}.$

$$\boxed{G = \begin{cases} \frac{\sin(1-\xi)\sin x}{\sin 1}, & x < \xi, \\ \frac{\sin(1-x)\sin\xi}{\sin 1}, & x > \xi. \end{cases}}$$

$$\boxed{y(x) = \int_0^1 G(x,\xi)F(\xi)\,d\xi = -\int_0^1 G(x,\xi)f(\xi)\,d\xi + \epsilon\int_0^1 G(x,\xi)y^2(\xi)\,d\xi}$$

$\square$

Problem (W'02, #1). *Consider the second order differential operator L defined by*

$$Ly = \frac{d^2y}{dx^2} - y.$$

Find the Green's function (= solution operator kernel) for the boundary value problem $Ly = f$ on $0 < x < 1$, $y(1) = y(0) = 0$.

Proof. ① $Ly = y'' - y = 0 \quad \Rightarrow \quad y = Ae^{-x} + Be^{x}$
$u(x) = ae^{-x} + be^{x}, \;\; v(x) = ce^{-x} + de^{x}$.
② $u(0) = 0 = a + b \qquad \Rightarrow \quad \underline{u(x) = a(e^{-x} - e^{x})}$.
$v(1) = 0 = ce^{-1} + de^{1} \quad \Rightarrow \quad \underline{v(x) = d(e^{x} - e^{2-x})}$.
Determine a and d in terms of ξ:
③ $u(\xi) = v(\xi)$,
$a(e^{-\xi} - e^{\xi}) = d(e^{\xi} - e^{2-\xi})$,
$a = d\frac{e^{\xi} - e^{2-\xi}}{e^{-\xi} - e^{\xi}}$.
④ $v'(\xi) - u'(\xi) = d(e^{\xi} + e^{2-\xi}) - a(-e^{-\xi} - e^{\xi}) = -\frac{1}{p(\xi)} = -1$.
Plugging in ③ into ④, we get

$$\begin{aligned}
d(e^{\xi} + e^{2-\xi}) - d\frac{e^{\xi} - e^{2-\xi}}{e^{-\xi} - e^{\xi}}(-e^{-\xi} - e^{\xi}) &= -1, \\
e^{\xi} + e^{2-\xi} + \frac{e^{\xi} - e^{2-\xi}}{e^{-\xi} - e^{\xi}}(e^{-\xi} + e^{\xi}) &= -\frac{1}{d}, \\
(e^{\xi} + e^{2-\xi})\frac{e^{-\xi} - e^{\xi}}{e^{-\xi} - e^{\xi}} + \frac{e^{\xi} - e^{2-\xi}}{e^{-\xi} - e^{\xi}}(e^{-\xi} + e^{\xi}) &= -\frac{1}{d}, \\
\frac{1 - e^{2\xi} + e^{2-2\xi} - e^{2}}{e^{-\xi} - e^{\xi}} + \frac{1 + e^{2\xi} - e^{2-2\xi} - e^{2}}{e^{-\xi} - e^{\xi}} &= -\frac{1}{d}, \\
\frac{2 - 2e^{2}}{e^{-\xi} - e^{\xi}} &= -\frac{1}{d}, \\
d &= \frac{e^{-\xi} - e^{\xi}}{2(e^{2} - 1)}.
\end{aligned}$$

$$a = d\frac{e^{\xi} - e^{2-\xi}}{e^{-\xi} - e^{\xi}} = \frac{e^{-\xi} - e^{\xi}}{2(e^{2} - 1)} \cdot \frac{e^{\xi} - e^{2-\xi}}{e^{-\xi} - e^{\xi}} = \frac{e^{\xi} - e^{2-\xi}}{2(e^{2} - 1)}.$$

$$\boxed{G = \begin{cases} \frac{e^{\xi} - e^{2-\xi}}{2(e^{2} - 1)}(e^{-x} - e^{x}), & x < \xi, \\ \frac{e^{-\xi} - e^{\xi}}{2(e^{2} - 1)}(e^{x} - e^{2-x}), & x > \xi. \end{cases}}$$

□

Example. *Show that the Green's function $G(x,\xi)$ associated with the expression $\frac{d^2y}{dx^2} - y$ over the infinite interval $(-\infty, \infty)$, subject to the requirement that y be bounded as $x \to \pm\infty$, is of the form*

$$G(x,\xi) = \frac{1}{2}e^{-|x-\xi|}.$$

Proof. ① $Ly = y'' - y = 0 \quad \Rightarrow \quad y = Ae^{-x} + Be^{x}$
$u(x) = ae^{-x} + be^{x}, \;\; v(x) = ce^{-x} + de^{x}$.
② Since y is bounded as $x \to -\infty$, $a = 0 \quad \Rightarrow \quad \underline{u(x) = be^{x}}$.
Since y is bounded as $x \to +\infty$, $d = 0 \quad \Rightarrow \quad \underline{v(x) = ce^{-x}}$.
Determine b and c in terms of ξ:

③ $u(\xi) = v(\xi)$,
$be^{\xi} = ce^{-\xi}$,
$b = ce^{-2\xi}$.
④ $v'(\xi) - u'(\xi) = -ce^{-\xi} - be^{\xi} = -\frac{1}{p(\xi)} = -1$.
$c = \frac{1-be^{\xi}}{e^{-\xi}} = \frac{1-ce^{-2\xi}e^{\xi}}{e^{-\xi}} = \frac{1-ce^{-\xi}}{e^{-\xi}} = e^{\xi} - c$,
$c = \frac{1}{2}e^{\xi} \quad \Rightarrow \quad b = \frac{1}{2}e^{-\xi}$. Thus,

$$
\begin{aligned}
G(x,\xi) &= \begin{cases} be^{x}, & x < \xi \\ ce^{-x}, & x > \xi \end{cases} = \begin{cases} \frac{1}{2}e^{-\xi}e^{x}, & x < \xi \\ \frac{1}{2}e^{\xi}e^{-x}, & x > \xi \end{cases} = \begin{cases} \frac{1}{2}e^{x-\xi}, & x < \xi \\ \frac{1}{2}e^{\xi-x}, & x > \xi \end{cases} \\
&= \begin{cases} \frac{1}{2}e^{-|x-\xi|}, & x < \xi \\ \frac{1}{2}e^{-|\xi-x|}, & x > \xi \end{cases} = \begin{cases} \frac{1}{2}e^{-|x-\xi|}, & x < \xi, \\ \frac{1}{2}e^{-|x-\xi|}, & x > \xi. \end{cases}
\end{aligned}
$$

$$
\boxed{G(x,\xi) = \frac{1}{2}e^{-|x-\xi|}}
$$

□

Problem (W'04, #7). *For the two-point boundary value problem $Lf = f_{xx} - f$ on $-\infty < x < \infty$ with $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$, the* ***Green's function*** *$G(x,x')$ solves $LG = \delta(x-x')$ in which L acts on the variable x.*
a) *Show that $G(x,x') = G(x-x')$.*
b) *For each x', show that*

$$
G(x,x') = \begin{cases} a_{-}e^{x} & \text{for } x < x', \\ a_{+}e^{-x} & \text{for } x' < x, \end{cases}
$$

in which $a_{\pm}$ are functions that depend only on x'.
c) *Using (a), find the x' dependence of $a_{\pm}$.*
d) *Finish finding $G(x,x')$ by using the jump conditions to find the remaining unknowns in $a_{\pm}$.*

Proof. **a)** We have

$$
\begin{aligned}
Lf &= f_{xx} - f, \\
LG &= G(x,x')_{xx} - G(x,x') = \delta(x-x'), \\
\textbf{???} &\Rightarrow \quad G(x,x') = G(x-x').
\end{aligned}
$$

b, c, d) ① $Lf = f'' - f = 0 \quad \Rightarrow \quad y = Ae^{-x} + Be^{x}$
$u(x) = ae^{-x} + be^{x}$, $v(x) = ce^{-x} + de^{x}$.
② Since $\lim_{x\to-\infty} f(x) = 0$, $a = 0 \quad \Rightarrow \quad \underline{u(x) = be^{x}}$.
Since $\lim_{x\to+\infty} f(x) = 0$, $d = 0 \quad \Rightarrow \quad \underline{v(x) = ce^{-x}}$.
Determine b and c in terms of ξ:
③ $u(\xi) = v(\xi)$,
$be^{\xi} = ce^{-\xi}$,
$b = ce^{-2\xi}$.
④ $v'(\xi) - u'(\xi) = -ce^{-\xi} - be^{\xi} = -\frac{1}{p(\xi)} = -1$.
$c = \frac{1-be^{\xi}}{e^{-\xi}} = \frac{1-ce^{-2\xi}e^{\xi}}{e^{-\xi}} = \frac{1-ce^{-\xi}}{e^{-\xi}} = e^{\xi} - c$,

$c = \frac{1}{2}e^{\xi} \quad \Rightarrow \quad b = \frac{1}{2}e^{-\xi}$. Thus,

$$\begin{aligned}
G(x,\xi) &= \begin{cases} be^{x}, & x < \xi \\ ce^{-x}, & x > \xi \end{cases} = \begin{cases} \frac{1}{2}e^{-\xi}e^{x}, & x < \xi \\ \frac{1}{2}e^{\xi}e^{-x}, & x > \xi \end{cases} = \begin{cases} \frac{1}{2}e^{x-\xi}, & x < \xi \\ \frac{1}{2}e^{\xi-x}, & x > \xi \end{cases} \\
&= \begin{cases} \frac{1}{2}e^{-|x-\xi|}, & x < \xi \\ \frac{1}{2}e^{-|\xi-x|}, & x > \xi \end{cases} = \begin{cases} \frac{1}{2}e^{-|x-\xi|}, & x < \xi, \\ \frac{1}{2}e^{-|x-\xi|}, & x > \xi. \end{cases}
\end{aligned}$$

$$\boxed{G(x,\xi) = \frac{1}{2}e^{-|x-\xi|}}$$

$\square$

9 Miscellaneous

Problem (F'98, #1). *Determine β such that the differential equation*

$$\frac{d^2\phi}{dx^2} + \phi = \beta + x^2, \tag{9.1}$$

with $\phi(0) = 0$ and $\phi(\pi) = 0$ has a solution.

Proof. Solve the **homogeneous** equation $\phi'' + \phi = 0$. Subsitution $\phi = e^{sx}$ gives $s^2 + 1 = 0$. Hence, $s_{1,2} = \pm i$ and the superposition principle gives the family of solutions:

$$\phi_h(x) = A\cos x + B\sin x.$$

Find a **particular** solution of the inhomogeneous equation $\phi'' + \phi = \beta + x^2$.
Try $\phi(x) = ax^2 + bx + c$. Substitution into (9.1) gives

$$ax^2 + bx + 2a + c = \beta + x^2.$$

By equating coefficients, $a = 1$, $b = 0$, $c = \beta - 2$. Thus,

$$\phi_p(x) = x^2 + \beta - 2.$$

Use the principle of the complementary function to form the family of solutions:

$$\begin{aligned}
\phi(x) &= \phi_h(x) + \phi_p(x) = A\cos x + B\sin x + x^2 + \beta - 2. \\
\phi(0) &= 0 = A + \beta - 2, \\
\phi(\pi) &= 0 = -A + \pi^2 + \beta - 2.
\end{aligned}$$

Thus, $A = \frac{\pi^2}{2}$, which gives $\beta = 2 - \frac{\pi^2}{2}$. $\square$

Problem (S'92, #5). *Consider the initial value problem for the ODEs*

$$y' = y - y^3, \qquad y' = y + y^3, \qquad t \geq 0,$$

with initial data

$$y(0) = \frac{1}{2}.$$

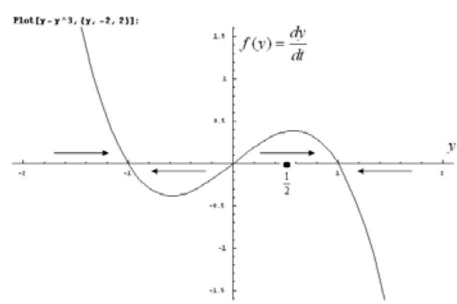
Investigate whether the solutions stay bounded for all times. If not compute the "blow-up" time.

Proof. **a)** We solve the initial value problem.

$$\begin{aligned}
\frac{dy}{dt} = y - y^3 = y(1-y^2) &= y(1-y)(1+y), \\
\frac{dy}{y(1-y)(1+y)} &= dt, \\
\Big(\frac{1}{y} + \frac{1}{2}\frac{1}{1-y} - \frac{1}{2}\frac{1}{1+y}\Big)dy &= dt, \\
\ln\, y - \frac{1}{2}\ln\,(1-y) - \frac{1}{2}\ln\,(1+y) &= t + c_1, \\
\ln\, y - \frac{1}{2}\ln\,((1-y)(1+y)) &= t + c_1, \\
\ln\, y - \ln\,((1-y)(1+y))^{\frac{1}{2}} &= t + c_1, \\
\ln\left(\frac{y}{((1-y)(1+y))^{\frac{1}{2}}}\right) &= t + c_1, \\
\frac{y}{((1-y)(1+y))^{\frac{1}{2}}} = c_2 e^t&, \\
\frac{y}{(1-y^2)^{\frac{1}{2}}} = c_2 e^t&.
\end{aligned}$$

Initial condition $y(0) = \frac{1}{2}$, we obtain $c_2 = \frac{1}{\sqrt{3}}$. Thus

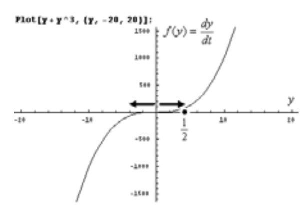
$$\begin{aligned}
\frac{y}{(1-y^2)^{\frac{1}{2}}} &= \frac{1}{\sqrt{3}}e^t, \\
\frac{y^2}{1-y^2} &= \frac{1}{3}e^{2t}, \\
y &= \frac{\pm 1}{\sqrt{3e^{-2t}+1}}.
\end{aligned}$$



As $t \to \infty \quad \Rightarrow \quad y \to \pm 1$. Thus, the solutions stay bounded for all times. We can also observe from the image above that at $y(0) = \frac{1}{2}$, $\frac{dx}{dt} > 0$. Thus $y \to 1$ as $t \to \infty$.

b) We solve the initial value problem.

$$\begin{aligned}
\frac{dy}{dt} &= y + y^3, \\
y^{-3}y' &= y^{-2} + 1.
\end{aligned}$$



Let $v = y^{-2}$, then $v' = -2y^{-3}y'$. We have

$$-\frac{1}{2}v' - v = 1 \qquad \Rightarrow \qquad v' + 2v = -2 \qquad \Rightarrow \qquad v = ce^{-2t} - 1,$$

$$\Rightarrow \quad y^{-2} = v = ce^{-2t} - 1 \qquad \Rightarrow \qquad y = \frac{\pm 1}{\sqrt{ce^{-2t}-1}} \qquad \Rightarrow \qquad y = \frac{\pm 1}{\sqrt{5e^{-2t}-1}}.$$

The solution blows up at $t = \frac{1}{2}\ln 5$. $\square$

Problem (S'94, #4).
Suppose that $\varphi_1(t)$ and $\varphi_2(t)$ are any two solutions of the linear differential equation

$$f'' + a_1(t)f' + a_2(t)f = 0. \tag{9.2}$$

a) *Show that*

$$\varphi_1(t)\varphi_2'(t) - \varphi_2(t)\varphi_1'(t) \;=\; ce^{-\int^t a_1(s)\,ds}$$

for some constant c.
b) *For any solution $\varphi_1(t)$, show that*

$$\psi(t) \;=\; \varphi_1(t)\int^t e^{-\int^s a_1(r)\,dr}\frac{1}{\varphi_1(s)^2}\,ds$$

is also a solution and is independent of φ_1, on any interval in which $\varphi_1(t) \neq 0$.

Proof. **a)** Suppose φ_1 and φ_2 are two solutions of (9.2). Then

$$\begin{aligned}
&\varphi_1'' + a_1\varphi_1' + a_2\varphi_1 = 0,\\
&\varphi_2'' + a_1\varphi_2' + a_2\varphi_2 = 0.\\
&\varphi_1[\varphi_2'' + a_1\varphi_2' + a_2\varphi_2] - \varphi_2[\varphi_1'' + a_1\varphi_1' + a_2\varphi_1] = 0,\\
&\varphi_1\varphi_2'' - \varphi_2\varphi_1'' + a_1[\varphi_1\varphi_2' - \varphi_2\varphi_1'] = 0.
\end{aligned}$$

Let $w = \varphi_1\varphi_2' - \varphi_2\varphi_1'$. Then, $w' = \varphi_1\varphi_2'' - \varphi_2\varphi_1''$. Thus,

$$w' + a_1(t)w = 0 \qquad \Rightarrow \qquad \frac{w'}{w} = -a_1(t) \qquad \Rightarrow \qquad w = c\,e^{-\int^t a_1(s)\,ds}.$$

$$\boxed{\varphi_1\varphi_2' - \varphi_2\varphi_1' = c\,e^{-\int^t a_1(s)\,ds}.}$$

b) Let $\psi(t) = \varphi_1(t)v(t)$ for some non-constant function $v(t)$, which we will find. Since $\psi(t)$ is a solution of (9.2), we have

$$\begin{aligned}
&\psi'' + a_1\psi' + a_2\psi = 0,\\
&(\varphi_1 v)'' + a_1(\varphi_1 v)' + a_2\varphi_1 v = 0,\\
&\varphi_1'' v + 2\varphi_1' v' + \varphi_1 v'' + a_1\varphi_1' v + a_1\varphi_1 v' + a_2\varphi_1 v = 0,\\
&\varphi_1 v'' + [2\varphi_1' + a_1\varphi_1]v' + [\underbrace{\varphi_1'' + a_1\varphi_1' + a_2\varphi_1}_{=0}]v = 0,\\
&\varphi_1 v'' + [2\varphi_1' + a_1\varphi_1]v' = 0,\\
&\frac{v''}{v'} = -\frac{2\varphi_1' + a_1\varphi_1}{\varphi_1} = -2\frac{\varphi_1'}{\varphi_1} - a_1,\\
&\ln v' = -2\ln\varphi_1 - \int^t a_1(s)\,ds + c_1,\\
&v' = c\frac{1}{\varphi_1^2}\,e^{-\int^t a_1(s)\,ds},\\
&v = c\int^t \frac{1}{\varphi_1^2}\,e^{-\int^s a_1(r)\,dr}ds.
\end{aligned}$$

$$\boxed{\psi(t) = \varphi_1(t)v(t) = c\varphi_1(t)\int^t \frac{1}{\varphi_1(s)^2}\,e^{-\int^s a_1(r)\,dr}ds.}$$

$\psi(t)$ is a solution independent of $\varphi_1(t)$. ∎

Problem (W'03, #7). *Under what conditions on g, continuous on $[0, L]$, is there a solution of*

$$\frac{\partial^2 u}{\partial x^2} = g,$$
$$u(0) = u(L/3) = u(L) = 0?$$

Proof. We have

$$\begin{aligned}
u_{xx} &= g(x), \\
u_x &= \int_0^x g(\xi)\, d\xi + C, \\
u(x) &= \int_0^x \int_0^\xi g(s)\, ds\, d\xi + Cx + D. \\
0 &= u(0) \,=\, D. \qquad\qquad \text{Thus,} \\
u(x) &= \int_0^x \int_0^\xi g(s)\, ds\, d\xi + Cx. \\
0 &= u(L) \,=\, \int_0^L \int_0^\xi g(s)\, ds\, d\xi + CL, \qquad\qquad \circledast \\
0 &= u(L/3) \,=\, \int_0^{\frac{L}{3}} \int_0^\xi g(s)\, ds\, d\xi + \frac{CL}{3}. \qquad \circledcirc
\end{aligned}$$

$\circledast$ and $\circledcirc$ are the conditions on g. $\square$

10 Dominant Balance

Problem (F'90, #4). *Use the* ***method of dominant balance*** *to find the asymptotic behavior at $t = \infty$ for solutions of the equation*

$$f_{tt} + t^3 f_t^2 - 4f = 0.$$

Proof. Assume $f = ct^n$ as $t \to \infty$, where need to find n and c. Then

$$n(n-1)ct^{n-2} + n^2c^2t^3t^{2n-2} - 4ct^n = 0,$$
$$n(n-1)ct^{n-2} + n^2c^2t^{2n+1} - 4ct^n = 0.$$

The 2^{nd} and the 3^{rd} terms are dominant. In order to satisfy the ODE for $t \to \infty$, set

$$2n + 1 = n \quad \Rightarrow \quad n = -1,$$
$$n^2c^2 = 4c \quad \Rightarrow \quad c^2 - 4c = 0 \quad \Rightarrow \quad c = 4.$$

$$\boxed{f \sim 4t^{-1}, \quad \text{as} \quad t \to \infty.}$$

□

Problem (S'91, #3). *Find the large time behavior for solutions of the equation*

$$\frac{d^2}{dt^2}f + \frac{d}{dt}f + f^3 = 0$$

using the ***method of dominant balance****.*

Proof. [23] Assume $f = ct^n$ as $t \to \infty$, where need to find n and c. Then

$$n(n-1)ct^{n-2} + nct^{n-1} + c^3t^{3n} = 0.$$

The 2^{nd} and the 3^{rd} terms are dominant. In order to satisfy the ODE for $t \to \infty$, set

$$n - 1 = 3n \quad \Rightarrow \quad n = -\frac{1}{2},$$
$$nc + c^3 = 0 \quad \Rightarrow \quad -\frac{1}{2}c + c^3 = 0 \quad \Rightarrow \quad c = \pm\frac{1}{\sqrt{2}}.$$

$$\boxed{f \sim \pm\frac{1}{\sqrt{2}}t^{-\frac{1}{2}}, \quad \text{as} \quad t \to \infty.}$$

□

[23] ChiuYen's solutions show a different approach, but they are wrong.

11 Perturbation Theory

Problem (F'89, #5a). *Solve the following ODE for $u(x)$ by* ***perturbation theory***

$$\begin{cases} u_{xx} = \varepsilon u^2 & 0 \le x \le 1 \\ u(0) = 0, \quad u(1) = 1 \end{cases} \tag{11.1}$$

for small ε. In particular, find the first two terms of u as an expansion in powers of the parameter ε.

Proof. We write $u = u_0(x) + \varepsilon u_1(x) + O(\varepsilon^2)$ as $\varepsilon \to 0$ and find the first two terms u_0 and u_1. We have

$$\begin{aligned} u &= u_0 + \varepsilon u_1 + O(\varepsilon^2), \\ u^2 &= \big(u_0 + \varepsilon u_1 + O(\varepsilon^2)\big)^2 = u_0^2 + 2\varepsilon u_0 u_1 + O(\varepsilon^2). \end{aligned}$$

Plugging this into (11.1), we obtain

$$\begin{aligned} u_{0xx} + \varepsilon u_{1xx} + O(\epsilon^2) &= \varepsilon\big(u_0^2 + 2\varepsilon u_0 u_1 + O(\varepsilon^2)\big), \\ u_{0xx} + \varepsilon u_{1xx} + O(\epsilon^2) &= \varepsilon u_0^2 + O(\varepsilon^2). \end{aligned}$$

$O(1)$ terms:

$$\begin{aligned} & u_{0xx} = 0, \\ & u_0 = c_0 x + c_1, \\ & u_0(0) = c_1 = 0, \\ & u_0(1) = c_0 = 1, \\ & \bullet\ u_0 = x. \end{aligned}$$

$O(\varepsilon)$ terms:

$$\begin{aligned} & \varepsilon u_{1xx} = \varepsilon u_0^2, \\ & u_{1xx} = u_0^2, \\ & u_{1xx} = x^2, \\ & u_1 = \frac{x^4}{12} + c_2 x + c_3, \\ & u_1(0) = c_3 = 0, \\ & u_1(1) = \frac{1}{12} + c_2 = 0 \quad \Rightarrow \quad c_2 = -\frac{1}{12}, \\ & \bullet\ u_1 = \frac{x^4}{12} - \frac{1}{12}x. \end{aligned}$$

$$\boxed{u(x) = x + \varepsilon\Big(\frac{x^4}{12} - \frac{1}{12}x\Big) + O(\varepsilon^2).}$$

$\square$

Problem (F'89, #5b). *For the differential equation*

$$u_{xx} = u^2 + x^3u^3 \tag{11.2}$$

look for any solution which are bounded for x near $+\infty$. Determine the behavior u for x near $+\infty$ for any such solutions.
Hint: Look for the dominant behavior of u to be in the form x^{-n}.

Proof. Let $u = cx^{-n}$. Plugging this into (11.2), we obtain

$$\begin{aligned} -n(-n-1)cx^{-n-2} &= c^2x^{-2n} + c^3x^3x^{-3n}, \\ n(n+1)cx^{-n-2} &= c^2x^{-2n} + c^3x^{3-3n}. \end{aligned}$$

Using the **method of dominant balance**, we want to cancel two terms such that the third term is 0 at $+\infty$ compared to the other two. Let

$$-n-2 = 3-3n,$$

$$\bullet \ n = \frac{5}{2}.$$

Also,

$$\frac{5}{2}\Big(\frac{5}{2}+1\Big)c = c^3,$$

$$\bullet \ c = \pm\frac{\sqrt{35}}{2}.$$

$$\boxed{u(x) = \pm\frac{\sqrt{35}}{2}x^{-\frac{5}{2}}.}$$

$\square$

Problem (F'03, #6a). *For the cubic equation*

$$\varepsilon^3 x^3 - 2\varepsilon x^2 + 2x - 6 = 0 \tag{11.3}$$

write the solutions x in the asymptotic expansion $x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$ as $\varepsilon \to 0$. Find the first two terms x_0 and x_1 for all solutions x.

Proof. As $\varepsilon \to 0$,

$$\begin{aligned}
x &= x_0 + \epsilon x_1 + O(\varepsilon^2), \\
x^2 &= \big(x_0 + \varepsilon x_1 + O(\varepsilon^2)\big)^2 = x_0^2 + 2\varepsilon x_0 x_1 + O(\varepsilon^2), \\
x^3 &= \big(x_0 + \varepsilon x_1 + O(\varepsilon^2)\big)^3 = \big(x_0^2 + 2\varepsilon x_0 x_1 + O(\varepsilon^2)\big)\big(x_0 + \varepsilon x_1 + O(\varepsilon^2)\big) \\
&= x_0^3 + 3\varepsilon x_0^2 x_1 + O(\varepsilon^2).
\end{aligned}$$

Plugging this into (11.3), we obtain

$$\varepsilon^3(x_0^3 + 3\varepsilon x_0^2 x_1 + O(\varepsilon^2)) - 2\varepsilon(x_0^2 + 2\varepsilon x_0 x_1 + O(\varepsilon^2)) + 2(x_0 + \varepsilon x_1 + O(\varepsilon^2)) - 6 = 0.$$

As $\varepsilon \to 0$, we ignore the $O(\varepsilon^2)$ terms:

$$-2\varepsilon x_0^2 - O(\varepsilon^2) + 2x_0 + 2\varepsilon x_1 + O(\varepsilon^2) - 6 = 0,$$

$$-\varepsilon x_0^2 + x_0 + \varepsilon x_1 - 3 + O(\varepsilon^2) = 0. \tag{11.4}$$

As $\varepsilon \to 0$, $\quad -\varepsilon x_0^2 + x_0 + \varepsilon x_1 - 3 + O(\varepsilon^2) \;\to\; x_0 - 3 = 0.$ Thus, $x_0 = 3$.
Plugging this value of x_0 into (11.4), we obtain

$$\begin{aligned}
-9\varepsilon + 3 + \varepsilon x_1 - 3 + O(\varepsilon^2) &= 0, \\
-9\varepsilon + \varepsilon x_1 + O(\varepsilon^2) &= 0, \\
x_2 &= 9.
\end{aligned}$$

$$\boxed{x = 3 + 9\varepsilon + O(\varepsilon^2).}$$

□

Problem (F'03, #6b). *For the ODE*

$$\begin{cases} u_t = u - \varepsilon u^3, \\ u(0) = 1, \end{cases} \tag{11.5}$$

write $u = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + O(\varepsilon^3)$ *as* $\varepsilon \to 0$. *Find the first three terms* u_0, u_1 *and* u_2.

Proof. We have $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^3)$ as $\varepsilon \to 0$.

$$u^3 = \big(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^3)\big)^3 = u_0^3 + 3\varepsilon u_0^2 u_1 + 3\varepsilon^2 u_0^2 u_2 + 3\varepsilon^2 u_0 u_1^2 + O(\varepsilon^3).$$

Plugging this into (11.5), we obtain

$$\begin{aligned} & u_{0t} + \varepsilon u_{1t} + \varepsilon^2 u_{2t} + O(\epsilon^3) \\ & = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^3) - \varepsilon\big(u_0^3 + 3\varepsilon u_0^2 u_1 + 3\varepsilon^2 u_0^2 u_2 + 3\varepsilon^2 u_0 u_1^2 + O(\varepsilon^3)\big), \end{aligned}$$

$$u_{0t} + \varepsilon u_{1t} + \varepsilon^2 u_{2t} + O(\epsilon^3) = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 - \varepsilon u_0^3 - 3\varepsilon^2 u_0^2 u_1 + O(\varepsilon^3),$$

$O(1)$ terms:

$$\begin{aligned} & u_{0t} = u_0, \\ & \bullet\ u_0 = c_0 e^t. \end{aligned}$$

$O(\varepsilon)$ terms:

$$\begin{aligned} & \varepsilon u_{1t} = \varepsilon u_1 - \varepsilon u_0^3, \\ & u_{1t} = u_1 - u_0^3, \\ & u_{1t} - u_1 = -c_0^3 e^{3t}, \\ & \bullet\ u_1 = c_1 e^t - \frac{1}{2} c_0^3 e^{3t}. \end{aligned}$$

$O(\varepsilon^2)$ terms: [24]

$$\begin{aligned} & \varepsilon^2 u_{2t} = \varepsilon^2 u_2 - 3\varepsilon^2 u_0^2 u_1, \\ & u_{2t} = u_2 - 3u_0^2 u_1, \\ & u_{2t} - u_2 = -3c_0^2 e^{2t}\big(c_1 e^t - \frac{1}{2} c_0^3 e^{3t}\big), \\ & u_{2t} - u_2 = -3c_0^2 c_1 e^t e^{2t} + \frac{3}{2} c_0^5 e^{2t} e^{3t}, \\ & \bullet\ u_2 = c_2 e^t - \frac{3}{2} c_0^2 c_1 e^t e^{2t} + \frac{3}{8} c_0^5 e^{2t} e^{3t}. \end{aligned}$$

Thus,

$$u(t) = c_0 e^t + \varepsilon\big(c_1 e^t - \frac{1}{2} c_0^3 e^{3t}\big) + \varepsilon^2\big(c_2 e^t - \frac{3}{2} c_0^2 c_1 e^t e^{2t} + \frac{3}{8} c_0^5 e^{2t} e^{3t}\big) + O(\varepsilon^3).$$

Initial condition gives

$$u(0) = c_0 + \varepsilon\big(c_1 - \frac{1}{2} c_0^3\big) + \varepsilon^2\big(c_2 - \frac{3}{2} c_0^2 c_1 + \frac{3}{8} c_0^5\big) + O(\varepsilon^3) = 1.$$

Thus, $c_0 = 1$, $c_1 = \frac{1}{2}$, $c_2 = \frac{3}{8}$, and

$$\boxed{u(t) = e^t + \varepsilon\frac{1}{2}\big(e^t - e^{3t}\big) + \varepsilon^2\big(\frac{3}{8} e^t - \frac{3}{4} e^t e^{2t} + \frac{3}{8} e^{2t} e^{3t}\big) + O(\varepsilon^3).}$$

□

[24] Solutions to ODEs in u_1 and u_2 are obtained by adding homogeneous and particular solutions.