CHAPTER 1

Compound Interest

1. Compound Interest

The simplest example of interest is a loan agreement two children might make: "I will lend you a dollar, but every day you keep it, you owe me one more penny." In this example, the interest rate is 1%/day and the amount owed after t days is

$$A(t) = 1 + .01t$$

In this formula, the quantity $.01t$ is the interest at time t. (In general, the interest is the difference between what was borrowed and what is owed.)

REMARK. In the above example, we can describe the interest rate as a percent (1%) or as a numeric value (.01). *When we state an interest rate we will always mean a numeric value, and not a percent, unless we indicate otherwise.*

If, as above, the interest is proportional to time, then we say that the interest is *simple interest.* Thus, if we borrow P at rate i simple interest, the amount owed at time t is

$$A(t) = P + itP = (1 + it)P$$

EXAMPLE 1. On Jan. 1 of a non-leap year, I borrow \$5,000 at 3% simple interest per year. How much do I owe on May 1? How much would I owe after 3 years?

SOLUTION. On May 1, I have had the money for $31 + 28 + 31 + 30 = 120$ days, which is 120/365th of a year. Hence, I owe

$$\left(1 + \frac{120}{365}.03\right) 5000 = 5049.32$$

dollars.

After 3 years, I owe

$$(1 + 3(.03))5000 = 5450.00$$

REMARK. In dealing with money, we will usually round our answers to the nearest penny. When reporting interest rates we will round to at least three significant figures, e.g. 6.13%.

REMARK. In computing interest, it is typically assumed that interest is earned only on either the first day the account is open or the last day, but not on both. Which day doesn't matter in computing the interest. Thus, in Example 1, it is correct not to count the interest earned on May 1.

The question of how many days are in a year is actually somewhat complicated. The most obvious answer is that a year will have either 365 or 366 days, depending on whether or not it is a leap year. It has to be remembered, however, that accounting practices became standardized long before even hand held calculators were available, not to mention personal computers. Thus, many schemes have been developed to simplify hand computations.

For example, it is common to not give interest on Feb. 29, in which case all years effectively have 365 days. Another method, referred to as *exact interest,* is to give interest on leap day, but still say that all years have 365 days. Thus, under this standard, at the nth day of the year, P dollars grows to

$$(1+\frac{n}{365}i)P$$

In particular, at the end of a leap year, you have

$$(1+\frac{366}{365}i)P$$

dollars.

There is another method, *ordinary interest,* (not to be confused with "simple interest") in which it is assumed that all months have 30 days and every year has 360 days! Thus, if you opened an 4% account on Jan. 1 1950 and closed it on May 10, 2002, you held your money for 52 years, 4 months and 10 days which, according to the rules of ordinary interest, is

$$52 \cdot 360 + 4 \cdot 30 + 10 = 18,850$$

days. Hence P dollars will have grown to

$$(1+\frac{18850}{360}.04)P$$

dollars. Ordinary interest has the feature that each month is 1/12 of a year.

There is also something called *Banker's rule,* in which every year has 360 days, but you count the exact number of days you have held the money in computing the interest. To use Banker's Rule on the preceding example, you would have to count the days between Jan. 1, 1950 and May 10, 2002 and use this number instead of the 18,850. Good luck!

The use of exact interest is common in Canada while the Banker's rule is common in the US and in international markets. In this class we will always assume that no interest is given on Feb. 29, in which case all years effectively have 365 days. When necessary, we will count the exact number of days, except Feb. 29.

Compound interest is much more common than simple interest. Suppose, for example, that I borrow P dollars at rate i, compounded yearly. As with simple interest, at the end of the year, I owe

$$A = (1+i)P$$

dollars.

With compound interest, however, I pay interest on the total amount owed at the beginning of the compounding period, not just the original principal. Hence, in another year, my debt will again grow by a factor of $(1+i)$. Hence at the end of year 2, I owe

$$A_2 = (1+i)A = (1+i)^2 P$$

dollars. In general, we denote the amount owed after n years by $A(n)$. Then

$$A_n = (1+i)^n P \tag{1}$$

dollars.

In interest theory, the difference between borrowing money and saving money is only in the point of view. When I open a bank account, I am in essence loaning the bank money. The interest I earn on the account is the interest the bank pays me on this loan. Thus, the only difference between a bank loan and a bank account is in who is doing the lending and who is doing the borrowing. In particular, we can analyze savings accounts using the same formulas.

Example 2. On Jan. 1, 1990, I deposited \$1,000 in an account that paid 7.3% interest, compounded yearly. How much did I have on Jan. 1, 2010.

Solution. My funds were on account for 20 years. Hence, I have

$$(1.073)^{20}1000 = 4,092.55$$

dollars.

Example 3. On Jan.1, 1998, I open an account with a \$1000 deposit. On Jan.1, 1999, I withdraw \$500 and on Jan.1, 2001 I deposit \$1,500. If the account earns 7.5% interest, compounded yearly, and no further deposits or withdraws are made, what was the balance on Jan.1, 2003?

Solution. There are two ways to solve this problem; easy and easier. First, the easy way:

The balance on Jan. 1, 1999 was one year interest on \$1000, minus \$500:

$$1000(1.075) - 500 = 575$$

The balance on Jan. 1, 2001 was 2 years interest on \$575, plus the \$1,500 deposit:

$$575(1.075)^2 + 1500 = 2164.48$$

My final balance is 2 years interest on \$2164.48:

$$2164.48(1.075)^2 = 2501.34$$

Now for the easier way. Without any further deposits, our \$1000 would have grown to

$$1000(1.075)^5 = 1435.63$$

Withdrawing \$500 caused us to loose both the \$500 as well as its interest for the next 4 years; a net loss of

$$500(1.075)^4 = 667.73$$

Finally, the \$1,500 deposit was on account for 2 years, yielding a total of

$$1500(1.075)^2 = 1733.44$$

Hence, our balance is

$$1435.63 - 667.73 + 1733.44 = 2501.34$$

as before.

In general, we may treat deposits and withdrawals separately.

Example 4. Ed borrows \$550 at 4% interest. At the end of year 1, he pays \$100, at the end of year 2 he pays \$300 and at the end of year 3 he borrows an additional \$50 at the same interest rate. He pays off the loan at the end of year 4. What was his final payment?

SOLUTION. We treat each payment and loan separately. The loans, together with interest, total to a debt of

$$(1.04)^4 550 + (1.04)50 = 695.42$$

Each payment results in an interest savings. Thus, the payments up to the end of year 4 reduce this debt by

$$(1.04)^3 100 + (1.04)^2 300 = 436.97$$

Thus, Ed still owes

$$695.42 - 436.97 = 258.45$$

which is his last payment.

What if, in Example 2, I were to close my account after having left my money on deposit for only 6 months; how much would I get? The answer depends on the rules of the bank. Some accounts charge a substantial penalty for early withdrawal, meaning that you could actually lose money. In some cases, the bank uses simple interest for partial periods, in which case you would get

$$\left(1 + \frac{.073}{2}\right) 1000 = 1,036.50$$

dollars since the money was on deposit for a half year. Finally, we might simply substitute $n = 1/2$ into formula (1) yielding

$$(1.073)^{1/2} 1000 = 1,035.86$$

In practice, this last method is probably the least common. However, in the mathematical theory of interest, *if we say that an account earns compound interest at a rate i, we are implicitly stating that we use formula (1) for partial periods as well:*

DEFINITION 1. *A quantity grows at a rate i compound interest if the amount at time t is given by*

$$A(t) = (1 + i)^t P \tag{2}$$

for some constant P.

EXAMPLE 5. Banks A and B both offer savings accounts that pay 5% interest per year. Bank A compounds yearly but uses simple interest for partial periods while bank B uses straight compound interest for all times. Compare the amount that you would have after 3 years and 2 months if you invested \$2,000 in bank A with the same investment in bank B.

SOLUTION. In bank A, at the end of 3 years, you have

$$(1.05)^3 2000 = 2315.25$$

dollars. For the next 2 months you earn 5% simple interest on \$2,315.25 dollars, yielding

$$\left(1 + .05\left(\frac{2}{12}\right)\right) 2315.25 = 2334.54$$

In bank B you have

$$(1.05)^{3\frac{2}{12}} 2000 = 2334.15$$

This example makes an important point: the difference between using simple interest for partial periods verses compound interest is slight.

The observation that for small time intervals, compound and simple interest are roughly the same is equivalent with saying that for small values of t

$$(1+i)^t \approx 1 + it \tag{3}$$

Example 6. The following chart is a record of the activity in a certain account that earns compound interest at rate i. The initial balance was \$50,000 and the final balance was \$48085.44. Approximate i.

Date	Deposit(+) or Withdraw(-)
Jan. 1	0
May 1	-5000
July. 1	1000
Jan. 1	0

Solution. Since we only need an approximate value of i, we assume that each month is 1/12 of a year. We may treat the effect of each deposit and withdrawal separately. We lost \$5,000, together with its interest for 8 months, and gained \$1,000, together with its interest for 6 months. Thus, using approximation (3),

$$\begin{aligned} 48085.44 &= (1+i)50000 - (1+i)^{8/12}5000 + (1+i)^{6/12}1000 \\ &\approx (1+i)50000 - (1+\frac{2}{3}i)5000 + (1+\frac{i}{2})1000 \\ 48085.44 - 50000 + 5000 - 1000 &\approx (50000 - \frac{2}{3}5000 + \frac{1}{2}1000)i \\ \frac{2085.44}{47166.67} &= .0442 \approx i \end{aligned} \tag{4}$$

Remark. If we have a calculator (or a computer) with a "solve" command, we can ask it to solve equation (4). Our computer produced $i = .04419677393$ which agrees favorably with our approximation.

At times, one hears of banks offering accounts which compound at intervals other than one year. For example, a bank might offer an account that pays 6% interest, compounded four times a year. What this means is that every quarter of a year, the account grows by $\frac{6}{4}$%. Thus, in one year, P dollars grows to

$$\left(1 + \frac{.06}{4}\right)^4 P = (1.0613)P$$

This is the same growth as an account at 6.13% interest, compounded annually. This 6.13% is called the *annual effective yield* while the "6%" interest rate is referred to as the *nominal* rate, in that it's the rate that the bank might name when describing the account.

In general, the symbol $i^{(n)}$ indicates a nominal interest rate i which is compounded n times a year. Thus, the discussion in the preceding paragraph says that an interest rate of $.06^{(4)}$ is the same as $.0613^{(1)}$. The rate $i^{(n)}$ is equivalent with the annual effective rate j where

$$\left(1+\frac{i^{(n)}}{n}\right)^n = 1+j$$

Example 7. A bank offers an account that yields a nominal rate of return of 3.3% per year, compounded quarterly. What is the annual effective rate of return? How many years will it take for the balance to double?

Solution. Since each year has 4 quarters, P dollars at the beginning of the year grows to

$$\left(1+\frac{.033}{4}\right)^4 P = (1.0334)P$$

by the end of the year. Hence, the annual effective rate of interest is 3.34%.

To compute how long it takes for the account to double, we can either work in quarters or years. In quarters, we seek n so that

$$\begin{aligned}(1+\frac{.033}{4})^n P &= 2P\\ (1+\frac{.033}{4})^n &= 2\\ n\ln(1.00825) &= \ln 2\\ n &= \frac{\ln 2}{\ln(1.00825)} = 84.36\end{aligned}$$

The number of years is $84.36/4 = 21.09$.

Since our effective rate of return is 3.34% per year, we can find the answer directly in years as follows:

$$\begin{aligned}(1.0334)^n P &= 2P\\ (1.0334)^n P &= 2\\ n\ln(1.0334) &= \ln 2\\ n &= \frac{\ln 2}{\ln(1.0334)} = 21.1\end{aligned}$$

The answer differs slightly from that found previously due to round off error. Specifically, 3.34% is only an approximation of the annual effective yield. A more exact value is 3.3410626%, which does yield the same answer as before.

Actually, both answers might be wrong. If the bank only credits interest each quarter, then the doubling would not occur until after the 85th quarter, in which case the correct answer is $21\frac{1}{4}$ years.

Example 8. Bank A offers a nominal rate of 5.2% interest per year, compounded twice a year. Bank B offers a nominal rate of 5.1% interest, compounded daily. Which is the better deal?

Solution. We convert each annual nominal rate into an annual effective rate:

Bank A

$$(1+\frac{.052}{2})^2 = 1.052676$$

for a 5.2676% annual effective rate of return.

Bank B

$$(1 + \frac{.051}{365})^{365} = 1.052319134$$

for a 5.2319134% annual effective rate of return. It's very close, but Bank A wins.

REMARK. Daily compounding is very common. Daily compounding eliminates the problem of partial periods: you get whatever the balance was at the end of the preceding day.

EXAMPLE 9. How much must I deposit today into an account that pays 6.4% annually to be able to pay you \$500 two years from now?

SOLUTION. Let the amount deposited be P. We need to solve the equation

$$(1.064)^2 P = 500$$

$$P = (1.064)^{-2} 500 = 441.66$$

dollars.

The preceding example makes an extremely important point: a promise to pay \$500, two years from today is not worth \$500 today: if we can invest money at 6.4%, \$500 two years from now is only worth \$441.66 today. We say that the *present value* of a promise to pay \$500 two years from now at 6.4% interest is \$441.66 today. Equivalently, at 6.4% interest, \$441.66 will grow to \$500. Hence, the *future value* of \$441.66 two years from now at 6.4% interest is \$500.

DEFINITION 2. The *future value* (FV) of P dollars at interest rate i, n years from now, is the amount that P dollars will grow to in n years. Hence

$$FV = (1 + i)^n P \tag{5}$$

The *present value* (PV) of a promise to pay P dollars n years from now at interest rate i, is

$$PV = (1 + i)^{-n} P \tag{6}$$

The quantity $(1 + i)^{-1}$ occurs so often that it has a special symbol:

$$(1 + i)^{-1} = \nu$$

Hence, Formula 6 is often written

$$PV = \nu^n P$$

EXAMPLE 10. On Jan. 1, you won a "\$400,000 sweepstakes." The prize is to be paid out in 4 yearly installments of \$100,000 each with the first paid immediately. Assuming that you can invest funds at 5% interest compounded annually, what was the prize worth when you won it?

SOLUTION. The question is asking for the present value of the prize at the time you won it, assuming a 5% interest rate. The present value of a series of payments is the sum of the present values of each of the individual payments. Since the first payment is received immediately, it is worth 100000. The second payment is only worth $(1.05)^{-1}100000 = 95238.10$ since we must wait a year to get it. Reasoning similarly we see that our answer is

$$100000 + (1.05)^{-1}100000 + (1.05)^{-2}100000 + (1.05)^{-3}100000 = 372324.80$$

which is considerably less than the advertized 4.000,000 value of the prize.

2. Annuities

The single most important theorem in interest theory is

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1} \tag{7}$$

The proof is simple:

$$\begin{aligned}(1 + x + x^2 + \cdots + x^n)(x - 1) &= -(1 + x + x^2 + \cdots + x^n)1 \\ &\quad + (1 + x + x^2 + \cdots + x^{n-1} + x^n)x \\ &= -1 - x - x^2 - \cdots - x^n \\ &\qquad + x + x^2 + \cdots + x^n + x^{n+1} = x^{n+1} - 1\end{aligned}$$

which is equivalent with formula (7).

EXAMPLE 11. I deposited \$300 at the end of each year from 1981 to 2000 into an account that yields 3% interest per year. How much did I have on Jan.1, 2001?

SOLUTION. I made a total of 20 deposits. My first deposit earned interest for 19 years and my last deposit earned no interest at all. Hence, I received

$$300(1.03)^{19} + 300(1.03)^{18} + \cdots + 300 = 300((1.03)^{19} + (1.03)^{18} + \cdots + 1)$$

dollars.

From formula (7), with $x = 1.03$, this equals

$$300\frac{(1.03)^{20} - 1}{.03} = 8061.11$$

dollars.

In general, if we make periodic deposits of D at the end of each period into an account that earns interest at the rate i per period, the balance after n deposits is

$$B_n = \frac{(1 + i)^n - 1}{i}D. \tag{8}$$

EXAMPLE 12. I deposited \$300 at the *beginning* of each year from 1981 to 2000 into an account that yields 3% interest per year. How much do I have at the end of 2000?

SOLUTION. Depositing at the beginning of a given year is the same as depositing at the end of the preceding year. Thus, we may consider that I deposited \$300 at the end of each year from 1980 to 1999 for a total of 20 deposits. From the reasoning of Example 11, at the end of 1999, we had \$8061.11. At the end of 2000 we gain one year's interest on this amount–i.e. we have

$$(1.03)8061.11 = 8302.95$$

dollars.

An account into which we make either periodic deposits (as in Examples 11 and 12) or periodic withdrawals is called an *annuity*. If the transactions always occur at the end of the compounding period, as in Example 11, the annuity is said to be an *annuity immediate* while if the transactions always occur at the beginning of the compounding period, as in Example 12, the annuity is said to be an *annuity due*.

If the deposits are made at the beginning of the year, then the formula is the same, except that we have an additional year's interest:

$$B_n = (1+i)\frac{(1+i)^n - 1}{i}D \tag{9}$$

dollars at the end of the nth year.

We may use formula (8) for accounts with non-yearly compounding. We only need to remember that i is the interest per period and n is the number of periods.

Example 13. At the beginning of 1992, I opened a bank account earning 4% interest compounded monthly with a \$5,000 deposit. I deposited \$100 at the end of each month from 1992 to 2001. What was my account worth at the end of 2001, after my last deposit?

Solution. Over the 10 years from Jan. 1, 1992 to Dec. 31, 2001 we compounded interest 12 times a year for 10 years. The rate of interest is $i = .05/12$ per month. Hence the original \$5000 grew to

$$\left(1+\frac{.04}{12}\right)^{120} 5000 = 7454.16$$

We also deposited \$100 each month. From formula (8), the monthly deposits accumulated to

$$\frac{(1+\frac{04}{12})^{120} - 1}{.04/12}100 = 14,724.98$$

Hence, the total is

$$7454.16 + 14724.98 = 22179.14$$

dollars.

Remark. Computations using short compounding periods over long periods of time are highly susceptible to round-off error. For example, had we rounded the monthly rate from Example (13) to .003, then we would compute the accumulation of the monthly deposits as

$$\frac{(1.003)^{120} - 1}{.003}100 = 14418.57$$

which is off by more than \$250! When doing compound interest problems, you should make full use of the memory of your calculator, writing as little on paper as possible and not rounding your answers until the end of the calculation.

3. The Present Value of an Annuity

Example 14. Today I won a prize that pays \$1,000,000 a year at the end of the year for the next 10 years. Assuming that I can invest funds at 3% interest, compounded annually, what is the actual value of the prize today?

Solution. The problem is asking for the present value of the prize at the time I won it. According to formula 8 if I had invested my winnings in an account earning 3% interest, at the end of the 10th year I would have

$$1000000\frac{(1.03)^{10} - 1}{.03}$$

dollars. The present value of this sum is obtained by multiplying by $(1.03)^{-10}$:

$$\begin{aligned} PV &= 1000000(1.03)^{-10}\frac{(1.03)^{10}-1}{.03} \\ &= 1000000\frac{1-(1.03)^{-10}}{.03} \\ &= 8,530,202.84 \end{aligned}$$

If the payments had been made at the beginning of the year, then from formula 8 the answer would be increased by multiplying by a factor of 1.03 yielding

$$PV = (1.03)8,530,202.84 = 8,786,108.93.$$

In general, if we make periodic deposits of D at the end of each period into an account that earns interest at the rate i per period, the **present value** of the account after n deposits is

$$PV_n = \frac{1-\nu^n}{i}D \tag{10}$$

where $\nu = (1+i)^{-1}$. If the deposits are made at the beginning of the period the present value is

$$PV_n = (1+i)\frac{1-\nu^n}{i}D. \tag{11}$$

4. Loans

In the typical loan we borrow an amount of money P, the "principal," at some interest rate i for a certain period of time n that could be days, months, years, etc. In paying a loan, one needs to remember that the total value of the payments, including interest, must, at the time the loan is paid off, equal the amount borrowed, including interest. (The lender would not be happy if we forgot to pay the interest!) In other words, *the future value of the payments must equal the future value of the amount borrowed.*

EXAMPLE 15. I borrow \$25,000 to buy a car on which I pay \$1000 down and make monthly payments at the end of the month over the next 5 years. If I pay 7% interest, compounded monthly, what are my monthly payments?

SOLUTION. Let P be my payment. After my down payment, I owe \$24,000. The future value of the amount borrowed must equal the future value of the payments. Since the payments are made at the end of the month, the future value of the payments is given by formula (8) on page 8 and the future value of the loan by formula (5) on page 7 where in both formulas i is the monthly interest rate and n is the number of payments. Hence

$$\begin{aligned} \left(1+\frac{.07}{12}\right)^{5\cdot 12} 24000 &= \frac{(1+\frac{.07}{12})^{12\cdot 5}-1}{.07/12}P \\ 34023.01 &= 71.59P \\ P &= 475.23 \end{aligned}$$

which is our monthly payment.

EXAMPLE 16. How much can I borrow at 4.7% interest compounded monthly if I can pay \$5,000 per month for 20 years?

SOLUTION. Let P be the amount you can borrow. The future value of the amount borrowed must equal the future value of the payments. Hence

$$\begin{aligned}\left(1+\frac{.047}{12}\right)^{240} P &= \frac{(1+\frac{.047}{12})^{240}-1}{\frac{.047}{12}}5000\\ 2.555285704P &= 1985471.11\\ P &= 777,005.53.\end{aligned}$$

An IRA is a special savings account that workers and/or their employers contribute to over the period of the worker's employment. At the time of retirement, the worker begins withdrawing from the IRA to pay his/her living expenses.

EXAMPLE 17. I plan to retire at age 70, at which time I will withdraw \$5,000 per month for 20 years from my IRA. Assuming that my funds are invested at 4.7% interest compounded monthly, how much must I have accumulated in my IRA?

SOLUTION. We think of the IRA as being a bank account into which we initially deposite P and from which we withdraw \$5000 per month. Since we may treat deposits and withdrawls separately, this will be possible if the future value of P equals the future value of the withdrawls. *This is exactly the same problem that we solved in Example 16.* Thus the answer is the same: $P = 777,005.53$.

5. Bonds

A coupon bond is an investment that typically yields a fixed sum (referred to as the *coupon*) several times a year for a period of years (the *term* of the bond) together with a lump sum payment (the *redemption value*) at the end of the term.

EXAMPLE 18. What is the most you should pay for a 9 year bond that pays \$250 every quarter and has a \$10,000 edemption value, given that you can invest funds at 7.1% interest compounded four times a year.

SOLUTION. Over 9 years we will receive 36 payments of \$250 and a final payment of \$10,000. If we were to invest the proceeds at 7.1% compounded quarter, immediately after the final \$10,000 payment, our balance would be

$$\frac{(1+\frac{.071}{4})^{36}-1}{.071/4}250+10000 = 22450.65.$$

This is the future value of the bond. Our answer is the present value of this number:

$$\left(1+\frac{.071}{4}\right)^{-36} 22450.65 = 11916.51$$

It is often said that as stock prices drop, bond prices rise. It is easy to understand why. Since we cannot invest money at as high of a rate of return, the idea of a steady income of \$250 together with \$10,000 at the end appears much more attractive and hence is worth more money.

EXAMPLE 19. Suppose that in Example 18, we were only able to invest money at 5% interest. How much should we pay for the same bond?

SOLUTION. Now the future value of the income stream is

$$\frac{(1+\frac{.05}{4})^{36}-1}{.05/4}250+10000 = 21278.88.$$

The value of the bond is the present value of this number which is

$$\left(1 + \frac{.05}{4})^{-36}\right) 21278.88 = 13605.91.$$

Hence, the value of the bond has increased by over $1,000.

Bonds are traded like stocks.

EXAMPLE 20. The bond in Example 18 is sold after two years immediately after payment of the 8th coupon to a buyer wanting a rate of return of 6% compounded four times a year. What should the price of the bond be?

SOLUTION. Eight coupons have already been paid. The buyer will receive the remaining 28 coupons plus the the $10,000 redemption value. The price of the bond will be the present value of all of these payments at 6% interest compounded quarterly.. Thus the price should be

$$\left(1 + \frac{.06}{4}\right)^{-28} \left(\frac{(1 + \frac{.06}{4})^{28} - 1}{.06/4} 250 + 10000\right) = 12272.67.$$

6. Rate of Return

In finance, one of the most fundamental problems is determining the rate of return on a given investment over a given year, the *annual effective rate of return.* In the simplest case, we put money in at the beginning of the year and don't touch it until the end of the year. Then the percentage of growth in the account over the year. As a fraction it is

$$i = \frac{B_1 - B_0}{B_0} \tag{12}$$

where B_1 is the end balance and B_0 is the initial balance.

The problem of determining the rate of return becomes more complicated when money is being added to or subtracted from the account throughout the year. This situation might arise, for example, in a large mutual fund in which individual investors periodically buy and sell shares. In this case, there are several different ways of defining the rate of return which do not necessarily yield the same answer. One of the most common is the following.

DEFINITION 3. The *annual effective rate of return* (or *dollar weighted rate of return*) on an investment over a given year is that interest rate i which would yield the same final balance for the same activity in the account. It is found by solving the equation

$$B_1 = (1 + i)B_0 + (1 + i)^{1-t_1} D_1 + (1 + i)^{1-t_2} D_2 + \cdots + (1 + i)^{1-t_k} D_k$$

for i, where B_0 is the initial balance, B_1 is the final balance, the D_k are the values of the deposits or withdrawals and the t_k are the times (in fractions of a year) of the deposits, where withdrawals are considered as negative deposits.

Thus, in Example 6 on page 5, we determined the annual effective rate of return on the account. Here is another example of the same concept.

EXAMPLE 21. The following chart is a record of the activity in an investment. The initial balance was \$10,000 and the final balance was \$10,176.22. Determine the annual effective rate of return.

Date	Deposit(+) or Withdraw(-)
Jan. 1	0
Apr. 1	1500
Sept. 1	-1000
Jan. 1	0

SOLUTION. We need to solve the following equation for i.

(13)
$$\begin{aligned}10176.22 &= (1+i)10000 + (1+i)^{9/12}1500 - (1+i)^{4/12}1000\\ &\approx (1+i)10000 + (1+\frac{3}{4}i)1500 - (1+\frac{i}{3})1000\\ 10176.22 - 10000 - 1500 + 1000 &\approx (10000 + \frac{3}{4}1500 - \frac{1}{3}1000)i\\ \frac{-323.78}{47166.67} &= -.030 \approx i\end{aligned}$$

Hence, the annual effective rate of return was -3%.

The annual effective rate of return is, in a way, the average interest rate on the investment over the year. Thus, in Example 21, if we invested \$1 at the beginning of the year, then we expect that our investment would only be worth around \$.97 at the end of the year. This will be exact if the interest rate stayed constant over the year–i.e. the value at time t is given by formula (2). In a typical investment, this will only be approximately correct. In fact, in an extremely volatile market where the return rates fluctuate wildly, it might be very far from true.

7. Discount and Force of Interest

According to formula 6, the value of money decreases as we look backward in time. For example, at 3% annual interest, P dollars today was worth

$$(1.03)^{-1}P \approx (0.9709)P$$

one year ago. Thus the value 1 year ago was

$$P - (0.9709)P = 0.0291P$$

less than it is today; it decreased by 2.91%. The annual rate of decrease as we look backward in time is called the *annual rate of discount.* It can be stated either as a percent or an a decimal.

More generally, at rate i, P dollars today was worth only $(1+i)^{-1}P$ dollars last year. The amount of decrease, then, is

$$P - (1+i)^{-1}P = (1 - (1+i)^{-1})P$$

The rate of decrease

(14)
$$d = 1 - (1+i)^{-1}$$

is called the *discount rate.* It is common to state the discount rate, instead of the interest rate.

We can also write

$$
\begin{aligned}
d &= 1 - \frac{1}{1+i} \\
&= \frac{i}{i+1}
\end{aligned} \tag{15}
$$

and

$$
\begin{aligned}
\frac{1}{1+i} &= 1 - d \\
1 + i &= \frac{1}{1-d} \\
i &= 1 - \frac{1}{1-d} \\
&= \frac{d}{1-d}.
\end{aligned} \tag{16}
$$

These formulas facilitate translating back and forth between discount and interest.

EXAMPLE 22. (a) What interest discount rate d corresponds to an annual interest rate of 5%? (b) What interest rate i corresponds to an annual discount rate of 5%?

SOLUTION. Part(a):
From formula (15)

$$d = \frac{.05}{1.05} = .0476.$$

As a percent, our answer is $d = 4.76\%$.
Part(b):
From formula (16)

$$i = \frac{.05}{1 - .05} = .0526.$$

As a percent, our answer is $i = 5.26\%$.

Just as with interest, there are also nominal discount rates. For example, nominal discount rate of 5% compounded 12 times a year means that each month we discount by a rate of .05/12. Formulas (15) and (16) do not require that the compounding period be a year. Thus we can use them for nominal rates as well.

EXAMPLE 23. How much will we have after 10 years if we invest \$ 2000 in an account that pays a discount rate of 4% per year, compounded monthly.

SOLUTION. The monthly discount rate is

$$d = \frac{.04}{12} = .0333\ldots.$$

Hence, from Formula (16), the monthly interest rate is computed from

$$
\begin{aligned}
i &= \frac{d}{1-d} \\
&= \frac{\frac{.04}{12}}{1 - \frac{.04}{12}} \\
&= 1.003344482
\end{aligned}
$$

Using Formula 5, our accumulation is

$$(1.003344482)^{12 \cdot 10} 2000 = 2985.64$$

dollars.

Another common measure of interest is what is called the "force of interest." From Formula 5, the amount $A(t)$ in the account at time t satisfies

$$\begin{aligned} A(t) &= (1+i)^t P \\ &= e^{\ln((1+i)^t)} P \\ &= e^{t\delta} P \end{aligned} \tag{17}$$

where

$$\begin{aligned} \delta &= \ln(1+i) \\ 1+i &= e^{\delta} \end{aligned} \tag{18}$$

The number δ is the *force of interest*.

EXAMPLE 24. An account grows with a force of interest of .04 per year. What is the interest rate?

SOLUTION. From Formula 18,

$$1 + i = e^{\delta} = e^{.04} = 1.0408$$

Thus, $i = .0408 = 4.08\%$.

EXAMPLE 25. An account grows with interest of .04 per year. What is the force of interest?

SOLUTION. From Formula 18,

$$\delta = \ln(1.04) = .03922$$

Thus, $\delta = 3.92\%$.

Note that differentiating Formula 17 produces

$$A'(t) = \delta A(t)$$

This tells us two important things:

(1) The rate of growth of the amount function is proportional to the amount of money in the account.
(2) The proportionality constant is the force of interest.

We can use the force of interest to describe circumstances in which the interest rate varies over time. Specifically, solving the preceding equation for δ produces

$$\delta = \frac{A'(t)}{A(t)} = \frac{d}{dt}(\ln A(t)) \tag{19}$$

DEFINITION 4. If $A(t)$ represents the amount in an account at time t, then the force of interest for this account is the quantity $\delta(t)$ defined by Formula 19.

Note that from Formula 19

$$\begin{aligned} \int_{t_0}^{t_1} \delta(t)\, dt &= \ln A(t_1) - \ln A(t_0) \\ &= \ln \frac{A(t_1)}{A(t_0)} \end{aligned}$$

This formula proves the following proposition.

PROPOSITION 1. *If an account grows with force of interest $\delta(t)$ over the interval $[t_0, t_1]$, then*

$$A(t_1) = e^{\int_{t_0}^{t_1} \delta(t)\, dt} A(t_0) \tag{20}$$

EXAMPLE 26. What is the annual effective rate of return on an account that grew at 4% interest per year for the first 2 years, a force of interest of $\delta_t = \frac{1}{2+t}$ for the next 3 years, and a discount rate of 4% for the last 2 years?

SOLUTION. From formulas (20) and (14) over the 7 year period P dollars will grow to

$$\begin{aligned}
&(1 - .04)^{-2} e^{\int_2^5 \frac{1}{2+t}\, dt} (1.04)^2 P \\
&= (1.085) e^{(\ln 7 - \ln 4)} (1.0816) P \\
&= (1.085)(\frac{7}{4})(1.0816) P \\
&= (2.054) P
\end{aligned}$$

The annual effective rate i is determined by solving the equation

$$(1 + i)^7 = 2.054$$

which yields $i = 10.82\%$.

Exercises

Calculate each of the following:

(1) You invest \$500 at 6.4% simple interest per year.
  (a) How much is in your account after 1 year?
  (b) How much is in your account after 5 years?
  (c) How much is in your account after 1/2 year?
  (d) How much is in your account after 5 years and 6 months?
(2) Redo Exercise 1 using *compound* interest instead of simple interest.
(3) Redo Exercise 1 assuming that the account earns *compound* interest for integral time periods and simple interest for fractional time periods.
(4) How long will it take for \$1,000 to accumulate to \$2000 at 5% annual compound interest?
(5) Value of \$1250 invested for 4 years at 5% simple interest.
(6) Value of \$1250 invested for 4 years at 5% compound interest.
(7) Value of \$624 invested for 3 years at 6% simple interest.
(8) Value of \$624 invested for 3 years at 6% compound interest .
(9) Value of \$624 invested for 3 years at 6% compounded quarterly.
(10) Value of \$3150 invested for 1 year at 4% simple interest.
(11) Value of \$3150 invested for 1 year at 4% compound interest.
(12) Value of \$8635 invested for 8 years at 5% compounded monthly.
(13) Amount you need to invest now to have \$5000 in 4 years if your account pays 6% simple interest.
(14) Amount you need to invest now to have \$5000 in 4 years if your account pays 6% compound interest .
(15) Amount you need to invest now to have \$5000 in 4 years if your account pays 6% compounded monthly.

(16) Amount you need to invest now to have \$100,000 in 15 years if your account pays 5% compounded monthly.
(17) Your account had \$486 in it on October 1, 1989 and \$743 in it on October 1, 1997. Assuming that no additions or withdrawals were made in the meantime, what annual effective interest rate accounts for the growth in the balance?
(18) What annual compound interest rate would account for a CD increasing in value from \$4000 on October 1, 1992 to a value of \$5431.66 on October 1, 1997? *Note:* In this problem you are commuting what is called the *annual rate of return* on the account.
(19) One bank is paying 4.8% compounded monthly. Another bank is paying 5% annual effective. Which is paying more?
(20) What is the annual effective rate on an investment that is paying 6% compounded quarterly?
(21) What is the present value of a payment of \$12,000 to made at the end of 6 years if the interest rate is 7% effective?
(22) What is the value in eight years (i.e. the future value) of a payment now of \$45,000 if the interest rate is 4.5% effective?
(23) George is borrowing \$20,000 and will pay 8.5% interest . He will pay off the loan in three annual installments, \$6,000 at the end of the first year, \$7,000 at the end of the second year, and a final payment at the end of the third year. What should the amount of the final payment be?
(24) Alice is saving money for a new car. She is putting the money in an account that earns 5.5% effective. At the beginning of this year, she deposited \$4,000, and the beginning of the second year, she expects to deposit \$5,000, and the beginning of the third year, she expects to deposit \$5,000 again. She anticipates the price of the car she wants to buy will be \$20,000. How much more will she need if she purchases the car at the beginning of the fourth year?
(25) Value of \$12,500 invested for 3 years at 6% compounded quarterly.
(26) Amount you need to invest now to have \$25,000 in 4 years if your account pays 6% compounded monthly.
(27) You can buy a \$25 Series EE savings bond for \$18.75; that is, you invest \$18.75 today and in 6 years, you get back \$25. What is the annual effective interest rate that you are getting.
(28) Amount you will have if you invest \$75 at the end of each month for 10 years if the account pays 7.5% compounded monthly.
(29) Amount you will have on January 31, 2000 if you invested \$150 at the end of each month starting in January 1996 in an account that pays 4.8% annual effective interest compounded monthly. Note: *You must first convert the given annual effective rate into a monthly rate.* Warning: Count the number of deposits very carefully!
(30) Amount you can borrow today if you are willing to pay \$300 at the end of each month for 5 years for a loan that charges 9% (compounded monthly).
(31) Amount you can borrow on January 15, 2001 if you are willing to pay \$1250 on the 15th of each month, starting February 15, 2001 with the final payment on July 15, 2013 if the interest rate is 6% compounded monthly.

(32) Amount you need to invest at the end of each month to have saved \$8000 at the end of 4 years if the account pays 5% compounded monthly.
(33) Amount of each payment if you borrow \$18000 on a 60 month auto loan that is charging 10.8% (compounded monthly).
(34) Amount of each monthly payment on a 15 year mortgage that charges 8.4% (compounded monthly) if the purchaser needs to borrow \$149,500.
(35) Amount of each monthly payment on a 30 year mortgage that charges 8.4% (compounded monthly) if the purchaser needs to borrow \$149,500. (Compare with previous problem.)
(36) You intend to start depositing \$100 into an account at the end of each month starting December 31, 2000. How long will it take you to save \$38,000 for your dream car if the account pays 6% compounded monthly?
(37) You intend to start depositing \$200 into an account at the end of each month starting December 31, 2000. How long will it take you to save \$38,000 for your dream car if the account pays 6% compounded monthly? (Compare with previous problem.)
(38) Bob Roarman will sell you a slightly used car for \$7,500 cash or you can buy the same car for 60 payments of only \$159 each (made at the end of each month). What rate of interest is Bob charging?
(39) Starting January 31, 1990, you saved \$100 at the end of each month in an account that paid 6% compounded monthly. Starting August 31, 1999, you increased the monthly contribution to \$150. How much will you have accumulated by December 31, 2000?
(40) Assuming that house prices have inflated at an average rate of 8% per year for each of the last 20 years, how much would a house that is currently worth \$100,000 have cost 20 years ago?
(41) On Jan. 1, 1998, I opened an account in a bank yielding 3.4% annual (compound) interest. On each subsequent Jan. 1, I made either a deposit or a withdraw according to the following chart. What is my balance on Jan. 1, 2003?

Year	1998	1999	2000	2001	2002
De./With.	2000	−700	600	−200	1500

(42) I borrow \$5,000 at 7.1% compound interest per year for 5 years with yearly payments starting at the end of the first year . My first 4 payments were: \$1,000, \$700, \$2,000, and \$1,000. What is my last payment?
(43) I want to be able to buy a \$25,000 car in ten years. If I can invest money at 8% annual effective interest, how much do I need to invest now?
(44) What is the present value of \$25,000 ten years from now at 8% interest?
(45) If I invest \$25,000 now at 8% annual effective interest, how much will I have in ten years?
(46) What is the future value of \$25,000 ten years from now at 8% interest?
(47) I buy a piano from Cheapside Music company on March 1, 1998. I pay \$1,000 immediately, then 3 more payments of \$1,000 on March 1 for each of the next 3 years. Finally, on March 1, 2002, I pay \$10,000. What was this deal worth to Cheapside on March 1, 1998, assuming that they can invest funds at 4% interest–i.e. what was the present value of all of my payments on March 1, 1998?

(48) In Exercise 47, assuming that Cheapside did invest all of my payments at 4% interest, how much did they have in their account on March 1, 2002?

(49) You just won the Publisher Clearing House grand prize which is $1,000,000 paid in 10 annual installments of $100,000 each. The first installment is paid immediately. Assuming that you can invest money at 3.9% annual effective interest, what is the present value of this prize?

(50) An account earning annual effective interest i had a balance of $ 1500 on Jan. 1, 2000 and a balance of $1773.63 on Jan. 1, 2001. The year's activity in the account consisted of (a) a deposit of $ 500 on June 1, a withdrawal of $ 600 on Aug. 1 and a deposit of $ 300 on Dec. 1. Approximate i using the technique from Example 6.

(51) An insurance company earns 7% on their investments. How much must they have on reserve (the present value of claims) on January 1, 2002 to cover the claims for the next 3 years, if they expect claims of $500,000 for 2002, $300,000 for 2003 and $250,000 for 2004. For sake of simplicity, assume that the claims are all paid on January 1 of the following year. Thus the 2002 claims are paid on January 1 of 2003, the 2003 claims are paid on January 1 of 2004, etc.

(52) On January 1, 1995, Susan put $5,000 into a bank account at Stingy's Bank which pays 3.5% compounded twice a year. On July1, 1996, she withdrew $500. On January 1, 1997 she deposited an additional $700. How much did she have on account on January 1, 2000?

(53) An interest rate of 6% compounded three times a year is equivalent to what rate of interest compounded twice a year.

(54) I have a bank account that initially has $6,000 invested at annual interest rate i. After 2 years I withdraw $4,000. After 2 more years (4 years total), I empty the account by withdrawing $3,000. Find i.

(55) Re-do Exercise 28 under the assumption that the account earns 7.5% ANNUAL EFFECTIVE. Hint: In this case, each month funds on deposit grow by a factor of $(1.075)^{(1/12)}$.

(56) Re-do exercise 36 under the assumption that the account earns 6% compounded quarterly.

(57) Re-do 37 under the assumption that the account earns 6% compounded quarterly.

(58) How much is a 15 year bond that pays quarterly coupons of $75 and has a redemption value of $5,000 worth, assuming that you can invest funds at 3%, compounded quarterly?–i.e. what is the present value at 3% compounded quarterly, of 60 quarterly payments of $75 plus a final payment of $5,000?

(59) How much can you borrow at 3% interest, compounded quarterly, if you can pay $75 each quarter for 15 years, together with a final "balloon" payment of $5,000?

(60) I deposit $1000 at the end of each month into an account that earns 6% interest compounded daily. How much do I have after 15 years? Hint: First compute how much interest you earn each monthit is NOT equal to .06/12. (Assume that each month has 365/12 days.) (Ans: $291,179.52)

(61) At the end of year 1 I deposit $1,000 into an account that is earning 6% interest compounded annually. At the end of each subsequent year I

deposit 2 % less than I did the previous year. Thus, for example, at the end of year 2 I deposit \$980 and at the end of year 3 I deposit \$960.40. How much do I have after the 30th deposit? (Ans: \$64975.09)

(62) I borrow \$500,000 for 30 years at 6% interest per year compounded monthly.
  (a) What are the monthly payments? (Ans: \$2997.75)
  (b) How much do I owe at the end of the 5th year? (Ans: \$465,271.78)
  (c) Immediately after the 5th year, I refinance the loan at 4% interest per year, compounded monthly, for 25 years. Find the new monthly payment. (Ans: \$2455.87)