Application: Area Between Curves

In this chapter we extend the notion of the area under a curve and consider the area of the region *between* two curves. To solve this problem requires only a minor modification of our point of view. We'll not need to develop any additional techniques of integration for the moment. However, we will also see that that we can think of the process used to find the area between two curves as an *accumulation process*, as we discussed earlier when we found the net distance traveled by integrating a velocity function. This theme of accumulation will be critical in the subsequent applications we carry out. Make sure you spend some time understanding this idea. Our objectives for this chapter are to

- Determine the area between two continuous curves using integration.
- Similarly, determine the area between two intersecting curves.
- Understand integration as an accumulation process.

6.1 *Area of a Region Between Two Curves*

With just a few modifications, we extend the application of definite integrals from finding the area of a region under a curve to finding the area of a region between two curves.

Consider two functions f and g that are continuous on the interval $[a, b]$.

In Figure 6.1, the graphs of both f and g lie above the x-axis, and the graph of g lies below the graph of f. There we can geometrically interpret the area of the region between the graphs as the area of the region under the graph of g subtracted from the area of the region under the graph of f, as shown in Figure 6.2

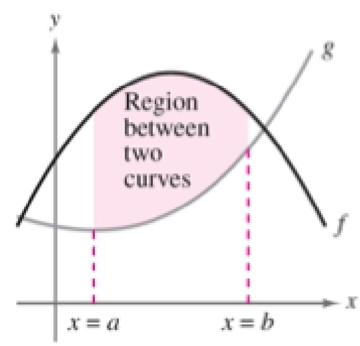


Figure 6.1: Find the area of the region between the curves f and g. (Diagram from Larson & Edwards)

The Riemann Sum Approach

Now let's step back and take a slightly different point of view on this. Remember that definite integrals are really limits of Riemann sums. So suppose we use a regular partition of $[a, b]$ into n equal subintervals of width Δx. We use the partition to subdivide the region between the two curves into n rectangles. We won't draw all of them, but rather we will draw a single *representative rectangle* (see Figure 6.3). The width of the rectangle is Δx and the height is $f(x_i) - g(x_i)$ where x_i is the right-hand endpoint of the ith subinterval.

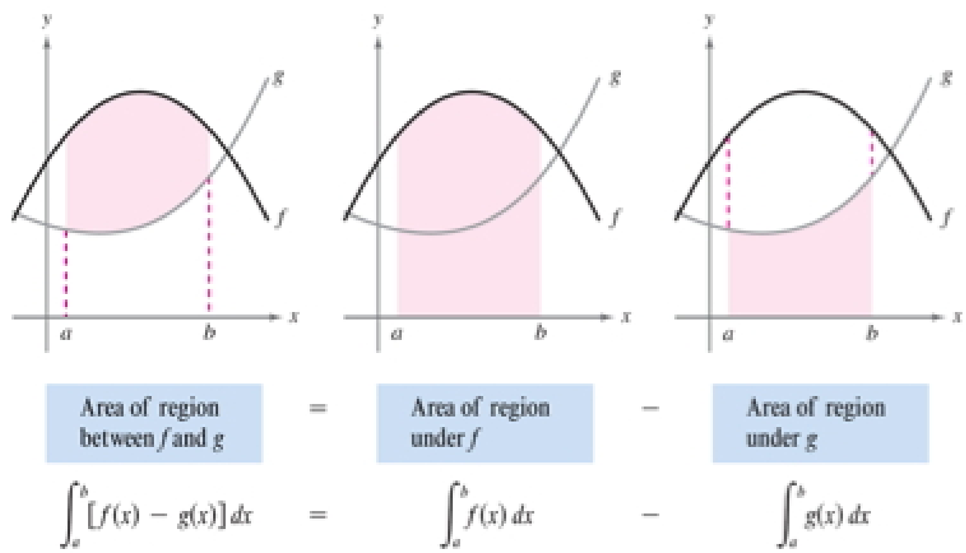


Figure 6.2: Find the area of the region between the curves f and g when both f and g lie above the x-axis and g lies below f. (Diagram from Larson & Edwards)

The area of the representative rectangle is

$$\text{height} \times \text{width} = [f(x_i) - g(x_i)]\Delta x.$$

We add up all the n rectangles to get an approximation to the total area between the curves:

$$\text{Approximate Area beween } f \text{ and } g = \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x.$$

To improve the approximation we take the limit as $n \to \infty$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x.$$

Now because both f and g are *continuous* we know that this limit exists and, in fact, equals a definite integral. Thus, the area of the given region is

$$\text{Area beween } f \text{ and } g = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x = \int_a^b [f(x) - g(x)]\,dx.$$

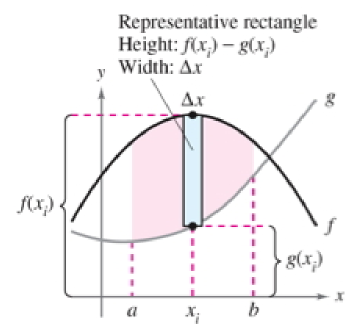


Figure 6.3: The area of the ith rectangle is $[f(x_i) - g(x_i)]\Delta x$. (Diagram from Larson & Edwards).

Let's summarize what we have found in a theorem.

THEOREM 6.1.1. If f and g are continuous on $[a, b]$ and $g(x) \le f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$\boxed{\text{Area beween } f \text{ and } g = \int_a^b [f(x) - g(x)]\,dx.}$$

Note: This area will always be non-negative.

Notice that the theorem gives the same answer as our earlier geometric argument in Figure 6.2 However, unlike in Figure 6.2, notice that the theorem does not say that both curves have to lie *above* the x-axis. The same integral

$$\int_a^b [f(x) - g(x)]\,dx$$

works as long as f and g are continuous on $[a, b]$ and $g(x) \leq f(x)$ for all x in the interval $[a, b]$. The reason this same integral remains valid when one or both curves dip below the x-axis is illustrated in Figure 6.4. The height of a representative rectangle is always $f(x) - g(x)$. This is the advantage of using Riemann sums and representative rectangles. It gives us a more general argument than a simple geometric one in this case.

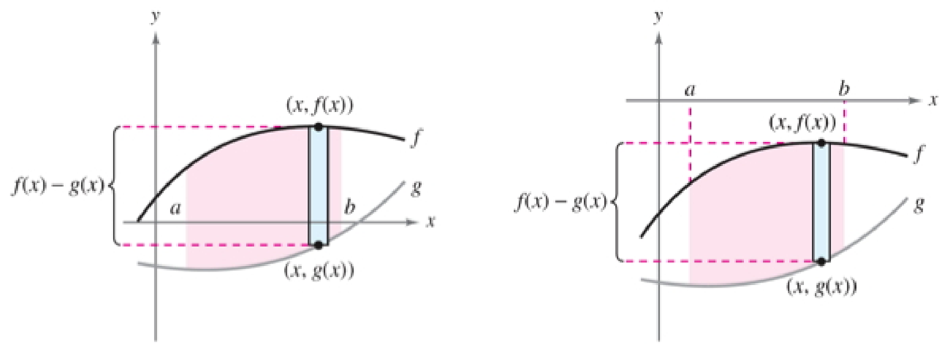


Figure 6.4: The height of a representative rectangle is $f(x) - g(x)$ whether or not one or both curves lie above or below the x-axis. (Diagram from Larson & Edwards)

Tip for Success

We will continue to use representative rectangles as we develop further applications. Drawing a figure with such representative rectangles will help you to write out the correct integral in these applications.

6.2 Examples

We now take a look at several examples.

EXAMPLE 6.2.1. Find the area of the region bounded by the graphs of $y = x^2 + 1$ and $y = x^3$ and the vertical lines $x = -1$ and $x = 1$.

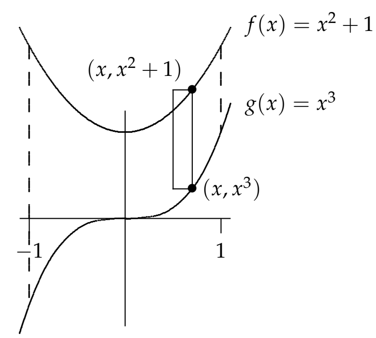


Figure 6.5: The area between f and g with a representative rectangle.

Solution. After quickly plotting the graphs we see that $x^2 + 1$ lies above x^3 on the interval. So let $f(x) = x^2 + 1$ and $g(x) = x^3$. Since both are continuous (polynomials) Theorem 6.1.1 applies and we have

$$\begin{aligned}\text{Area beween } f \text{ and } g = \int_a^b [f(x) - g(x)]\,dx &= \int_{-1}^1 [(x^2+1) - x^3]\,dx \\ &= \frac{x^3}{3} + x - \frac{x^4}{4}\bigg|_{-1}^1 \\ &= \left(\frac{1}{3} + 1 - \frac{1}{4}\right) - \left(-\frac{1}{3} - 1 - \frac{1}{4}\right) = \frac{8}{3}.\end{aligned}$$

Area Enclosed by Two Intersecting Curves

In Example 6.2.1 we found the area below one curve but above another curve on a given interval. A more common problem is a slight variation on this. Find the region enclosed by two intersecting curves. Usually the points of intersection are not provided and that becomes the first step in solving such a problem.

EXAMPLE 6.2.2 (Two Intersecting Curves). Find the area of the region enclosed by the graphs of $y = x^2 - 2$ and $y = x$. (In a typical problem, not even the graph is given.)

Solution. Let $f(x) = x^2 - 2$ and $g(x) = x$. First we find the intersections of the two graphs:

$$x^2 - 2 = x \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x+1)(x-2) = 0 \Rightarrow x = -1, 2.$$

Which curve lies above the other on the interval $[-1, 2]$? We can test an intermediate point. The point $x = 0$ is convenient: Notice $f(0) = -2$ and $g(0) = 0$. Or we can can quickly plot the graphs (see Figure 6.6) and see that x lies above $x^2 - 2$ on the interval $[-1, 2]$. Since both are continuous (polynomials) Theorem 6.1.1 applies and we have (notice that g is 'on top').

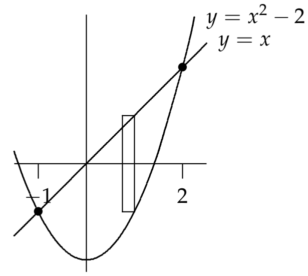


Figure 6.6: The area enclosed by $y = x^2 - 2$ and $y = x$ with a representative rectangle.

$$\begin{aligned}\text{Area enclosed by } g \text{ and } f = \int_a^b [g(x) - f(x)]\,dx &= \int_{-1}^{2} [x - (x^2 - 2)]\,dx \\ &= \frac{x^2}{2} - \frac{x^3}{3} + 2x\Big|_{-1}^{2} \\ &= \left(2 - \frac{8}{3} + 4\right) - \left(\frac{1}{2} + \frac{1}{3} - 2\right) = \frac{9}{2}.\end{aligned}$$

EXAMPLE 6.2.3 (Division into Two Regions). Find the area of the region enclosed by the graphs of $y = x^3$ and $y = x$.

Solution. Let $f(x) = x^3$ and $g(x) = x$. First we find the intersections of the two graphs:

$$\begin{aligned}x^3 = x \Rightarrow x^3 - x = 0 \Rightarrow x(x^2 - 1) = 0 &\Rightarrow x(x+1)(x-1) = 0 \\ &\Rightarrow x = -1, 0, 1.\end{aligned}$$

Since there are three points of intersection, we need to determine which curve lies above the other on each subinterval. On $[-1, 0]$, we can test an intermediate point $x = -\frac{1}{2}$: $f(-\frac{1}{2}) = -\frac{1}{8}$ and $g(-\frac{1}{2}) = -\frac{1}{2}$. So f lies above g. On $[0, 1]$, we test at the intermediate point $x = \frac{1}{2}$: $f(\frac{1}{2}) = \frac{1}{8}$ and $g(\frac{1}{2}) = \frac{1}{2}$. So g lies above f. Also we can quickly plot the graphs (see Figure 6.7) and the same behavior. Since both are continuous (polynomials) Theorem 6.1.1 applies. However, we will have to split the integration into two pieces since the top and bottom curves change at the point $x = 0$ in the interval $[-1, 1]$.

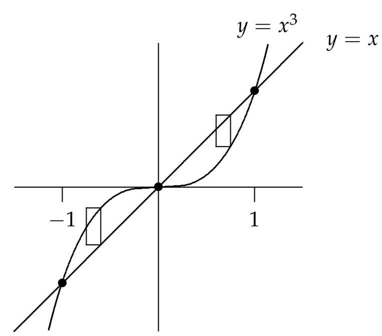


Figure 6.7: The area enclosed by $y = x^3$ and $y = x$. The top and bottom curve switch at $x = 0$. There are two different representative rectangles.

$$\begin{aligned}\text{Area enclosed by } g \text{ and } f &= \int_{-1}^{0} [x^3 - x]\,dx + \int_0^1 [x - x^3]\,dx \\ &= \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_{-1}^{0} + \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_{0}^{1} \\ &= \left([0] - \left[\frac{1}{4} - \frac{1}{2}\right]\right) - \left(\left[\frac{1}{2} - \frac{1}{4}\right] - [0]\right) = \frac{1}{2}.\end{aligned}$$

EXAMPLE 6.2.4. Find the area of the region enclosed by the graphs of $y = x\sqrt{x+1}$ and $y = 2x$.

Solution. Let $f(x) = x\sqrt{x+1}$ and $g(x) = 2x$. First we find the intersections of the two graphs:

$$x\sqrt{x+1} = 2x \Rightarrow x^2(x+1) = 4x^2 \Rightarrow x^3 - 3x^2 = 0 \Rightarrow x^2(x-3) = 0$$
$$\Rightarrow x = 0, 3.$$

To determine which curve lies above the other on On $[0, 3]$, we can test an intermediate point, say $x = 1$: $f(1) = \sqrt{2}$ and $g(1) = 2$. So g lies above f. We can quickly plot the graphs (see Figure 6.8). Since both are continuous Theorem 6.1.1 applies.

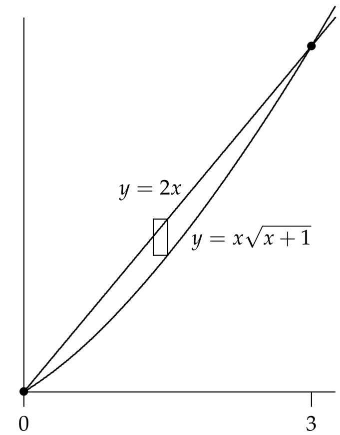


Figure 6.8: The area enclosed by $y = x\sqrt{x+1}$ and $y = 2x$ and a representative rectangle.

$$\text{Area enclosed by } g \text{ and } f = \int_0^3 [2x - x\sqrt{x+1}]\,dx = \int_0^3 2x\,dx - \int_0^3 x\sqrt{x+1}\,dx.$$

For the second integral we use the substitution

$$u = \sqrt{x+1} \Rightarrow u^2 = x+1 \Rightarrow u^2 - 1 = x \Rightarrow 2u\,du = dx$$

and change the limits:

$$\text{when } x = 0,\ u = \sqrt{0+1} = 1; \text{ when } x = 3,\ u = \sqrt{3+1} = 2.$$

So

$$\begin{aligned}
\int_0^3 2x\,dx - \int_0^3 x\sqrt{x+1}\,dx &= \int_0^3 2x\,dx - \int_1^2 (u^2-1)\cdot u \cdot 2u\,du \\
&= x^2\Big|_0^3 - \int_1^2 2u^4 - u^2\,du \\
&= (9-0) - \left(\frac{2u^5}{5} - \frac{u^3}{3}\right)\Big|_1^2 \\
&= 9 - \left(\left[\frac{64}{5} - \frac{8}{3}\right] - \left[\frac{1}{5} - \frac{1}{3}\right]\right) = \frac{19}{15}.
\end{aligned}$$

Variations

Here are some additional 'variations on the theme' of Theorem 6.1.1.

EXAMPLE 6.2.5 (Multiple Curves, Multiple Regions). Find the area of the region enclosed by the graphs of $y = 8 - x^2$, $y = 7x$, and $y = 2x$ in the first quadrant.

Solution. This time there are three curves to contend with. Since the curves are relatively simple (an upside-down parabola and two lines through the origin, it is relatively easy to make a sketch of the region. See Figure 6.9. Let $f(x) = 8 - x^2$, $g(x) = 7x$, and $h(x) = 2x$. A wedge-shaped region is determined by all three curves. Notice that the 'top' curve of the region switches from $g(x)$ to $f(x)$. We find the intersections of the pairs of graphs:

$f(x) = g(x) \Rightarrow 8 - x^2 = 7x \Rightarrow x^2 + 7x - 8 = 0 \Rightarrow (x-1)(x+8) = 0 \Rightarrow x = 1$ (not -8).

$f(x) = h(x) \Rightarrow 8 - x^2 = 2x \Rightarrow x^2 + 2x - 8 = 0 \Rightarrow (x-2)(x+4) = 0 \Rightarrow x = 2$ (not -4).

$g(x) = h(x) \Rightarrow 7x = 2x \Rightarrow 5x = 0 \Rightarrow x = 0.$

The region is thus divided into two subregions and the graph gives the relative positions of the curves. Since all the functions are continuous Theorem 6.1.1 applies.

$$\begin{aligned}
\text{Area enclosed by } f, g, \text{ and } h &= \int_0^1 [7x - 2x]\,dx + \int_1^2 [(8 - x^2) - 2x\,dx \\
&= \int_0^1 [5x]\,dx + \left(8x - \frac{x^3}{3} - x^2\right)\Big|_1^2 \\
&= \left(\frac{5x^2}{2}\right)\Big|_0^1 + \left(\left[16 - \frac{8}{3} - 4\right] - \left[8 - \frac{1}{3} - 1\right]\right) \\
&= \left(\frac{5}{2} - 0\right) + \left(\frac{8}{3}\right) = \frac{31}{6}.
\end{aligned}$$

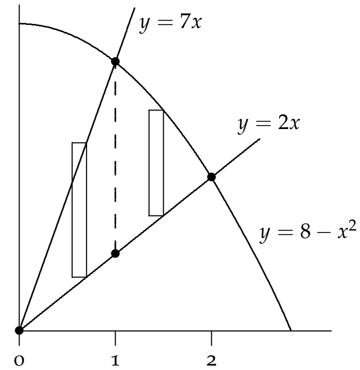
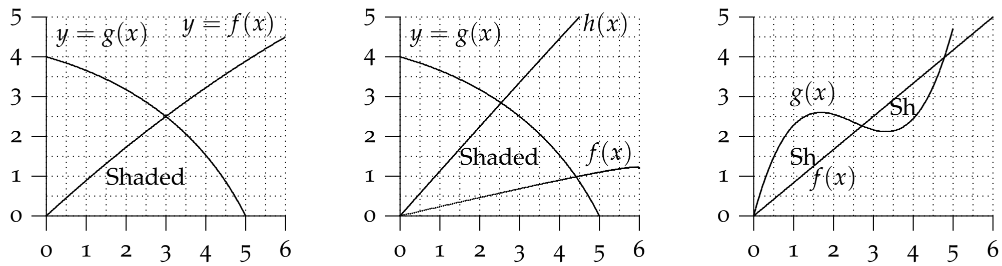


Figure 6.9: The area enclosed by $y = 8 - x^2$, $y = 7x$, and $y = 2x$ in the first quadrant. There are two representative rectangles because the top curve changes.

YOU TRY IT 6.1. Set up the integrals using the functions $f(x)$, $g(x)$, and $h(x)$ and their points of intersection that would be used to find the shaded areas in the three regions below.



YOU TRY IT 6.2. Sketch the regions for each of the following problems before finding the areas.

(*a*) Find the area enclosed by the curves $y = x^3$ and $y = x^2$.

(*b*) Find the area enclosed by the curves $y = x^3 + x$ and $y = 3x^2 - x$.

(*c*) Find the area between the curves $f(x) = \cos x + \sin x$ and $g(x) = \cos x - \sin x$ over $[0, 2\pi]$.

ANSWER TO YOU TRY IT 6.2. (a) $1/12$; (b) $1/2$; (c) 8.

YOU TRY IT 6.3. Sketch each region before finding its area:

(*a*) The area in the first quadrant enclosed by $y = \cos x$, $y = \sin x$, and the y axis.

(*b*) The area enclosed by $y = x^3$ and $y = \sqrt[3]{x}$.

(*c*) The area enclosed by $y = x^3 + 1$ and $y = (x + 1)^2$.

(*d*) Harder integration: The area enclosed by $y = x\sqrt{2x + 3}$ and $y = x^2$.

ANSWER TO YOU TRY IT 6.3. (a) $\sqrt{2} - 1$; (b) 1; (c) $37/12$; (d) $\frac{6}{5}\sqrt{3} + \frac{26}{15}$.

WEBWORK: Click to try Problems 75 through 80. Use GUEST login, if not in my course.

EXAMPLE 6.2.6. Find the area of the region in the first quadrant enclosed by the graphs of $y = 1$, $y = \ln x$, and the x- and y-axes.

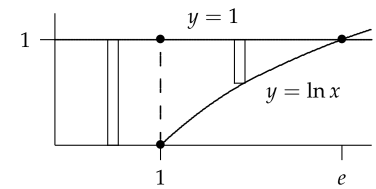


Figure 6.10: The region in the first quadrant enclosed by the graphs of $y = 1$, $y = \ln x$, and the x- and y-axes. There are two representative rectangles because the bottom curve changes.

Solution. It is easy to sketch the region. See Figure 6.10. The curve $y = \ln x$ intersects the x-axis at $x = 1$ and the line $y = 1$ at $x = e$. Notice that the 'bottom' curve of the region switches from x-axis to $y = \ln x$ at $x = 1$. The region is divided into two subregions (one is a square!) and the graph gives the relative positions of the curves. Since both the functions are continuous Theorem 6.1.1 applies.

$$\text{Area} = \int_0^1 1\,dx + \int_1^e 1 - \ln x\,dx$$

We can rewrite the integral in a more convenient way. Notice that the area the we are trying to find is really just the rectangle of height 1 minus the area under $y = \ln x$ on the interval $[1, e]$. (Yet another way of saying this is that we are splitting $\int_1^e 1 - \ln x\,dx$ into two integrals $\int_1^e 1\,dx$ and $\int_1^e -\ln x\,dx$ and then combining the two integrals $\int_0^1 1\,dx + \int_1^e 1\,dx$ into one leaving $-\int_1^e \ln x\,dx$.) We get

$$\text{Area} = \int_0^e 1\,dx - \int_1^e \ln x\,dx = e + ???$$

The problem is that we do not know an antiderivative for $\ln x$. So we need another way to attack the problem. We describe this below.

6.3 *Point of View: Integrating along the y-axis*

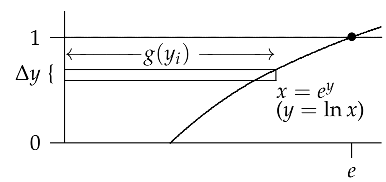


Figure 6.11: The region in the first quadrant enclosed by the graphs of $y = 1$, $y = \ln x$, and the x- and y-axes. There are two representative rectangles because the bottom curve changes.

Reconsider Example 6.2.6 and change our point of view. Suppose that we drew our representative rectangles horizontally instead of vertically as in Figure 6.11. The integration now takes place along the y-axis on the interval $[0, 1]$. Using inverse functions, the function $y = \ln x$ is viewed as $x = g(y) = e^y$. Now the 'width' of a representative rectangle is Δy and the (horizontal) 'height' of the ith such rectangle is given by $g(y_i)$.

As we saw earlier in the term with integration along the x-axis, since g is continuous, the exact area of the region is given by

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} g(y_i)\Delta y = \int_c^d g(y)\,dy.$$

In our particular case, the interval $[c, d] = [0, 1]$ along the y-axis. The function $g(y) = e^y$. So the area of the region is in Figure 6.11 (or equivalently 6.10) is

$$\text{Area} = \int_c^d g(y)\,dy = \int_0^1 e^y\,dy = e^y\Big|_0^1 = e - 1.$$

We can generalize the argument we just made and state the equivalent of Theorem 6.1.1 for finding areas between curves by integrating along the y-axis.

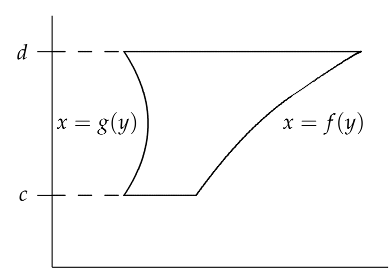


Figure 6.12: The region bounded by the graphs of $x = f(y)$ and $x = g(y)$ and the horizontal lines $y = c$ and $y = d$.

THEOREM 6.3.1 (Integration along the y-axis). If $f(y)$ and $g(y)$ are continuous functions on $[c, d]$ and $g(y) \le f(y)$ for all y in $[c, d]$, then the area of the region bounded by the graphs of $x = f(y)$ and $x = g(y)$ and the horizontal lines $y = c$ and $y = d$ is

$$\boxed{\text{Area beween } f \text{ and } g = \int_c^d [f(y) - g(y)]\,dy.}$$

6.4 More Examples

Here are a few more examples of area calculations, this time involving integrals along the y-axis.

EXAMPLE 6.4.1. Find the area of the region in the first quadrant enclosed by the graphs of $x = y^2$ and $x = y + 2$.

Solution. The intersections of the two curves are easily determined:

$$y^2 = y + 2 \Rightarrow y^2 - y - 2 = 0 \Rightarrow (y+1)(y-2) = 0 \Rightarrow y = -1, 2.$$

It is easy to sketch the region since one curve is a parabola and the other a straight line. See Figure 6.13. Since both the functions are continuous Theorem 6.3.1 applies.

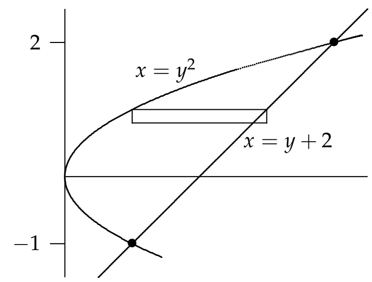


Figure 6.13: The region in the first quadrant enclosed by the graphs of $x = y^2$ and $x = y + 2$.

$$\begin{aligned}\text{Area beween } f \text{ and } g = \int_c^d [f(y) - g(y)]\,dy &= \int_{-1}^{2} [y + 2 - y^2]\,dx \\ &= \frac{y^2}{2} + 2y - \frac{y^3}{3}\Big|_{-1}^{2} \\ &= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = \frac{9}{2}.\end{aligned}$$

EXAMPLE 6.4.2. Find the area of the region enclosed by the graphs of $y = \arctan x$, the x-axis, and $x = 1$.

Solution. The region is given to us in a way that simply requires sketching $\arctan x$. The area is described by

$$\text{Area} = \int_0^1 \arctan x\,dx.$$

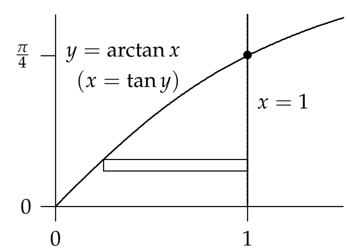


Figure 6.14: The region in the first quadrant enclosed by the graphs of $y = \arctan x$, the x-axis, and $x = 1$. Integrate along the y-axis and use $x = \tan y$.

However, we don't yet know an antiderivative for the arctangent function. We could develop that now (or look it up in a reference table), or we can switch that axis of integration. Notice that

$$y = \arctan x \iff x = \tan y.$$

The old limits were $x = 0$ and $x = 1$, so the new limits for y are $\arctan 0 = 0$ and $\arctan 1 = \frac{\pi}{4}$. Notice the function $x = 1$ is the 'top' curve and $x = \tan y$ is the 'bottom' curve (reading from left to right). Since both the functions are continuous Theorem 6.3.1 applies.

$$\begin{aligned}\text{Area} = \int_0^{\pi/4} 1 - \tan y\,dy &= y - \ln|\sec y|\Big|_0^{\pi/4} \\ &= \left(\frac{\pi}{4} - \ln\sqrt{2}\right) - (0 - \ln 1) = \frac{\pi}{4} - \frac{1}{2}\ln 2.\end{aligned}$$

EXAMPLE 6.4.3. Find the area of the region enclosed by the three graphs $y = x^2$, $y = \frac{8}{x}$, and $y = 1$. (The region is enclosed by all three curves at the same time.)

Solution. Determine where the three curves meet:
$x^2 = \frac{8}{x} \Rightarrow x^3 = 8 \Rightarrow x = 2.$

$x^2 = 1 \Rightarrow x = 1$ (not -1, see Figure 6.15).
$\frac{8}{x} = 1 \Rightarrow x = 8$.

Notice from Figure 6.15 that if we were to find the area by integrating along the x-axis, we would need to split the integral into two pieces because the top curve of the region changes at $x = 2$. We can avoid the two integrations and all of the corresponding evaluations by integrating along the y-axis. We need to convert the functions to functions of x in terms of y:

$$y = x^2 \Rightarrow x = \sqrt{y} \text{ and } y = \frac{8}{x} \Rightarrow x = \frac{8}{y}.$$

Remember to change the limits: At $x = 2$, $y = 4$ and at $x = 1$ or $x = 8$, $y = 1$. Notice the function $x = \frac{8}{y}$ is the 'top' curve and $x = \sqrt{y}$ is the 'bottom' curve (reading from left to right). Since both the functions are continuous Theorem 6.3.1 applies.

$$\text{Area} = \int_1^4 \frac{8}{y} - \sqrt{y}\,dy = 8\ln|y| - \frac{2x^{3/2}}{3}\Big|_1^4$$
$$= \left(8\ln 4 - \frac{16}{3}\right) - \left(\ln 1 - \frac{2}{3}\right) = 8\ln 4 - \frac{14}{3}.$$

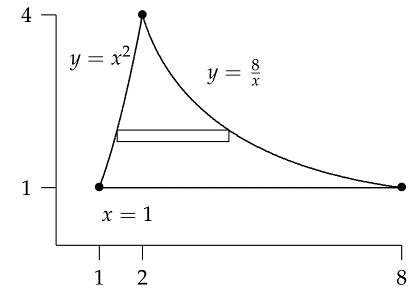
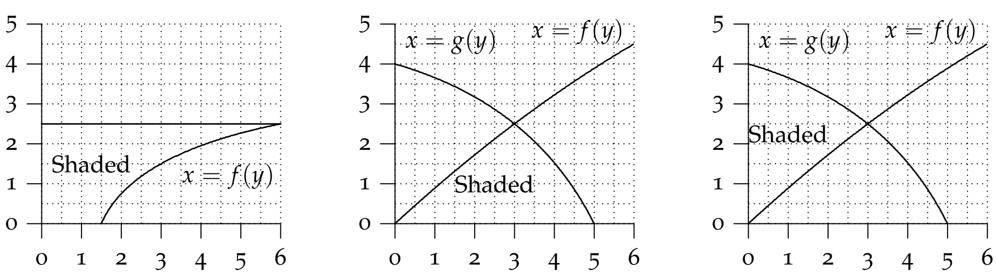


Figure 6.15: The region enclosed by the three graphs $y = x^2$, $y = \frac{8}{x}$, and $y = 1$. Integrating along the y-axis uses only a single integral.

YOU TRY IT 6.4. Redo Example 6.4.3 using integration along the x-axis. Verify that you get the same answer. Which method seemed easier to you?

YOU TRY IT 6.5. Set up the integrals that would be used to find the shaded areas bounded by the curves in the three regions below *using integration along the y-axis*. You will need to use appropriate notation for inverse functions, e.g., $x = f^{-1}(y)$.



YOU TRY IT 6.6. Sketch the regions for each of the following problems before finding the areas.

(*a*) Find the area enclosed by $x = y^2 + 1$ and $x = 2y + 9$. Integrate along the y-axis.

(*b*) Along the y-axis (more in the next problem). The area enclosed by $y = x - 4$ and $y^2 = 2x$.

(*c*) Find the area in the first quadrant enclosed by the curves $y = \sqrt{x-1}$, $y = 3 - x$, the x-axis, and the y-axis by using definite integrals along the y-axis.

(*d*) Find the area of the wedge-shaped region *below* the curves $y = \sqrt{x-1}$, $y = 3 - x$, and above the x-axis. Integrate along either axis: your choice!

ANSWER TO YOU TRY IT 6.6. (a) 36; (b) 18; (c) 10/3, if you get 9/2, you have the wrong region; (d) 7/6, not the same as (b)).

Solution. We do part (c). The curves are easy to sketch; remember $y = \sqrt{x-1}$ is the graph of $y = \sqrt{x}$ shifted to the right 1 unit. To integrate along the y axis, solve for x in each equation.

$$y = \sqrt{x-1} \Rightarrow y^2 = x - 1 \Rightarrow x = y^2 + 1$$
$$y = 3 - x \Rightarrow x = 3 - y$$

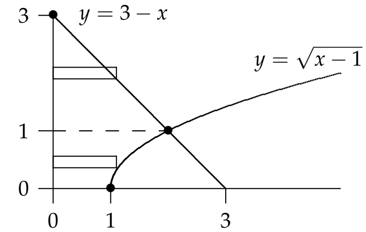


Figure 6.16: The region enclosed by $y = \sqrt{x-1}$, $y = 3 - x$, the x-axis and the y-axis.

These curves intersect when

$$y^2 - 1 = 3 - y \Rightarrow y^2 + y - 2 = (y-1)(y+2) = 0 \Rightarrow y = 1 \text{ (not } -2).$$

Of course $x = 3 - y$ intersects the y-axis at 3. So

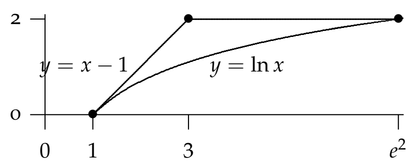
$$\text{Area} = \int_0^1 y^2 + 1\,dy + \int_1^3 3 - y\,dy = \frac{y^3}{3} + y\Big|_0^1 + 3y - \frac{y^2}{2}\Big|_1^3 = \frac{10}{3}.$$

YOU TRY IT 6.7. Sketch the region (use your calculator?) and find the area under $y = \arcsin x$ on the interval $[0, 1]$. Hint: switch axes.

ANSWER TO YOU TRY IT 6.7. $\pi/2 - 1$.

WEBWORK: Click to try Problems 81 through 82. Use GUEST login, if not in my course.

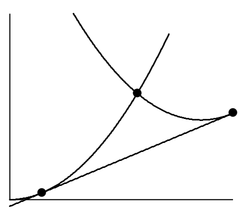
YOU TRY IT 6.8. **[From a test in a previous year]** Consider the region bounded by $y = \ln x$, $y = 2$, and $y = x - 1$ shown below. Find the area of this region.



ANSWER TO YOU TRY IT 6.8. $e^2 - 5$.

YOU TRY IT 6.9. Find the area of the region in the first quadrant enclosed by $y = 9 - x$, $y = x\sqrt{x+1}$, and the y-axis. Hint: The two curves meet at the point $(3, 6)$.

YOU TRY IT 6.10. Find the region enclosed by the three curves $y = x^2$, $y = x^2 - 12x + 48$, and $y = 2x - 1$. You will need to find three intersections.

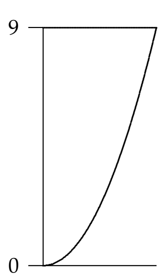


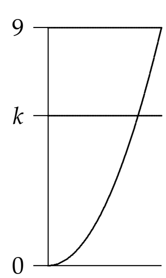
ANSWER TO YOU TRY IT 6.10. 18.

YOU TRY IT 6.11. Extra Fun.

(a) (Easy.) The region R in the first quadrant enclosed by $y = x^2$, the y-axis, and $y = 9$ is shown in the graph on the left below. Find the area of R.

(b) A horizontal line $y = k$ is drawn so that the region R is divided into two pieces of equal area. Find the value of k. (See the graph on the right below). Hint: It might be easier to integrate along the y-axis now.





ANSWER TO YOU TRY IT 6.11. (b) $(13.5)^{2/3}$.

YOU TRY IT 6.12. Let R be the region enclosed by $y = x$, $y = \frac{2}{x+1}$, and the y axis in the first quadrant. Find its area. *Be careful to use the correct region: One edge is the y axis.*

YOU TRY IT 6.13. Two ways

(*a*) Find the area in the first quadrant enclosed by $y = \sqrt{x-1}$, the line $y = 7 - x$, and the x-axis by integrating along the x-axis. Draw the figure.

(*b*) Do it instead by integrating along the y-axis.

(*c*) Which method was easier for you?

ANSWER TO YOU TRY IT 6.13. $\frac{22}{3}$.

YOU TRY IT 6.14. Find the area of the region R enclosed by $y = \sqrt{x}$, $y = \sqrt{12 - 2x}$, and the x-axis in the first quadrant *by integrating along the y axis*. Be careful to use the correct region: *One edge is the x axis.*

ANSWER TO YOU TRY IT 6.14. 8.

YOU TRY IT 6.15 (Good Problem, Good Review). Find the area in the first quadrant bounded by $y = x^2$, $y = 2$, the tangent to $y = x^2$ at $x = 2$ and the x-axis. Find the tangent line equation. Draw the region. Does it make sense to integrate along the y-axis? Why? (Answer: $\frac{2}{3}$.)

ANSWER TO YOU TRY IT 6.15. $\frac{2}{3}$.

YOU TRY IT 6.16 (Extra Credit). Find the number k so that the horizontal line $y = k$ divides the area enclosed by $y = \sqrt{x}$, $y = 2$, and the y axis into two equal pieces. Draw it first. This is easier if you integrate along the y axis.

YOU TRY IT 6.17 (Real Extra Credit). There is a line $y = mx$ through the origin that divides the area between the parabola $y = x - x^2$ and the x axis into two equal regions. Find the slope of this line. Draw it first. The answer is not a simple number.

6.5 An Application of Area Between Curves to Economics: Lorenz Curves

During the 2012 presidential election, there was much talk about "the 1%" meaning "the wealthiest 1% of the people in the country," and the rest of us, "we are the 99%." Such labels were intended to highlight the income and wealth inequality in the United States. Consider the following from the *New York Times*.

- The top 1 percent of earners in a given year receives just under a fifth of the country's pretax income, about double their share 30 years ago. (from `http://www.nytimes.com/2012/01/15/business/the-1-percent-paint-a-more-nuanced-portrait-of-the-rich.html`
- The wealthiest 1 percent took in about 16 percent of overall income—8 percent of the money earned from salaries and wages, but 36 percent of the income earned from self-employment.
- They controlled nearly a third of the nation's financial assets (investment holdings) and about 28 percent of non-financial assets (the value of property, cars, jewelry, etc.). (See `http://economix.blogs.nytimes.com/2012/01/17/measuring-the-top-1-by-wealth-not-income/`)

Statements such as "x% of the population has y% of the wealth in the country," actually describe points on what economists call a Lorenz curve.

DEFINITION 6.5.1. The **Lorenz Curve** $L(x)$ gives the proportion of the total income earned by the lowest proportion x of the population. It can also be used to show distribution of assets (total wealth, rather than income). Economists consider it to be a measure of social inequality. It was developed by Max O. Lorenz in 1905 for representing inequality of the wealth distribution.

EXAMPLE 6.5.2. $L(0.25) = 0.10$ would mean that the poorest 25% of households earns 10% of the total income. $L(0.90) = 0.55$ would mean that the poorest 90% earns 55% of the total income. Equivalently, the richest 10% households earn 45% of the total income.

Focusing on wealth rather than income, if the top 1% households control about a third of the nation's financial assets as the *New York Times* indicates, then the bottom 99% control about two-thirds of the nation's wealth. This would be represented on the Lorenz curve by the point $L(0.99) = 0.67$.

Basic Properties of the Lorenz Curve. There are a couple of simple observations about the Lorenz curve.

- The domain of Lorenz curve is $[0, 1]$; any percent is expressed as a decimal in this interval. For the same reason, the range of Lorenz curve is $[0, 1]$. So the graph of a Lorenz curve lies inside the unit square in the first quadrant.
- $L(0) = 0$ since no money is earned by 0 households.
- $L(1) = 1$ because all of the income is earned by the entire population.
- $L(x)$ is an increasing function. More of the total income is earned by more of the households.

Extreme Cases. Two extreme cases that help us understand the Lorenz curve.

- *Absolute Equality of Income*. Everyone earns exactly the same amount of money. In this situation $L(x) = x$, that is, x% of the people earn x% of the income.
- *Absolute Inequality of Income*. Nobody earns any income except one person (who earns it all). In this situation $L(x) = \begin{cases} 0, & \text{for } 0 \le x < 1 \\ 1, & \text{for } x = 1. \end{cases}$

Let's think about this a bit. The lowest paid $x\%$ of the population cannot earn more than $x\%$ of the income, therefore, $L(x) \leq x$. (If they did, the remaining $(1-x)\%$ would earn less than $(1-x)\%$ of the total income and would be lower paid than the $x\%$.) This means that the Lorenz curve $L(x)$ lies at or under the diagonal line $y = x$ in the unit square. A typical Lorenz curve is shown to the right in Figure 6.18.

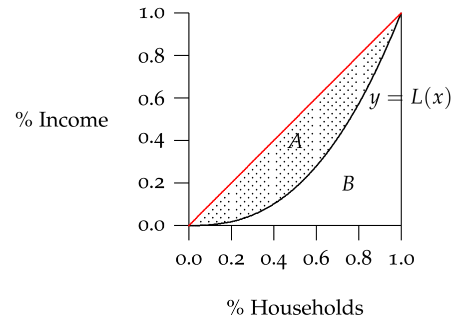


Figure 6.17: A Lorenz curve $L(x)$ compared to the diagonal line which represents perfect income equality. The region A represents the difference between perfect income equality and actual income distribution. The larger A is, the more unequal the income distribution is.

The information in a Lorenz curve can be summarized in a single measure called the Gini index.

DEFINITION 6.5.3. Let A be the area *between* the line $y = x$ representing perfect income equality and the Lorenz curve $y = L(x)$. (This is the shaded area in Figure 6.18.) Let B denote the region *under* the Lorenz curve. Then the **Gini index** is

$$\boxed{G = \frac{A}{A+B}.}$$

The area $A + B = \frac{1}{2}$ because it is half of the unit square. So

$$G = \frac{A}{\frac{1}{2}} = 2A.$$

Using definite integrals to calculate the shaded area of A in Figure 6.18, we find

$$\boxed{G = 2A = 2\int_0^1 x - L(x)\,dx.}$$

YOU TRY IT 6.18. Using properties of the integral prove that

$$\boxed{G = 1 - 2\int_0^1 L(x)\,dx.}$$

Notice that when there is perfect income equality, $L(x) = x$ is the diagonal and we have $A = 0$, so $G = 2A = 0$. When there is absolute income inequality, then $B = 0$ and $A = \frac{1}{2}$, so $G = 1$. G always falls in the range of 0 to 1 with values closer to 0 representing more equally distributed income.

Problems Several of these problems are taken almost word-for-word directly from `http://f10.middlebury.edu/MATH0122C/Lorenz%20Curves.pdf`.

1. A very simple function used to model a Lorenz curve is $L(x) = x^p$, where $p \geq 1$.

(*a*) Check that any such function $L(x) = x^p$, where $p \geq 1$ is a valid Lorenz curve. (That is, check that $L(0) = 0$, $L(1) = 1$ and $L(x) \leq x$ on $[0, 1]$.)

(*b*) Use a graphing calculator or computer to examine the graphs of $L(x) = x^p$ on the interval $[0, 1]$ for $p = 1.2, 1.5, 2, 3$, and 4. Which value of p gives the most equitable distribution of income? The least?

2. A country in Northern Europe has a Lorenz curve for household incomes given by the function

$$L(x) = \frac{e^x - 1}{e - 1} \quad \text{for } 0 \leq x \leq 1.$$

(*a*) Show that this function is a valid candidate to be a Lorenz curve. (That is, check that $L(0) = 0$, $L(1) = 1$ and $L(x) \leq x$.)

(*b*) Determine $L(0.5)$ and interpret what it means.

(*c*) What is the Gini coefficient for this country? Give your interpretation and commentary.

(*d*) The CIA website reports the Gini index for the distribution of family income in the United States to be 0.45. Roughly how does the income inequality you computed compare to that in the U.S. at the present time?

3. Find the Gini index corresponding to the Lorenz curve $f(x) = x^3$.

4. Find the Gini index corresponding to the Lorenz curve $f(x) = \frac{1}{4}x + \frac{3}{4}x^3$.

5. Prove that a Lorenz curve of the form $L(x) = x^p$ has a Gini index of $G = \frac{p-1}{p+1}$.

6. The CIA website reports the Gini index for the distribution of family income in the United States to be 0.45.

(*a*) Determine the number p so that the Gini index is 0.45 if the Lorenz curve has the form of a power function $f(x) = x^p$

(*b*) According to this model, how much of the family income is earned by the top 5% of families?

7. One type of function often used to model Lorenz curves is

$$L(x) = ax + (1-a)x^p.$$

(*a*) Suppose that $a = \frac{1}{4}$ and that the Gini index for the distribution of wealth in a country is known to be $\frac{9}{16}$. Find the value of p that fits this situation.

(*b*) According to this model, how much of the wealth is owned by the wealthiest 5% of the population?

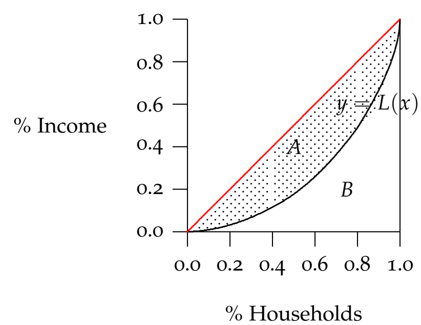


Figure 6.18: The Lorenz curve for the United States based on data from `http://assets.opencrs.com/rpts/RS20811_20121113.pdf`

6.6 *An Application of Area Between Curves: Consumer Surplus*

The material up to Figure 6.19 is taken almost word-for-word directly from `http://tutor2u.net/economics/revision-notes/as-markets-consumer-surplus.html`.

In this note we look at the importance of willingness to pay for different goods and services. When there is a difference between the price that you actually pay in the market and the price or value that you place on the product, then the concept of consumer surplus is useful.

Defining consumer surplus

Consumer surplus is a measure of the welfare that people gain from the consumption of goods and services, or a measure of the benefits they derive from the exchange of goods.

DEFINITION 6.1. Consumer surplus is the difference between the total amount that consumers are *willing and able to pay* for a good or service (indicated by the demand curve) and the total amount that they *actually do pay* (i.e., the actual market price for the product). The level of consumer surplus is shown by the area below the demand curve and above the ruling market price line as illustrated in Figure 6.19. **Producer surplus** is the difference between the total amount that producers of a good receive and the minimum amount that they would be willing to accept for the good (lighter shading).

Consumer and producer surpluses are relatively easy to calculate if the supply and demand curves are straight lines. However, in realistic models of the economy supply and demand generally do not behave in this way.

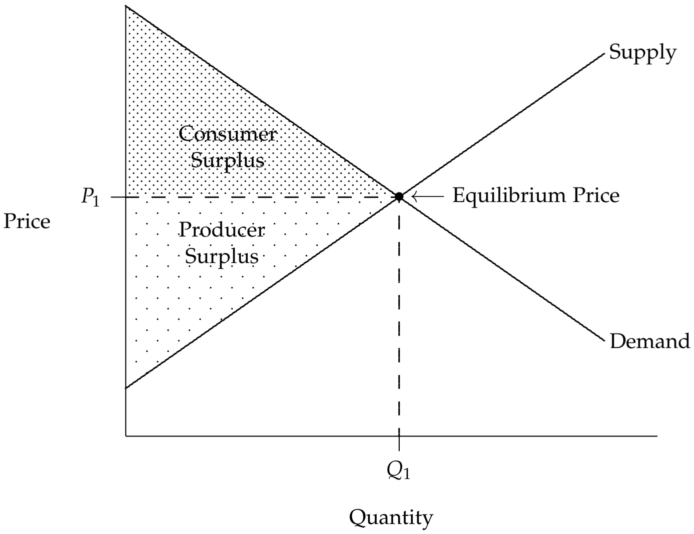


Figure 6.19: **Consumer surplus** is the difference between the total amount that consumers are willing and able to pay for a good or service and the total amount that they actually do pay (darker shading). **Producer surplus** is the difference between the total amount that producers of a good receive and the minimum amount that they would be willing to accept for the good (lighter shading).

To determine either surplus for a product, we first need to determine the **equilibrium price** for that product, that is, the price for a good at which the suppliers are willing to supply an amount of the good equal to the amount demanded by consumers. Typically, as in Figure 6.19, consumers will demand more of a good only if the price (vertical axis) is lowered. So the demand curve has a negative slope. On the other hand, producers will be willing to make and sell more of a good if the price paid increases. So the supply curve generally has a positive slope. The two curves meet at some quantity Q_1 and some corresponding price P_1.

There are examples of so-called 'backward-bending' supply curves, where the supply curve increases for awhile and then when a particular price is reached suppliers actually are willing to 'produce' less supply so the curve continues upward but bends back to the left. Extra credit if you can think of and justify an example where this might be true.

EXAMPLE 6.6.1. Suppose that the demand curve for students wanting to attend Hobart and William Smith each year is Demand Price $= \frac{180}{q+2}$ where q is measured in thousands of students and demand price is measured in thousands of dollars. (E.g., no students are interested in attending if the tuition is $\frac{180}{0+2} = 90$ thousand dollars, whereas 4 thousand students are interested in attending if the tuition is $\frac{180}{4+2} = 30$ thousand dollars.) The Colleges are willing to accept students according to the formula for Supply Price $= \frac{56q}{q+1}$. Determine the equilibrium price, make a quick sketch of the graphs, and then determine the consumer surplus.

Solution. The equilibrium price is where the demand price equals the supply price,

$$\frac{180}{q+2} = \frac{56q}{q+1} \Rightarrow 56q^2 + 112q = 180q + 180 \Rightarrow 14q^2 - 17q - 45 = 0.$$

Using the quadratic formula we find that the only positive root is $q = 2.5$ and the corresponding price is $p = \frac{180}{2.5+2} = 40$ thousand dollars.

The consumer surplus is the area below the demand curve and above the equilibrium price $p = 40$. So

$$\begin{aligned}\text{Consumer Surplus} = \int_0^{2.5} \frac{180}{q+2} - 40\,dq &= 180\ln|q+2| - 40q\Big|_0^{2.5} \\ &= (180\ln 4.5 - 100) - (180\ln 2 - 0) \\ &\approx 45.96743891\end{aligned}$$

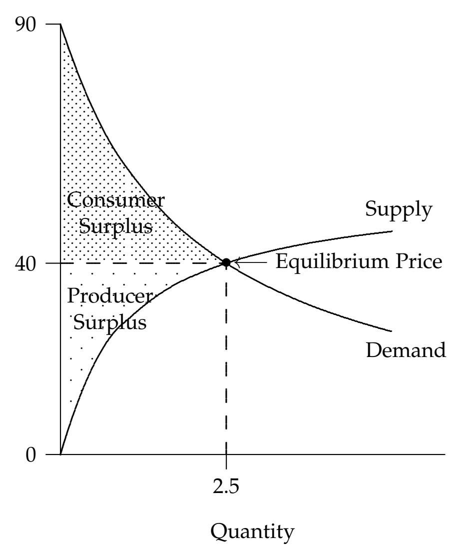


Figure 6.20: The supply and demand curves for enrollment at HWS (in thousands of dollars and thousands of students).

The consumer surplus is approximately \$45,967,438.91 (since the units are thousands times thousands).

YOU TRY IT 6.19 (Extra Credit). Find the producer surplus for the situation in Example 6.6.1.

YOU TRY IT 6.20 (Extra Credit). Suppose that the Federal Government wants to increase the number of students able to attend college and offers every such student \$2,000 per year (in the form of a 'coupon' payable to the the student's college). This means that students have another \$2,000 to spend per year, so they are willing to accept 'higher prices'. Their new Demand Price $= \frac{180}{q+2} + 2$. (The $+2$ represents the extra \$2,000.) The Colleges' supply price remains the same.

(*a*) How many students now would attend the Colleges? You will need to use the quadratic formula. Note: Round your answers for q to the nearest one-thousandth (and your final answer to the nearest student.)

(*b*) Find the new equilibrium price (tuition) at HWS. Did it go up \$2000?

(*c*) Find the new consumer surplus. Did it go up or down?

(*d*) Find the new producer surplus. Did it go up or down?